Iranian Journal of Fuzzy Systems

Volume 20, Number 7, (2023), pp. 85-99

Original paper

Analyzing of process capability indices under uncertain information and hesitancy by using Pythagorean fuzzy sets

İ. Kaya[1] and S. Yalçın[2]

[1]*Department of Industrial Engineering, Yıldız Technical University, 34349, Yıldız, Istanbul, Turkey*
[2]*Department of Industrial Engineering, Beykent University, 34398, Saryer, Istanbul, Turkey*

ihkaya@yildiz.edu.tr, selinyalcin@beykent.edu.tr

Abstract

Process capability analysis (PCA) is a completely effective statistical tool for ability of a process to meet predetermined specification limits (SLs). Unfortunately, especially the real case problems include many uncertainties, it is one of the critical necessities to define the parameters of PCIs by using crisp numbers. So, the results obtained may be incorrect, if the PCIs are calculated without taking into account the uncertainty. To overcome this problem, the fuzzy set theory (FST) has been successfully used to design of PCA. We also know that fuzzy set extensions have an important role in modelling the case that include uncertainty, incomplete and inconsistent information and they are more powerful than traditional FST to model uncertainty. Defining of main parameters of PCIs such as SLs, mean (μ) and variance (σ^2) by using the flexible of fuzzy set extensions rather than precise values due to uncertainty, time, cost, inspectors hesitancy and the results based on fuzzy sets for PCIs contain more, flexible and sensitive information. In this study, two of well-known PCIs called C_p and C_{pk} have been re-designed at the first time by using one of fuzzy set extensions named Pythagorean fuzzy sets (PFSs). Defining PCIs with more than one membership function instead of an only one membership function is enabling to evaluate the process more broadly more flexibility. For this aim, the main parameters of PCIs have been defined and analyzed by using PFSs. Finally, four new PCIs based on PFSs such as $\widetilde{C}_{sp}$, $\widetilde{C}_{spk}$, $\widetilde{C}_{fp}$ and $\widetilde{C}_{fpk}$ have been derived. The proposed new PCIs based on PFSs have been also applied on manufacturing process and capability for gears have been analyzed. It is shown that the flexibility of the PFSs on PCIs enables the PCA to give more realistic, more sensitive, and more comprehensive results.

Keywords: Process capability analysis, process capability indices, the fuzzy set theory, Pythagorean fuzzy sets.

1 Introduction

Process capability analysis (PCA) is a statistical method that enables the analysis of how a manufacturing process performs against product requirements or specification limits (SLs) [4]. One of the most effective methods called process capabiliy indices (PCIs) have been widely used to measure process performance [27]. It is a very effective method to analyze capability of quality characteristic that is expressed as input variables with acceptable ranges. The minimum allowable value for a quality characteristic is called the lower specification limit (LSL) and the maximum allowable value is called the upper specification limit (USL). In this analysis, outputs produced within the specification limits (SLs) are considered conforming, and the outputs outside that because of at least one characteristic fails to conform to the SLs are considered non-conforming. A non-defective part or product has all quality characteristics within SLs. Manufacturers use the term nonconformity to describe any situation where the characteristic value of a part or product falls outside the relevant SLs [5]. The fact that the process is within the SLs means that the product is of the desired quality. Companies should regularly analyze the capability of the process and interpret results obtained

Corresponding Author: İ. Kaya
Received: February 2023; Accepted: Invited paper.
https://doi.org/10.22111/IJFS.2023.7632

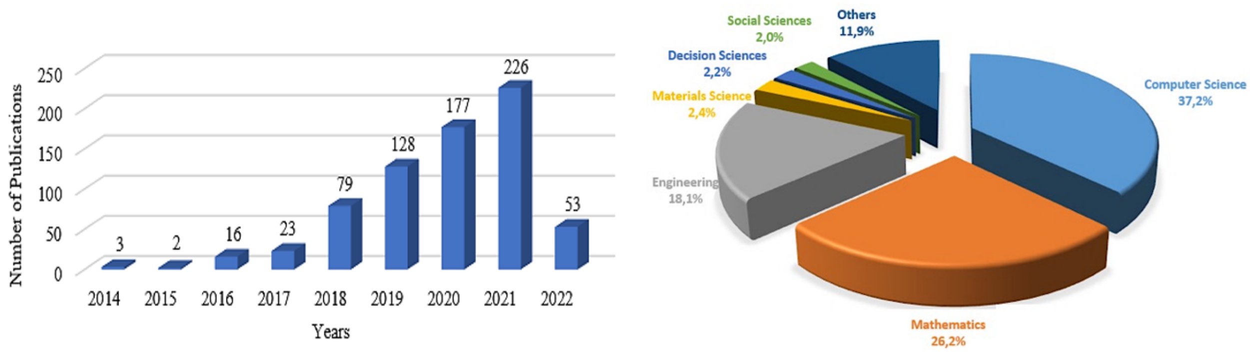
correctly to produce output with desired quality and to improve process quality. For this aim, the PCIs that are widely used to measure process performance are obtained for evaluation.

We also know that uncertainty prevails in the vast majority of real-case problems, and traditional approaches can not effective in this situation. Due to the subjective thinking of the quality inspectors who play a role in determining the SLs be of a product, or some measurement errors arising from the uncertainty in the measurement values or the inspector factor during the quality control of the samples obtained from each process in production, the process cannot be analyzed and followed up effectively. Although, the fuzzy set theory (FST) has been integrated with these uncertainities, in case of lack of sufficient information, type-1 fuzzy sets (traditional fuzzy sets) may cause the opinions of quality inspectors not be conveyed fully or to have sufficient information about the process. Although these sets are very effective tools for modeling uncertainty, in some cases they may be insufficient and/or require more effective tools for modeling. The quality inspectors may not be able to fully express their evaluations about quality process during data collection, limit definitions or may remain hesitant about evaluation for process quality evaluations. Additionally, there may be some uncertainties arising from the measurement system, process characteristics or factors based on quality inspectors such as their hesitancies and reliability level on their assessments. Because of these uncertainties, defining process parameters with precise numbers does not yield effective results in PCA. So, the classical PCA methods can not produce some remarkable results for the process if they include uncertainties. It is known that the FST has been succesfully used to model complex problems with uncertainty. The fuzzy logic approach can be effectively used in considering the uncertainty in the process or defining the limit values as approximate values. Also, it is not easy to define SLs precisely, especially in sensitive situations on real case studies. It is also clear that sometimes it is quite difficult to define these parameters as exact or crisp values. These identification studies can lead to both loss of time and increase in costs. In these cases, using FST provides both an important definition advantage and flexibility on PCA.

Recently, the developed fuzzy set extensions can be used more effectively than traditional fuzzy sets in modeling uncertainties. The fuzzy set extenisions can be used very effectively both in modeling uncertainties and in adding uncertainties in expert evaluations to the process. So, it is more appropriate to use a fuzzy set extension that has a huge capability to represent unceratinity for modelling the process. There are various extensions of fuzzy sets in the literature such as type-2 fuzzy sets (T2FSs), hesitant fuzzy sets (HFSs), intuitionistic fuzzy sets (IFSs), pythagorean fuzzy sets (PFSs), neutrosophic sets (NSs).

Fuzzy set extensions have been used in some studies because of their advantages in modeling uncertainty. Kaya and olak [17] examined the studies on PCA that used FST. They concluded that the majority of these studies are on traditional fuzzy sets. It has been observed that studies using fuzzy set extensions are quite limited in the literature. Yaln and Kaya [33] defined the SLs with NSs and introduced the neutrosophic PCIs to the literature. Chen and Hung [10] proposed fuzzy process incapability index where the SLs are T2FSs. Haktanr and Kahraman [11] developed Pythagorean fuzzy sets, which are an extension of IFSs and applied them on PCIs. Haktanr and Kahraman [12] also examined the PCA of two machines used in the production of surgical masks with Penthagorean fuzzy $\widetilde{C}_p$ and $\widetilde{C}_{pk}$ indices. Aslam and Albassam [6] presented a sampling plan by using neutrosophic statistics. They developed the $\widehat{C}_{Npk}$ index, taking into account the neutrosophic mean and standard deviation. Hesamian and Akbari [4] calculated the index $\widehat{C}_{pm}$ by defining SLs, process mean (μ) and target value (T) with the help of IFSs. Yaln and Kaya [38-39] analyzed the effects of PFSs on PCA. They analyzed the PCIs based on PFSs. Kahraman et al. [15] obtained the indices $\widetilde{C}_p$, $\widetilde{C}_{pk}$ and $\widetilde{C}_{pm}$ when SLs were defined using IFSs. Parchami et al. [25] developed the $\widetilde{\widetilde{C}}_p$, $\widetilde{\widetilde{C}}_{pk}$ and $\widetilde{\widetilde{C}}_{pm}$ indices by defining the SLs with interval T2FSs to calculate process capability. Cao et al. [9] calculated the multivariate process capability index with the help of IFSs. Senvar and Kahraman [29] developed $\widetilde{\widetilde{C}}_p$ and $\widetilde{\widetilde{C}}_{pk}$ indices by using interval T2FSs to analyze the capability of non-normal processes. Kaya and olak [17] concluded that the most frequently used indices in studies on fuzzy based PCIs are $\widetilde{C}_p$, $\widetilde{C}_{pk}$, $\widetilde{C}_{pm}$ and $\widetilde{C}_{pmk}$, respectively. Yaln and Kaya [34] analyzed the effects of neutrosofic sets (NSs) on PCA. They analyzed the indices C_p and C_{pk} based on NSs.

Altough FST has been succesfully applied on PCA, there is a limited number of studies have been done in the literature by using fuzzy set extensions. In this direction, the effects of fuzzy set extensions on PCA have been analyzed in order to fill the gap in the literature. Additionally, the fuzzy set extensions have been used to reflect all of unceratinity in process by including instructors hesitancy. For this aim, one of the fuzzy set extensions that were developed based on IFSs and named Pythagorean fuzzy sets (PFSs) has been used in this paper. While FST developed by Zadeh [37] includes only the degree of membership defined in the range [0,1], the IFSs put forward by Atanassov [7] are expressed by the degree of membership as well as the degree of non-membership in the range of [0,1]. In IFSs, the sum of membership and non-membership degrees should be less than 1. Yager [32] introduced PFSs, which are a more advanced form of IFSs, to the literature. The sum of membership and non-membership degrees of PFSs can be greater than 1, but the sum of their squares must be less than 1.00. We also know that in some real case problems such as calculating of process capability, the sum of membership and non-membership degrees that summarize alternative evaluations for any criteria decided by decision makers (DMs) may be bigger than 1, but their square sum is less than or equal to 1. Accordingly, the Pythagorean membership degrees are larger than the intuitionistic membership degrees space. Thus, using PFSs instead of IFSs provides a better expression of uncertainty. For this reason, the effects of PFSs, which are more successful in handling uncertainty than IFSs, on the PCA are investigated. In the literature, there are studies that take advantage of the PFSs. A summary of 707 studies on PFSs in the Scopus database is summarized in Fig. 1.



(a) (b)

Figure 1: Distribution of the PFSs publications (a) based on years (b) by type of application area

In Fig. 1, it is seen that the studies using PFSs are increasing day by day and they are frequently discussed in fields such as computers, engineering and mathematics. This shows that PFSs are very successfully and frequently used in modeling uncertainty. PFSs are used in many studies because of their advantages. For example, Shakeel et al. [30] introduced interval-valued Pythagorean trapezoidal fuzzy numbers (IVPTFNs) and some operations on IVPTFNs. They defined different types of operators for aggregating IVPTFNs and developed a general algorithm for group decision making problem. Shakeel et al. [31] defined interval-valued Pythagorean trapezoidal fuzzy (IVPTF) aggregation operators. Finally, the proposed methods were applied to deal with multiple attribute group decision-making problems. Aghamohagheghi et al. [1] provided the multi-criteria decision analysis (MCDA) part of the framework with uncertainty, an IVPFS is used as an uncertain modeling tool. Aydn et al. [8] developed harmonic aggregation operators for PTFNs and used them in order to rank the alternatives. Luqman et al. [23] defined Pythagorean triangular fuzzy numbers (PTrFNs) and used digraphs and matrix techniques for risk evaluations. Akram et al. [3] extended the linear programming problem based on PTrFNs. Kahraman et al. [14] modeled facial expressions of a humanoid robot with PFSs. Rani et al. [28] developed the entropy measure and score function using the PFSs and applied them in pharmacological therapy selection for type 2 diabetes. Ak and Gul [2] developed a combined approach based on AHP and TOPSIS methods using PFSs for information security risk analysis. Kumar et al. [22] introduced a novel approach for the transportation problem under Pythagorean fuzzy environment. The literature review shows that PFSs have been widely studied and successfully applied to many application areas such as aggregation operators [8, 30, 31], decision-making [1, 2, 23, 28], modelling [3, 14] and transportation management [22]. So, the PFSs have been utilized to deal with uncertainty in this study.

PFSs can be successfully used to handle uncertainty in some situations where traditional fuzzy sets cannot model. Therefore, PFSs provide an effective approach on PCA compared to IFSs in terms of flexibility. It is clear that belonging and non-belonging, which are the basic characteristics of PFSs, are very suitable for PCA. It is thought that a definition below the limit values will not be very accurate in some cases and it would be wrong to evaluate the product as completely inadequate. Similarly, it would not be correct to say that it is completely suitable for a product that is just above the limit value. Therefore, for evaluations that are around the limit values, it will be more appropriate to evaluate them both within and outside the limits. As a result, it will be useful to model this approach with PFSs and to situate it in PCA. In this study, these abilities of PFSs to define and model uncertainties is integrated with PCA. Thus, new PCIs have been developed that can better reflect and represent process uncertainty. The obtained PFSs based PCIs show all possible values of PCIs with functions of membership and non-membership degrees. Since quality inspectors take into account the degree of membership as well as the degree of non-membership while defining SLs, it reduces the subjectivity to a minimum level and increases the effectiveness of PCA. Hereby, the number of defective products outside the SLs is also reduced. Furthermore, analyzing the process parameters using PFSs will enable us to make more flexible, more realistic and more accurate decisions in evaluating the capability of the process. Defining process parameters as PFSs allows to reduce uncertainty and give more accurate results for PCA.

The rest of this paper has been organized as follows: The indices C_p and C_{pk} have been briefly summarized into Section 2. The PFSs have been generally introduced in Section 3. The PCIs have been analyzed based on PFSs and design of these indices is detailed into Section 4. The proposed approach has been applied on a real case application from manufacturing industry in Section 5. Finally, obtained results, conclusions and future directions have been presented in Section 6.

2 Process capability indices

Process performance can be effectively analyzed by using PCIs that are summary statistics and able to measure the actual or the potential performance of the process characteristics relative to the target and SLs by considering process location and dispersion [18]. PCIs, which provide numerical measures on whether a process meets the customer expectations or not, have been popularly applied for evaluating process performance. Several PCIs such as C_p, C_{pk}, C_{pm} and C_{pmk} are frequently used to estimate the capability of a process [17], [19]-[21]. The index C_p is defined as the ratio of specification width over the process spread. The specification width represents customer and/or product requirements. The process variations are represented by the specification width. It can be obtained by using $C_p = (USL - LSL)/6\sigma$ [19]-[24]. where σ is the standard deviation of the process, USL and LSL represent the upper and lower SLs, respectively [20]. Therefore, Kane [16] introduced the index C_{pk} which is used to provide an indication of the variability associated with a process. It shows how a process confirms to its specification. The index is usually used to relate the $naturaltolerance(3\sigma)$ to the SLs and describes how well the process fits within these limits, taking into account the location of the process mean. The index C_{pk} is calculated by using $C_{pk} = \min\{C_{pl}, C_{pu}\} = \min\{USL-\mu, LSL-\mu\}/3\sigma$ [19]-[24].

In this study, two of well-known PCIs named C_p and C_{pk} have been re-considered with respect to PFSs. For this aim, the use of PFSs has been examined in order to take into account the uncertainties of PCA more effectively and more accurately. Then, we carried out on Pythagorean fuzzy set integration in the design and mathematical infrastructure of process competence indices. The main design principles of these indices have been analyzed and mathematical formulations have been derived by using PFSs. Finally, four new PCIs based on PFSs such as $\widetilde{C}_{sp}$, $\widetilde{C}_{spk}$, $\widetilde{C}_{fp}$ and $\widetilde{C}_{fpk}$ have been derived.

3 Pythagorean fuzzy sets

After Zadeh [37] proposed FST to the literature, many different fuzzy set extensions have been developed to deal with uncertainty. We know that traditional fuzzy sets can be insufficient in solving some problems that includes uncertainties. This has led to the emergence of various fuzzy set extensions. As an extension of fuzzy sets, Yager [32] developed PFSs that deals with fuzziness with both membership and non-membership functions. PFSs have more powerful ability than IFSs to model the uncertainty in the real case problems. It is clean that the space of membership grades of PFSs are bigger than the space of membership grades of IFSs. This makes PFSs more advantageous than traditional fuzzy sets in modeling uncertainty [38]. PFSs are useful for solving the real case problems because of their flexibility. Some basic definitions of PFSs are explained as following [30]-[32], [1], [26]-[38]:

Definition 3.1. *Let a set X be a universe of discourse. A pythagorean fuzzy set $\tilde{P}$ in X is shown as:*

$$\tilde{P} = \{\langle x, P(\mu_{\tilde{p}}(x), v_{\tilde{p}}(x))\rangle \mid x \in X\}, \tag{1}$$

where $\mu_{\tilde{p}} : X \to [0,1]$ and $v_{\tilde{p}} : X \to [0,1]$ define degree of membership and the degree of non-membership of the element $x \in X$ to $\tilde{P}$, respectively, and for every $x \in X$, it holds that,

$$0 \leq (\mu_{\tilde{p}}(x))^2 + (v_{\tilde{p}}(x))^2 \leq 1. \tag{2}$$

The degree of hesitancy is calculated as follows:

$$\pi_{\tilde{p}}(x) = \sqrt{1 - (\mu_{\tilde{p}}(x))^2 - (v_{\tilde{p}}(x))^2}. \tag{3}$$

Pythagorean trapezoidal fuzzy numbers (PTFNs) are defined by inspiration of similar concepts in the interval-valued PTFNs [1].

Definition 3.2. $\widetilde{P}_1 = \left[(p_1, q_1, r_1, s_1); \mu_{\widetilde{P}_1}, v_{\widetilde{P}_1}\right]$ *and* $\widetilde{P}_2 = \left[(p_2, q_2, r_2, s_2); \mu_{\widetilde{P}_2}, v_{\widetilde{P}_2}\right]$ *be two PTFNs and* $\lambda \geq 0$*, then the arithmatical operations for PTFNs are denoted by Eq.* (4)-(9)*:*

$$\tilde{P}_1 \oplus \tilde{P}_2 = \left[\begin{array}{c} (p_1 + p_2, q_1 + q_2, r_1 + r_2, s_1 + s_2) \\ \sqrt{\left(\mu_{\widetilde{P}_1}\right)^2 + \left(\mu_{\widetilde{P}_2}\right)^2 - \left(\mu_{\widetilde{P}_1}\right)^2\left(\mu_{\widetilde{P}_2}\right)^2}, v_{\widetilde{P}_1} v_{\widetilde{P}_2} \end{array}\right]. \tag{4}$$

$$\tilde{P}_1 \otimes \tilde{P}_2 = \left[\begin{array}{c} (p_1 x p_2, q_1 x q_2, r_1 x r_2, s_1 x s_2) \\ \mu_{\widetilde{P}_1}\mu_{\widetilde{P}_2}, \sqrt{\left(v_{\widetilde{P}_1}\right)^2 + \left(v_{\widetilde{P}_2}\right)^2 - \left(v_{\widetilde{P}_1}\right)^2\left(v_{\widetilde{P}_2}\right)^2} \end{array}\right]. \tag{5}$$

$$\tilde{P}_1 \ominus \tilde{P}_2 = \left[\begin{array}{c} (p_1 - s_2, q_1 - r_2, r_1 - q_2, s_1 - p_2); \\ \sqrt{\frac{\left(\mu_{\widetilde{P}_1}\right)^2 - \left(\mu_{\widetilde{P}_2}\right)^2}{1 - \left(\mu_{\widetilde{P}_2}\right)^2}}, \frac{v_{\widetilde{P}_1}}{v_{\widetilde{P}_2}} \end{array}\right], \tag{6}$$

if $\mu_{\widetilde{P}_1} \geq \mu_{\widetilde{P}_2}$ *and* $v_{\widetilde{P}_1} \leq v_{\widetilde{P}_2}, \mu_{\widetilde{P}_2} \neq 1, v_{\widetilde{P}_2} \neq 0$, $\left(\mu_{\widetilde{P}_1}\right)^2\left(v_{\widetilde{P}_2}\right)^2 - \left(\mu_{\widetilde{P}_2}\right)^2\left(v_{\widetilde{P}_1}\right)^2 \leq \left(v_{\widetilde{P}_2}\right)^2 - \left(v_{\widetilde{P}_1}\right)^2$.

$$\tilde{P}_1 \oslash \tilde{P}_2 = \left[\begin{array}{c} (p_1/s_2, q_1/r_2, r_1/q_2, s_1/p_2); \\ \frac{\mu_{\widetilde{P}_1}}{\mu_{\widetilde{P}_2}}, \sqrt{\frac{\left(v_{\widetilde{P}_1}\right)^2 - \left(v_{\widetilde{P}_2}\right)^2}{1 - \left(v_{\widetilde{P}_2}\right)^2}}, \end{array}\right], \tag{7}$$

if $\mu_{\widetilde{P}_1} \leq \mu_{\widetilde{P}_2}$ *and* $v_{\widetilde{P}_1} \geq v_{\widetilde{P}_2}, \mu_{\widetilde{P}_2} \neq 0, v_{\widetilde{P}_2} \neq 1$, $\left(\mu_{\widetilde{P}_1}\right)^2\left(v_{\widetilde{P}_2}\right)^2 - \left(\mu_{\widetilde{P}_2}\right)^2\left(v_{\widetilde{P}_1}\right)^2 \geq \left(\mu_{\widetilde{P}_1}\right)^2 - \left(\mu_{\widetilde{P}_2}\right)^2$.

$$\lambda\tilde{P}_1 = \left[\begin{array}{c} (\lambda p_1, \lambda q_1, \lambda r_1, \lambda s_1); \\ \sqrt{1 - \left(1 - \left(\mu_{\widetilde{P}_1}\right)^2\right)^\lambda}, \left(v_{\widetilde{P}_1}\right)^\lambda \end{array}\right], \tag{8}$$

$$\tilde{P}_1^\lambda = \left[\begin{array}{c} (p_1^\lambda, q_1^\lambda, r_1^\lambda, s_1^\lambda); \\ \left(\mu_{\widetilde{P}_1}\right)^\lambda, \sqrt{1 - \left(1 - \left(v_{\widetilde{P}_1}\right)^2\right)^\lambda} \end{array}\right]. \tag{9}$$

Definition 3.3. $\widetilde{P} = \left[(p, q, r, s); \mu_{\widetilde{P}}, v_{\widetilde{P}}\right]$ *be a PTFN. The score function* $s(\widetilde{P})$ *of* $\widetilde{P}$ *is defined by Eq.* (10)*:*

$$s(\widetilde{P}) = \frac{p + q + r + s}{4}\left(\mu_{\widetilde{P}}{}^2 - v_{\widetilde{P}}{}^2\right), s(\widetilde{P}) \in [-1, 1]. \tag{10}$$

Definition 3.4. *Let* $\widetilde{P}_1$ *and* $\widetilde{P}_2$ *be two PTFNs. According to the score and accuracy functions, comparison approach is as follows:*

(i) If $s\left(\widetilde{P}_1\right) > s\left(\widetilde{P}_2\right)$*, then* $\widetilde{P}_1 > \widetilde{P}_2$*;*

(ii) If $s\left(\widetilde{P}_1\right) < s\left(\widetilde{P}_2\right)$*, then* $\widetilde{P}_1 < \widetilde{P}_2$*;*

(iii) If $s\left(\widetilde{P}_1\right) = s\left(\widetilde{P}_2\right)$*, then*

- *(a) If* $h\left(\widetilde{P}_1\right) > h\left(\widetilde{P}_2\right)$*, then* $\widetilde{P}_1 > \widetilde{P}_2$*,*
- *(b) If* $h\left(\widetilde{P}_1\right) < h\left(\widetilde{P}_2\right)$*, then* $\widetilde{P}_1 < \widetilde{P}_2$*,*
- *(c) If* $h\left(\widetilde{P}_1\right) = h\left(\widetilde{P}_2\right)$*, then* $\widetilde{P}_1 \sim \widetilde{P}_2$*.*

4 Process capability indices based on Pythagorean fuzzy set theory

For the first time in the literature, fuzzy PCIs based on PFSs (PFPCIs) named $\widetilde{C}_{sp}$ and $\widetilde{C}_{fp}$ that are derived the index C_p; $\widetilde{C}_{spk}$ and $\widetilde{C}_{fpk}$ that are derived the index C_{pk} have been obtained. These indices have been re-designed by using Pythagorean fuzzy numbers (PFNs) in this paper. The flexibility of PCIs provides by using PFSs convenience in evaluating the capability of the process under uncertainty. Since, PFSs are more suitable for real-case problems because they contain the non-member function and are more successful in dealing with uncertainty than traditional fuzzy sets. So, they have more flexibility that are used for PCA to define uncertainty. The application of the definitions based on PFSs in different forms on PCIs allows us to have more information about the capability of process. The case of process parameters being PFNs and flexible PFNs can be defined as follows and the indices $\widetilde{C}_p$ and $\widetilde{C}_{pk}$ have been obtained as detailed below.

4.1 The case that the process parameters are defined as PFNs

In this subsection, we firstly consider that the main parameters of PCIs are defined by using PTrFNs. For this aim, suppose that some critical process parameters such as specification limits ($\widetilde{SLs}$), process mean ($\widetilde{\mu}_p$), standard deviation ($\widetilde{\sigma}$) and target value ($\widetilde{T}$) can be defined as follows based on PTrFNs: $\widetilde{LSL} = \left[(lsl_1, lsl_2, lsl_3); \mu_{\widetilde{\text{LSL}}}, v_{\widetilde{\text{LSL}}}\right]$, $\widetilde{USL} = \left[(usl_1, usl_2, usl_3); \mu_{\widetilde{\text{USL}}}, v_{\widetilde{\text{USL}}}\right]$, $\tilde{\mu}_p = \left[(\mu_1, \mu_2, \mu_3); \mu_{\tilde{\mu}_p}, v_{\tilde{\mu}_p}\right]$ and $\widetilde{T} = \left[(t_1, t_2, t_3); \mu_{\widetilde{\text{T}}}, v_{\widetilde{\text{T}}}\right]$. Similarly, now suppose that the main parameters of PCIs can be defined by using PTFNs as follows: $\widetilde{LSL} = \left[(lsl_1, lsl_2, lsl_3, lsl_4); \mu_{\widetilde{\text{LSL}}}, v_{\widetilde{\text{LSL}}}\right]$, $\widetilde{USL} = \left[(usl_1, usl_2, usl_3, usl_4); \mu_{\widetilde{\text{USL}}}, v_{\widetilde{\text{USL}}}\right]$, $\tilde{\mu}_p = \left[(\mu_1, \mu_2, \mu_3, \mu_4); \mu_{\tilde{\mu}_p}, v_{\tilde{\mu}_p}\right]$, $\tilde{\sigma} = \left[(\sigma_1, \sigma_2, \sigma_3, \sigma_4); \mu_{\tilde{\sigma}}, v_{\tilde{\sigma}}\right]$ and $\widetilde{T} = \left[(t_1, t_2, t_3, t_4); \mu_{\widetilde{\text{T}}}, v_{\widetilde{\text{T}}}\right]$ [35]-[36]. Then, the index $\widetilde{C}_{sp}$ is obtained using PTrFNs as $\widetilde{C}_{sp} = \widetilde{USL} \ominus \widetilde{LSL}/6\tilde{\sigma}$. The index $\widetilde{C}_{sp}$ is obtained by using Eq. (11):

$$\widetilde{C}_{sp} = \frac{\left[(usl_1, usl_2, usl_3); \mu_{\widetilde{USL}}, v_{\widetilde{USL}}\right] \Theta \left[(lsl_1, lsl_2, lsl_3); \mu_{\widetilde{LSL}}, v_{\widetilde{LSL}}\right]}{6\left[(\sigma_1, \sigma_2, \sigma_3); \mu_{\tilde{\sigma}}, v_{\tilde{\sigma}}\right]}, \tag{11}$$

$$\widetilde{C}_{sp} = \left[\left(\frac{usl_1 - lsl_3}{6\sigma_3}, \frac{usl_2 - lsl_2}{6\sigma_2}, \frac{usl_3 - lsl_1}{6\sigma_1}\right); \sqrt{\frac{\frac{\mu_{\widetilde{USL}}{}^2 - \mu_{\widetilde{LSL}}{}^2}{1-\mu_{\widetilde{LSL}}{}^2}}{1 - \left(1 - \mu_{\tilde{\sigma}}{}^2\right)^6}}, \sqrt{\frac{\left(\frac{v_{\widetilde{USL}}}{v_{\widetilde{LSL}}}\right)^2 - v_{\tilde{\sigma}}{}^{12}}{1 - v_{\tilde{\sigma}}{}^{12}}}\right]. \tag{12}$$

The index $\widetilde{C}_{sp}$ is also derived by using the arithmetical operations of PTFNs as follows:

$$\tilde{C}_{sp} = \frac{\left[(usl_1, usl_2, usl_3, (usl_4); \mu_{\widetilde{USL}}, v_{\widetilde{USL}}\right] \Theta \left[(lsl_1, lsl_2, lsl_3, lsl_4); \mu_{\widetilde{LSL}}, v_{\widetilde{LSL}}\right]}{6\left[(\sigma_1, \sigma_2, \sigma_3, \sigma_4); \mu_{\tilde{\sigma}}, v_{\tilde{\sigma}}\right]}, \tag{13}$$

$$\widetilde{C}_{sp} = \left[\left(\frac{usl_1 - lsl_4}{6\sigma_4}, \frac{usl_2 - lsl_3}{6\sigma_3}, \frac{usl_3 - lsl_2}{6\sigma_2}, \frac{usl_4 - lsl_1}{6\sigma_1}\right); \sqrt{\frac{\frac{\mu_{\widetilde{USL}}{}^2 - \mu_{\widetilde{LSL}}{}^2}{1-\mu_{\widetilde{LSL}}{}^2}}{1 - \left(1 - \mu_{\tilde{\sigma}}{}^2\right)^6}}, \sqrt{\frac{\left(\frac{v_{\widetilde{USL}}}{v_{\widetilde{LSL}}}\right)^2 - v_{\tilde{\sigma}}{}^{12}}{1 - v_{\tilde{\sigma}}{}^{12}}}\right]. \tag{14}$$

Then, membership functions (MFs) of the index $\widetilde{C}_{sp}$ for PTrFNs and PTFNs are shown in Fig. 4.1, respectively. According to Fig. 2, both the degrees of membership function and non-membership function of the index $\widetilde{C}_{sp}$ take values between 0 and 1. But, the sum of their squares is lower than 1.

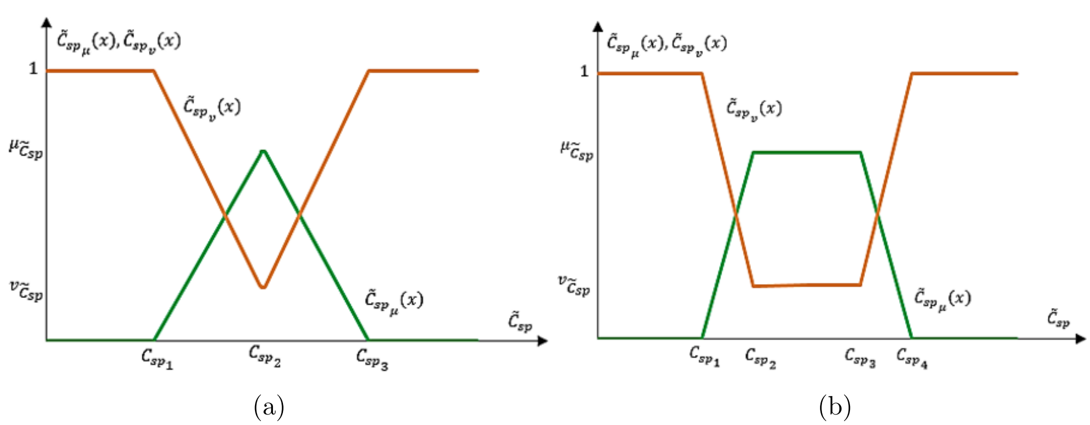


Figure 2: The MFs of the index $\widetilde{C}_{sp}$ by using (a) PTrFNs (b) PTFNs

Similarly, the index $\widetilde{C}_{spk}$ is obtained using PTrFNs as follows:

$$\widetilde{C}_{spk} = \min\left\{\widetilde{C}_{spl}, \widetilde{C}_{spu}\right\} = \min\left\{\frac{\widetilde{\mu}_p \ominus \widetilde{LSL}}{3\widetilde{\sigma}}, \frac{\widetilde{USL} \ominus \widetilde{\mu}_p}{3\widetilde{\sigma}}\right\}. \tag{15}$$

The index $\widetilde{C}_{spk}$ is obtained as in Eq. (16):

$$\widetilde{C}_{spk} = \min\left\{\begin{array}{c} \frac{\left[(\mu_1,\mu_2,\mu_3);\mu_{\widetilde{\mu}_p},v_{\widetilde{\mu}_p}\right]\Theta\left[(lsl_1,lsl_2,lsl_3);\mu_{\widetilde{LSL}},v_{\widetilde{LSL}}\right]}{3[(\sigma_1,\sigma_2,\sigma_3);\mu_{\widetilde{\sigma}},v_{\widetilde{\sigma}}]}, \\ \frac{\left[(usl_1,usl_2,usl_3,);\mu_{\widetilde{USL}},v_{\widetilde{USL}}\right]\Theta\left[(\mu_1,\mu_2,\mu_3);\mu_{\widetilde{\mu}_p},v_{\widetilde{\mu}_p}\right]}{3[(\sigma_1,\sigma_2,\sigma_3,);\mu_{\widetilde{\sigma}},v_{\widetilde{\sigma}}]} \end{array}\right\}. \tag{16}$$

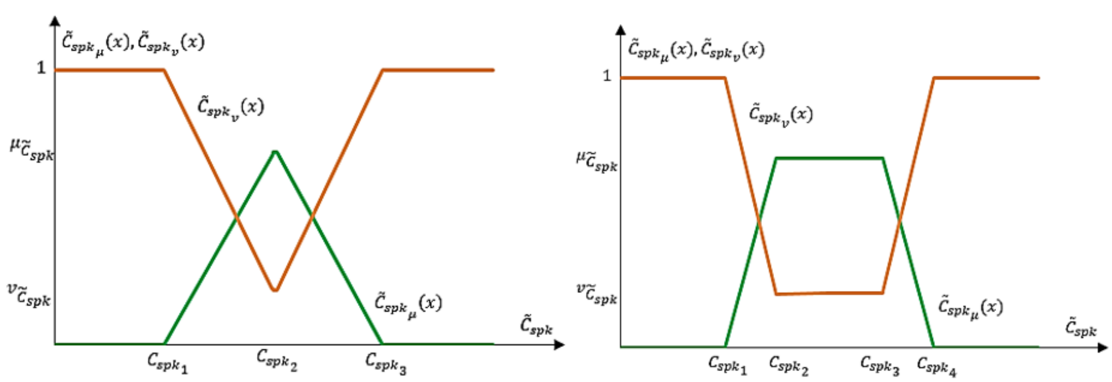
$$\tilde{C}_{spk} = \min\left\{\begin{array}{c} \left[\left(\frac{\mu_1-lsl_3}{3\sigma_3}, \frac{\mu_2-lsl_2}{3\sigma_2}, \frac{\mu_3-lsl_1}{3\sigma_1},\right); \sqrt{\frac{\frac{\mu_{\widetilde{\mu}_p}{}^2-\mu_{\widetilde{LSL}}{}^2}{1-\mu_{\widetilde{LSL}}{}^2}}{1-(1-\mu_{\widetilde{\sigma}}{}^2)^3}}, \sqrt{\frac{\left(\frac{v_{\widetilde{\mu}_p}}{v_{\widetilde{LSL}}}\right)^2-v_{\widetilde{\sigma}}{}^6}{1-v_{\widetilde{\sigma}}{}^6}}\right] \\ \left[\left(\frac{usl_1-\mu_3}{3\sigma_3}, \frac{usl_2-\mu_2}{3\sigma_2}, \frac{usl_3-\mu_1}{3\sigma_1}\right); \sqrt{\frac{\frac{\mu_{\widetilde{USL}}{}^2-\mu_{\widetilde{\mu}_p}{}^2}{1-\mu_{\widetilde{\mu}_p}{}^2}}{1-(1-\mu_{\widetilde{\sigma}}{}^2)^3}}, \sqrt{\frac{\left(\frac{v_{\widetilde{USL}}}{v_{\widetilde{\mu}_p}}\right)^2-v_{\widetilde{\sigma}}{}^6}{1-v_{\widetilde{\sigma}}{}^6}}\right] \end{array}\right\}. \tag{17}$$

In the same way, the index $\widetilde{C}_{spk}$ is handled by using the arithmetical operations of PTFNs based as follows:

$$\tilde{C}_{spk} = \min\left\{\begin{array}{c} \frac{\left[(\mu_1,\mu_2,\mu_3,\mu_4);\mu_{\widetilde{\mu}_p},v_{\widetilde{\mu}_p}\right]\Theta\left[(lsl_1,lsl_2,lsl_3,lsl_4);\mu_{\widetilde{LSL}},v_{\widetilde{LSL}}\right]}{3[(\sigma_1,\sigma_2,\sigma_3,\sigma_4);\mu_{\widetilde{\sigma}},v_{\widetilde{\sigma}}]}, \\ \frac{\left[(usl_1,usl_2,usl_3,usl_4);\mu_{\widetilde{USL}},v_{\widetilde{USL}}\right]\Theta\left[(\mu_1,\mu_2,\mu_3,\mu_4);\mu_{\widetilde{\mu}_p},v_{\widetilde{\mu}_p}\right]}{3[(\sigma_1,\sigma_2,\sigma_3,\sigma_4);\mu_{\widetilde{\sigma}},v_{\widetilde{\sigma}}]} \end{array}\right\}. \tag{18}$$

$$\tilde{C}_{spk} = \min\left\{\begin{array}{c} \left[\left(\frac{\mu_1-lsl_4}{3\sigma_4}, \frac{\mu_2-lsl_3}{3\sigma_3}, \frac{\mu_3-lsl_2}{3\sigma_2}, \frac{\mu_4-lsl_1}{3\sigma_1}\right); \sqrt{\frac{\frac{\mu_{\widetilde{\mu}_p}{}^2-\mu_{\widetilde{LSL}}{}^2}{1-\mu_{\widetilde{LSL}}{}^2}}{1-(1-\mu_{\widetilde{\sigma}}{}^2)^3}}, \sqrt{\frac{\left(\frac{v_{\widetilde{\mu}_p}}{v_{\widetilde{LSL}}}\right)^2-v_{\widetilde{\sigma}}{}^6}{1-v_{\widetilde{\sigma}}{}^6}}\right] \\ \left[\left(\frac{usl_1-\mu_4}{3\sigma_4}, \frac{usl_2-\mu_3}{3\sigma_3}, \frac{usl_3-\mu_2}{3\sigma_2}, \frac{\mu_4-lsl_1}{3\sigma_1}\right); \sqrt{\frac{\frac{\mu_{\widetilde{USL}}{}^2-\mu_{\widetilde{\mu}_p}{}^2}{1-\mu_{\widetilde{\mu}_p}{}^2}}{1-(1-\mu_{\widetilde{\sigma}}{}^2)^3}}, \sqrt{\frac{\left(\frac{v_{\widetilde{USL}}}{v_{\widetilde{\mu}_p}}\right)^2-v_{\widetilde{\sigma}}{}^6}{1-v_{\widetilde{\sigma}}{}^6}}\right] \end{array}\right\}. \tag{19}$$

The MFs of the indices $\widetilde{C}_{spl}$ and $\widetilde{C}_{spu}$ have been derived. Then the MFs of the Pythagorean $\widetilde{C}_{spk}$ are shown in Fig. 3.



(a) (b)

Figure 3: The MFs of the index $\widetilde{C}_{spk}$ by using (a) PTrFNs (b) PTFNs

4.2 The case that the process parameters are defined as flexible structure of PFNs

Various arithmetic operations have been developed in the literature due to the definition of PFSs with two-dimensional membership functions and allowing experts to express their views more flexibly in modeling problems involving uncertainty. In addition to the literature, in this study, the effect of $\widetilde{A}$ as PFN on C_p and C_{pk} indices have been analyzed by defining it more flexible together with membership ($\mu_{\widetilde{A}}$) and non-membership ($v_{\widetilde{A}}$) functions. Defining PFNs as $\widetilde{A} = \left[(a_1, a_2, a_3, a_4), (a'_1, a_2, a_3, a'_4); \mu_{\widetilde{A}}, v_{\widetilde{A}}\right]$ provides more flexibility to PCIs. In this subsection, process parameters are defined with flexible PFNs and flexible PFPCIs are obtained. Suppose that the main parameters of PCIs are defined by using the flexible PTrFNs as $\widetilde{LSL} = \left[(lsl_1, lsl_2, lsl_3), (lsl'_1, lsl_2, lsl'_3); \mu_{\widetilde{LSL}}, v_{\widetilde{LSL}}\right]$, $\widetilde{USL} = \left[(usl_1, usl_2, usl_3), (usl'_1, usl_2, usl'_3); \mu_{\widetilde{USL}}, v_{\widetilde{USL}}\right]$, $\widetilde{\mu}_p = \left[(\mu_1, \mu_2, \mu_3), (\mu'_1, \mu_2, \mu'_3); \mu_{\widetilde{\mu}_p}, v_{\widetilde{\mu}_p}\right]$, $\widetilde{\sigma} = [(\sigma_1, \sigma_2, \sigma_3), (\sigma'_1, \sigma_2, \sigma'_3); \mu_{\widetilde{\sigma}}, v_{\widetilde{\sigma}}]$ and $\widetilde{T} = \left[(t_1, t_2, t_3), (t'_1, t_2, t'_3); \mu_{\widetilde{\mathrm{T}}}, v_{\widetilde{\mathrm{T}}}\right]$. Similarly, suppose that the main parameters of PCIs are defined by using the flexible PTFNs as $\widetilde{LSL} = \left[(lsl_1, lsl_2, lsl_3, lsl_4), (lsl'_1, lsl_2, lsl_3, lsl'_4); \mu_{\widetilde{LSL}}, v_{\widetilde{LSL}}\right]$, $\widetilde{USL} = \left[(usl_1, usl_2, usl_3, usl_4), (usl'_1, usl_2, usl_3, usl'_4); \mu_{\widetilde{USL}}, v_{\widetilde{USL}}\right]$, $\widetilde{\mu}_p = \left[(\mu_1, \mu_2, \mu_3, \mu_4), (\mu'_1, \mu_2, \mu_3, \mu'_4); \mu_{\widetilde{\mu}_p}, v_{\widetilde{\mu}_p}\right]$, $\widetilde{\sigma} = [(\sigma_1, \sigma_2, \sigma_3, \sigma_4), (\sigma'_1, \sigma_2, \sigma_3, \sigma'_4); \mu_{\widetilde{\sigma}}, v_{\widetilde{\sigma}}]$ and $\widetilde{T} = \left[(t_1, t_2, t_3, t_4), (t'_1, t_2, t_3, t'_4); \mu_{\widetilde{\mathrm{T}}}, v_{\widetilde{\mathrm{T}}}\right]$. The index $\widetilde{C}_{fp}$ for a flexible structure of PFNs that is obtained by using the process parameters are calculated as follows:

$$\widetilde{C}_{fp} = \frac{\left[(usl_1, usl_2, usl_3), (usl'_1, usl_2, usl'_3); \mu_{\widetilde{USL}}, v_{\widetilde{USL}}\right] \Theta \left[(lsl_1, lsl_2, lsl_3), (lsl'_1, lsl_2, lsl'_3); \mu_{\widetilde{LSL}}, v_{\widetilde{LSL}}\right]}{6\left[(\sigma_1, \sigma_2, \sigma_3), (\sigma'_1, \sigma_2, \sigma'_3); \mu_{\widetilde{\sigma}}, v_{\widetilde{\sigma}}\right]}, \tag{20}$$

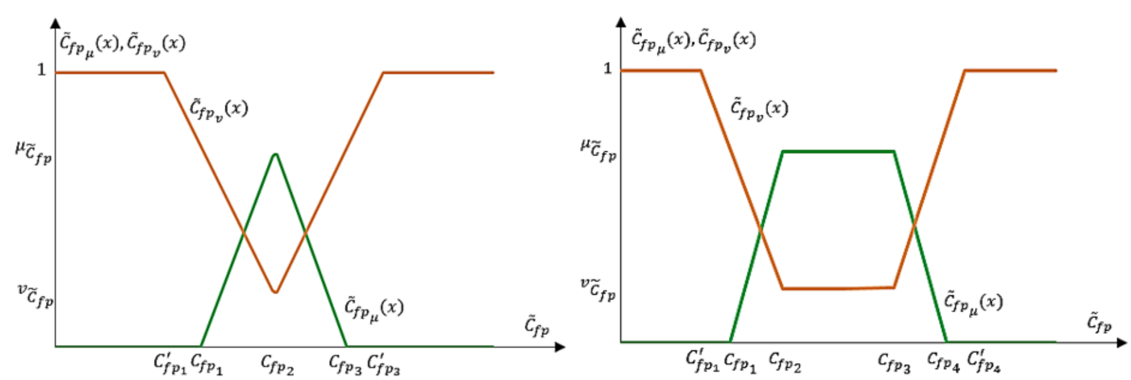
$$\widetilde{C}_{fp} = \left[\left(\frac{usl_1-lsl_3}{6\sigma_3}, \frac{usl_2-lsl_2}{6\sigma_2}, \frac{usl_3-lsl_1}{6\sigma_1}\right), \left(\frac{usl'_1-lsl'_3}{6\sigma'_3}, \frac{usl_2-lsl_2}{6\sigma'_2}, \frac{usl'_3-lsl'_1}{6\sigma'_1}\right); \sqrt{\frac{\frac{\mu_{\widetilde{USL}}{}^2-\mu_{\widetilde{LSL}}{}^2}{1-\mu_{\widetilde{LSL}}{}^2}}{1-(1-\mu_{\widetilde{\sigma}}{}^2)^6}}, \sqrt{\frac{\left(\frac{v_{\widetilde{USL}}}{v_{\widetilde{LSL}}}\right)^2-v_{\widetilde{\sigma}}{}^{12}}{1-v_{\widetilde{\sigma}}{}^{12}}}\right]. \tag{21}$$

The flexible structure of the index $\widetilde{C}_{fp}$ is derivate based on PTFNs as follows:

$$\widetilde{C}_{fp} = \frac{\left[(usl_1,usl_2,usl_3,usl_4),\left(usl'_1,usl_2,usl_3,usl'_4\right);\mu_{\widetilde{USL}},v_{\widetilde{USL}}\right]\Theta\left[(lsl_1,lsl_2,lsl_3,lsl_4),\left(lsl'_1,lsl_2,lsl_3,lsl'_4\right);\mu_{\widetilde{LSL}},v_{\widetilde{LSL}}\right]}{6\left[(\sigma_1,\sigma_2,\sigma_3,\sigma_4),\left(\sigma'_1,\sigma_2,\sigma_3,\sigma'_4\right);\mu_{\widetilde{\sigma}},v_{\widetilde{\sigma}}\right]}, \tag{22}$$

$$\widetilde{C}_{fp} = \left[\left(\frac{usl_1-lsl_4}{6\sigma_4}, \frac{usl_2-lsl_3}{6\sigma_3}, \frac{usl_3-lsl_2}{6\sigma_2}, \frac{usl_4-lsl_1}{6\sigma_1}\right), \left(\frac{usl'_1-lsl'_4}{6\sigma'_4}, \frac{usl_2-lsl_3}{6\sigma_3}, \frac{usl_3-lsl_2}{6\sigma_2}, \frac{usl'_4-lsl'_1}{6\sigma'_1}\right); \sqrt{\frac{\frac{\mu_{\widetilde{USL}}{}^2-\mu_{\widetilde{LSL}}{}^2}{1-\mu_{\widetilde{LSL}}{}^2}}{1-(1-\mu_{\widetilde{\sigma}}{}^2)^6}}, \sqrt{\frac{\left(\frac{v_{\widetilde{USL}}}{v_{\widetilde{LSL}}}\right)^2-v_{\widetilde{\sigma}}{}^{12}}{1-v_{\widetilde{\sigma}}{}^{12}}}\right]. \tag{23}$$

The MFs of the Pythagorean $\widetilde{C}_{fp}$ based on PFNs for flexible structure are shown in Fig. 4.



(a) (b)

Figure 4: The MFs of the index $\widetilde{C}_{fp}$ by using (a) PTrFNs for flexible structure (b) PTFNs for flexible structure

Based on PTFNs, the index $\widetilde{C}_{fpk}$ is obtained for a flexible structure of PFNs as in Eq. (24):

$$\widetilde{C}_{fpk} = \min\left\{\begin{array}{c} \frac{\left[(\mu_1,\mu_2,\mu_3),\left(\mu'_1,\mu_2,\mu'_3\right);\mu_{\widetilde{\mu}_p},v_{\widetilde{\mu}_p}\right]\Theta\left[(lsl_1,lsl_2,lsl_3),\left(lsl'_1,lsl_2,lsl'_3\right);\mu_{\widetilde{LSL}},v_{\widetilde{LSL}}\right]}{3\left[(\sigma_1,\sigma_2,\sigma_3),\left(\sigma'_1,\sigma_2,\sigma'_3\right);\mu_{\widetilde{\sigma}},v_{\widetilde{\sigma}}\right]}, \\ \frac{\left[(usl_1,usl_2,usl_3),\left(usl'_1,usl_2,usl'_3\right);\mu_{\widetilde{USL}},v_{\widetilde{USL}}\right]\Theta\left[(\mu_1,\mu_2,\mu_3),\left(\mu'_1,\mu_2,\mu'_3\right);\mu_{\widetilde{\mu}_p},v_{\widetilde{\mu}_p}\right]}{3\left[(\sigma_1,\sigma_2,\sigma_3),\left(\sigma'_1,\sigma_2,\sigma'_3\right);\mu_{\widetilde{\sigma}},v_{\widetilde{\sigma}}\right]} \end{array}\right\}. \tag{24}$$

$$\widetilde{C}_{fpk} = \min \left\{ \begin{array}{c} \left[\left(\frac{\mu_1 - lsl_3}{3\sigma_3}, \frac{\mu_2 - lsl_2}{3\sigma_2}, \frac{\mu_3 - lsl_1}{3\sigma_1}, \right), \left(\frac{\mu_1 - lsl_3'}{3\sigma_3'}, \frac{\mu_2 - lsl_2}{3\sigma_2}, \frac{\mu_3 - lsl_1'}{3\sigma_1'} \right); \sqrt{\frac{\frac{\mu_{\widetilde{\mu}_p}{}^2 - \mu_{\widetilde{LSL}}{}^2}{1-\mu_{\widetilde{LSL}}{}^2}}{1-(1-\mu_{\widetilde{\sigma}}{}^2)^3}}, \sqrt{\frac{\left(\frac{v_{\widetilde{\mu}_p}}{v_{\widetilde{LSL}}}\right)^2 - v_{\widetilde{\sigma}}{}^6}{1-v_{\widetilde{\sigma}}{}^6}} \right] \\ \left[\left(\frac{usl_1 - \mu_3}{3\sigma_3}, \frac{usl_2 - \mu_2}{3\sigma_2}, \frac{usl_3 - \mu_1}{3\sigma_1} \right), \left(\frac{usl_1' - \mu_3}{3\sigma_3'}, \frac{usl_2 - \mu_2}{3\sigma_2}, \frac{usl_3' - \mu_1}{3\sigma_3'} \right); \sqrt{\frac{\frac{\mu_{\widetilde{USL}}{}^2 - \mu_{\widetilde{\mu}_p}{}^2}{1-\mu_{\widetilde{\mu}_p}{}^2}}{1-(1-\mu_{\widetilde{\sigma}}{}^2)^3}}, \sqrt{\frac{\left(\frac{v_{\widetilde{USL}}}{v_{\widetilde{\mu}_p}}\right)^2 - v_{\widetilde{\sigma}}{}^6}{1-v_{\widetilde{\sigma}}{}^6}} \right] \end{array} \right\}. \quad (25)$$

The index $\widetilde{C}_{fpk}$ based on PTFNs for a flexible structure is re-designed by using the arithmetical operations that indicated into Eqs. (26)-(27) as detailed in below:

$$\widetilde{C}_{fpk} = \min \left\{ \begin{array}{c} \frac{\left[(\mu_1,\mu_2,\mu_3,\mu_4),(\mu_1',\mu_2,\mu_3,\mu_4');\mu_{\widetilde{\mu}_p},v_{\widetilde{\mu}_p}\right] \ominus \left[(lsl_1,lsl_2,lsl_3,lsl_4),(lsl_1',lsl_2,lsl_3,lsl_4');\mu_{\widetilde{LSL}},v_{\widetilde{LSL}}\right]}{3\left[(\sigma_1,\sigma_2,\sigma_3,\sigma_4),(\sigma_1',\sigma_2,\sigma_3,\sigma_4');\mu_{\widetilde{\sigma}},v_{\widetilde{\sigma}}\right]}, \\ \frac{\left[(usl_1,usl_2,usl_3,usl_4),(usl_1',usl_2,usl_3,usl_4');\mu_{\widetilde{USL}},v_{\widetilde{USL}}\right] \ominus \left[(\mu_1,\mu_2,\mu_3,\mu_4),(\mu_1',\mu_2,\mu_3,\mu_4');\mu_{\widetilde{\mu}_p},v_{\widetilde{\mu}_p}\right]}{3\left[(\sigma_1,\sigma_2,\sigma_3,\sigma_4),(\sigma_1',\sigma_2,\sigma_3,\sigma_4');\mu_{\widetilde{\sigma}},v_{\widetilde{\sigma}}\right]} \end{array} \right\}, \quad (26)$$

$$\widetilde{C}_{fpk} = \min \left\{ \begin{array}{c} \left[\left(\frac{\mu_1 - lsl_4}{3\sigma_4}, \frac{\mu_2 - lsl_3}{3\sigma_3}, \frac{\mu_3 - lsl_2}{3\sigma_2}, \frac{\mu_4 - lsl_1}{3\sigma_1} \right), \left(\frac{\mu_1' - lsl_4'}{3\sigma_4'}, \frac{\mu_2 - lsl_3}{3\sigma_3}, \frac{\mu_3 - lsl_2}{3\sigma_2}, \frac{\mu_4' - lsl_1'}{3\sigma_1'} \right); \sqrt{\frac{\frac{\mu_{\widetilde{\mu}_p}{}^2 - \mu_{\widetilde{LSL}}{}^2}{1-\mu_{\widetilde{LSL}}{}^2}}{1-(1-\mu_{\widetilde{\sigma}}{}^2)^3}}, \sqrt{\frac{\left(\frac{v_{\widetilde{\mu}_p}}{v_{\widetilde{LSL}}}\right)^2 - v_{\widetilde{\sigma}}{}^6}{1-v_{\widetilde{\sigma}}{}^6}} \right] \\ \left[\left(\frac{usl_1 - \mu_4}{3\sigma_4}, \frac{usl_2 - \mu_3}{3\sigma_3}, \frac{usl_3 - \mu_2}{3\sigma_2}, \frac{\mu_4 - lsl_1}{3\sigma_1} \right), \left(\frac{usl_1' - \mu_4'}{3\sigma_4'}, \frac{usl_2 - \mu_3}{3\sigma_3}, \frac{usl_3 - \mu_2}{3\sigma_2}, \frac{usl_4' - \mu_1'}{3\sigma_1'} \right); \sqrt{\frac{\frac{\mu_{\widetilde{USL}}{}^2 - \mu_{\widetilde{\mu}_p}{}^2}{1-\mu_{\widetilde{\mu}_p}{}^2}}{1-(1-\mu_{\widetilde{\sigma}}{}^2)^3}}, \sqrt{\frac{\left(\frac{v_{\widetilde{USL}}}{v_{\widetilde{\mu}_p}}\right)^2 - v_{\widetilde{\sigma}}{}^6}{1-v_{\widetilde{\sigma}}{}^6}} \right] \end{array} \right\}. \quad (27)$$

The one-sided capability indices named C_{pl} and C_{pu} are re-formulated and the indices $\widetilde{C}_{fpl}$ and $\widetilde{C}_{fpu}$ are obtained based on PTFNs. The MFs of the Pythagorean $\widetilde{C}_{fpk}$ based on PFNs for flexible structure are also shown in Fig. 5.

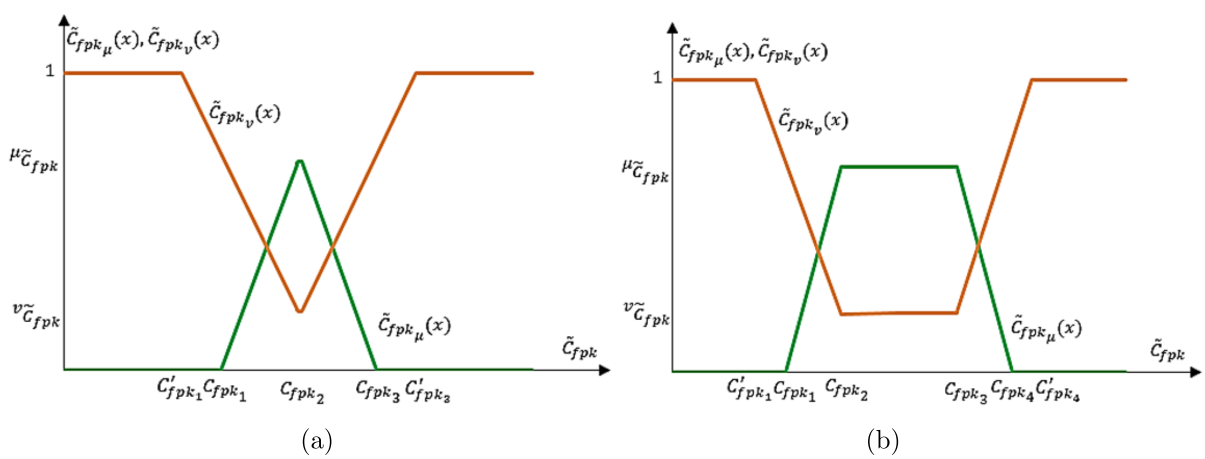


Figure 5: The MFs of the index $\widetilde{C}_{fpk}$ by using (a) PTrFNs for flexible structure (b) PTFNs for flexible structure

After, to calculate the minimum value of the flexible Pythagorean $\widetilde{C}_{fpk}$, the score function has been defined as follows is used:

$$s(\tilde{A}) = \frac{a + a' + 2b + 2c + d + d'}{8} \left(w_{\tilde{A}}^2 - u_{\tilde{A}}^2 \right), s(\tilde{A}) \in [-1, 1]. \quad (28)$$

5 A real case application in manufacturing process

The less likely the process is out of SLs, the fewer the number of times the process produces defective parts. For this reason, the definition of a more comprehensive the main parameters of PCIs using PFNs rather than crisp numbers is of great importance in analyzing process capability to address the gap into literature. In this section, two cases are analyzed in order to evaluate the performance of the proposed PCIs based on PFSs by considering the measurements of a gear. For this aim, a real case example is considered from an automotive company. The process capability of the inside diameter of the gear is evaluated. The mean of the gear is 10 mm and the standard deviation is 0.2 mm [15]. In application stage, the PCIs have been evaluated in terms of flexibility and flexible PFCIs have been derived by using PFSs. The flexibility of PCIs facilitates the evaluation of process capability under uncertainty. The main parameters of the gear such as specification limits ($\widetilde{SLs}$), process mean ($\widetilde{\mu}_p$), standard deviation ($\widetilde{\sigma}$) and target value ($\widetilde{T}$) have been defined by using PFNs for the PCIs to give more sensitive and more accurate results. Then, the PFPCIs for the two proposed cases have been evaluated as follows:

5.1 The case that the process parameters are defined by using PFNs

In the first stage, the performance of the proposed PFPCIs have been evaluated on the application when the process parameters are defined as PFNs. Thus, hesitancies of the quality inspectors and uncertainties of the process have been handled. For this aim, $\widetilde{SLs}$, $\widetilde{\mu}_p$, $\widetilde{\sigma}$, and $\widetilde{T}$ values based on PTrFNs for this dimension can be defined as follows: $\widetilde{LSL} = [(9.2, 9.4, 9.6); 0.90, 0.22]$, $\widetilde{USL} = [(10.2, 10.4, 10, 6); 0.98, 0.05]$, $\widetilde{\mu}_p = [(9.95, 10.00, 10.05); 0.95, 0.10]$, $\widetilde{\sigma} = [(0.19, 0.20, 0.21); 0.95, 0.22]$ and $\widetilde{T} = [(9.75, 9.80, 9.85); 0.92, 0.15]$. These parameters can be also defined as follows: $\widetilde{LSL} = [(9.2, 9.4, 9.6, 9.8); 0.90, 0.22]$, $\widetilde{USL} = [(10.2, 10.4, 10, 6, 10.8); 0.98, 0.05]$, $\widetilde{\mu}_p = [(9.90, 9.95, 10.00, 10.05); 0.95, 0.10]$, $\widetilde{\sigma} = [(0.18, 0.19, 0.20, 0.21); 0.95, 0.22]$ and $\widetilde{T} = [(9.70, 9.75, 9.80, 9.85); 0.92, 0.15]$.

Then, the index $\widetilde{C}_{sp}$ and $\widetilde{C}_{spk}$ are calculated by using PTrFNs as follows:

$$\widetilde{C}_{sp} = \left[\left(\frac{10.20-9.60}{6\times0.21}, \frac{10.40-9.40}{6\times0.20}, \frac{10.60-9.20}{6\times0.19}\right); \sqrt{\frac{\frac{0.98^2-0.90^2}{1-0.90^2}}{1-(1-0.95^2)^6}}, \sqrt{\frac{\left(\frac{0.05}{0.22}\right)^2-0.22^{12}}{1-0.22^{12}}}\right].$$

$\widetilde{C}_{sp} = [(0.48, 0.83, 1.23); 0.89, 0.23]$.

The process capability of this process within the based on crisp parameters is approximately 0.83. However, the quality inspectors may not be able to define these parameters precisely. In order to eliminate this problem, the index $\widetilde{C}_{sp}$ is defined as PTrFNs and the capability of the process is analyzed. When the index $\widetilde{C}_{sp}$ is analyzed with a crisp value, the capability of the process is calculated as 0.62. Consequently, the non-membership degree of the process is expressed as 0.23. According to the result, the process is incapability. It also gives more information about the capability of the process.

$$\widetilde{C}_{spk} = \min\left\{\begin{array}{c}\left[\left(\frac{9.95-9.60}{3\times0.21}, \frac{10.00-9.40}{3\times0.20}, \frac{10.05-9.20}{3\times0.19}\right); \sqrt{\frac{\frac{0.95^2-0.90^2}{1-0.90^2}}{1-(1-0.95^2)^3}}, \sqrt{\frac{\left(\frac{0.10}{0.22}\right)^2-0.22^6}{1-0.22^6}}\right] \\ \left[\left(\frac{10.20-10.05}{3\times0.21}, \frac{10.40-10.00}{3\times0.20}, \frac{10.60-9.95}{3\times0.19}\right); \sqrt{\frac{\frac{0.98^2-0.95^2}{1-0.95^2}}{1-(1-0.95^2)^3}}, \sqrt{\frac{\left(\frac{0.05}{0.10}\right)^2-0.22^6}{1-0.22^6}}\right]\end{array}\right\}.$$

$\widetilde{C}_{spk} = \min\{[(0.56, 1.00, 1.49); 0.70, 0.45]; [(0.24, 0.67, 1.14); 0.77, 0.50]\}$.

$\widetilde{C}_{spk} = [(0.24, 0.67, 1.14); 0.77, 0.50]$.

The score values of indices $\widetilde{C}_{spl}$ and $\widetilde{C}_{spu}$ are calculated as 0.28 and 0.23, respectively. Since the index $\widetilde{C}_{spu}$ is smaller than the index $\widetilde{C}_{spl}$, it is chosen as the index $\widetilde{C}_{spk}$. The indices $\widetilde{C}_{sp}$ and $\widetilde{C}_{spk}$ are also calculated by using PTFNs as follows:

$$\widetilde{C}_{sp} = \left[\left(\frac{10.20-9.80}{6\times0.21}, \frac{10.40-9.60}{6\times0.20}, \frac{10.60-9.40}{6\times0.19}, \frac{10.80-9.20}{6\times0.18}\right); \sqrt{\frac{\frac{0.98^2-0.90^2}{1-0.90^2}}{1-(1-0.95^2)^6}}, \sqrt{\frac{\left(\frac{0.05}{0.22}\right)^2-0.22^{12}}{1-0.22^{12}}}\right].$$

$\widetilde{C}_{sp} = [(0.32, 0.67, 1.05, 1.48); 0.89, 0.23]$.

$$\widetilde{C}_{spk} = \min\left\{\begin{array}{c}\left[\left(\frac{9.90-9.80}{3\times0.21}, \frac{9.95-9.60}{3\times0.20}, \frac{10.00-9.40}{3\times0.19}, \frac{10.05-9.20}{3\times0.18}\right); \sqrt{\frac{\frac{0.95^2-0.90^2}{1-0.90^2}}{1-(1-0.95^2)^3}}, \sqrt{\frac{\left(\frac{0.10}{0.22}\right)^2-0.22^6}{1-0.22^6}}\right] \\ \left[\left(\frac{10.20-10.05}{3\times0.21}, \frac{10.40-10.00}{3\times0.20}, \frac{10.60-9.95}{3\times0.19}, \frac{10.80-9.90}{3\times0.18}\right); \sqrt{\frac{\frac{0.98^2-0.95^2}{1-0.95^2}}{1-(1-0.95^2)^3}}, \sqrt{\frac{\left(\frac{0.05}{0.10}\right)^2-0.22^6}{1-0.22^6}}\right]\end{array}\right\}.$$

$\widetilde{C}_{spk} = \min\{[(0.16, 0.58, 1.05, 1.57); 0.70, 0.45]; [(0.24, 0.67, 1.14, 1.67); 0.77, 0.50]\}$.

$\widetilde{C}_{spk} = [(0.16, 0.58, 1.05, 1.57); 0.70, 0.45]$.

The score values of indices $\widetilde{C}_{spl}$ and $\widetilde{C}_{spu}$ are calculated as 0.24 and 0.32, respectively. Since the index $\widetilde{C}_{spl}$ is smaller than the index $\widetilde{C}_{spu}$, it was chosen as the index $\widetilde{C}_{spk}$.

Compared to other fuzzy set extensions, PCIs analyzed with PFNs allowed a more realistic interpretation of the process as it takes into account all possible values. Type-1 fuzzy sets do not successfully take into account the subjective thinking of the quality inspector in defining SLs. It is clearly seen in Tab. 1 that the capability of the process can be between 0.48 and 1.23 with a membership degree of 0.89. And by means of the 0.23 non-membership degree, it can be inferenced that the process will not be between these limits. The index $\widetilde{C}_{sp}$ shows us that the process capability has changed significantly as a result of the change in the degree of membership and non-membership of the process. As a result, it has been observed that PFSs give very effective results on real-case problems.

Table 1: The Obtained Results for the indices $\widetilde{C}_{sp}$ and $\widetilde{C}_{spk}$

	PTrFNs	Crisp Value
$\widetilde{\boldsymbol{C}}_{sp}$	$[(0.48, 0.83, 1.23); 0.89, 0.23]$	0.62
$\widetilde{\boldsymbol{C}}_{spk}$	$[(0.24, 0.67, 1.14); 0.77, 0.50]$	0.23
	PTFNs	Crisp Value
$\widetilde{\boldsymbol{C}}_{sp}$	$[(0.32, 0.67, 1.05, 1.45); 0.89, 0.23]$	0.65
$\widetilde{\boldsymbol{C}}_{spk}$	$[(0.16, 0.58, 1.05, 1.67); 0.70, 0.45]$	0.24

5.2 The case that the process parameters are defined as the flexible structure of PFNs

The more flexible version of the PCIs can also be defined, the easier it will be to apply them to real-case problems. In this subsection, this case is analyzed. This allows us to examine the capability of the process more accurately and more flexibly while also saving time and money. For this reason, the parameters of the indices C_p and C_{pk} are handled differently from the traditional approach by providing flexibility as follows. In this subsection, the case that the process parameters are PFNs for flexible structure are handled. Therefore, the performance of the proposed PFPCIs as flexible structure have been evaluated on the similar application for PCA of gear. The process parameters are defined under the flexible PTrFNs as follows: $\widetilde{LSL} = [(9.20, 9.40, 9.60), (9.10, 9.40, 9.70); 0.90, 0.22]$, $\widetilde{USL} = [(10.20, 10.40, 10.60), (10.10, 10.40, 10.70); 0.98, 0.05]$, $\widetilde{\mu}_p = [(9.95, 10.00, 10.05), (9.93, 10.00, 10.07); 0.95, 0.10]$, $\widetilde{\sigma} = [(0.19, 0.20, 0.21), (0.18, 0.20, 0, 22); 0.95, 0.22]$ and $\widetilde{T} = [(9.75, 9.80, 9.85), (9.73, 9.80, 9.87); 0.92, 0.15]$. The process parameters are defined in a flexible structure of based on PTFNs for the gear as follows:

$$\widetilde{LSL} = [(9.20, 9.40, 9.60, 9.80), (9.15, 9.40, 9.60, 9.85); 0.90, 0.22],$$

$$\widetilde{USL} = [(10.20, 10.40, 10.60, 10.80), (10.10, 10.40, 10.60, 10.90); 0.98, 0.05],$$

$$\widetilde{\mu}_p = [(9.90, 9.95, 10.00, 10.05), (9.88, 9.95, 10.00, 10.07); 0.95, 0.10],$$

$$\widetilde{\sigma} = [(0.18, 0.19, 0.20, 0.21), (0.17, 0.19, 0.20, 0.22); 0.95, 0.22] \text{ and}$$

$$\widetilde{T} = [(9.70, 9.75, 9.80, 9.85), (9.68, 9.75, 9.80, 9.87); 0.92, 0.15].$$

The indices $\widetilde{C}_{fp}$ and $\widetilde{C}_{fpk}$ as flexible PFNs are calculated by using PTrFNs as follows:

$$\widetilde{C}_{fp} = \left[\ \left(\frac{10.20-9.60}{6x0.21}, \frac{10.40-9.40}{6x0.20}, \frac{10.60-9.20}{6\times 0.19}\right), \left(\frac{10.10-9.70}{6x0.22}, \frac{10.40-9.40}{6\times 0.20}, \frac{10.70-9.10}{6\times 0.18}\right); \sqrt{\frac{\frac{0.98^2-0.90^2}{1-0.90^2}}{1-(1-0.95^2)^6}}, \sqrt{\frac{\left(\frac{0.05}{0.22}\right)^2-0.22^{12}}{1-0.22^{12}}}\ \right].$$

$$\widetilde{C}_{fp} = [(0.48, 0.83, 1.23), (0.30, 0.83, 1.48); 0.89, 0.23].$$

According to classic method, the capability of this process is determined as 0.83.The index $\widetilde{C}_{fp}$ as a flexible structure has been calculated. The capability of the process is calculated as 0.63. The index $\widetilde{C}_{fp}$ which is a flexible structure shows that the capability of the process cannot be less than 0.30 and more than 1.48. In addition, it allows us to have more information about the capability of the process as it includes membership and non-membership degrees. We know that if the index C_{fp} is lower than 1.00, the process can be incapable. In similar way, the index $\widetilde{C}_{fp}$ is calculated as follows:

$$\widetilde{C}_{fpk} = \min \left\{ \begin{array}{c} \left[\left(\frac{9.95-9.60}{3\times0.21}, \frac{10.00-9.40}{3\times0.20}, \frac{10.05-9.20}{3\times0.19} \right), \left(\frac{9.93-9.70}{3\times0.22}, \frac{10.00-9.40}{3\times0.20}, \frac{10.07-9.10}{3\times0.18} \right); \sqrt{\frac{\frac{0.95^2-0.90^2}{1-0.90^2}}{1-(1-0.95^2)^3}}, \sqrt{\frac{\left(\frac{0.10}{0.22}\right)^2-0.22^6}{1-0.22^6}} \right] \\ \left[\left(\frac{10.20-10.05}{3\times0.21}, \frac{10.40-10.00}{3\times0.20}, \frac{10.60-9.95}{3\times0.19} \right), \left(\frac{10.10-10.07}{3\times0.22}, \frac{10.40-10.00}{3\times0.20}, \frac{10.70-9.93}{3\times0.18} \right); \sqrt{\frac{\frac{0.98^2-0.95^2}{1-0.95^2}}{1-(1-0.95^2)^3}}, \sqrt{\frac{\left(\frac{0.05}{0.10}\right)^2-0.22^6}{1-0.22^6}} \right] \end{array} \right\}.$$

$$\widetilde{C}_{fpk} = \min \left\{ \begin{array}{l} [(0.56, 1.00, 1.49), (0.35, 1.00, 1.80); 0.70, 0.45], \\ [(0.24, 0.67, 1, 14), (0.05, 0.67, 1.43); 0.77, 0.50] \end{array} \right\}.$$

$$\widetilde{C}_{fpk} = [(0.24, 0.67, 1.14), (0.05, 0.67, 1.43); 0.77, 0.50].$$

The score values of indices $\widetilde{C}_{fpl}$ and $\widetilde{C}_{fpu}$ as flexible are calculated as 0.29 and 0.24, respectively. Since the index $\widetilde{C}_{fpu}$ is smaller than the index $\widetilde{C}_{fpl}$, it is selected as the index $\widetilde{C}_{fpk}$. Based on the result of the flexible index $\widetilde{C}_{fpk}$, the process is incapable. The indices $\widetilde{C}_{fp}$ and $\widetilde{C}_{fpk}$ as flexible are calculated by using PTFNs:

$$\widetilde{C}_{fp} = \left[\left(\frac{10.20-9.80}{6x0.21}, \frac{10.40-9.60}{6x0.20}, \frac{10.60-9.40}{6x0.19}, \frac{10.80-9.20}{6x0.18} \right), \left(\frac{10.10-9.85}{6x0.22}, \frac{10.40-9.60}{6x0.20}, \frac{10.60-9.40}{6x0.19}, \frac{10.90-9.15}{6x0.17} \right); \sqrt{\frac{\frac{0.98^2-0.90^2}{1-0.90^2}}{1-(1-0.95^2)^6}}, \sqrt{\frac{\left(\frac{0.05}{0.22}\right)^2-0.22^{12}}{1-0.22^{12}}} \right].$$

Similar comments are also valid for the index $\widetilde{C}_{fp}$ as a flexible based on PTFNs.

$$\widetilde{C}_{fpk} = \min \left\{ \begin{array}{c} \left[\left(\frac{9.90-9,80}{3\times0.21}, \frac{9.95-9.60}{3\times0.20}, \frac{10.00-9.40}{3\times0.19}, \frac{10.05-9.20}{3\times0.18} \right), \left(\frac{9.88-9.85}{3\times0.22}, \frac{9.95-9.60}{3\times0.20}, \frac{10.00-9.40}{3\times0.19}, \frac{10.07-9.15}{3\times0.17} \right); \sqrt{\frac{\frac{0.95^2-0.90^2}{1-0.90^2}}{1-(1-0.95^2)^3}}, \sqrt{\frac{\left(\frac{0.10}{0.22}\right)^2-0.22^6}{1-0.22^6}} \right] \\ \left[\left(\frac{10.20-10.05}{3\times0.21}, \frac{10.40-10.00}{3\times0.20}, \frac{10.60-9.95}{3\times0.19}, \frac{10.80-9.90}{3\times0.18} \right), \left(\frac{10.10-10.07}{3\times0.22}, \frac{10.40-10.00}{3\times0.20}, \frac{10.60-9.95}{3\times0.19}, \frac{10.90-9.88}{3\times0.17} \right); \sqrt{\frac{\frac{0.98^2-0.95^2}{1-0.95^2}}{1-(1-0.95^2)^3}}, \sqrt{\frac{\left(\frac{0.05}{0.10}\right)^2-0.22^6}{1-0.22^6}} \right] \end{array} \right\}.$$

$$\widetilde{C}_{fpk} = \min \left\{ \begin{array}{l} [(0.16, 0.58, 1.05, 1.57), (0.05, 0.58, 1.05, 1.80); 0.70, 0.45], \\ [(0.24, 0.67, 1.14, 1.67), (0.05, 0.67, 1, 14, 2.00); 0.77, 0.50] \end{array} \right\}.$$

$$\widetilde{C}_{fpk} = [(0.16, 0.58, 1.05, 1.57), (0.05, 0.58, 1.05, 1.80); 0.70, 0.45].$$

The minimum value of the index $\widetilde{C}_{fpk}$ as a flexible is calculated as 0.24. It is also seen that the PFPCIs developed in the evaluation of process capability give more informative results. The obtained results are summarized in Tab. 2.

Table 2: The Obtained Results for the indices $\widetilde{C}_{fp}$ and $\widetilde{C}_{fpk}$

	PTrFNs as flexible	Crisp Value
$\widetilde{\boldsymbol{C}}_{fp}$	$[(0.48, 0.83, 1.23), (0.30, 0.83, 1.48); 0.89, 0.23]$	0.63
$\widetilde{\boldsymbol{C}}_{fpk}$	$[(0.24, 0.67, 1.14), (0.05, 0.67, 1.43); 0.77, 0.50]$	0.24
	PTFNs as flexible	Crisp Value
$\widetilde{\boldsymbol{C}}_{fp}$	$[(0.32, 0.67, 1.05, 1.48), (0.19, 0.67, 1.05, 1.72); 0.89, 0.23]$	0.66
$\widetilde{\boldsymbol{C}}_{fpk}$	$[(0.16, 0.58, 1.05, 1.67), (0.05, 0.58, 1.05, 1.80); 0.70, 0.45]$	0.24

Tab. 2 summarizes that the derived values for PCIs when the process parameters are PFNs. The flexible PFPCIs values that are generated when flexible PFNs are defined as process parameters are listed in Tab. 2. It is clearly seen that the PCIs produced by using PFSs contain more information than the crisp values. It is clear that more realistic results are obtained because it also addresses the uncertainty in the thinking of the quality inspector. The flexible PCIs attained with PFSs in Tab. 2 contain more information about the process than the classical method in Tab. 1, as well as the interval value of the results obtained from the PCA. For example, the index $\widetilde{C}_{fp}$ as flexible structure indicates that the capability of the process cannot be less than 0.30 and higher than 1.48. Thus, the PCIs obtained by using PFNs enable us to make more realistic and detailed analyzes about the process.

6 Conclusion and future research suggestions

Process capability analysis (PCA) is a statistical method that measures the capability of the process to meet customer demands. The sensitivity of the production processes or the measurement errors caused by the inspector factors make it difficult to determine the SLs by the quality inspectors. If the process includes uncertainties, traditional PCIs cannot certainly explain process performance. Since the data obtained from the process may contain uncertainty, the classical methods may not yield effective results on PCA. For this reason, the fact that FST and fuzzy set extensions are more flexible and easier to implement is very important in analyzing process capability. The successful results of fuzzy set extensions will enable us to obtain more accurate and realistic information in the analysis of the capability of the process under uncertainty and in cases where the quality inspectors are hesitant while making the evaluation. In this paper the PFSs one of fuzzy set extensions is integrated with PCA. Expressing the PFSs in terms of membership and non-membership degrees will allow quality inspector to evaluate the process more realistically. Evaluating the process with more than one membership function instead of a single membership function will provide more accurate results. Handling the process parameters with PFSs adds more precision or sensitivity to the process. For this aim, PCIs have been re-designed by considering the state of being Pythagorean of the process parameters. The main process parameters have been defined by using PFSs. For this aim, the indices C_p and C_{pk} have been re-designed and four new PCIs based on PFSs such as $\widetilde{C}_{sp}$, $\widetilde{C}_{spk}$, $\widetilde{C}_{fp}$ and $\widetilde{C}_{fpk}$ have been derived. The proposed new PCIs based on PFSs have been also applied on manufacturing process and capability for gears have been analyzed. The results obtained show that fuzzy set extensions give PCA more flexibility and more precision. It has also contributed to yielding more informative results. It was also concluded how important the membership degrees are on the PCA in the definition of fuzzy set extensions. When the results obtained in both Tab. 1 and Tab. 2 are evaluated, it is noticeably realized that the PFSs are more flexible than other fuzzy sets and that the non-membership function is crucial for modeling uncertainty. As a future study, the effects of fuzzy set extensions on different PCIs can be analyzed and the obtained results can be compared. The effects of different fuzzy set extensions on PCA can be evaluated. In cases where the process does not show a normal distribution, fuzzy set extensions can be examined. By making comparisons with different score functions in the literature, it can be analyzed which score function gives better results.

References

[1] M. Aghamohagheghi, S. M. Hashemi, R. Tavakkoli-Moghaddam, *An advanced decision support framework to assess sustainable transport projects using a new uncertainty modeling tool: Interval-valued Pythagorean trapezoidal fuzzy numbers*, Iranian Journal of Fuzzy Systems, **18**(1) (2021), 53-73.

[2] M. F. Ak, M. Gul, *AHP–TOPSIS integration extended with Pythagorean fuzzy sets for information security risk analysis*, Complex and Intelligent Systems, **5**(2) (2019), 113-126.

[3] M. Akram, I. Ullah, T. Allahviranloo, S. A. Edalatpanah, *Fully Pythagorean fuzzy linear programming problems with equality constraints*, Computational and Applied Mathematics, **40**(4) (2021), 1-20.

[4] M. Albing, *Process capability analysis with focus on indices for one-sided specification limits*, Doctoral Dissertation, Luleå University of Technology, 2006.

[5] T. T. Allen, *Introduction to engineering statistics and six sigma: Statistical quality control and design of experiments and systems*, Berlin/Heidelberg: Springer Science and Business Media, 2006.

[6] M. Aslam, M. Albassam, *Inspection plan based on the process capability index using the neutrosophic statistical method*, Mathematics, **7**(7) (2019), 1-10.

[7] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, **20**(1) (1986), 87-96.

[8] S. Aydin, C. Kahraman, M. Kabak, *Development of harmonic aggregation operator with trapezoidal Pythagorean fuzzy numbers*, Soft Computing, **24**(15) (2020), 11791-11803.

[9] Y. Cao, Z. Wu, T. Liu, Z. Gao, J. Yang, *Multivariate process capability evaluation of cloud manufacturing resource based on intuitionistic fuzzy set*, The International Journal of Advanced Manufacturing Technology, **84**(1-4) (2016), 227-237.

[10] S. M. Chen, T. M. Hung, *What can fuzziness do for capability analysis based on fuzzy data*, Scientia Iranica, **28**(2) (2021), 1049-1064.

[11] E. Haktanr, C. Kahraman, *Design for six sigma and process capability using penthagorean fuzzy sets*, In Intelligent and Fuzzy Techniques: Smart and Innovative Solutions: Proceedings of the INFUS 2020 Conference, Istanbul, Turkey, July 2020.

[12] E. Haktanr, C. Kahraman, *Process design and capability analysis using penthagorean fuzzy sets: Surgical mask production machines comparison*, Journal of Intelligent and Fuzzy Systems, **42**(1) (2022), 477-489.

[13] G. Hesamian, M. G. Akbari, *A process capability index for normal random variable with intuitionistic fuzzy information*, Operational Research, **21**(2) (2021), 951-964.

[14] C. Kahraman, E. Boltürk, S. C. Onar, B. Oztaysi, *Modeling humanoid robots facial expressions using Pythagorean fuzzy sets*, Journal of Intelligent and Fuzzy Systems, **39**(5) (2020), 6507-6515.

[15] C. Kahraman, A. Parchami, S. Cevik Onar, B. Oztaysi, *Process capability analysis using intuitionistic fuzzy sets*, Journal of Intelligent and Fuzzy Systems, **32**(3) (2017), 1659-1671.

[16] V. E. Kane, *Process capability indices*, Journal of Quality Technology, **18**(1) (1986), 41-52.

[17] İ. Kaya, M. Çolak, *A literature review on fuzzy process capability analysis*, Journal of Testing and Evaluation, **48**(5) (2020), 3963-3985.

[18] İ. Kaya, C. Kahraman, *A new perspective on fuzzy process capability indices: Robustness*, Expert Systems with Applications, **37**(6) (2010), 4593-4600.

[19] İ. Kaya, C. Kahraman, *Fuzzy process capability indices with asymmetric tolerances*, Expert Systems with Application, **38**(12) (2011), 14882-14890.

[20] İ. Kaya, C. Kahraman, *Process capability analyses based on fuzzy measurements and fuzzy control charts*, Expert Systems with Application, **38**(4) (2011), 3172-3184.

[21] S. Kotz, N. L. Johnson, *Process capability indices-A review, 1992-2000*, Journal of Quality Technology, **34**(1) (2002), 2-19.

[22] R. Kumar, S. A. Edalatpanah, S. Jha, R. Singh, *A Pythagorean fuzzy approach to the transportation problem*, Complex and Intelligent Systems, **5**(2) (2019), 255-263.

[23] A. Luqman, M. Akram, J. C. R. Alcantud, *Digraph and matrix approach for risk evaluations under Pythagorean fuzzy information*, Expert Systems with Applications, **170**(1) (2021), 114518.

[24] D. C. Montgomery, *Introduction to statistical quality control*, Fifth Edition, John Wiley and Sons, 2005.

[25] A. Parchami, S. Ç. Onar, B. Öztayşi, K. Kahraman, *Process capability analysis using interval type-2 fuzzy sets*, International Journal of Computational Intelligence Systems, **10**(1) (2017), 721-733.

[26] X. Peng, Y. Yang, *Some results for Pythagorean fuzzy sets*, International Journal of Intelligent Systems, **30**(11) (2015), 1133-1160.

[27] L. J. Porter, J. S. Oakland, *Process capability indices-an overview of theory and practice*, Quality and Reliability Engineering International, **7**(6) (1991), 437-448.

[28] P. Rani, A. R. Mishra, A. Mardani, *An extended Pythagorean fuzzy complex proportional assessment approach with new entropy and score function: Application in pharmacological therapy selection for type 2 diabetes*, Applied Soft Computing, **94** (2020), 106441.

[29] O. Senvar, C. Kahraman, *Type-2 fuzzy process capability indices for non-normal processes*, Journal of Intelligent and Fuzzy Systems, **27**(2) (2014), 769-781.

[30] M. Shakeel, S. Abdullah, M. S. Ali Khan, K. Rahman, *Averaging aggregation operators with interval Pythagorean trapezoidal fuzzy numbers and their application to group decision making*, Punjab University Journal of Mathematics, **50**(2) (2020), 147-170.

[31] M. Shakeel, S. Abdullah, M. Shahzad, N. Siddiqui, *Geometric aggregation operators with interval-valued Pythagorean trapezoidal fuzzy numbers based on Einstein operations and their application in group decision making*, International Journal of Machine Learning and Cybernetics, **10**(10) (2019), 2867-2886.

[32] R. R. Yager, *Pythagorean fuzzy subsets*, in Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting, IFSA/NAFIPS 2013, Edmonton, AB, Canada, 2013.

[33] S. Yalçın, İ. Kaya, *Design and analysis of process capability indices Cpm and Cpmk by neutrosophic sets*, Iranian Journal of Fuzzy Systems, **19**(1) (2022), 13-30.

[34] S. Yalçın, İ. Kaya, *Analyzing of process capability indices based on neutrosophic sets*, Computational and Applied Mathematics, **41** (2022), 287.

[35] S. Yalçın, İ. Kaya, *Two-dimensional uncertainty analysis for Cp and Cpk process capability indices*, 2022 International Conference on Innovation and Intelligence for Informatics, Computing, and Technologies (3ICT), Sakheer, Bahrain, (2022), 419-423.

[36] S. Yalçın, İ. Kaya, *Design and analysis of Cpm and Cpmk indices for uncertainty environment by using Pythagorean fuzzy sets*, 2022 International Conference on Innovation and Intelligence for Informatics, Computing, and Technologies (3ICT), Sakheer, Bahrain, (2022), 293-297.

[37] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8**(3) (1965), 338-353.

[38] X. Zhang, Z. Xu, *Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets*, International Journal of Intelligent Systems, **29**(12) (2014), 1061-1078.