Iranian Journal of Fuzzy Systems
Volume 20, Number 7, (2023), pp. 71-84
Original paper

Completions of $\top$-quasi-Cauchy spaces

G. Jäger[1]

[1] *University of Applied Sciences Stralsund, Stralsund, Germany.*

gunther.jaeger@hochschule-stralsund.de

Abstract

In the category of $\top$-quasi-Cauchy spaces, completeness and completion can be studied in a non-symmetric framework encompassing $\top$-quasi-uniform (limit) spaces. Based on constructions by E.E. Reed in the category of Cauchy spaces and, recently, by L. Reid and G. Richardson in the category of $\top$-Cauchy spaces, we give a family of completions for a non-complete $\top$-quasi Cauchy space. As particular instances we study pretopological and topological completions of $\top$-quasi-Cauchy spaces.

Keywords: Fuzzy topology, pair $\top$-filter, Cauchy pair $\top$-filter, $\top$-quasi-Cauchy space, $\top$-quasi-uniform space, $\top$-quasi-uniform limit space, L-metric space.

1 Introduction

Since Reed's fundamental paper [16] it is well-known that Cauchy spaces [13] form an appropriate framework for studying completeness and for the construction of completions of uniform limit spaces. Lattice-valued counterparts of Reed's approaches were initiated in [17] and [18]. In order to capture also non-symmetric spaces, like $\top$-quasi-uniform spaces [21] or $\top$-quasi-uniform limit spaces [9, 12], in [11] we introduced the category of $\top$-quasi-Cauchy spaces. Such a space has as structure a set of pair $\top$-filters satisfying suitable axioms. It was demonstrated in [11] that $\top$-quasi-uniform (limit) spaces have an underlying $\top$-quasi-Cauchy structure and that completeness and completions can be defined and studied in the category of $\top$-quasi-Cauchy spaces.

This paper continues the study of completions in this non-symmetric framework by extending approaches of Reed [16] and of Reid and Richardson [17] to the non-symmetric case. We define in particular pretopological and topological $\top$-quasi-Cauchy spaces and construct completions that preserve these properties.

The paper is organized as follows. In the second section we collect the necessary theory and results about L-sets and $\top$-filters that are needed in the paper. Section 3 reviews the main definitions and concepts about $\top$-quasi-Cauchy spaces. In Section 4, a general approach for constructing completions is given and Sections 5 treats the special case of pretopological completions. In Section 6 we introduce a diagonal axiom and show that this axioms carries over from a space satisfying the diagonal axiom to its pretopological completion. We show that the pretopological completion of Section 5 is the coarsest completion that preserves the diagonal axiom and that the finest completion of a $\top$-quasi-Cauchy space also preserves the diagonal axiom. Finally, in the last section, we draw some conclusions.

2 Preliminaries

A *commutative and integral quantale* $\mathsf{L} = (L, \leq, *)$ consists of a complete lattice $(L, \leq)$ with order relation $\leq$ and with distinct top and bottom elements $\top \neq \bot$, and a commutative semigroup operation on L for which the top element of L acts as the unit, that is, $\alpha * \top = \alpha$ for all $\alpha \in L$, and which is distributive over arbitrary joins, that is, we have $(\bigvee_{i\in J} \alpha_i) * \beta = \bigvee_{i\in J}(\alpha_i * \beta)$, see e.g. [8]. Important examples of such quantales are the unit interval $[0, 1]$ with a

Corresponding Author: G. Jäger
Received: February 2023; Accepted: Invited paper.
https://doi.org/10.22111/IJFS.2023.7631

left-continuous *t-norm* [19], or *Lawvere's quantale*, the interval $[0,\infty]$ with the opposite order and addition $\alpha*\beta=\alpha+\beta$, extended by $\alpha+\infty=\infty+a=\infty$, see e.g. [4]. A further important example is the quantale $(\Delta^+,\leq,*)$ of distance distribution functions [4] which is used in the theory of probabilistic metric spaces [19].

In a quantale, we can define an *implication* by defining, for $\alpha,\beta\in L$, $\alpha\to\beta=\bigvee\{\delta\in L\ :\ \alpha*\delta\leq\beta\}$. The implication is characterized by $\delta\leq\alpha\to\beta$ if, and only if, $\delta*\alpha\leq\beta$.

We list some of the properties of the implication that we will use later on.

Lemma 2.1. [7] *Let $\alpha,\beta,\gamma,\delta,\alpha_j,\beta_j\in L$ for $j\in J$. The following assertions hold.*

1. *If $\alpha\leq\beta$, then $\alpha\to\gamma\geq\beta\to\gamma$ and $\gamma\to\alpha\leq\gamma\to\beta$.*
2. $(\alpha\to\beta)*(\beta\to\delta)\leq(\alpha*\gamma)\to(\beta*\delta)$.
3. $\bigwedge_{j\in J}(\alpha_j\to\beta)=(\bigvee_{j\in J}\alpha_j)\to\beta$.
4. $\bigwedge_{j\in J}(\alpha_j\to\beta_j)\leq(\bigvee_{j\in J}\alpha_j)\to(\bigvee_{j\in J}\beta_j)$.

The set of *L-sets in X*, or, more precisely, *L-subsets of X*, $a,b,c,...$ is denoted by $L^X=\{a:X\longrightarrow L\}$. In particular, we denote for $A\subseteq X$, the *characteristic function* $\top_A\in L^X$ by $\top_A(x)=\top$ if $x\in A$ and $\top_A(x)=\bot$ otherwise. The lattice operations are extended pointwisely from L to L^X. If $a\in L^X$, $b\in L^Y$ and $\varphi:X\longrightarrow Y$ is a mapping, then we define $\varphi(a)\in L^Y$ by $\varphi(a)(y)=\bigvee_{x:\varphi(x)=y}a(x)$ for $y\in Y$ and $\varphi^{\leftarrow}(b)=b\circ\varphi$.

For $a\in L^X$ and $b\in L^Y$ we define $a\otimes b\in L^{X\times Y}$ by $(a\otimes b)(x,y)=a(x)*b(y)$ for all $(x,y)\in X\times Y$ and for L-sets $b,c\in L^{X\times X}$ we define the *composition*, $b\circ c\in L^{X\times X}$, by $b\circ c(x,y)=\bigvee_{z\in X}c(x,z)*b(z,y)$ for all $x,y\in X$.

On L^X the *fuzzy inclusion order* [2] is defined by $[b,d]=\bigwedge_{x\in X}(b(x)\to d(x))$ for $b,d\in L^X$. We collect properties that we will often use later.

Lemma 2.2. *Let $a,a',b,b',c\in L^X$, $d\in L^Y$ and let $\varphi:X\longrightarrow Y$ be a mapping. Then*

(i) *$a\leq b$ if and only if $[a,b]=\top$;*

(ii) *$a\leq a'$ implies $[a',b]\leq[a,b]$ and $b\leq b'$ implies $[a,b]\leq[a,b']$;*

(iii) $[a,c]\wedge[b,c]=[a\vee b,c]$;

(iv) $[\varphi(a),d]=[a,\varphi^{\leftarrow}(d)]$;

(v) $[a,b]*[b,c]\leq[a,c]$.

Definition 2.3. [5, 6, 20] *A subset $\mathbb{F}\subseteq L^X$ is called a* $\top$-filter *on X if*

$(TF1)$ *$\bigvee_{x\in X}b(x)=\top$ for all $b\in\mathbb{F}$;*

$(TF2)$ *$a,b\in\mathbb{F}$ implies $a\wedge b\in\mathbb{F}$;*

$(TF3)$ *$\bigvee_{b\in\mathbb{F}}[b,d]=\top$ implies $d\in\mathbb{F}$.*

We denote the set of all $\top$-filters on X by $\mathsf{F}_{\mathsf{L}}^{\top}(X)$.

Example 2.4. *If $a(x)=\top$ for some $x\in X$, then $[a]=\{b\in L^X\ :\ a\leq b\}$ is a $\top$-filter. We write, for $x\in X$, for short $[x]=[\top_{\{x\}}]=\{a\in L^X\ :\ a(x)=\top\}$ and call this $\top$-filter a* point $\top$-filter.

Definition 2.5. [5, 6, 20] *A subset $\mathbb{B}\subseteq L^X$ is called a* $\top$-filter basis *if*

$(TB1)$ *$\bigvee_{x\in X}b(x)=\top$ for all $b\in\mathbb{B}$;*

$(TB2)$ *$a,b\in\mathbb{B}$ implies $\bigvee_{c\in\mathbb{B}}[c,a\wedge b]=\top$.*

For a $\top$-filter basis $\mathbb{B}$, $[\mathbb{B}]=\{a\in L^X\ :\ \bigvee_{b\in\mathbb{B}}[b,a]=\top\}$ is a $\top$-filter, the $\top$-filter generated by $\mathbb{B}$. For example, for an ordinary filter $\mathcal{F}$ on X, the set $\{\top_F\ :\ F\in\mathcal{F}\}$ is a $\top$-filter basis.

For the following definitions and properties, we refer to [6, 20, 9]. The set $\mathsf{F}_{\mathsf{L}}^{\top}(X)$ is ordered by $\mathbb{F}\leq\mathbb{G}$ if $\mathbb{F}\subseteq\mathbb{G}$. The meet of a non-empty family $(\mathbb{F}_j)_{j\in J}$ of $\top$-filters on X is given by $\bigwedge_{j\in J}\mathbb{F}_j=\bigcap_{j\in J}\mathbb{F}_j$ and a $\top$-filter base for $\bigwedge_{j\in J}\mathbb{F}_j$ is given by $\{\bigvee_{j\in J}f_j\ :\ f_j\in\mathbb{F}_j\ \forall j\in J\}$.

For a ⊤-filter $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$ and a mapping $\varphi : X \longrightarrow Y$, the set $\{\varphi(f) \ : \ f \in \mathbb{F}\}$ is a ⊤-filter basis and we call the generated ⊤-filter, $\varphi(\mathbb{F}) \in \mathsf{F}_{\mathsf{L}}^{\top}(Y)$, the *image of* $\mathbb{F}$ *under* φ. We then have $\varphi([x]) = [\varphi(x)]$. For a ⊤-filter $\mathbb{G} \in \mathsf{F}_{\mathsf{L}}^{\top}(Y)$, the set $\{\varphi^{\leftarrow}(g) \ : \ g \in \mathbb{G}\}$ is a ⊤-filter basis if, and only if, $\bigvee_{y\in\varphi(X)} g(y) = \top$ for all $g \in \mathbb{G}$. In this case, we denote the generated ⊤-filter by $\varphi^{\leftarrow}(\mathbb{G}) \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$ and call it the *inverse image of* $\mathbb{G}$ *under* φ. We also say that $\varphi^{\leftarrow}(\mathbb{G})$ *exists*. In the special case of a subset $A \subseteq X$ and the embedding $i_A : A \longrightarrow X$, we denote for $\mathbb{G} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$, in case of existence, $i_A^{\leftarrow}(\mathbb{G}) = \mathbb{G}_A$ and call it the *trace of* $\mathbb{G}$ *on* A.

For $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\top}(X), \mathbb{G} \in \mathsf{F}_{\mathsf{L}}^{\top}(Y)$ we define $\mathbb{F} \otimes \mathbb{G}$ as the ⊤-filter on $X \times U$ generated by the ⊤-filter basis $\{f \otimes g \ : \ f \in \mathbb{F}, g \in \mathbb{G}\}$.

Finally, for ⊤-filters Φ, Ψ on $X \times X$, we define the *composition* $\Phi \circ \Psi$ as the ⊤-filter generated by the ⊤-filter basis $\{b \circ c \ : \ b \in \Phi, c \in \Psi\}$, whenever $\bigvee_{(x,y)\in X\times X} b \circ c(x,y) = \top$ for all $b \in \Phi, c \in \Psi$.

For notions from category theory, we refer to [1].

3 ⊤-quasi Cauchy spaces

Following [15] and [21], we call, for $\mathbb{F}, \mathbb{G} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$, $(\mathbb{F}, \mathbb{G})$ a *pair ⊤-filter* if for all $f \in \mathbb{F}, g \in \mathbb{G}$ we have $\bigvee_{x\in X} f(x) * g(x) = \top$. If this condition is satisfied, we also say that $\mathbb{F}$ and $\mathbb{G}$ are *linked*. The set of all pair ⊤-filters on X is denoted by $\mathsf{PF}_{\mathsf{L}}^{\top}(X)$. For $(\mathbb{F}, \mathbb{G}), (\mathbb{F}', \mathbb{G}') \in \mathsf{PF}_{\mathsf{L}}^{\top}(X)$ we define $(\mathbb{F}, \mathbb{G}) \leq (\mathbb{F}', \mathbb{G}')$ if $\mathbb{F} \leq \mathbb{F}'$ and $\mathbb{G} \leq \mathbb{G}'$. Then $\bigwedge_{j\in J}(\mathbb{F}_j, \mathbb{G}_j) = (\bigwedge_{j\in J} \mathbb{F}_j, \bigwedge_{j\in J} \mathbb{G}_j)$.

We showed in [11] that a mapping $\varphi : X \longrightarrow X'$ extends to a mapping (that we again denote by φ) $\varphi : \mathsf{PF}_{\mathsf{L}}^{\top}(X) \longrightarrow \mathsf{PF}_{\mathsf{L}}^{\top}(X')$ defining $\varphi((\mathbb{F}, \mathbb{G})) = (\varphi(\mathbb{F}), \varphi(\mathbb{G}))$. It is straightforward to show that for $(\mathbb{F}_j, \mathbb{G}_j) \in \mathsf{PF}_{\mathsf{L}}^{\top}(X)$ for all $j \in J$, we have $\varphi(\bigwedge_{j\in J}(\mathbb{F}_j, \mathbb{G}_j)) = \bigwedge_{j\in J} \varphi((\mathbb{F}_j, \mathbb{G}_j))$.

A ⊤-quasi-Cauchy space $(X, \mathcal{CP})$ is a pair of a set X and a ⊤-*quasi-Cauchy structure* $\mathcal{CP} \subseteq \mathsf{PF}_{\mathsf{L}}^{\top}(X)$ satisfying the axioms

(TQC1) $([x], [x]) \in \mathcal{CP}$ for all $x \in X$;

(TQC2) $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ and $(\mathbb{F}, \mathbb{G}) \leq (\mathbb{F}', \mathbb{G}') \in \mathsf{PF}_{\mathsf{L}}^{\top}(X)$ implies $(\mathbb{F}', \mathbb{G}') \in \mathcal{CP}$;

(TQC3) if $(\mathbb{F}, \mathbb{G}), (\mathbb{F}', \mathbb{G}') \in \mathcal{CP}$ and $\mathbb{F}$ and $\mathbb{G}'$ as well as $\mathbb{F}'$ and $\mathbb{G}$ are linked, then $(\mathbb{F} \wedge \mathbb{F}', \mathbb{G} \wedge \mathbb{G}') \in \mathcal{CP}$.

A mapping $\varphi : (X, \mathcal{CP}) \longrightarrow (X', \mathcal{CP}')$ is called *Cauchy continuous* if $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ implies $\varphi((\mathbb{F}, \mathbb{G})) \in \mathcal{CP}'$. We denote the category which has as objects the ⊤-quasi-Cauchy spaces and as morphisms the Cauchy continuous mappings by ⊤-QChy.

A motivating example is given by ⊤-*quasi-uniform spaces* (*fuzzy* L-*quasi-uniform space* [6], *probabilistic quasi-uniform space* [21]), that is, by pairs $(X, \mathcal{U})$ of a set X and a ⊤-filter $\mathcal{U} \in \mathsf{F}_{\mathsf{L}}^{\top}(X \times X)$ with the properties (TU1) $\mathcal{U} \leq [(x, x)]$ for all $x \in X$, (TU2) $\mathcal{U} \leq \mathcal{U} \circ \mathcal{U}$. A mapping $\varphi : (X, \mathcal{U}) \longrightarrow (X', \mathcal{U}')$ between two ⊤-quasi-uniform spaces $(X, \mathcal{U}), (X', \mathcal{U}')$ is called *uniformly continuous* if $\mathcal{U}' \leq (\varphi \times \varphi)(\mathcal{U})$. We define $\mathcal{CP}^{\mathcal{U}}$ as the set of *Cauchy pair ⊤-filters* in $(X, \mathcal{U})$, that is, a pair ⊤-filter $(\mathbb{F}, \mathbb{G})$ is in $\mathcal{CP}^{\mathcal{U}}$ if $\mathcal{U} \leq \mathbb{G} \otimes \mathbb{F}$, [21]. Then $(X, \mathcal{CP}^{\mathcal{U}})$ is a ⊤-quasi-Cauchy space and a uniformly continuous mapping $\varphi : (X, \mathcal{U}) \longrightarrow (X', \mathcal{U}')$ is Cauchy continuous as a mapping $\varphi : (X, \mathcal{CP}^{\mathcal{U}}) \longrightarrow (X', \mathcal{CP}^{\mathcal{U}'})$. This example encompasses L-*metric spaces* (also called *continuity spaces* [4], L-*categories* [14, 8] or L-*preordered sets* [22]), see [11].

We showed in [11] that ⊤-QChy is a well-fibred and topological category. Initial constructions are done as follows. For a source $(\varphi_j : X \longrightarrow (X_j, \mathcal{CP}_j))_{j\in J}$ we define the initial ⊤-quasi-Cauchy structure $\mathcal{CP}$ on X by $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ if for all $j \in J$ we have $\varphi_j((\mathbb{F}, \mathbb{G})) \in \mathcal{CP}_j$.

Important examples of initial constructions are *subspaces* and *product spaces*. For subspaces, let $(X, \mathcal{CP})$ be a ⊤-quasi-Cauchy space and let $A \subseteq X$. The initial ⊤-quasi-Cauchy structure on A for the embedding $i_A : A \longrightarrow X, a \longmapsto a$, $\mathcal{CP}_A$, is defined, for $(\mathbb{F}, \mathbb{G}) \in \mathsf{PF}_{\mathsf{L}}^{\top}(A)$, by

$$(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}_A \iff ([\mathbb{F}], [\mathbb{G}]) = (i_A(\mathbb{F}), i_A(\mathbb{G})) = i_A((\mathbb{F}, \mathbb{G})) \in \mathcal{CP},$$

and $(A, \mathcal{CP}_A)$ is called a *subspace* of $(X, \mathcal{CP})$.

For product spaces, let $(X_j, \mathcal{CP}_j)$ be ⊤-quasi-Cauchy spaces for all $j \in J$. The initial ⊤-quasi-Cauchy structure on the Cartesian product $\prod_{j\in J} X_j$ with respect to the projects $pr_i : \prod_{j\in J} X_j \longrightarrow X_i$, $\pi - \mathcal{CP}$, is defined by

$$(\mathbb{F}, \mathbb{G}) \in \pi - \mathcal{CP} \iff pr_j((\mathbb{F}, \mathbb{G})) \in \mathcal{CP}_j \ \forall j \in J,$$

and $(\prod_{j\in X} X_j, \pi - \mathcal{CP})$ is called the *product space.*

We showed in [11] that if the quantale L has the property that $\bigvee A = \top$ for $A \subseteq L$ implies $\bigvee_{\alpha\in A} \alpha * \alpha = \top$, then the category $\top$-QChy is Cartesian closed.

For a $\top$-quasi-Cauchy space $(X, \mathcal{CP})$ we say that a pair $\top$-filter $(\mathbb{F},\mathbb{G}) \in \mathsf{PF}^{\top}_{\mathsf{L}}(X)$ *converges to* $x \in X$, and we write $x \in q^{\mathcal{CP}}((\mathbb{F},\mathbb{G}))$, if $(\mathbb{F}\wedge[x], \mathbb{G}\wedge[x]) \in \mathcal{CP}$. We note that convergent pair $\top$-filters are Cauchy pair $\top$-filters by (TQCS2).

For a $\top$-quasi-uniform space $(X,\mathcal{U})$ Yue and Fang [21] defined that a pair $\top$-filter $(\mathbb{F},\mathbb{G})$ converges to x if and only if $[x]\otimes\mathbb{F} \geq \mathcal{U}$ and $\mathbb{G}\otimes[x] \geq \mathcal{U}$. In [10] it was shown that this requirement is equvalent to $(\mathbb{F}\wedge[x], \mathbb{G}\wedge[x])$ being a Cauchy pair $\top$-filter. Hence convergence in $(X,\mathcal{U})$ coincides with convergence in $(X, \mathcal{CP}^{\mathcal{U}})$.

It is shown in [11] that this convergence concept has the following properties. Let $(X, \mathcal{CP})$ be a $\top$-quasi-Cauchy space. For all $(\mathbb{F},\mathbb{G}), (\mathbb{F}',\mathbb{G}') \in \mathsf{PF}^{\top}_{\mathsf{L}}(X)$ and all $x \in X$ we have:

(TQL1) $x \in q^{\mathcal{CP}}(([x],[x]))$;

(TQL2) $(\mathbb{F},\mathbb{G}) \leq (\mathbb{F}',\mathbb{G}')$ implies $q^{\mathcal{CP}}((\mathbb{F},\mathbb{G})) \subseteq q^{\mathcal{CP}}((\mathbb{F}',\mathbb{G}'))$;

(TQL3) $q^{\mathcal{CP}}((\mathbb{F},\mathbb{G})) \cap q^{\mathcal{CP}}((\mathbb{F}',\mathbb{G}')) \subseteq q^{\mathcal{CP}}((\mathbb{F}\wedge\mathbb{F}', \mathbb{G}, \mathbb{G}'))$.

Furthermore, a Cauchy continuous mapping $\varphi : (X, \mathcal{CP}) \longrightarrow (X', \mathcal{CP}')$ between the $\top$-quasi-Cauchy spaces $(X, \mathcal{CP})$ and $(X', \mathcal{CP}')$ is *continuous* as a mapping from $(X, q^{\mathcal{CP}})$ to $(X', q^{\mathcal{CP}'})$ in the sense that $x \in q^{\mathcal{CP}}((\mathbb{F},\mathbb{G}))$ implies $\varphi(x) \in q^{\mathcal{CP}'}((\varphi(\mathbb{F}), \varphi(\mathbb{G})))$.

For a $\top$-quasi-Cauchy space $(X, \mathcal{CP})$ and a subset $A \subseteq X$ the *closure of* A, $\overline{A} = \overline{A}^{\mathcal{CP}}$, is defined by $x \in \overline{A}$ if there is a pair $\top$-filter $(\mathbb{F},\mathbb{G}) \in \mathsf{PF}^{\top}_{\mathsf{L}}(A)$ such that $x \in q^{\mathcal{CP}}(([\mathbb{F}],[\mathbb{G}]))$, see [11].

We call a $\top$-quasi-Cauchy space $(X, \mathcal{CP})$ *separated* if $x, y \in q^{\mathcal{CP}}((\mathbb{F},\mathbb{G}))$ implies $x = y$.

We say that a pair $\top$-filter $(\mathbb{F},\mathbb{G})$ is *convergent* in $(X, \mathcal{CP})$ if is converges to some $x \in X$. Otherwise, we call $(\mathbb{F},\mathbb{G})$ *non-convergent.* We denote $\mathcal{N}_X = \{(\mathbb{F},\mathbb{G}) \in \mathcal{CP} \text{ non-convergent}\}$. A $\top$-quasi-Cauchy space $(X, \mathcal{CP})$ is called *complete* if every $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ is convergent, that is, if $\mathcal{N}_X = \emptyset$.

A completion $((X^+, \mathcal{CP}^+), \phi)$ of a non-complete $\top$-quasi-Cauchy space $(X, \mathcal{CP})$ is a complete $\top$-quasi-Cauchy space $(X^+, \mathcal{CP}^+)$ and a dense Cauchy embedding $\phi : (X, \mathcal{CP}) \longrightarrow (X^+, \mathcal{CP}^+)$. This means that ϕ is injective and that we have $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ if, and only if, $(\phi(\mathbb{F}), \phi(\mathbb{G})) \in \mathcal{CP}^+$ and that $\overline{\phi(X)}^{\mathcal{CP}^+} = X^+$. When the embedding ϕ is clear from the context, for simplicity we also call the space $(X^+, \mathcal{CP}^+)$ a completion of $(X, \mathcal{CP})$.

For two completions $((X^+, \mathcal{CP}^+), \phi), ((X^\sim, \mathcal{CP}^\sim), \psi)$ of $(X, \mathcal{CP})$ we call $((X^+, \mathcal{CP}^+), \phi)$ *finer than* $((X^\sim, \mathcal{CP}^\sim), \psi)$, equivalently $((X^\sim, \mathcal{CP}^\sim), \phi)$ *coarser than* $((X^+, \mathcal{CP}^+), \psi)$, if there is a Cauchy continuous mapping $h : (X^+, \mathcal{CP}^+) \longrightarrow (X^\sim, \mathcal{CP}^\sim)$ such that $h \circ \phi = \psi$.

In [11] we constructed a completion of a non-complete $\top$-quasi-Cauchy space $(X, \mathcal{CP})$ as follows. Let $(\mathbb{F},\mathbb{G}), (\mathbb{F}',\mathbb{G}') \in \mathcal{CP}$. We define the relation

$$(\mathbb{F},\mathbb{G}) \sim (\mathbb{F}',\mathbb{G}') \iff (\mathbb{F}\wedge\mathbb{F}', \mathbb{G}\wedge\mathbb{G}') \in \mathcal{CP}.$$

It is clear that for $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ we have $(\mathbb{F},\mathbb{G}) \sim ([x],[x])$ if and only if $x \in q^{\mathcal{CP}}((\mathbb{F},\mathbb{G}))$. Furthermore, the relation $\sim$ is an equivalence relation [11] and we denote the equivalence class of $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ by $\langle(\mathbb{F},\mathbb{G})\rangle$.

We define now $X^* = X \cup \{\langle(\mathbb{F},\mathbb{G})\rangle : (\mathbb{F},\mathbb{G}) \in \mathcal{N}_X\}$ and we denote $j : X \longrightarrow X^*, x \longmapsto x$ the embedding injection of X into X^*. We define $\mathcal{CP}^* \subseteq \mathsf{PF}^{\top}_{\mathsf{L}}(X^*)$ as follows. We have $(\mathbb{H},\mathbb{K}) \in \mathcal{CP}^*$ if there is $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ convergent such that $j(\mathbb{F}) \leq \mathbb{H}$ and $j(\mathbb{G}) \leq \mathbb{K}$ or if there is $(\mathbb{F},\mathbb{G}) \in \mathcal{N}_X$ such that $j(\mathbb{F}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \leq \mathbb{H}$ and $j(\mathbb{G}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \leq \mathbb{K}$.

It is shown in [11] that $((X^*, \mathcal{CP}^*), j)$ is a completion of $(X, \mathcal{CP})$ and that it is the finest completion of $(X, \mathcal{CP})$. Furthermore, with $(X, \mathcal{CP})$ separated, also $(X^*, \mathcal{CP}^*)$ is separated [11].

4 Completions via selection mappings

In this section, we adapt an approach by Reid and Richardson [17] to our non-symmetric setting. The origin of the idea goes back to Reed [16]. Most of the proofs parallel the corresponding proofs in [17] and we include them to make the paper self-contained.

Let $(X, \mathcal{CP})$ be a $\top$-quasi-Cauchy space. A *selection mapping* is a mapping $\alpha : X^* \longrightarrow \mathcal{CP}$ such that $\alpha(x) = ([x],[x])$ for $x \in X$ and $\alpha(\langle(\mathbb{F},\mathbb{G})\rangle) \in \langle(\mathbb{F},\mathbb{G})\rangle$ for $(\mathbb{F},\mathbb{G}) \in \mathcal{N}_X$. For simplicity, we use the notation $\alpha(\langle(\mathbb{F},\mathbb{G})\rangle) = (\mathbb{F}_\alpha, \mathbb{G}_\alpha)$. We also write $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1(x) = \alpha_2(x) = [x]$ for $x \in X$ and $\alpha_1(\langle(\mathbb{F},\mathbb{G})\rangle) = \mathbb{F}_\alpha$ and $\alpha_2(\langle(\mathbb{F},\mathbb{G})\rangle) = \mathbb{G}_\alpha$.

For $a \in L^X$ we define two L-sets $a^{\alpha_1}, a^{\alpha_2} \in L^{X^*}$ by $a^{\alpha_1}(x) = a(x)$ if $x \in X$ and $a^{\alpha_1}(\langle(\mathbb{F},\mathbb{G})\rangle) = \bigvee_{f\in\mathbb{F}_\alpha}[f,a]$ for $(\mathbb{F},\mathbb{G}) \in \mathcal{N}_X$, and similarly, $a^{\alpha_2}(x) = a(x)$ if $x \in X$ and $a^{\alpha_2}(\langle(\mathbb{F},\mathbb{G})\rangle) = \bigvee_{g\in\mathbb{G}_\alpha}[g,a]$ for $(\mathbb{F},\mathbb{G}) \in \mathcal{N}_X$. The value $a^{\alpha_k}(x^*)$ can be interpreted as the grade to which a belongs to $\alpha_k(x^*)$ for $x^* \in X^*$, $k = 1, 2$.

Proposition 4.1. *We have for $a, b \in L^X$ and for $k = 1, 2$:*

1. $[a^{\alpha_k}, b^{\alpha_k}] = [a, b]$;
2. $j(a) \le a^{\alpha_k}$;
3. $j^{\leftarrow}(a^{\alpha_k}) = a$;
4. $(a \wedge b)^{\alpha_k} = a^{\alpha_k} \wedge b^{\alpha_k}$.

Proof. We show the claims for $k = 1$, the case $k = 2$ is analogous. (1) We have, for $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$,

$$\begin{aligned} a^{\alpha_1}(\langle\langle(\mathbb{F},\mathbb{G})\rangle\rangle) \to b^{\alpha_1}(\langle\langle(\mathbb{F},\mathbb{G})\rangle\rangle) &= \bigvee_{f\in\mathbb{F}_\alpha}[f,a] \to \bigvee_{f\in\mathbb{F}_\alpha}[f,b] \\ &\ge \bigwedge_{f\in\mathbb{F}_\alpha}([f,a]\to[f,b]) \ \ge \bigwedge_{f\in\mathbb{F}_\alpha}[a,b] \ = \ [a,b]. \end{aligned}$$

We conclude

$$[a^{\alpha_1}, b^{\alpha_1}] = [a,b] \wedge \bigwedge_{(\mathbb{F},\mathbb{G})\in\mathcal{N}_X} (a^{\alpha_1}(\langle\langle(\mathbb{F},\mathbb{G})\rangle\rangle) \to b^{\alpha_1}(\langle\langle(\mathbb{F},\mathbb{G})\rangle\rangle)) = [a,b].$$

(2) For $x \in X$ we have $j(a)(x) = a(x) = a^{\alpha_1}(x)$ and for $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$ we have $j(a)(\langle\langle(\mathbb{F},\mathbb{G})\rangle\rangle) = \bot \le a^{\alpha_1}(\langle\langle(\mathbb{F},\mathbb{G})\rangle\rangle)$.

(3) For $x \in X$ we have $j^{\leftarrow}(a)(x) = a(j(x)) = a(x) = a^{\alpha_1}(x)$.

(4) From (1) we see that $a^{\alpha_1} \le b^{\alpha_1}$ whenever $a \le b$. Hence we have $(a \wedge b)^{\alpha_1} \le a^{\alpha_1} \wedge b^{\alpha_1}$. We need to show the converse. For $x \in X$ we have by definition $(a \wedge b)^{\alpha_1}(x) = a^{\alpha_1}(x) \wedge b^{\alpha_1}(x) = (a^{\alpha_1} \wedge b^{\alpha_1})(x)$. For $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$ we have

$$\begin{aligned} (a^{\alpha_1} \wedge b^{\alpha_1})(\langle\langle(\mathbb{F},\mathbb{G})\rangle\rangle) &= \bigvee_{f_1,f_2\in\mathbb{F}_\alpha}[f_1,a]\wedge[f_2,b] \ \le \bigvee_{f\in\mathbb{F}_\alpha}[f,a]\wedge[f,b] \\ &= \bigvee_{f\in\mathbb{F}_\alpha}[f,a\wedge b] \ = \ (a\wedge b)^{\alpha_1}(\langle\langle(\mathbb{F},\mathbb{G})\rangle\rangle). \end{aligned}$$

□

From Proposition 4.1(4) it follows that for $\mathbb{F} \in \mathsf{F}_{\mathsf{L}}^{\top}(X)$ the sets $\{f^{\alpha_1} : f \in \mathbb{F}\}$ and $\{f^{\alpha_2} : f \in \mathbb{F}\}$ are $\top$-filter bases on X^* and we denote the generated $\top$-filters by $\mathbb{F}^{\alpha_1}$ and $\mathbb{F}^{\alpha_2}$, respectively.

Proposition 4.2. *Let $\alpha : X^* \longrightarrow \mathcal{CP}$ be a selection mapping and let $\mathbb{B}$ be a $\top$-filter basis for the $\top$-filter $\mathbb{F}$. Then $\mathbb{B}^{\alpha_k} = \{b^{\alpha_k} : b \in \mathbb{B}\}$ is a $\top$-filter basis of $\mathbb{F}^{\alpha_k}$, $k = 1, 2$.*

Proof. Again, we only prove the case $k = 1$. We first show that $\mathbb{B}^{\alpha_1}$ is a $\top$-filter basis on X^*. We have $\bigvee_{x^*\in X^*} b^{\alpha_1}(x^*) \ge \bigvee_{x\in X} b^{\alpha_1}(x) = \bigvee_{x\in X} b(x) = \top$ and hence the property (TB1) is shown. For (TB2), let $b^{\alpha_1}, c^{\alpha_1} \in \mathbb{B}^{\alpha_1}$. Then, with (4) and (1) of Proposition 4.1, we have $\bigvee_{d\in\mathbb{B}}[d^{\alpha_1}, b^{\alpha_1} \wedge c^{\alpha_1}] \ge \bigvee_{d\in\mathbb{B}}[d^{\alpha_1}, (b \wedge c)^{\alpha_1}] = \bigvee_{d\in\mathbb{B}}[b, b \wedge c] = \top$.

From $\mathbb{B} \subseteq \mathbb{F}$ we get $[\mathbb{B}^{\alpha_1}] \le \mathbb{F}^{\alpha_1}$. Let now $h \in \mathbb{F}^{\alpha_1}$. Then we have $\bigvee_{f\in\mathbb{F}}[f^{\alpha_1}, h] = \top$. For $f \in \mathbb{F}$ we moreover have $\top = \bigvee_{b\in\mathbb{B}}[b, f] = \bigvee_{b\in\mathbb{B}}[b^{\alpha_1}, f^{\alpha_1}]$. We conclude $\top = \bigvee_{f\in\mathbb{F}}\bigvee_{b\in\mathbb{B}}[b^{\alpha_1}, f^{\alpha_1}] * [f^{\alpha_1}, h] \le \bigvee_{b\in\mathbb{B}}[b^{\alpha_1}, h]$ and hence $h \in [\mathbb{B}^{\alpha_1}]$. □

Proposition 4.3. *Let $\mathbb{F}, \mathbb{G}$ be $\top$-filters. Then $(\mathbb{F}, \mathbb{G}) \in \mathsf{PF}_{\mathsf{L}}^{\top}(X)$ if, and only if, $(\mathbb{F}^{\alpha_1}, \mathbb{G}^{\alpha_2}) \in \mathsf{PF}_{\mathsf{L}}^{\top}(X^*)$.*

Proof. If $\mathbb{F}$ and $\mathbb{G}$ are linkend then from $\bigvee_{x^*\in X^*} f^{\alpha_1}(x^*) * g^{\alpha_2}(x^*) \ge \bigvee_{x\in X} f(x) * g(x) = \top$, we conclude that $\mathbb{F}^{\alpha_1}$ and $\mathbb{G}^{\alpha_2}$ are linked.

If $\mathbb{F}^{\alpha_1}$ and $\mathbb{G}^{\alpha_2}$ are linked, then we have for $a \in \mathbb{F}, b \in \mathbb{G}$ and $(\mathbb{L}, \mathbb{M}) \in \mathcal{N}_X$,

$$\begin{aligned} a^{\alpha_1}(\langle\langle(\mathbb{L},\mathbb{M})\rangle\rangle) * b^{\alpha_2}(\langle\langle(\mathbb{L},\mathbb{M})\rangle\rangle) &\le \bigvee_{f\in\mathbb{L}_\alpha, g\in\mathbb{M}_\alpha}[f*g, a*b] \\ &\le \bigvee_{f\in\mathbb{L}_\alpha, g\in\mathbb{M}_\alpha}(\bigvee_{x\in X}(f(x)*g(x)) \to (\bigvee_{x\in X} a(x)*b(x)) \\ &= \bigvee_{x\in X} a(x)*b(x), \end{aligned}$$

as $\mathbb{L}_\alpha$ and $\mathbb{M}_\alpha$ are linked. Hence,

$$\top = \bigvee_{x^*\in X^*} a^{\alpha_1}(x^*) * b^{\alpha_2}(x^*) = \bigvee_{x\in X} a(x) * b(x) \vee \bigvee_{(\mathbb{L},\mathbb{M})\in\mathcal{N}_X} a^{\alpha_1}(\langle(\mathbb{L},\mathbb{M})\rangle) * b^{\alpha_2}(\langle(\mathbb{L},\mathbb{M})\rangle) = \bigvee_{x\in X} a(x) * b(x),$$

and $\mathbb{F}$ and $\mathbb{G}$ are linked. □

Proposition 4.4. *Let $(X, \mathcal{CP})$ be a $\top$-quasi-Cauchy space and $\alpha : X^* \longrightarrow \mathcal{CP}$ be a selection mapping. Then for $x \in X$ we have $[x]^{\alpha_k} \le [j(x)]$, $k = 1, 2$ and for $(\mathbb{F},\mathbb{G}) \in \mathcal{N}_X$ we have $(\mathbb{F}_\alpha)^{\alpha_1}, (\mathbb{G}_\alpha)^{\alpha_2} \le [\langle(\mathbb{F},\mathbb{G})\rangle]$.*

Proof. If $f \in [x]^{\alpha_1}$, then $\top = \bigvee_{a\in[x]}[a^{\alpha_1}, f] \le \bigvee_{a\in[x]}[j(a), f] = \bigvee_{a\in[x]}[a, j^{\leftarrow}(f)]$ and hence $j^{\leftarrow}(f) \in [x]$, that is, $f \in j([x]) = [j(x)]$.

For the second part, we first note that for $f \in \mathbb{F}_\alpha$ we have $f^{\alpha_1}(\langle(\mathbb{F},\mathbb{G})\rangle) = \top$. If $h \in (\mathbb{F}_\alpha)^{\alpha_1}$, then $\top = \bigvee_{f\in\mathbb{F}_\alpha}[f^{\alpha_1}, h] \le \bigvee_{f\in\mathbb{F}_\alpha} f^{\alpha_1}(\langle(\mathbb{F},\mathbb{G})\rangle) \to h(\langle(\mathbb{F},\mathbb{G})\rangle) = h(\langle(\mathbb{F},\mathbb{G})\rangle)$. This means, $h \in [\langle(\mathbb{F},\mathbb{G})\rangle]$. □

Let now $(X, \mathcal{CP})$ be a $\top$-quasi-Cauchy space and $\alpha : X^* \longrightarrow \mathcal{CP}$ be a selection mapping. We define

$$\mathcal{CP}^\alpha = \{(\mathbb{H},\mathbb{K}) \in \mathsf{PF}^\top_\mathsf{L}(X^*) \ : \ \exists(\mathbb{F},\mathbb{G}) \in \mathcal{CP} \text{ such that } (\mathbb{F}^{\alpha_1}, \mathbb{G}^{\alpha_2}) \le (\mathbb{H},\mathbb{K})\}.$$

Theorem 4.5. *Let $(X, \mathcal{CP})$ be a non-complete $\top$-quasi-Cauchy space and $\alpha : X^* \longrightarrow \mathcal{CP}$ be a selection mapping. Then $((X^*, \mathcal{CP}^\alpha), j)$ is a completion of $(X, \mathcal{CP})$.*

Proof. We show first, that $(X^*, \mathcal{CP}^\alpha)$ is a $\top$-quasi-Cauchy space.

(TQC1) follows from Proposition 4.4 and (TQC2) is obvious. In order to show (TQC3), let $(\mathbb{H}_1,\mathbb{K}_1), (\mathbb{H}_2,\mathbb{K}_2) \in \mathcal{CP}^\alpha$ and let $\mathbb{H}_1$ and $\mathbb{K}_2$ be linked and also $\mathbb{H}_2$ and $\mathbb{K}_1$ be linked. Then there are $(\mathbb{F}_1,\mathbb{G}_1), (\mathbb{F}_2,\mathbb{G}_2) \in \mathcal{CP}$ such that $(\mathbb{F}_1^{\alpha_1}, \mathbb{G}_1^{\alpha_2}) \le (\mathbb{H}_1,\mathbb{K}_1)$ and $(\mathbb{F}_2^{\alpha_1}, \mathbb{G}_2^{\alpha_2}) \le (\mathbb{H}_2,\mathbb{K}_2)$. Then also $\mathbb{F}_1^{\alpha_1}$ and $\mathbb{G}_2^{\alpha_2}$ are linked and hence, by Proposition 4.3, also $\mathbb{F}_1$ and $\mathbb{G}_2$ are linked. Similarly, $\mathbb{F}_2$ and $\mathbb{G}_1$ are linked and hence $(\mathbb{F}_1 \wedge \mathbb{F}_2, \mathbb{G}_1 \wedge \mathbb{G}_2) \in \mathcal{CP}$. As $(\mathbb{F}_1 \wedge \mathbb{F}_2)^{\alpha_1} \le \mathbb{F}_1^{\alpha_1} \wedge \mathbb{F}_2^{\alpha_1} \le \mathbb{H}_1 \wedge \mathbb{H}_2$ and $(\mathbb{G}_1 \wedge \mathbb{G}_2)^{\alpha_2} \le \mathbb{G}_1^{\alpha_2} \wedge \mathbb{G}_2^{\alpha_2} \le \mathbb{K}_1 \wedge \mathbb{K}_2$ we see that $(\mathbb{H}_1 \wedge \mathbb{H}_2, \mathbb{K}_1 \wedge \mathbb{K}_2) \in \mathcal{CP}^\alpha$.

Next we show that $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ if, and only if, $(j(\mathbb{F}), j(\mathbb{G})) \in \mathcal{CP}^\alpha$. If $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$, then, as $\mathbb{F}^{\alpha_1} \le j(\mathbb{F})$ and $\mathbb{G}^{\alpha_2} \le j(\mathbb{G})$, we conclude $(j(\mathbb{F}), j(\mathbb{G})) \in \mathcal{CP}^\alpha$. On the other hand, if $(j(\mathbb{F}), j(\mathbb{G})) \in \mathcal{CP}^\alpha$, then there is $(\mathbb{L},\mathbb{M}) \in \mathcal{CP}$ such that $\mathbb{L}^{\alpha_1} \le j(\mathbb{F})$ and $\mathbb{M}^{\alpha_2} \le j(\mathbb{G})$. We conclude $\mathbb{L} = j^{\leftarrow}(\mathbb{L}^{\alpha_1}) \le j^{\leftarrow}(j(\mathbb{F})) = \mathbb{F}$ and, similarly, $\mathbb{M} \le \mathbb{G}$. Therefore, $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$.

We now show that $\overline{j(X)}^{\mathcal{CP}^\alpha} = X^*$. Let $x^* \in X^*$. If $x^* = j(x) = x$ with $x \in X$, then $([j(x)], [j(x)])$ converges to $j(x)$ in $(X^*, \mathcal{CP}^\alpha)$ and as clearly $\top_{j(X)} \in [j(x)]$ we have $j(x) \in \overline{j(X)}^{\mathcal{CP}^\alpha}$. If $x^* = \langle(\mathbb{F},\mathbb{G})\rangle$ with $(\mathbb{F},\mathbb{G}) \in \mathcal{N}_X$, then we have seen above that $(\mathbb{F}_\alpha)^{\alpha_1} \le [\langle(\mathbb{F},\mathbb{G})\rangle]$. As, by Proposition 4.1(3) also $(\mathbb{F}_\alpha)^{\alpha_1} \le j(\mathbb{F}_\alpha)$ we conclude $(\mathbb{F}_\alpha)^{\alpha_1} \le j(\mathbb{F}_\alpha) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]$. Similarly, we see $(\mathbb{G}_\alpha)^{\alpha_2} \le j(\mathbb{G}_\alpha) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]$. Again, as $\top_{j(X)} \in j(\mathbb{F}_\alpha), j(\mathbb{G}_\alpha)$, this shows that $\langle(\mathbb{F},\mathbb{G})\rangle \in \overline{j(X)}^{\mathcal{CP}^\alpha}$.

Finally, we show that $(X^*, \mathcal{CP}^\alpha)$ is complete. To this end, let $(\mathbb{H},\mathbb{K}) \in \mathcal{CP}^\alpha$. Then there is $(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ such that $(\mathbb{F}^{\alpha_1}, \mathbb{G}^{\alpha_2}) \le (\mathbb{H},\mathbb{K})$. If $(\mathbb{F},\mathbb{G})$ converges to $x \in X$ in $(X, \mathcal{CP})$, then $(\mathbb{F} \wedge [x], \mathbb{G} \wedge [x]) \in \mathcal{CP}$. From $(\mathbb{F} \wedge [x])^{\alpha_1} \le \mathbb{F}^{\alpha_1} \wedge [x]^{\alpha_1} \le \mathbb{H} \wedge [j(x)]$ and, similarly, $(\mathbb{G} \wedge [x])^{\alpha_2} \le \mathbb{K} \wedge [j(x)]$, we see that $(\mathbb{H} \wedge [j(x)], \mathbb{K} \wedge [j(x)]) \in \mathcal{CP}^\alpha$, that is, $(\mathbb{H},\mathbb{K})$ converges to $j(x)$ in $(X^*, \mathcal{CP}_\alpha)$. If $(\mathbb{F},\mathbb{G}) \in \mathcal{N}_X$, then $(\mathbb{F} \wedge \mathbb{F}_\alpha)^{\alpha_1} \le \mathbb{F}^{\alpha_1} \wedge (\mathbb{F}_\alpha)^{\alpha_1} \le \mathbb{H} \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]$ and, similarly, $(\mathbb{G} \wedge \mathbb{G}_\alpha)^{\alpha_2} \le \mathbb{K} \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]$. As $(\mathbb{F}_\alpha, \mathbb{G}_\alpha) \sim (\mathbb{F},\mathbb{G})$ we see that $(\mathbb{F} \wedge \mathbb{F}_\alpha, \mathbb{G} \wedge \mathbb{G}_\alpha) \in \mathcal{CP}$ and therefore $(\mathbb{H} \wedge [\langle(\mathbb{F},\mathbb{G})\rangle], \mathbb{K} \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]) \in \mathcal{CP}^\alpha$ and $(\mathbb{H},\mathbb{K})$ converges to $\langle(\mathbb{F},\mathbb{G})\rangle$ in $(X^*, \mathcal{CP}^\alpha)$. □

The construction of the completion preserves the separation property.

Proposition 4.6. *Let $(X, \mathcal{CP})$ be a non-complete $\top$-quasi-Cauchy space and $\alpha : X^* \longrightarrow \mathcal{CP}$ be a selection mapping. If $(X, \mathcal{CP})$ is separated. then $(X^*, \mathcal{CP}^\alpha)$ is separated.*

Proof. Let $x^*, y^* \in q^{\mathcal{CP}^\alpha}((\mathbb{H},\mathbb{K}))$. We distinguish three cases.

Case 1: $x^* = \langle(\mathbb{F}_1,\mathbb{G}_1)\rangle, y^* = \langle(\mathbb{F}_2,\mathbb{G}_2)\rangle$ with $(\mathbb{F}_1,\mathbb{G}_1), (\mathbb{F}_2,\mathbb{G}_2) \in \mathcal{N}_X$. Then there are $(\mathbb{L}_1,\mathbb{M}_1), (\mathbb{L}_2,\mathbb{M}_2) \in \mathcal{CP}$ such that $(\mathbb{L}_i^{\alpha_1}, \mathbb{M}_i^{\alpha_2}) \le (\mathbb{H} \wedge [\langle(\mathbb{F}_i,\mathbb{G}_i)\rangle], \mathbb{K} \wedge [\langle(\mathbb{F}_i,\mathbb{G}_i)\rangle])$ for $i = 1, 2$. As $\mathbb{L}_1^{\alpha_1}, \mathbb{M}_1^{\alpha_2} \le [\langle(\mathbb{F}_1,\mathbb{G}_1)\rangle]$ and also $(\mathbb{F}_{1\alpha})^{\alpha_1}, (\mathbb{G}_{1\alpha})^{\alpha_2} \le [\langle(\mathbb{F}_1,\mathbb{G}_1)\rangle]$ we see that $\mathbb{L}_1^{\alpha_1}$ and $(\mathbb{G}_{1\alpha})^{\alpha_2}$ as well as $\mathbb{M}_1^{\alpha_2}$ and $(\mathbb{F}_{1\alpha})^{\alpha_1}$ are linked and hence also $\mathbb{L}_1$ and $\mathbb{G}_{1\alpha}$ as well as $\mathbb{M}_1$ and $\mathbb{F}_{1\alpha}$ are linked. We conclude that $(\mathbb{L}_1 \wedge \mathbb{M}_1, \mathbb{F}_{1\alpha} \wedge \mathbb{G}_{1\alpha}) \in \mathcal{CP}$, that is, we have $(\mathbb{L}_1,\mathbb{M}_1) \sim (\mathbb{F}_{1\alpha},\mathbb{G}_{1\alpha}) \sim (\mathbb{F}_1,\mathbb{G}_1)$. In a similar way we can show that $(\mathbb{L}_2,\mathbb{M}_2) \sim (\mathbb{F}_2,\mathbb{G}_2)$. As moreover $(\mathbb{L}_1^{\alpha_1}, \mathbb{M}_1^{\alpha_2}), (\mathbb{L}_2^{\alpha_1}, \mathbb{M}_2^{\alpha_2}) \le (\mathbb{H},\mathbb{K})$, then $\mathbb{L}_1^{\alpha_1}$ and $\mathbb{M}_2^{\alpha_2}$ as well as $\mathbb{L}_2^{\alpha_1}$ and $\mathbb{M}_1^{\alpha_2}$ are linked and hence $(\mathbb{L}_1 \wedge \mathbb{L}_2, \mathbb{M}_1 \wedge \mathbb{M}_2) \in \mathcal{CP}$, that is $(\mathbb{L}_1,\mathbb{M}_1) \sim (\mathbb{L}_2,\mathbb{M}_2)$. By transitivity we obtain $(\mathbb{F}_1,\mathbb{G}_1) \sim (\mathbb{F}_2,\mathbb{G}_2)$, and we have $x^* = \langle(\mathbb{F}_1,\mathbb{G}_1)\rangle = \langle(\mathbb{F}_2,\mathbb{G}_2)\rangle = y^*$.

Case 2: $x^* = j(x), y^* = j(y)$ with $x, y \in X$. Then there are $(\mathbb{L}_1,\mathbb{M}_1), (\mathbb{L}_2,\mathbb{M}_2) \in \mathcal{CP}$ such that $(\mathbb{L}_1^{\alpha_1}, \mathbb{M}_1^{\alpha_2}) \le (\mathbb{H} \wedge [j(x)], \mathbb{K} \wedge [j(x)])$ and $(\mathbb{L}_2^{\alpha_1}, \mathbb{M}_2^{\alpha_2}) \le (\mathbb{H} \wedge [j(y)], \mathbb{K} \wedge [j(y)])$. As $(\mathbb{L}_1^{\alpha_1}, \mathbb{M}_1^{\alpha_2}) \le ([j(x)], [j(x)])$ we conclude $(\mathbb{L}_1^{\alpha_1} \wedge$

$[j(x)], \mathbb{M}_1^{\alpha} \wedge [j(x)]) \in \mathcal{CP}^{\alpha}$. Also we have $(\mathbb{L}_1^{\alpha} \wedge \mathbb{L}_2^{\alpha_1}, \mathbb{M}_1^{\alpha_2} \wedge \mathbb{M}_2^{\alpha_2}) \in \mathcal{CP}^{\alpha}$, because $(\mathbb{L}_i^{\alpha_1}\mathbb{M}_i^{\alpha_2}) \leq (\mathbb{H}, \mathbb{K})$, $i = 1, 2$. We conclude that $(\mathbb{L}_1^{\alpha_1} \wedge \mathbb{L}_2^{\alpha_1} \wedge [j(x)], \mathbb{M}_1^{\alpha_2} \wedge \mathbb{M}_2^{\alpha_2} \wedge [j(x)]) \in \mathcal{CP}^{\alpha}$ and hence, noting that in general $\mathbb{F}^{\alpha_i} \leq j(\mathbb{F})$, we have $(j(\mathbb{L}_1 \wedge \mathbb{L}_2 \wedge [x]), j(\mathbb{M}_1 \wedge \mathbb{M}_2 \wedge [x])) \in \mathcal{CP}^{\alpha}$. Therefore, $(\mathbb{L}_1 \wedge \mathbb{L}_2, \mathbb{M}_1 \wedge \mathbb{M}_2)$ converges to x in $(X, \mathcal{CP})$. In the same way we see that $(\mathbb{L}_1 \wedge \mathbb{L}_2, \mathbb{M}_1 \wedge \mathbb{M}_2)$ converges to y in $(X, \mathcal{CP})$ and hence, by separatedness of $(X, \mathcal{CP})$, $x = y$, that is $x^* = y^*$.

Case 3: $x^* = \langle(\mathbb{F}, \mathbb{G})\rangle, y^* = j(y)$ with $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$ and $y \in Y$. Then there are $(\mathbb{L}_1, \mathbb{M}_1), (\mathbb{L}_2, \mathbb{M}_2) \in \mathcal{CP}$ such that $(\mathbb{L}_1^{\alpha_1}, \mathbb{M}_1^{\alpha_2}) \leq (\mathbb{H} \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle], \mathbb{K} \wedge [\langle(\mathbb{F}, \mathbb{G})\rangle])$ and $(\mathbb{L}_2^{\alpha_1}, \mathbb{M}_2^{\alpha_2}) \leq (\mathbb{H} \wedge [j(y)], \mathbb{K} \wedge [j(y)])$. This implies that $(\mathbb{L}_1^{\alpha_1} \wedge \mathbb{L}_2^{\alpha_1}, \mathbb{M}_1^{\alpha_2} \wedge \mathbb{M}_2^{\alpha_2}) \in \mathcal{CP}^{\alpha}$ and hence $(j(\mathbb{L}_1 \wedge \mathbb{L}_2), j(\mathbb{M}_1 \wedge \mathbb{M}_2)) \in \mathcal{CP}^{\alpha}$. We conclude $(\mathbb{L}_1 \wedge \mathbb{L}_2, \mathbb{M}_1 \wedge \mathbb{M}_2) \in \mathcal{CP}$. As in Case 1 we see that $(\mathbb{L}_1 \wedge \mathbb{L}_2, \mathbb{M}_1 \wedge \mathbb{M}_2) \sim (\mathbb{F}, \mathbb{G})$ and as in Case 2 we see that $(\mathbb{L}_1 \wedge \mathbb{L}_2, \mathbb{M}_1 \wedge \mathbb{M}_2)$ converges to y in $(X, \mathcal{CP})$, that is $(\mathbb{L}_1 \wedge \mathbb{L}_2, \mathbb{M}_1 \wedge \mathbb{M}_2) \sim ([j(y)], [j(y)])$. Transitivity implies $(\mathbb{F}, \mathbb{G}) \sim ([j(y)], [j(y)])$, a contradiction to $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$. Hence, this case cannot occur. □

5 Pretopological completions

Again, we refer to [17] for an analogous construction in the symmetric case.

We call a $\top$-quasi Cauchy space $(X, \mathcal{CP})$ *pretopological* if for each $x \in X$ there is a coarsest pair $\top$-filter $(\mathbb{U}_x^l, \mathbb{U}_x^r)$ converging to x. That is, we have $x \in q^{\mathcal{CP}}((\mathbb{F}, \mathbb{G}))$ if, and only if, $(\mathbb{F}, \mathbb{G}) \geq (\mathbb{U}_x^l, \mathbb{U}_x^r)$.

In terms of the equivalence relation on $\mathcal{CP}$, we can say that $(X, \mathcal{CP})$ is pretopological if, and only if, for all $x \in X$, $\langle([x], [x])\rangle$ has a smallest member.

It is not difficult to see that $(X, \mathcal{CP})$ is pretopological if, and only if, $x \in q^{\mathcal{CP}}(\bigwedge_{j \in J}(\mathbb{F}_j, \mathbb{G}_j))$ whenever $x \in q^{\mathcal{CP}}((\mathbb{F}_j, \mathbb{G}_j))$ for all $j \in J$.

Example 5.1. *Let $(X, \mathcal{U})$ be a $\top$-quasi-uniform space. Then $x \in q^{\mathcal{CP}^{\mathcal{U}}}((\mathbb{F}, \mathbb{G}))$ if, and only if, $[x] \otimes \mathbb{F} \geq \mathcal{U}$ and $\mathbb{G} \otimes [x] \geq \mathcal{U}$. This is equivalent to $\mathbb{F} \geq \mathcal{U}(\cdot, x)$ and $\mathbb{G} \geq \mathcal{U}(x, \cdot)$ where $\mathcal{U}(\cdot, x)$ is the $\top$-filter on X generated by the $\top$-filter basis $\{u(\cdot, x) \;:\; u \in \mathcal{U}\}$ and $\mathcal{U}(x, \cdot)$ is the $\top$-filter on X generated by the $\top$-filter basis $\{u(x, \cdot) \;:\; u \in \mathcal{U}\}$, see* [10]. *Hence, $\mathbb{U}_x^l = \mathcal{U}(\cdot, x)$ and $\mathbb{U}_x^r = \mathcal{U}(x, \cdot)$ and $(X, \mathcal{CP}^{\mathcal{U}})$ is pretopological.*

Lemma 5.2. *Let $(X_j, \mathcal{CP}_j)$ be pretopological $\top$-quasi-Cauchy spaces for all $j \in J$ and let $(f_j : X \longrightarrow (X_j, \mathcal{CP}_j))_{j \in J}$ be a source and let $\mathcal{CP}$ be the initial $\top$-quasi-Cauchy structure on X. Then $(X, \mathcal{CP})$ is pretopological.*

Proof. Let $x \in q^{\mathcal{CP}}((\mathbb{F}_k, \mathbb{G}_k))$ for all $k \in K$. Then $(\mathbb{F}_k \wedge [x], \mathbb{G}_k \wedge [x]) \in \mathcal{CP}$ for all $k \in K$ and hence, for all $j \in J$, $f_j((\mathbb{F}_k \wedge [x], \mathbb{G}_k \wedge [x])) \in \mathcal{CP}_j$ for all $k \in K$. By pretopologicalness of the $(X_j, \mathcal{CP}_j)$ this implies that, for all $j \in J$, we have $f_j(\bigwedge_{k \in K}(\mathbb{F}_k \wedge [x], \mathbb{G}_k \wedge [x])) = \bigwedge_{k \in K} f_j((\mathbb{F}_k \wedge [x], \mathbb{G}_k \wedge [x])) \in \mathcal{CP}_j$, that is $\bigwedge_{k \in K}(\mathbb{F}_k \wedge [x], \mathbb{G}_k \wedge [x]) \in \mathcal{CP}$. Hence, $x \in q^{\mathcal{CP}}(\bigwedge_{k \in K}(\mathbb{F}_k, \mathbb{G}_k))$ and $(X, \mathcal{CP})$ is pretopological. □

We are interested in completions of a $\top$-quasi-Cauchy space that are pretopological. Clearly, as we can consider a space as a subspace of its completion, by Lemma 5.2 the space itself needs to be pretopological. However, we need to demand a bit more. We call a $\top$-quasi-Cauchy space $(X, \mathcal{CP})$ *full* if it is pretopological and if for each $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$ the equivalence class $\langle(\mathbb{F}, \mathbb{G})\rangle$ has a smallest member $(\mathbb{F}_{\min}, \mathbb{G}_{\min})$, that is, we have $(\mathbb{F}', \mathbb{G}') \in \langle(\mathbb{F}, \mathbb{G})\rangle$ if, and only if, $(\mathbb{F}', \mathbb{G}') \geq (\mathbb{F}_{\min}, \mathbb{G}_{\min})$. Clearly, a complete pretopological $\top$-quasi-Cauchy space is full.

Noting that a pair $\top$-filter $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ is either convergent or non-convergent, we have that $(X, \mathcal{CP})$ is full if, and only if, for all $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$, $\langle(\mathbb{F}, \mathbb{G})\rangle$ has a smallest member.

Example 5.3. *For a $\top$-quasi-uniform space $(X, \mathcal{U})$, the $\top$-quasi-Cauchy space $(X, \mathcal{CP}^{\mathcal{U}})$ is full. To see this, let $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$. It was shown in [21] that $(\mathcal{U} \circ \mathbb{F}, \mathbb{G} \circ \mathcal{U}) \in \mathcal{CP}^{\mathcal{U}}$ and is $\leq (\mathbb{F}, \mathbb{G})$ and minimal. Here, $\mathcal{U} \circ \mathbb{F}$ has the $\top$-filter basis $\{u \circ f \;:\; u \in \mathcal{U}, f \in \mathbb{F}\}$ with $u \circ f(x) = \bigvee_{z \in X} f(z) * u(z, x)$ for all $x \in X$ and, similarly, $\mathbb{G} \circ \mathcal{U}$ has the $\top$-filter basis $\{g \circ u \;:\; g \in \mathbb{G}, u \in \mathcal{U}\}$ with $g \circ u(x) = \bigvee_{z \in X} u(x, z) * g(z)$ for all $x \in X$. If $(\mathbb{L}, \mathbb{M}) \sim (\mathbb{F}, \mathbb{G})$, then $(\mathbb{L} \wedge \mathbb{F}, \mathbb{M} \wedge \mathbb{G}) \in \mathcal{CP}^{\mathcal{U}}$ and therefore again $(\mathcal{U} \circ (\mathbb{L} \wedge \mathbb{F}), (\mathbb{M} \wedge \mathbb{G}) \circ \mathcal{U}) \in \mathcal{CP}^{\mathcal{U}}$ and is $\leq (\mathcal{U} \circ \mathbb{F}, \mathbb{G} \circ \mathcal{U})$. Hence, from the minimality, we conclude $(\mathcal{U} \circ (\mathbb{L} \wedge \mathbb{F}), (\mathbb{M} \wedge \mathbb{G}) \circ \mathcal{U}) = (\mathcal{U} \circ \mathbb{F}, \mathbb{G} \circ \mathcal{U})$. Furthermore, $(\mathcal{U} \circ (\mathbb{L} \wedge \mathbb{F}), (\mathbb{M} \wedge \mathbb{G}) \circ \mathcal{U}) \leq (\mathbb{L} \wedge \mathbb{F}, \mathbb{M} \wedge \mathbb{G}) \leq (\mathbb{L}, \mathbb{M})$ and hence also $(\mathcal{U} \circ \mathbb{F}, \mathbb{G} \circ \mathcal{U}) \leq (\mathbb{L}, \mathbb{M})$ and we conclude $(\mathcal{U} \circ \mathbb{F}, \mathbb{G} \circ \mathcal{U}) = (\mathbb{F}_{\min}, \mathbb{G}_{\min}) \in \mathcal{CP}^{\mathcal{U}}$ and therefore, as we have seen above that $(X, \mathcal{CP}^{\mathcal{U}})$ is pretopological, $(X, \mathcal{CP}^{\mathcal{U}})$ is full.*

Example 5.4. *We produce an example of a $\top$-quasi-Cauchy space that is pretopological but not full. Let X be an infinite set. We choose infinitely many ultrafilters that are not point filters, and we denote the set of finite intersections of these ultrafilters by Λ. This choice is possible because the cardinality of the set of ultrafilters on X is the cardinality of $\mathsf{P}(\mathsf{P}(X))$. We define now $\mathcal{CP} \subseteq \mathsf{PF}_{\mathsf{L}}^{\top}(X)$ by: $([x], [x]) \in \mathcal{CP}$ for all $x \in X$ and $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ if there are $\mathcal{F}, \mathcal{G} \in \Lambda$ such that $\mathbb{F} \geq [\top_{\mathcal{F}}]$ and $\mathbb{G} \geq [\top_{\mathcal{G}}]$. Then $(X, \mathcal{CP})$ is a $\top$-quasi-Cauchy space. The axiom (TQC1) is clear by definition. Also (TQC2) follows easily. We show (TQC3). Let $(\mathbb{F}, \mathbb{G}), (\mathbb{F}', \mathbb{G}') \in \mathcal{CP}$ and let $\mathbb{F}, \mathbb{G}'$ and $\mathbb{G}, \mathbb{F}'$ be linked. We distinuish three cases.*

Case 1: $(\mathbb{F},\mathbb{G})=([x],[x])$, $(\mathbb{F}',\mathbb{G}')=([y],[y])$ *with* $x,y\in X$. *From* $\top=\bigvee_{z\in X}\top_{\{x\}}(z)*\top_{\{y\}}$ *we conclude* $x=y$ *and hence* $(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')=([x],[x])\in\mathcal{CP}$.

Case 2: $(\mathbb{F},\mathbb{G})=([x],[x])$ *with* $x\in X$ *and* $(\mathbb{F}',\mathbb{G}')\geq([\top_{\mathcal{F}}],[\top_{\mathcal{G}}])$ *with* $\mathcal{F},\mathcal{G}\in\Lambda$. *If* $\mathcal{G}=\mathcal{U}_1\wedge\mathcal{U}_2\wedge...\wedge\mathcal{U}_n$ *with ultrafilters* $\mathcal{U}_k\in\Lambda$ *for* $k=1,2,...,n$, *then there are* $U_k\in\mathcal{U}_k$ *such that* $x\notin U_k$ *for all* $k=1,2,...,n$. *Hence* $x\notin U_1\cup U_2\cup...\cup U_n$ *in contradiction to* $[x],[\top_{\mathcal{G}}]$ *being linked, that is,* $\top=\bigvee_{z\in X}\top_{\{x\}}(z)*T_{U_1\cup U_2\cup...\cup U_n}(z)$. *Hence this case cannot occur.*

Case 3: $(\mathbb{F},\mathbb{G})\geq([\top_{\mathcal{F}}],[\top_{\mathcal{G}}]),(\mathbb{F}',\mathbb{G}')\geq([\top_{\mathcal{F}'}],[\top_{\mathcal{G}'}])$ *with* $\mathcal{F},\mathcal{G},\mathcal{F}',\mathcal{G}'\in\Lambda$. *Then also* $\mathcal{F}\wedge\mathcal{F}',\mathcal{G}\wedge\mathcal{G}'\in\Lambda$ *and* $(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')\geq([\top_{\mathcal{F}\wedge\mathcal{F}'}],[\top_{\mathcal{G}\wedge\mathcal{G}'}])$, *that is,* $(\mathbb{F}\wedge\mathbb{F}',\mathbb{G}\wedge\mathbb{G}')\in\mathcal{CP}$.

Now we note that for $\mathcal{F},\mathcal{G}\in\Lambda$, *the pair* $\top$*-filter* $([\top_{\mathcal{F}}],[\top_{\mathcal{G}}])$ *is not convergent. Otherwise we had* $([\top_{\mathcal{F}}]\wedge[x],[\top_{\mathcal{G}}]\wedge[x])\in\mathcal{CP}$ *for some* $x\in X$. *If* $([\top_{\mathcal{F}}]\wedge[x],[\top_{\mathcal{G}}]\wedge[x])\geq([y],[y])$ *for some* $y\in X$, *then* $[x]\leq[y]$ *and hence* $x=y$. *This implies* $[x]\leq[\top_{\mathcal{F}}]\wedge[x]$, *which is not possible. If* $([\top_{\mathcal{F}}]\wedge[x],[\top_{\mathcal{G}}]\wedge[x])\geq([\top_{\mathcal{F}'}],[\top_{\mathcal{G}'}])$ *with* $\mathcal{F}',\mathcal{G}'\in\Lambda$, *then* $[\top_{\mathcal{F}'}]\leq[x]$ *which is not possible as we have seen in Case 2 above.*

Hence, for $\mathcal{F},\mathcal{G}\in\Lambda$ *we have that* $([\top_{\mathcal{F}}],[\top_{\mathcal{G}}])\in\mathcal{N}_X$. *It is not difficult to see that any* $([\top_{\mathcal{F}'}],[\top_{\mathcal{G}'}])$ *with* $\mathcal{F}',\mathcal{G}'\in\Lambda$ *is in the equivalence class of* $([\top_{\mathcal{F}}],[\top_{\mathcal{G}}])$. *Therefore the meet* $\bigwedge_{(\mathbb{F},\mathbb{G})\sim([\top_{\mathcal{F}}],[\top_{\mathcal{G}}])}(\mathbb{F},\mathbb{G})=([\top_{\bigwedge_{\mathcal{F}\in\Lambda}\mathcal{F}}],[\top_{\bigwedge_{\mathcal{F}\in\Lambda}\mathcal{F}}])$ *is not in* $\mathcal{CP}$, *as* $\bigwedge_{\mathcal{F}\in\Lambda}\mathcal{F}$ *is not in* Λ.

We note that the only convergent pair $\top$*-filters are* $([x],[x])$ *for* $x\in X$ *and therefore* $(X,\mathcal{CP})$ *is pretopological.*

If $(X,\mathcal{CP})$ is a non-complete and full $\top$-quasi-Cauchy space, then we define the selection mapping $\alpha:X^*\longrightarrow\mathcal{CP}$ by $\alpha(x)=([x],[x])$ for $x\in X$ and $\alpha(\langle(\mathbb{F},\mathbb{G})\rangle)=(\mathbb{F}_{\min},\mathbb{G}_{\min})$ for $(\mathbb{F},\mathbb{G})\in\mathcal{N}_X$.

Following [17] we use for $a\in L^X$ and $\mathbb{F}\in\mathsf{F}_L^{\top}(X)$ the notation $\widetilde{a}^k=a^{\alpha_k}$, $\widetilde{\mathbb{F}}^k=\mathbb{F}^{\alpha_k}$, $k=1,2$, and $\widetilde{\mathcal{CP}}=\mathcal{CP}^{\alpha}$, for this selection mapping.

Theorem 5.5. *Let* $(X,\mathcal{CP})$ *be a non-complete and full* $\top$*-quasi-Cauchy space. Then* $(X^*,\widetilde{\mathcal{CP}})$ *is pretopological.*

Proof. Let first $x^*=j(x)\in X^*$ for $x\in X$. We denote the coarsest pair $\top$-filter converging to x in $(X,\mathcal{CP})$ by $(\mathbb{U}_x^l,\mathbb{U}_x^r)$. We will show that $j(x)\in q^{\widetilde{\mathcal{CP}}}((\widetilde{\mathbb{U}_x^l}^1,\widetilde{\mathbb{U}_x^r}^2))$ and that $j(x)\in q^{\widetilde{\mathcal{CP}}}((\mathbb{H},\mathbb{K}))$ implies $(\mathbb{H},\mathbb{K})\geq(\widetilde{\mathbb{U}_x^l}^1,\widetilde{\mathbb{U}_x^r}^2)$.

We have $(\mathbb{U}_x^l\wedge[x],\mathbb{U}_x^r\wedge[x])\in\mathcal{CP}$ and hence $(\widetilde{\mathbb{U}_x^l\wedge[x]}^1,\widetilde{\mathbb{U}_x^r\wedge[x]}^2)\in\widetilde{\mathcal{CP}}$. As $\widetilde{\mathbb{U}_x^l\wedge[x]}^1\leq\widetilde{\mathbb{U}_x^l}^1\wedge\widetilde{[x]}^1\leq\widetilde{\mathbb{U}_x^l}^1\wedge[j(x)]$ and, similarly, $\widetilde{\mathbb{U}_x^r\wedge[x]}^2\leq\widetilde{\mathbb{U}_x^r}^2\wedge[j(x)]$, it follows that $(\widetilde{\mathbb{U}_x^l}^1\wedge[j(x)],\widetilde{\mathbb{U}_x^r}^2\wedge[j(x)])\in\widetilde{\mathcal{CP}}$, that is, $(\widetilde{\mathbb{U}_x^l}^1,\widetilde{\mathbb{U}_x^r}^2)$ converges to $j(x)$ in $(X^*,\widetilde{\mathcal{CP}})$.

Let now $(\mathbb{H},\mathbb{K})$ converge to $j(x)$ in $(X^*,\widetilde{\mathcal{CP}})$. Then there is $(\mathbb{F},\mathbb{G})\in\mathcal{CP}$ such that $\widetilde{\mathcal{CP}}\ni(\widetilde{\mathbb{F}}^1,\widetilde{\mathbb{G}}^2)\leq(\mathbb{H}\wedge[j(x)],\mathbb{K}\wedge[j(x)])$. Together with $\widetilde{\mathbb{F}}^1\leq j(\mathbb{F})$ and $\widetilde{\mathbb{G}}^2\leq j(\mathbb{G})$ we conclude $(j(\mathbb{F}\wedge[x]),j(\mathbb{G}\wedge[x]))=(j(\mathbb{F})\wedge[j(x)],j(\mathbb{G})\wedge[j(x)])\in\widetilde{\mathcal{CP}}$. Hence, $(\mathbb{F}\wedge[x],\mathbb{G}\wedge[x])\in\mathcal{CP}$ and, by assumption, $(\mathbb{F},\mathbb{G})\geq(\mathbb{U}_x^l,\mathbb{U}_x^r)$. We conclude $(\mathbb{H},\mathbb{K})\geq(\widetilde{\mathbb{F}}^1,\widetilde{\mathbb{G}}^2)\geq(\widetilde{\mathbb{U}_x^l}^1,\widetilde{\mathbb{U}_x^r}^2)$.

Consider now $(\mathbb{F},\mathbb{G})\in\mathcal{N}_X$. We will show that $(\widetilde{\mathbb{F}_{\min}}^1,\widetilde{\mathbb{G}_{\min}}^2)$ converges to $\langle(\mathbb{F},\mathbb{G})\rangle$ in $(X^*,\widetilde{\mathcal{CP}})$ and that the convergence of $(\mathbb{H},\mathbb{K})$ to $\langle(\mathbb{F},\mathbb{G})\rangle$ in $(X^*,\widetilde{\mathcal{CP}})$ implies $(\mathbb{H},\mathbb{K})\geq(\widetilde{\mathbb{F}_{\min}}^1,\widetilde{\mathbb{G}_{\min}}^2)$.

We note that we have seen before that $\widetilde{\mathbb{F}_{\min}}^1,\widetilde{\mathbb{G}_{\min}}^2\leq[\langle(\mathbb{F},\mathbb{G})\rangle]$ and from $(\mathbb{F}_{\min},\mathbb{G}_{\min})\in\mathcal{CP}$ we obtain $(\widetilde{\mathbb{F}_{\min}}^1,\widetilde{\mathbb{G}_{\min}}^2)=(\widetilde{\mathbb{F}_{\min}}^1\wedge[\langle(\mathbb{F},\mathbb{G})\rangle],\widetilde{\mathbb{G}_{\min}}^2\wedge[\langle(\mathbb{F},\mathbb{G})\rangle])\in\widetilde{\mathcal{CP}}$, that is $(\widetilde{\mathbb{F}_{\min}}^1,\widetilde{\mathbb{G}_{\min}}^2)$ converges to $\langle(\mathbb{F},\mathbb{G})\rangle$ in $(X^*,\widetilde{\mathcal{CP}})$.

Let now $(\mathbb{H},\mathbb{K})$ converge to $\langle(\mathbb{F},\mathbb{G})\rangle$ in $(X^*,\widetilde{\mathcal{CP}})$. Then $(\mathbb{H}\wedge[\langle(\mathbb{F},\mathbb{G})\rangle],\mathbb{K}\wedge[\langle(\mathbb{F},\mathbb{G})\rangle])\in\widetilde{\mathcal{CP}}$ and hence there is $(\mathbb{L},\mathbb{M})\in\mathcal{CP}$ such that $(\widetilde{\mathbb{L}}^1,\widetilde{\mathbb{M}}^2)\leq(\mathbb{H}\wedge[\langle(\mathbb{F},\mathbb{G})\rangle],\mathbb{K}\wedge[\langle(\mathbb{F},\mathbb{G})\rangle])$. We conclude $\widetilde{\mathcal{CP}}\ni(\widetilde{\mathbb{L}}^1,\widetilde{\mathbb{M}}^2)\leq(j(\mathbb{L})\wedge[\langle(\mathbb{F},\mathbb{G})\rangle],j(\mathbb{M}\wedge[\langle(\mathbb{F},\mathbb{G})\rangle]))$. Furthermore, $\widetilde{\mathcal{CP}}\ni(\widetilde{\mathbb{F}_{\min}}^1\wedge[\langle(\mathbb{F},\mathbb{G})\rangle],\widetilde{\mathbb{G}_{\min}}^2\wedge[\langle(\mathbb{F},\mathbb{G})\rangle])\leq(j(\mathbb{F})\wedge[\langle(\mathbb{F},\mathbb{G})\rangle],j(\mathbb{G})\wedge[\langle(\mathbb{F},\mathbb{G})\rangle])$ and hence $(j(\mathbb{L}\wedge\mathbb{F}),j(\mathbb{M}\wedge\mathbb{G}))=(j(\mathbb{F})\wedge j(\mathbb{F}),j(\mathbb{M})\wedge j(\mathbb{G}))\in\widetilde{\mathcal{CP}}$. We conclude $(\mathbb{L}\wedge\mathbb{F},\mathbb{M}\wedge\mathbb{G})\in\mathcal{CP}$, that is $(\mathbb{L},\mathbb{M})\sim(\mathbb{F},\mathbb{G})$ and thus $(\mathbb{L},\mathbb{M})\geq(\mathbb{F}_{\min},\mathbb{G}_{\min})$. This is implies $(\mathbb{H},\mathbb{K})\geq(\widetilde{\mathbb{L}}^1,\widetilde{\mathbb{M}}^2)\geq(\widetilde{\mathbb{F}_{\min}}^1,\widetilde{\mathbb{G}_{\min}}^2)$ and the proof is complete. □

We will now show that to demand the fullness of $(X,\mathcal{CP})$ in Theorem 5.3 is not too much. To this end, we need the following concept. We say that a completion $((X^+,\mathcal{CP}^+),\psi)$ of $(X,\mathcal{CP})$ is in *standard form* if $X^+=X^*$, $\psi=j$ and $j((\mathbb{F},\mathbb{G}))$ converges to $\langle(\mathbb{F},\mathbb{G})\rangle$ in $(X^*,\mathcal{CP}^+)$ for any $(\mathbb{F},\mathbb{G})\in\mathcal{N}_X$. We note that all completions considered in this paper are in standard form. Furthermore it can be shown, adapting the arguments of the proof of Theorem 5 in [16], that every separated completion of a $\top$-quasi-Cauchy space is equivalent to one in standard form.

Proposition 5.6. *Let* $(X,\mathcal{CP})$ *be a non-complete* $\top$*-quasi-Cauchy space. If* $(X^*,\mathcal{CP}^+),j)$ *is a pretopological completion of* $(X,\mathcal{CP})$ *in standard form, then* $(X,\mathcal{CP})$ *is full.*

Proof. Let first $x^*=j(x)\in X^*$ with $x\in X$. Then $j(\bigwedge_{x\in q^{\mathcal{CP}}((\mathbb{F},\mathbb{G}))}(\mathbb{F},\mathbb{G}))=\bigwedge_{x\in q^{\mathcal{CP}}((\mathbb{F},\mathbb{G}))}(j(\mathbb{F}),j(\mathbb{G}))\geq\bigwedge_{j(x)\in q^{\mathcal{CP}^+}((\mathbb{H},\mathbb{K}))}(\mathbb{H},\mathbb{K})$ which converges to $j(x)$ in $(X^*,\mathcal{CP}^+)$ because $(X^*,\mathcal{CP}^+)$ is pretopological. Hence,

$j(\bigwedge_{x\in q^{\mathcal{CP}}((\mathbb{F},\mathbb{G}))}(\mathbb{F},\mathbb{G}) \wedge ([x],[x])) = j(\bigwedge_{x\in q^{\mathcal{CP}}((\mathbb{F},\mathbb{G}))}(\mathbb{F},\mathbb{G})) \wedge ([j(x)],[j(x)]) \in \mathcal{CP}^+$ and, j being a Cauchy embedding, we conclude that $(\mathbb{U}^l_x,\mathbb{U}^r_x) = \bigwedge_{x\in q^{\mathcal{CP}}((\mathbb{F},\mathbb{G}))}(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ and converges to x in $(X,\mathcal{CP})$.

For $x^* = \langle(\mathbb{F},\mathbb{G})\rangle$ with $(\mathbb{F},\mathbb{G}) \in \mathcal{N}_X$ we argue similarly. If $(\mathbb{F}',\mathbb{G}') \in \langle(\mathbb{F},\mathbb{G})\rangle$, then $j((\mathbb{F}',\mathbb{G}'))$ converges to $\langle(\mathbb{F}',\mathbb{G}')\rangle = \langle(\mathbb{F},\mathbb{G})\rangle$ because the completion is in standard form. Hence, $j(\bigwedge_{(\mathbb{F}',\mathbb{G}')\in\langle(\mathbb{F},\mathbb{G})\rangle}(\mathbb{F}',\mathbb{G}')) = \bigwedge_{(\mathbb{F}',\mathbb{G}')\in\langle(\mathbb{F},\mathbb{G})\rangle} j((\mathbb{F}',\mathbb{G}'))$ converges to $\langle(\mathbb{F},\mathbb{G})\rangle$. This implies $j(\bigwedge_{(\mathbb{F}',\mathbb{G}')\in\langle(\mathbb{F},\mathbb{G})\rangle}(\mathbb{F}',\mathbb{G}')) \in \mathcal{CP}^+$ and again j being a Cauchy embedding yields $(\mathbb{F}_{\min},\mathbb{G}_{\min}) = \bigwedge_{(\mathbb{F}',\mathbb{G}')\in\langle(\mathbb{F},\mathbb{G})\rangle}(\mathbb{F}',\mathbb{G}') \in \mathcal{CP}$ and clearly $(\mathbb{F}_{\min},\mathbb{G}_{\min}) \in \langle(\mathbb{F},\mathbb{G})\rangle$. □

6 Topological completions

Let J be a set and consider a mapping $\sigma = (\sigma_1,\sigma_2) : J \longrightarrow \mathsf{PF}^\top_\mathsf{L}(X)$ with $\sigma(j) = (\sigma_1(j),\sigma_2(j))$ for all $j \in J$. Hence, $\sigma_k : J \longrightarrow \mathsf{F}^\top_\mathsf{L}(X)$ for $k = 1,2$ such that $(\sigma_1(j),\sigma_2(j)) \in \mathsf{PF}^\top_\mathsf{L}(X)$.

Following [3] we define for $a \in L^X$ and $k = 1,2$ the L-sets $\widehat{\sigma_k}(a) \in L^J$ by $\widehat{\sigma_k}(a)(j) = \bigvee_{d\in\sigma_k(j)}[d,a]$, $(j \in J)$. For $(\mathbb{F},\mathbb{G}) \in \mathsf{PF}^\top_\mathsf{L}(J)$ we define two $\top$-filters on X, $\kappa\sigma_1\mathbb{F}$ and $\kappa\sigma_2\mathbb{G}$ by $a \in \kappa\sigma_1\mathbb{F}$ if $\widehat{\sigma_1}(a) \in \mathbb{F}$ and $b \in \kappa\sigma_2\mathbb{G}$ if $\widehat{\sigma_2}(b) \in \mathbb{F}$. It was shown in [3] that $\kappa\sigma_1\mathbb{F},\kappa\sigma_2\mathbb{G} \in \mathsf{F}^\top_\mathsf{L}(X)$. We now define the *diagonal pair* $\top$*-filter* of $((\mathbb{F},\mathbb{G}),\sigma)$

$$\kappa\sigma(\mathbb{F},\mathbb{G}) = (\kappa\sigma_1\mathbb{F},\kappa\sigma_2\mathbb{G}),$$

and show with the next proposition that it is in fact a pair $\top$-filter.

Proposition 6.1. *For a set J and a mapping $\sigma = (\sigma_1,\sigma_2) : J \longrightarrow \mathsf{PF}^\top_\mathsf{L}(X)$ and $(\mathbb{F},\mathbb{G}) \in \mathsf{PF}^\top_\mathsf{L}(J)$ we have $\kappa\sigma(\mathbb{F},\mathbb{G}) \in \mathsf{PF}^\top_\mathsf{L}(X)$.*

Proof. Let $a \in \kappa\sigma_1\mathbb{F}$ and $b \in \kappa\sigma_2\mathbb{G}$. Then $\widehat{\sigma_1}(a) \in \mathbb{F}$ and $\widehat{\sigma_2}(b) \in \mathbb{G}$ and hence $\top = \bigvee_{j\in J}\widehat{\sigma_1}(a)(j) * \widehat{\sigma_2}(b)(j)$. As $\sigma_1(j)$ and $\sigma_2(j)$ are linked for all $j \in J$, we have

$$\widehat{\sigma_1}(a)(j) * \widehat{\sigma_2}(b)(j) = \bigvee_{d\in\sigma_1(j),e\in\sigma_2(j)} [d,a] * [e,b]$$

$$\leq \bigvee_{d\in\sigma_1(j),e\in\sigma_2(j)} (\underbrace{\bigvee_{x\in X} d(x) * e(x))}_{=\top} \rightarrow (\bigvee_{x\in X} a(x) * b(x)) = \bigvee_{x\in X} a(x) * b(x),$$

and we obtain $\top = \bigvee_{x\in X} a(x) * b(x)$. □

Let now $\sigma : J \longrightarrow \mathsf{PF}^\top_\mathsf{L}(X)$ and $\varphi : X \longrightarrow Y$. We define $\varphi(\sigma) : J \longrightarrow \mathsf{PF}^\top_\mathsf{L}(Y)$ by $\varphi(\sigma)(j) = \varphi(\sigma(j)) = (\varphi(\sigma_1(j)),\varphi(\sigma_2(j)))$. Using Lemma 4.2 in [3] we have $\varphi(\kappa\sigma(\mathbb{F},\mathbb{G})) = (\varphi(\kappa\sigma_1\mathbb{F}),\varphi(\kappa\sigma_2\mathbb{G})) = (\kappa\varphi(\sigma_1)\mathbb{F},\kappa\varphi(\sigma_2)\mathbb{G}) = \kappa\varphi(\sigma)(\mathbb{F},\mathbb{G})$.

We call a $\top$-quasi-Cauchy space $(X,\mathcal{CP})$ *topological* if for all sets J, mappings $\psi : J \longrightarrow X$, $\sigma = (\sigma_1,\sigma_2) : J \longrightarrow \mathsf{PF}^\top_\mathsf{L}(X)$ with $\sigma(j) \wedge ([\psi(j)],[\psi(j)]) \in \mathcal{CP}$ for all $j \in J$ and $(\mathbb{F},\mathbb{G}) \in \mathsf{PF}^\top_\mathsf{L}(J)$ we have $\kappa\sigma(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$ whenever $\psi((\mathbb{F},\mathbb{G})) \in \mathcal{CP}$.

We note that this definition deviates from the related concept in the symmetric case in [17], as we demand $\sigma(j)$ to be convergent in $(X,\mathcal{CP})$ and do not include non-convergent pair $\top$-filters, which would make σ to have values in X^* and we would have to involve the finest completion $(X^*,\mathcal{CP}^*)$ in the definition. We avoid this, however, we have to pay the price that for some results, we will have to demand the fullness of the space (which is implied by the topologicalness in [17], whereas the definition employed here only implies pretopologicalness).

We denote the subcategory of $\top$-QChy with objects the topological $\top$-quasi-Cauchy spaces by $\top$-TQChy.

Proposition 6.2. *The category $\top$-TQChy allows initial constructions.*

Proof. Let $(X_k,\mathcal{CP}_k)$ be topological $\top$-quasi-Cauchy spaces for all $k \in K$ and let $(\varphi_k : X \longrightarrow (X_k,\mathcal{CP}_k))_{k\in K}$ be a source in $\top$-QChy and denote $\mathcal{CP}$ the initial $\top$-quasi-Cauchy structure on X for this source. We show that $(X,\mathcal{CP})$ is topological. Let J be a set and consider mappings $\psi : J \longrightarrow X$, $\sigma = (\sigma_1,\sigma_2) : J \longrightarrow \mathsf{PF}^\top_\mathsf{L}(X)$ with $\sigma(j) \wedge ([\psi(j)],[\psi(j)]) \in \mathcal{CP}$ for all $j \in J$. Let further $(\mathbb{F},\mathbb{G}) \in \mathsf{PF}^\top_\mathsf{L}(X)$ and $\psi((\mathbb{F},\mathbb{G})) \in \mathcal{CP}$. By definition of the initial structure, then $\varphi_k(\sigma(j)) \wedge ([\varphi_k(\psi(j))],[\varphi_k(\psi(j))]) \in \mathcal{CP}_k$ for each $k \in K$. We define for each $k \in K$, $\varphi_k(\sigma)$ by $\varphi_k(\sigma(j)) = (\varphi_k(\sigma_1(j)),\varphi_k(\sigma_2(j)))$ for $j \in J$ and $\psi_k = \varphi_k \circ \psi : J \longrightarrow X_k$. Then for each $k \in K$ we have $\varphi_k(\sigma)(j) \wedge ([\psi_k(j)],[\psi_k(j)]) \in \mathcal{CP}_k$ for all $j \in J$ and $\psi_k((\mathbb{F},\mathbb{G})) \in \mathcal{CP}_k$. Hence, for each $k \in K$ we have $\varphi_k(\kappa\sigma(\mathbb{F},\mathbb{G})) = \kappa\varphi_k(\sigma)(\mathbb{F},\mathbb{G}) \in \mathcal{CP}_k$ and by definition of the initial structure we conclude $\kappa\sigma(\mathbb{F},\mathbb{G}) \in \mathcal{CP}$. □

Proposition 6.3. *If the $\top$-quasi-Cauchy space $(X, \mathcal{CP})$ is topological, then $(X, q^{\mathcal{CP}})$ satisfies the following axiom: For all sets J and mappings $\psi : J \longrightarrow X, \sigma = (\sigma_1, \sigma_2) : J \longrightarrow \mathsf{PF}^{\top}_{\mathsf{L}}(X)$ with $\psi(j) \in q^{\mathcal{CP}}(\sigma(j))$ for all $j \in J$ and $(\mathbb{F}, \mathbb{G}) \in \mathsf{PF}^{\top}_{\mathsf{L}}(J)$ we have $x \in q^{\mathcal{CP}}(\kappa\sigma(\mathbb{F}, \mathbb{G}))$ whenever $x \in q^{\mathcal{CP}}(\psi(\mathbb{F}, \mathbb{G}))$.*

Proof. We have $\psi(j) \in q^{\mathcal{CP}}(\sigma(j))$ if, and only if, $\sigma(j) \wedge ([\psi(j)], [\psi(j)]) \in \mathcal{CP}$ and we have $x \in q^{\mathcal{CP}}(\psi((\mathbb{F}, \mathbb{G})))$ if, and only if, $(\psi(\mathbb{F}) \wedge [x], \psi(\mathbb{G}) \wedge [x]) \in \mathcal{CP}$. The latter implies that $(\psi(\mathbb{F}), \psi(\mathbb{G})) \in \mathcal{CP}$. We define $\sigma^* = (\sigma_1^*, \sigma_2^*) : J \longrightarrow \mathsf{PF}^{\top}_{\mathsf{L}}(X)$ by $\sigma_k^*(j) = \sigma_k(j) \wedge [\psi(j)]$ for $k = 1, 2$. Then also $\sigma^*(j) \wedge ([\psi(j)], [\psi(j)]) \in \mathcal{CP}$ for all $j \in J$ and hence $\kappa\sigma^*(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$. We show that $\kappa\sigma^*(\mathbb{F}, \mathbb{G}) \le \kappa\sigma(\mathbb{F}, \mathbb{G}) \wedge (\psi(\mathbb{F}), \psi(\mathbb{G}))$. Let $a \in \kappa\sigma_1^*\mathbb{F}$. Then $\widehat{\sigma_1}^*(a) \in \mathbb{F}$. We have $\widehat{\sigma}_1^*(a) \le \widehat{\sigma_1}(a)$. Also, for $j \in J$, we have

$$\widehat{\sigma_1}^*(a)(j) \le \bigvee_{d \in [\psi(j)]} [d, a] \le \bigvee_{d \in [\psi(j)]} (\underbrace{d(\psi(j))}_{=\top} \to a(\psi(j))) = \psi^{\leftarrow}(a)(j).$$

Hence we have $\widehat{\sigma_1}(a) \in \mathbb{F}$, that is, $a \in \kappa\sigma_1\mathbb{F}$ and $\psi^{\leftarrow}(a) \in \mathbb{F}$, that is, $a \in \psi(\mathbb{F})$ and we have shown that $\kappa\sigma_1^*\mathbb{F} \le \kappa\sigma_1\mathbb{F} \wedge \psi(\mathbb{F})$. Similarly, we can show that $\kappa\sigma_2^*\mathbb{G} \le \kappa\sigma_2\mathbb{G} \wedge \psi(\mathbb{F})$. Hence we conclude $(\psi(\mathbb{F}) \wedge \kappa\sigma_1\mathbb{F}, \psi(\mathbb{G}) \wedge \kappa\sigma_2\mathbb{G}) \in \mathcal{CP}$ and $(\psi(\mathbb{F}) \wedge [x], \psi(\mathbb{G}) \wedge [x]) \in \mathcal{CP}$ which implies $(\psi(\mathbb{F}) \wedge \kappa\sigma_1\mathbb{F} \wedge [x], \psi(\mathbb{G}) \wedge \kappa\sigma_2\mathbb{G} \wedge [x]) \in \mathcal{CP}$ and therefore $\kappa\sigma(\mathbb{F}, \mathbb{G}) \wedge ([x], [x]) \in \mathcal{CP}$, that is, $x \in q^{\mathcal{CP}}(\kappa\sigma(\mathbb{F}, \mathbb{G}))$. □

Proposition 6.4. *A topological $\top$-quasi-Cauchy space $(X, \mathcal{CP})$ is pretopological.*

Proof. Let $x \in X$ and define $J = \{(\mathbb{F}, \mathbb{G}) \in \mathsf{PF}^{\top}_{\mathsf{L}}(X) \ : \ x \in q^{\mathcal{CP}}((\mathbb{F}, \mathbb{G}))\}$. Let $\psi : J \longrightarrow X$ be defined by $\psi(j) = x$ for all $j \in J$ and $\sigma = (\sigma_1, \sigma_2) : J \longrightarrow \mathsf{PF}^{\top}_{\mathsf{L}}(X)$ be defined by $\sigma((\mathbb{F}, \mathbb{G})) = (\mathbb{F}, \mathbb{G})$. We consider $([\top_J], [\top_J]) \in \mathsf{PF}^{\top}_{\mathsf{L}}(X)$. From $a \in \psi([\top_J])$ if, and only if $\top = \bigwedge_{j \in J} \top_J(j) \to \psi^{\leftarrow}(a)(j) = \bigwedge_{j \in J} a(\psi(j)) = a(x)$ if, and only if, $a \in [x]$ we obtain $\psi(([\top_J], [\top_J])) = ([x], [x]) \in \mathcal{CP}$. Furthermore, we have $\sigma(j) \wedge ([\psi(j)], [\psi(j)]) = (\mathbb{F}, \mathbb{G}) \wedge ([x], [x]) \in \mathcal{CP}$ for all $j = (\mathbb{F}, \mathbb{G}) \in J$. Hence, as $(X, \mathcal{CP})$ is topological, we have $\kappa\sigma([\top_J], [\top_J]) \in \mathcal{CP}$. We have $a \in \kappa\sigma_1[\top_J]$ if, and only if, $\widehat{\sigma_1}(a) \in [\top_J]$. This is equivalent to $\top = \bigvee_{d \in \sigma_1(j)} [d, a]$ for all $j \in J$, that is, to $a \in \bigwedge_{j \in J} \sigma_1(j)$. This shows with the definition of J that $\kappa\sigma([\top_J], [\top_J]) = \bigwedge_{(\mathbb{F}, \mathbb{G}) \in q^{\mathcal{CP}}((\mathbb{F}, \mathbb{G}))} (\mathbb{F}, \mathbb{G}) = (\mathbb{U}^l_x, \mathbb{U}^r_x) \in \mathcal{CP}$ and it is clear that this pair $\top$-filter converges to x in $(X, \mathcal{CP})$. □

Using this proposition, we can characterize topological $\top$-quasi-Cauchy spaces $(X, \mathcal{CP})$ in another, simpler way. To this end, we define $\sigma_{\mathbb{U}} : X \longrightarrow \mathsf{PF}^{\top}_{\mathsf{L}}(X)$ by $\sigma_{\mathbb{U}}(x) = (\sigma_{\mathbb{U}^l}, \sigma_{\mathbb{U}^r}) = (\mathbb{U}^l_x, \mathbb{U}^r_x)$ for all $x \in X$, where $(\mathbb{U}^l_x, \mathbb{U}^r_x) = \bigwedge_{x \in q^{\mathcal{CP}}((\mathbb{F}, \mathbb{G}))} (\mathbb{F}, \mathbb{G})$, and denote, for $(\mathbb{F}, \mathbb{G}) \in \mathsf{PF}^{\top}_{\mathsf{L}}(X)$, the diagonal pair $\top$-filter by $\mathbb{U}(\mathbb{F}, \mathbb{G}) = (\mathbb{U}^l(\mathbb{F}), \mathbb{U}^r(\mathbb{G})) = \kappa\sigma_{\mathbb{U}}(\mathbb{F}, \mathbb{G})$.

Proposition 6.5. *A $\top$-quasi-Cauchy space is topological if, and only if, $\mathbb{U}(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ whenever $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$.*

Proof. As a topological $\top$-quasi-Cauchy space is pretopological, we have $\sigma_{\mathbb{U}}(x) \wedge ([x], [x]) \in \mathcal{CP}$ for all $x \in X$. Hence, with $J = X, \psi = id_X$ and $\sigma = \sigma_{\mathbb{U}}$, the stated condition is true.

To show the converse, let J be a set, $\psi : J \longrightarrow X$ and $\sigma = (\sigma_1, \sigma_2) : J \longrightarrow \mathsf{PF}^{\top}_{\mathsf{L}}(X)$ with $\sigma(j) \wedge ([\psi(j)], [\psi(j)]) \in \mathcal{CP}$ for all $j \in J$. Then $\sigma(j) \ge (\mathbb{U}^l_{\psi(j)}, \mathbb{U}^r_{\psi(j)})$ for all $j \in J$. If $\psi((\mathbb{F}, \mathbb{G})) \in \mathcal{CP}$, then $\mathbb{U}(\psi(\mathbb{F}), \psi(\mathbb{G})) \in \mathcal{CP}$ and we show that $\mathbb{U}(\psi(\mathbb{F}), \psi(\mathbb{G})) \le \kappa\sigma(\mathbb{F}, \mathbb{G})$. Let $a \in \kappa\sigma_{\mathbb{U}^l}(\psi(\mathbb{F}))$. Then $\widehat{\sigma_{\mathbb{U}^l}}(a) \in \psi(\mathbb{F})$. For $j \in J$ we have

$$\psi^{\leftarrow}(\widehat{\sigma_{\mathbb{U}^l}}(a))(j) = \bigvee_{d \in \mathbb{U}^l_{\psi(j)}} [d, a] \le \bigvee_{d \in \sigma_1(j)} [d, a] = \widehat{\sigma_1}(a)(j),$$

and hence $\widehat{\sigma_1}(a) \in \mathbb{F}$, that is, $a \in \kappa\sigma_1\mathbb{F}$. Similarly, we can show $\kappa\sigma_{\mathbb{U}^r}(\psi(\mathbb{G})) \le \kappa\sigma_2\mathbb{G}$ and we conclude $\kappa\sigma(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ and $(X, \mathcal{CP})$ is topological. □

For the next result we remember that the existence of a topological completion for a non-complete $\top$-Cauchy space implies the fullness of the space. Hence we will not demand too much.

Theorem 6.6. *If $(X, \mathcal{CP})$ is a non-complete, full and topological $\top$-quasi-Cauchy space, then $(X^*, \widetilde{\mathcal{CP}})$ is topological.*

Proof. Let $(\mathbb{H}, \mathbb{K}) \in \widetilde{\mathcal{CP}}$. Then there is $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ such that $(\widetilde{\mathbb{F}}^1, \widetilde{\mathbb{G}}^2) \le (\mathbb{H}, \mathbb{K})$. As $(X, \mathcal{CP})$ is topological, we have $\mathbb{U}(\mathbb{F}, \mathbb{G}) = (\mathbb{U}^l(\mathbb{F}), \mathbb{U}^r(\mathbb{G})) \in \mathcal{CP}$ and hence $(\widetilde{\mathbb{U}^l(\mathbb{F})}^1, \widetilde{\mathbb{U}^r(\mathbb{G})}^2) \in \widetilde{\mathcal{CP}}$. We denote the coarsest pair $\top$-filter converging to x^* in $(X^*, \widetilde{\mathcal{CP}})$ by $(\mathbb{U}^{*l}_{x^*}, \mathbb{U}^{*r}_{x^*})$ and we show that $\widetilde{\mathbb{U}^l(\mathbb{F})}^1 \le \mathbb{U}^{*l}(\widetilde{\mathbb{F}}^1)$ and $\widetilde{\mathbb{U}^r(\mathbb{G})}^2 \le \mathbb{U}^{*r}(\widetilde{\mathbb{G}}^2)$. In Theorem 5.3 we found $\mathbb{U}^*_x = (\mathbb{U}^{*l}_x, \mathbb{U}^{*r}_x) = (\widetilde{\mathbb{U}^l_x}^1, \widetilde{\mathbb{U}^r_x}^2)$ for $x \in X$ and $\mathbb{U}^*_{\langle(\mathbb{F}, \mathbb{G})\rangle} = (\mathbb{U}^{*l}_{\langle(\mathbb{F}, \mathbb{G})\rangle}, \mathbb{U}^{*r}_{\langle(\mathbb{F}, \mathbb{G})\rangle}) = (\widetilde{\mathbb{F}_{\min}}^1, \widetilde{\mathbb{G}_{\min}}^2)$ for $\langle(\mathbb{F}, \mathbb{G})\rangle \in X^* \setminus X$. Let now $a \in \widetilde{\mathbb{U}^l(\mathbb{F})}^1$. Then we have $\top = \bigvee_{h \in \mathbb{U}^l(\mathbb{F})} [\widetilde{h}^1, a]$. We note that $h \in \mathbb{U}^l(\mathbb{F})$ is equivalent to $\widehat{\sigma_{\mathbb{U}^l}}(h) \in \mathbb{F}$ and hence implies $\widetilde{\widehat{\sigma_{\mathbb{U}^l}}(h)}^1 \in \widetilde{\mathbb{F}}^1$. For $x \in X$ we have

$$\widetilde{\widehat{\sigma_{\mathbb{U}^l}}(h)}^1(x) = \bigvee_{d \in \mathbb{U}^l_x} [d, h] = \bigvee_{d \in \mathbb{U}^l_x} [\widetilde{d}^1, \widetilde{h}^1] \le \bigvee_{\widetilde{d}^1 \in \widetilde{\mathbb{U}^l_x}^1} [\widetilde{d}^1, \widetilde{h}^1] \le \widehat{\sigma_{\mathbb{U}^{*l}}}(\widetilde{h}^1)(x).$$

For $\langle(\mathbb{F}, \mathbb{G})\rangle \in X^* \setminus X$ we have

$$\widetilde{\widehat{\sigma_{\mathbb{U}^l}}(h)}^1(\langle(\mathbb{F}, \mathbb{G})\rangle) = \bigvee_{f \in \mathbb{F}_{\min}} [f, \widehat{\sigma_{\mathbb{U}^l}}(h)] \le \bigvee_{f \in \mathbb{F}_{\min}} [f, h] = \bigvee_{f \in \mathbb{F}_{\min}} [\widetilde{f}^1, \widetilde{h}^1] \le \widehat{\sigma_{\mathbb{U}^{*l}}}(\widetilde{h}^1)(\langle(\mathbb{F}, \mathbb{G})\rangle).$$

Hence, $a \in \widetilde{\mathbb{U}^l(\mathbb{F})}^1$ implies $\top = \bigvee_{\widetilde{h}^1 \in \mathbb{U}^{*l}(\widetilde{\mathbb{F}}^1)} [\widetilde{h}^1, a]$, that is, it implies $a \in \mathbb{U}^{*l}(\widetilde{\mathbb{F}}^1)$ and we conclude $\widetilde{\mathbb{U}^l(\mathbb{F})}^1 \le \mathbb{U}^{*l}(\widetilde{\mathbb{F}}^1) \le \mathbb{U}^{*l}(\mathbb{H})$. In a similar way we can show $\widetilde{\mathbb{U}^l(\mathbb{G})}^2 \le \mathbb{U}^{*r}(\widetilde{\mathbb{G}}^2) \le \mathbb{U}^{*r}(\mathbb{K})$ and therefore $\mathbb{U}^*(\mathbb{H}, \mathbb{K}) \in \widetilde{\mathcal{CP}}$. Proposition 6.5 yields that $(X^*, \widetilde{\mathcal{CP}})$ is topological. □

Combining Theorem 6.6 and Propositions 6.4, 6.2, 5.4 we conclude that a $\top$-quasi-Cauchy space has a topological completion if, and only if, it is topological and full.

Among the topological completions in standard form of a (necessarily topological and full) $\top$-quasi-Cauchy space, the completion $(X^*, \widetilde{\mathcal{CP}})$ is the coarsest. This is the content of the next proposition.

Proposition 6.7. *Let $(X, \mathcal{CP})$ be a non-complete, full and topological $\top$-quasi-Cauchy space. If $((X^*, \mathcal{CP}^+), j)$ is a topological completion of $(X, \mathcal{CP})$ in standard form, then $\mathcal{CP}^+ \subseteq \widetilde{\mathcal{CP}}$.*

Proof. We denote, for $x \in X$, the coarsest converging pair $\top$-filter in $(X, \mathcal{CP})$ by $(\mathbb{U}^l_x, \mathbb{U}^r_x)$ and the ones in $(X^*, \mathcal{CP}^+)$, for $x^* \in X^*$, by $(\mathbb{U}^{+l}_{x^*}, \mathbb{U}^{+r}_{x^*})$.

First we note that from $(\mathbb{U}^l_x \wedge [x], \mathbb{U}^r_x \wedge [x]) \in \mathcal{CP}$ we conclude $(j(\mathbb{U}^l_x) \wedge [j(x)], j(\mathbb{U}^r_x) \wedge [j(x)]) \in \mathcal{CP}^+$ and hence $(\mathbb{U}^{+l}_{j(x)}, \mathbb{U}^{+r}_{j(x)}) \le (j(\mathbb{U}^l_x), j(\mathbb{U}^r_x)) \le ([j(x)], [j(x)])$.

Also, as $((X^*, \mathcal{CP}^+), j)$ is in standard form, for $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$, we know that $(j(\mathbb{F}_{\min}), j(\mathbb{G}_{\min}))$ converges to $\langle(\mathbb{F}_{\min}, \mathbb{G}_{\min})\rangle = \langle(\mathbb{F}, \mathbb{G})\rangle$ and hence $(\mathbb{U}^{+l}_{\langle(\mathbb{F},\mathbb{G})\rangle}, \mathbb{U}^{+r}_{\langle(\mathbb{F},\mathbb{G})\rangle}) \le (j(\mathbb{F}_{\min}), j(\mathbb{G}_{\min}))$.

We use this and show that for $c \in L^{X^*}$ we have $\widehat{\sigma_{\mathbb{U}^{+l}}}(c) \le \widetilde{j^{\leftarrow}(c)}^1$ and $\widehat{\sigma_{\mathbb{U}^{+r}}}(c) \le \widetilde{j^{\leftarrow}(c)}^2$. For $x \in X$ we have

$$\widehat{\sigma_{\mathbb{U}^{+l}}}(c)(j(x)) \le \bigvee_{d \in [j(x)]} [d, c] = c(j(x)) = j^{\leftarrow}(c)(x) = \widetilde{j^{\leftarrow}(c)}^1(j(x)),$$

and for $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$ we have

$$\widehat{\sigma_{\mathbb{U}^{+l}}}(c)(\langle(\mathbb{F}, \mathbb{G})\rangle) \le \bigvee_{d \in j(\mathbb{F}_{\min})} [d, c] = \bigvee_{f \in \mathbb{F}_{\min}} [j(f), c] = \bigvee_{f \in \mathbb{F}_{\min}} [f, j^{\leftarrow}(c)] = \widetilde{j^{\leftarrow}(c)}^1(\langle(\mathbb{F}, \mathbb{G})\rangle).$$

This implies that for $\mathbb{H} \in \mathsf{F}^{\top}_{\mathsf{L}}(X)$ we have that $c \in \mathbb{U}^{+l}(\mathbb{H})$ implies $\widetilde{j^{\leftarrow}(c)}^1 \in \mathbb{H}$.

Let now $(\mathbb{H}, \mathbb{K}) \in \mathcal{CP}^+$. As $(X^*, \mathcal{CP}^+)$ is topological, $\mathbb{U}^+(\mathbb{H}, \mathbb{K}) = (\mathbb{U}^{+l}(\mathbb{H}), \mathbb{U}^{+r}(\mathbb{K})) \in \mathcal{CP}^+$. We show that $j^{\leftarrow}(\mathbb{U}^{+l}(\mathbb{H}))$ exists. We have for $k \in \mathbb{U}^{+l}(\mathbb{H})$ that $\widehat{\sigma_{\mathbb{U}^{+l}}}(k) \in \mathbb{H}$ and hence

$$\top = \bigvee_{x^* \in X^*} \widehat{\sigma_{\mathbb{U}^{+l}}}(k)(x^*) \le \bigvee_{x \in X} \bigvee_{d \in [j(x)]} [d, k] \vee \bigvee_{(\mathbb{F},\mathbb{G}) \in \mathcal{N}_X} \bigvee_{d \in j(\mathbb{F}_{\min})} [d, k] \le \bigvee_{x \in X} k(j(x)).$$

Similarly, $j^{\leftarrow}(\mathbb{U}^{+r}(\mathbb{K}))$ exists. From $((j(j^{\leftarrow}(\mathbb{U}^{+l}(\mathbb{H}))), j(j^{\leftarrow}(\mathbb{U}^{+r}(\mathbb{K})))) \ge (\mathbb{U}^{+l}(\mathbb{H}), \mathbb{U}^{+r}(\mathbb{K}))$ we deduce $(j^{\leftarrow}(\mathbb{U}^{+l}(\mathbb{H})), j^{\leftarrow}(\mathbb{U}^{+r}(\mathbb{K}))) \in \mathcal{CP}$. Therefore $(\widetilde{j^{\leftarrow}(\mathbb{U}^{+l}(\mathbb{H}))}^1, \widetilde{j^{\leftarrow}(\mathbb{U}^{+r}(\mathbb{K}))}^2) \in \widetilde{\mathcal{CP}}$. For $a \in \widetilde{j^{\leftarrow}(\mathbb{U}^{+l}(\mathbb{H}))}^1$ we have

$$\top = \bigvee_{c \in \mathbb{U}^{+l}(\mathbb{H})} [\widetilde{j^{\leftarrow}(c)}^1, a] \le \bigvee_{\widetilde{j^{\leftarrow}(c)}^1 \in \mathbb{H}} [\widetilde{j^{\leftarrow}(c)}^1, a] \le \bigvee_{h \in \mathbb{H}} [h, a],$$

and hence $a \in \mathbb{H}$, that is, $\widetilde{j^{\leftarrow}(\mathbb{U}^{+l}(\mathbb{H}))}^1 \le \mathbb{H}$. Similarly, $\widetilde{j^{\leftarrow}(\mathbb{U}^{+r}(\mathbb{K}))}^2 \le \mathbb{K}$ and this implies $(\mathbb{H}, \mathbb{K}) \in \widetilde{\mathcal{CP}}$. □

In order to show that there is always also a finest completion, we will show that for a topological and full $\top$-quasi-Cauchy space $(X, \mathcal{CP})$, the finest completion $(X^*, \mathcal{CP}^*)$ is topological.

We first need some preparations.

Proposition 6.8. *Let $(X, \mathcal{CP})$ be full and not complete and denote, for $x \in X$ the coarsest pair $\top$-filter converging to x in $(X, \mathcal{CP})$ by $(\mathbb{U}^l_x, \mathbb{U}^r_x)$. Let further $(X^*, \mathcal{CP}^*)$ be the finest completion of $(X, \mathcal{CP})$ and denote for $x^* \in X^*$ the coarsest pair $\top$-filter converging to x^* in $(X^*, \mathcal{CP}^*)$ by $(\mathbb{U}^{*l}_{x^*}, \mathbb{U}^{*r}_{x^*})$. Then, for $x \in X$ we have $(\mathbb{U}^{*l}_{j(x)}, \mathbb{U}^{*r}_{j(x)}) = (j(\mathbb{U}^l_x), j(\mathbb{U}^r_x))$ and for $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$ we have $(\mathbb{U}^{*l}_{\langle(\mathbb{F},\mathbb{G})\rangle}, \mathbb{U}^{*r}_{\langle(\mathbb{F},\mathbb{G})\rangle}) = (j(\mathbb{F}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle], j(\mathbb{G}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle])$.*

Proof. Let $x \in X$. Then $\mathbb{U}^l_x \wedge [x] \in \mathcal{CP}$ and hence $j(\mathbb{U}^l_x) \wedge [j(x)] \in \mathcal{CP}^*$ which implies $\mathbb{U}^{*l}_{j(x)} \le j(\mathbb{U}^l_x)$.

To show the converse inequality, let $(\mathbb{H}, \mathbb{K})$ converge to $j(x)$ in $(X^*, \mathcal{CP}^*)$. If there were $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$ such that $(j(\mathbb{F}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle], j(\mathbb{G}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]) \le (\mathbb{H} \wedge [j(x)], \mathbb{K} \wedge [j(x)])$, then as in the proof of Proposition 6.1 in [11] we see $\mathbb{F}, \mathbb{G} \le [x]$ and hence $(\mathbb{F}, \mathbb{G})$ would converge to x in $(X, \mathcal{CP})$, a contradiction. Hence there are $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ convergent such that $(j(\mathbb{F}, j(\mathbb{G})) \le (\mathbb{H} \wedge [j(x)], \mathbb{K} \wedge [j(x)])$. Hence $(j(\mathbb{F}) \wedge [j(x)], j(\mathbb{G}) \wedge [j(x)]) \in \mathcal{CP}^*$ which implies $(\mathbb{F} \wedge [x], \mathbb{G} \wedge [x]) \in \mathcal{CP}$. Hence $(\mathbb{U}^l_x, \mathbb{U}^r_x) \le (\mathbb{F}, \mathbb{G})$ and we conclude $(j(\mathbb{U}^l_x), j(\mathbb{U}^r_x)) \le (j(\mathbb{F}), j(\mathbb{G})) \le (\mathbb{H}, \mathbb{K})$. This implies $(j(\mathbb{U}^l_x), j(\mathbb{U}^r_x)) \le (\mathbb{U}^{*l}_{j(x)}, \mathbb{U}^{*r}_{j(x)})$.

Let now $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$. As $(X^*, \mathcal{CP}^*)$ is in standard form, then $(j(\mathbb{F}_{\min}), j(\mathbb{G}_{\min}))$ converges to $\langle(\mathbb{F}_{\min}, \mathbb{G}_{\min}) = \langle(\mathbb{F}, \mathbb{G})\rangle$ and hence $(\mathbb{U}^{*l}_{\langle(\mathbb{F},\mathbb{G})\rangle}, \mathbb{U}^{*r}_{\langle(\mathbb{F},\mathbb{G})\rangle}) \le (j(\mathbb{F}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle], j(\mathbb{G}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle])$, as we always have $(\mathbb{U}^{*l}_{\langle(\mathbb{F},\mathbb{G})\rangle}, \mathbb{U}^{*r}_{\langle(\mathbb{F},\mathbb{G})\rangle}) \le ([\langle(\mathbb{F},\mathbb{G})\rangle], [\langle(\mathbb{F},\mathbb{G})\rangle])$. To show the converse inequality, let $(\mathbb{H}, \mathbb{K})$ converge to $\langle(\mathbb{F}, \mathbb{G})\rangle$ in $(X^*, \mathcal{CP}^*)$. If there were $(\mathbb{L}, \mathbb{M}) \in \mathcal{CP}$ convergent such that $(j(\mathbb{L}), j(\mathbb{M})) \le (\mathbb{H} \wedge [\langle(\mathbb{F},\mathbb{G})\rangle], \mathbb{K} \wedge [\langle(\mathbb{F},\mathbb{G})\rangle])$, then we had for $a \in \mathbb{L}$, $\top = \bigvee_{h \in \mathbb{H}} [h \vee \top_{\{\langle(\mathbb{F},\mathbb{G})\rangle\}}, j(a)] \le \top \to j(a)(\langle(\mathbb{F},\mathbb{G})\rangle) = \top \to \bot = \bot$, a contradiction. Hence there are $(\mathbb{L}, \mathbb{M}) \in \mathcal{N}_X$ such that $(j(\mathbb{L}) \wedge \langle(\mathbb{L},\mathbb{M})\rangle, j(\mathbb{M}) \wedge [\langle(\mathbb{L},\mathbb{M})\rangle]) \le (\mathbb{H} \wedge [\langle(\mathbb{F},\mathbb{G})\rangle], \mathbb{K} \wedge [\langle(\mathbb{F},\mathbb{G})\rangle])$. As in the proof of Proposition 6.1 in [11] then $\langle(\mathbb{L},\mathbb{M})\rangle = \langle(\mathbb{F},\mathbb{G})\rangle = \langle(\mathbb{F}_{\min},\mathbb{G}_{\min})\rangle$ and hence $\mathbb{F}_{\min} \le \mathbb{L}$ and $\mathbb{G}_{\min} \le \mathbb{M}$. This implies $j(\mathbb{F}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \le \mathbb{H} \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \le \mathbb{H}$ and, likewise, $j(\mathbb{G}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \le \mathbb{K}$. Hence we have also $(j(\mathbb{F}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle], j(\mathbb{G}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]) \le (\mathbb{U}^{*l}_{\langle(\mathbb{F},\mathbb{G})\rangle}, \mathbb{U}^{*r}_{\langle(\mathbb{F},\mathbb{G})\rangle})$. □

Lemma 6.9. *Under the assumptions of Proposition 6.8 we have, for $b \in L^X$, $j(\widehat{\sigma_{\mathbb{U}^l}}(b)) \le \widehat{\sigma_{\mathbb{U}^{*l}}}(j(b))$ and $j(\widehat{\sigma_{\mathbb{U}^r}}(b)) \le \widehat{\sigma_{\mathbb{U}^{*r}}}(j(b))$.*

Proof. For $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$ we have $j(\widehat{\sigma_{\mathbb{U}^l}}(b))(\langle(\mathbb{F},\mathbb{G})\rangle) = \bot$. For $x \in X$, we have $j(\widehat{\sigma_{\mathbb{U}^l}}(b))(j(x)) = \widehat{\sigma_{\mathbb{U}^l}}(b)(x) \le \bigvee_{j(d) \in j(\mathbb{U}^l_x)} [j(d), j(b)] = \bigvee_{j(d) \in \mathbb{U}^{*l}_{j(x)}} [j(d), j(b)] \le \widehat{\sigma_{\mathbb{U}^{*l}}}(j(b))(j(x))$. □

Lemma 6.10. *Under the assumptions of Proposition 6.8 we have, for $\mathbb{F} \in \mathsf{F}^\top_\mathsf{L}(X)$, $j(\mathbb{U}^l(\mathbb{F})) \le \mathbb{U}^{*l}(j(\mathbb{F}))$ and $j(\mathbb{U}^r(\mathbb{F})) \le \mathbb{U}^{*r}(j(\mathbb{F}))$.*

Proof. For $b \in \mathbb{U}^l(\mathbb{F})$ we have $\widehat{\sigma_{\mathbb{U}^l}}(b) \in \mathbb{F}$ and hence, $j(\widehat{\sigma_{\mathbb{U}^l}}(b)) \in j(\mathbb{F})$. With the previous Lemma this implies $\widehat{\sigma_{\mathbb{U}^{*l}}} \in j(\mathbb{F})$ which is equivalent to $j(b) \in \mathbb{U}^{*l}(j(\mathbb{F}))$. We conclude, for $a \in j(\mathbb{U}^l(\mathbb{F}))$ that $\top = \bigvee_{b \in \mathbb{U}^l(\mathbb{F})} [j(b), a] \le \bigvee_{h \in \mathbb{U}^{*l}(j(\mathbb{F}))} [h, a]$ and hence $a \in \mathbb{U}^{*l}(j(\mathbb{F}))$. □

Lemma 6.11. *Under the assumptions of Proposition 6.8 we have, for $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$, that $j(\mathbb{F}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \le \mathbb{U}^{*l}(j(\mathbb{F}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle])$ and $j(\mathbb{G}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \le \mathbb{U}^{*r}(j(\mathbb{G}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle])$.*

Proof. From $\mathbb{F}_{\min} \le \mathbb{F}$ we obtain $\mathbb{U}^l(\mathbb{F}_{\min}) \le \mathbb{U}^l(\mathbb{F}) \le \mathbb{F}$ and, similarly, $\mathbb{U}^r(\mathbb{G}_{\min}) \le \mathbb{G}$ and hence $(\mathbb{U}^l(\mathbb{F}_{\min}), \mathbb{U}^r(\mathbb{G}_{\min})) \sim (\mathbb{F}, \mathbb{G})$, which implies $\mathbb{F}_{\min} = \mathbb{U}^l(\mathbb{F}_{\min})$ and $\mathbb{G}_{\min} = \mathbb{U}^r(\mathbb{G}_{\min})$. We conclude

$$j(\mathbb{F}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \le j(\mathbb{U}^l(\mathbb{F})) \wedge j(\mathbb{F}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \le \mathbb{U}^{*l}(j(\mathbb{F})) \wedge \mathbb{U}^{*l}_{\langle(\mathbb{F},\mathbb{G})\rangle}$$

$$= \mathbb{U}^{*l}(j(\mathbb{F})) \wedge \mathbb{U}^{*l}([\langle(\mathbb{F},\mathbb{G})\rangle]) = \mathbb{U}^{*l}(j(\mathbb{F}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]).$$

The other inequality can be shown in a similar way. □

Theorem 6.12. *Let $(X, \mathcal{CP})$ be a full, topological and non-complete $\top$-quasi-Cauchy space. Then the finest completion $(X^*, \mathcal{CP}^*)$ is topological.*

Proof. Let $(\mathbb{H}, \mathbb{K}) \in \mathcal{CP}^*$. We distinguish two cases.

Case 1: There is $(\mathbb{F}, \mathbb{G}) \in \mathcal{CP}$ such that $(j(\mathbb{F}), j(\mathbb{G})) \le (\mathbb{H}, \mathbb{K})$. Then we have $(\mathbb{U}^l(\mathbb{F}), \mathbb{U}^r(\mathbb{G})) \in \mathcal{CP}$ and, as $(j(\mathbb{U}^l(\mathbb{F})), j(\mathbb{U}^r(\mathbb{G}))) \le (\mathbb{U}^{*l}(j(\mathbb{F})), \mathbb{U}^{*r}(j(\mathbb{G}))) \le (\mathbb{U}^{*l}(\mathbb{H}), \mathbb{U}^{*r}(\mathbb{K}))$ we obtain $\mathbb{U}^*(\mathbb{H}, \mathbb{K}) \in \mathcal{CP}^*$.

Case 2: There are $(\mathbb{F}, \mathbb{G}) \in \mathcal{N}_X$ such that $(j(\mathbb{F}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle], j(\mathbb{G}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]) \le (\mathbb{H}, \mathbb{K})$. Then $(\mathbb{U}^+(\mathbb{F}), \mathbb{U}^r(\mathbb{G})) \in \mathcal{CP}$ and, noting that also $(\mathbb{F}_{\min}, \mathbb{G}_{\min}) \in \mathcal{CP}$, we conclude $j(\mathbb{F}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \le \mathbb{U}^{*l}(j(\mathbb{F}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle]) \le \mathbb{U}^{*l}(\mathbb{H})$ and, similarly, $j(\mathbb{G}_{\min}) \wedge [\langle(\mathbb{F},\mathbb{G})\rangle] \le \mathbb{U}^{*l}(\mathbb{K})$. Hence also in this case we have $\mathbb{U}^*(\mathbb{H}, \mathbb{K}) \in \mathcal{CP}^*$. Therefore, $(X^*, \mathcal{CP}^*)$ is topological. □

7 Conclusions

In this paper, we studied a general method for obtaining completions of $\top$-quasi-Cauchy spaces. As special cases, we studied pretopological and topological completions. Requiring (pre-)topologicalness is related to similar properties of the underlying convergence structures (see Proposition 6.3), that we would like to term $\top$-*biconvergence structures*. In the topological case, such $\top$-biconvergence structures may be the same as strong L-bitopological spaces, a concept that has not been studied so far. However, it is well-known that the study of non-symmetric structures like quasi-metrics can benefit from considering simultaneously two topologies on them. In the same vein it seems interesting to study $\top$-biconvergence structures also.

A further point that is of interest are completions of $\top$-quasi-uniform (limit) spaces. In the symmetric case, Reed [16] showed that Cauchy spaces are an appropriate framework for studying completions for uniform limit spaces and Reid and Richardson [17] started this line of research in the many-valued setting of $\top$-Cauchy spaces and $\top$-uniform limit spaces. In the non-symmetric case, Yue and Fang [21] studied the completion of $\top$-quasi-uniform spaces. We believe that the study of completions of $\top$-quasi-uniform (limit) spaces could benefit from $\top$-quasi-Cauchy spaces as well.

References

[1] J. Adámek., H. Herrlich, G. E. Strecker, *Abstract and concrete categories*, Wiley, New York, 1989.

[2] R. Bělohlávek, *Fuzzy relation systems, foundation and principles*, Kluwer Academic/Plenum Publishers, New York, Boston, Dordrecht, London, Moscow, 2002.

[3] J. Fang, Y. Yue, $\top$-*diagonal conditions and continuous extension theorem*, Fuzzy Sets and Systems, **321** (2017), 73-89.

[4] R. C. Flagg, *Quantales and continuity spaces*, Algebra Universalis, **37** (1997), 257-276.

[5] J. Gutierrez-Garcia, *On stratified L-valued filters induced by* $\top$*-filters*, Fuzzy Sets and Systems, **157** (2006), 813-819.

[6] U. Höhle, *Probabilistic topologies induced by L-fuzzy uniformities*, Manuscripta Mathematica, **38** (1982), 289-323.

[7] U. Höhle, *Commutative, residuated l-monoids*, in: *Non-classical logics and their applications to fuzzy subsets* (U. Höhle, E. P. Klement, eds.), Kluwer, Dordrecht, 1995, 53-106.

[8] D. Hofmann, G. J. Seal, W. Tholen, *Monoidal topology*, Cambridge University Press, 2014.

[9] G. Jäger, *Diagonal conditions and uniformly continuous extension in* $\top$*-uniform limit spaces*, Iranian Journal of Fuzzy Systems, **19**(5) (2022), 131-145.

[10] G. Jäger, *Sequential completeness for* $\top$*-quasi-uniform spaces and a fixed point theorem*, Mathematics, **10** (2022), 2285.

[11] G. Jäger, $\top$*-quasi-Cauchy spaces–a non-symmetric theory of completeness and completion*, Applied General Topology, **24**(1) (2023), 205-227.

[12] G. Jäger, Y. Yue, $\top$*-uniform convergence spaces*, Iranian Journal of Fuzzy Systems, **19**(2) (2022), 133-149.

[13] H. H. Keller, *Die limes-uniformisierbarkeit der Limesräume*, Mathematische Annalen, **176** (1968), 334-341.

[14] F. W. Lawvere, *Metric spaces, generalized logic, and closed categories*, Rendiconti del Seminario Matematico e Fisico di Milano, **43** (1973), 135-166. Reprinted in: Reprints in Theory and Applications of Categories, **1** (2002) 1-37.

[15] W. F. Lindgren, P. Fletcher, *A construction of the pair completion of a quasi-uniform space*, Canadian Mathematical Bulletin, **21** (1978), 53-59.

[16] E. E. Reed, *Completion of uniform convergence spaces*, Mathematische Annalen, **194** (1971), 83-108.

[17] L. Reid, G. Richardson, *Lattice-valued spaces:* $\top$*-completions*, Fuzzy Sets and Systems, **369** (2019), 1-19.

[18] L. Reid, G. Richardson, *Strict* $\top$*-embeddings*, Quaestiones Mathematicae, **43** (2020), 903-917.

[19] B. Schweizer, A. Sklar, *Probabilistic metric spaces*, North Holland, New York, 1983.

[20] Q. Yu, J. Fang, *The category of* $\top$*-convergence spaces and its Cartesian-closedness*, Iranian Journal of Fuzzy Systems, **14**(3) (2017), 121-138.

[21] Y. Yue, J. Fang, *Completeness in probabilistic quasi-uniform spaces*, Fuzzy Sets and Systems, **370** (2019), 34-62.

[22] D. Zhang, *An enriched category approach to many valued topology*, Fuzzy Sets and Systems, **158** (2007), 349-366.