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Equivalent axioms of M-fuzzifying convex matroids

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Abstract

In this paper, our aim is to present some characterizations of M-fuzzifying convex matroids. First we discuss the relation between M-fuzzifying convex matroids and M-fuzzy families of dependent sets. Secondly, we give characterizations of M-fuzzifying convex matroids by M-fuzzifying rank functions. Finally we discuss the relation between two concept of M-fuzzifying hull (closure) operators.

Keywords: M-fuzzifying convex matroid, M-fuzzifying hull operator, M-fuzzy family of dependent sets, M-fuzzifying circuit map, M-fuzzifying rank function.

1 Introduction

Matroid theories are widely used in combinatorial optimization, game theory, graph theory, information security and so on. Matroids are precisely the structures for which the very simple and efficient greedy algorithm works.

In fact, there are two different definitions about matroids. The earliest definition of matroids was introduced by Whitney in [21]. In a Whitney matroid, the basic set is finite. Another definition of matroids can be found in [19]. Van De Vel introduced the definition of matroids by means of convex structure, where the basic set of a matroid need not be finite. Thus a matroid can be regarded as a special convexity space.

In the classical problem, it is assumed that all the weights are precisely known. However, this assumption may be a serious restriction since in many practical applications the exact values of the weights are not known in advance. For example, they can be fuzzy numbers.

In 1988, R. Goetschel and W. Voxman introduced the notion of fuzzy matroids in [4], which is a generalization of Whitney matroid in [14, 20, 21]. In a GV-fuzzy matroid, the family of independent sets is crisp and its independent sets are fuzzy sets. On this basis, many scholars have done a lot of follow-up work [6, 9, 10, 12, 13, 23, 24].

From a completely different point of view, Shi introduced the notions of M-fuzzifying matroids and (L, M)-fuzzy matroids in [15, 16], where L and M are completely distributive lattices. Just like the theory of crisp matroids, M-fuzzifying matroids have many good properties and characteristics. In particular, it is proved that for a fuzzy set $\mathscr{I} : 2^E \to [0, 1]$ with the property that if $B \subseteq C$, then $\mathscr{I}(B) \geq \mathscr{I}(C)$, and for all fuzzy weight functions ω on E, there exists an optimal set to some extent if and only if $\mathscr{I}$ is a $[0, 1]$-fuzzifying matroid [17, 28].

The fuzzifying matroids and the fuzzy matroids in the sense of [4, 15, 16] are generalizations of Whitneys matroids.

In 2014, F. G. Shi and Z. Y. Xiu introduced the notion of M-fuzzifying convex structures. An M-fuzzifying convex structure is a mapping $\mathfrak{C}$ from 2^X to a complete lattice M satisfying a set of axioms [18]. Based on the notion of M-fuzzifying convex structure, X. Y. Wu, B. Davvaz, S. Z. Bai, S. J. Yang and F. G. Shi respectively generalized the notions of independence structures and matroids to fuzzy setting. They respectively introduced the notions of strong M-fuzzifying independence structures and M-fuzzifying matroids [22, 27] which will be called M-fuzzifying convex matroids.

In fact, there are many equivalent ways to characterize an M-fuzzifying matroid. Significant characterizations of M-fuzzifying matroids include those in terms of M-fuzzy family of dependent sets, M-fuzzifying base maps, M-fuzzifying

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circuit maps, M-fuzzifying rank functions, and so on [2, 7, 8, 11, 17, 23, 25, 26, 27, 28, 29]. As we know, the notion of closure operators is very important in matroid theory. A matroid can be characterized by its closure operator, and the closure operator can be characterized by its rank function. Then a natural problem is: Can an M-fuzzifying matroid be characterized by a fuzzy closure operator? In [17], Shi and Wang presented a notion of closure operators. But its applications are limited and its condition is cumbersome. If we define a fuzzifying matroid from a fuzzifying convex space, then this fuzzifying matroid naturally has a fuzzifying closure operator, but this naturally leads to another question, can this fuzzifying closure operator be represented by a fuzzy rank function?

In this paper, we shall answer the above problem. We shall present some new characterizations of M-fuzzifying convex matroids in the sense of [27]. First we discuss the relation among M-fuzzifying convex matroids, M-fuzzy families of dependent sets and M-fuzzifying circuit map. Then we give characterizations of the fuzzifying hull operator of an M-fuzzifying convex matroid by M-fuzzifying rank functions. Finally we discuss the relation two definitions of closure operators in an M-fuzzifying convex matroid.

2 Preliminaries

Let us first recall some basic concepts in fuzzifying matroid theory and fuzzifying convex spaces [15, 16, 18, 27].

Given a non-empty finite set E, we denote the set of all subsets of E by 2^E. A complete Heyting algebra [3] is a complete lattice $(M, \bigvee, \bigwedge)$ satisfying the following:

$$a \wedge \left(\bigvee_{i\in\Omega} b_i \right) = \bigvee_{i\in\Omega} (a \wedge b_i) .$$

In a complete Heyting algebra M, there exists a binary operation $\mapsto$. Explicitly the implication is given by

$$a \mapsto b = \bigvee \{c \in L \mid a \wedge c \leq b\}.$$

We list some of its properties.

(1) $(a \mapsto b) \geq c \Leftrightarrow a \wedge c \leq b$;

(2) $a \mapsto b = \top \Leftrightarrow a \leq b$;

(3) $a \mapsto (\bigwedge_i b_i) = \bigwedge_i (a \mapsto b_i)$;

(4) $(\bigvee_i a_i) \mapsto b = \bigwedge_i (a_i \mapsto b)$;

(5) $(a \mapsto c) \wedge (c \mapsto b) \leq a \mapsto b$;

(6) $a \leq b \Rightarrow c \mapsto a \leq c \mapsto b$;

(7) $a \leq b \Rightarrow b \mapsto c \leq a \mapsto c$;

(8) $(a \mapsto b) \wedge (c \mapsto d) \leq a \wedge c \mapsto b \wedge d$.

A completely distributive lattice must be a complete Heyting algebra.

Throughout this paper, $(M, \bigvee, \bigwedge, \ ')$ denotes a completely distributive lattice with an order-reversing involution $'$. The smallest element (or zero element) and the largest element (or unit element) in M are denoted by $\perp$ and $\top$ respectively.

An element a in M is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. a in M is called co-prime if $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$ [3]. The set of non-unit prime elements in M is denoted by $P(M)$. The set of non-zero co-prime elements in M is denoted by $J(M)$. For $\emptyset \subseteq M$, we define $\bigvee \emptyset = \perp$ and $\bigwedge \emptyset = \top$.

The binary relation $\prec$ in M is defined as follows: for $a, b \in M$, $a \prec b$ if and only if for every subset $D \subseteq M$, the relation $b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leq d$ [1].

An M-fuzzy set on a nonempty set X is a map $A : X \to M$. All M-fuzzy sets on X are denoted by M^X. For $A \in M^X$ and $a \in M$ we can define

$$A_{[a]} = \{x \in X \mid A(x) \geq a\}, \quad A^{(a)} = \{x \in X \mid A(x) \not\leq a\}.$$

Some properties of these cut sets can be found in [5, 15, 16, 17].

Definition 2.1. [15, 16] *Let $\mathbb{N}$ denote the set of all natural numbers. An M-fuzzy natural number is an antitone map $\lambda : \mathbb{N} \to M$ satisfying*

$$\lambda(0) = \top, \quad \bigwedge_{n\in\mathbb{N}} \lambda(n) = \bot.$$

The set of all M-fuzzy natural numbers is denoted by $\mathbb{N}(M)$.

$\forall\lambda, \mu \in \mathbb{N}(M)$, we define $\lambda \le \mu$ if and only if $\lambda(n) \le \mu(n)$ for all $n \in \mathbb{N}(M)$.

Remark 2.2. [15] *For any $n \in \mathbb{N}$, define $\underline{n} \in \mathbb{N}(M)$ such that*

$$\underline{n}(t) = \begin{cases} \top, & \text{if } t \le n; \\ \bot, & \text{if } t \ge n+1; \end{cases}$$

Then $\mathbb{N}$ can be regarded as a subset of $\mathbb{N}(M)$.

In the sequel, we will not distinguish n and $\underline{n}$.

In [15, 16], the notion of M-fuzzy family of independent sets is presented. It was defined as follows:

Definition 2.3. [15, 16] *Let E be a finite set. A map $\mathscr{I} : 2^E \to M$ is called an M-fuzzy family of independent sets on E if it satisfies the following three conditions:*

(FI1) $\mathscr{I}(\emptyset) = \top$.

(FI2) *For any $A, B \in 2^E$, $A \subseteq B \Rightarrow \mathscr{I}(A) \ge \mathscr{I}(B)$.*

(FI3) *If $A, B \in 2^E$ and $|B| > |A|$, then $\bigvee_{e\in B-A} \mathscr{I}(A \cup \{e\}) \ge \mathscr{I}(A) \wedge \mathscr{I}(B)$.*

If $\mathscr{I}$ is an M-fuzzy family of independent sets on E, then the pair $(E, \mathscr{I})$ is called an M-fuzzifying matroid.

Theorem 2.4. [15, 16] *Let E be a finite set and let $\mathscr{I} : 2^E \to M$ be a map. Then the following conditions are equivalent:*

(1) *$(E, \mathscr{I})$ is an M-fuzzifying matroid;*

(2) *For each $a \in J(M)$, $(E, \mathscr{I}_{[a]})$ is a matroid;*

(3) *For each $a \in P(M)$, $\left(E, \mathscr{I}^{(a)}\right)$ is a matroid.*

In order to characterize M-fuzzifying matroid, the notion of M-fuzzifying rank functions was introduced in [15]. Its definition is as follows.

Definition 2.5. [15] *Let $(E, \mathscr{I})$ be an M-fuzzifying matroid. The map $R : 2^E \to \mathbb{N}(M)$ defined by*

$$R(A)(n) = \bigvee \{\mathscr{I}(B) : B \subseteq A, |B| \ge n\},$$

is called the M-fuzzifying rank function of $(E, \mathscr{I})$.

Definition 2.6. *Let $R : 2^E \to \mathbb{N}(M)$ be a map satisfying* (FR1), (FR2) *and* (FR3).

(FR1) *For any $A \in 2^E$, $\underline{0} \leqslant R(A) \leqslant \underline{|A|}$.*

(FR2) *If $A, B \in 2^E$ and $A \subseteq B$, then $R(A) \leqslant R(B)$.*

(FR3) *For any $A, B \in 2^E$, $R(A) + R(B) \geqslant R(A \cap B) + R(A \cup B)$.*

Then R is the M-fuzzifying rank function for an M-fuzzifying matroid.

In [27], based on the notion of M-fuzzifying convex structures, Yang and Shi presented a new notion of M-fuzzifying matroids by means of M-fuzzifying hull operators (or M-fuzzifying closure operators).

Definition 2.7. [18] *A map $\mathfrak{C} : 2^X \to M$ is called an M-fuzzifying convex structure, and the pair $(X, \mathfrak{C})$ is called an M-fuzzifying convexity space if $\mathfrak{C}$ satisfies the following three conditions:*

($CS1$) $\mathfrak{C}(\emptyset) = \mathfrak{C}(X) = \top$.

(*CS*2) $\mathfrak{C}$ *is stable for intersections, that is, if* $\mathscr{A} \subseteq 2^X$*, then* $\mathfrak{C}(\bigcap \mathscr{A}) \geq \bigwedge\{\mathfrak{C}(A) \mid A \in \mathscr{A}\}$.

(*CS*3) $\mathfrak{C}$ *is satble for nested unions, that is, if* $\mathscr{A} \subseteq 2^X$ *is totally ordered by inclusion, then* $\mathfrak{C}(\bigcup \mathscr{A}) \geq \bigwedge\{\mathfrak{C}(A) \mid A \in \mathscr{A}\}$.

Lemma 2.8. [18] *There is a one-to-one correspondence between M-fuzzifying convex structures and M-fuzzifying hull operators* $\mathrm{co} : 2^X \to M^X$ *satisfying the following conditions:*

(CL1) $\mathrm{co}(\emptyset) = \emptyset$.

(CL2) $\mathrm{co}(A)(x) = \top, \forall x \in A$.

(CL3) $A \subseteq B$ *implies* $\mathrm{co}(A) \leq \mathrm{co}(B)$ *(that is,* $\mathrm{co}(A)(x) \leq \mathrm{co}(B)(x), \forall x \in X$*).*

(CL4) $\mathrm{co}(A)(x) = \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} \mathrm{co}(B)(y)$, $\forall A \in 2^X$, $\forall x \in X$.

(CL5) $\mathrm{co}(A) = \bigvee\{\mathrm{co}(F) \mid F \in 2^A_{fin}\}$, $\forall A \in 2^X$, *where* 2^A_{fin} *is the set of all finite subsets of A.*

In [22, 27], based on M-fuzzifying convex structures, the notion of M-fuzzifying convex matroids are respectively introduced. In fact, they are equivalent to each other. Now we use the definition in [27], because it seems relatively simple.

Definition 2.9. [27] *An M-fuzzifying convex space* $(X, \mathfrak{C})$ *is called an M-fuzzifying convex matroid if the hull operator* co *of* $\mathfrak{C}$ *satisfy the following condition:*

(*CL*6) $\mathrm{co}(A \cup \{x\})(y) \leq \mathrm{co}(A)(y) \vee \mathrm{co}(A \cup \{y\})(x)$, $\forall A \in 2^E$, $\forall x, y \in X$.

Based on Lemma 2.8 and Definition 2.9, an M-fuzzifying convex matroid is also denoted by (X, co), where co is the hull operator satisfying conditions (CL1)–(CL6).

From the results in [27], an M-fuzzifying convex matroid must be an M-fuzzifying matroid.

3 M-fuzzy family of dependent sets and M-fuzzifying circuit maps

In this section, we will present the relation between the hull operator and M-fuzzy family of dependent sets in an M-fuzzifying convex matroid.

In the sequel, we suppose that E is a finite set.

Theorem 3.1. *Let* $\mathrm{co} : 2^E \to M^E$ *be a map. Define a map* $\mathrm{co}_{[a]} : 2^E \to 2^E$ *such that* $\mathrm{co}_{[a]}(A) = \mathrm{co}(A)_{[a]}$*. Then* (E, co) *is an M-fuzzifying convex matroid if and only if for all* $a \in J(M)$, $(E, \mathrm{co}_{[a]})$ *is a crisp convex matroid.*

Proof. (⇒) Let (E, co) be an M-fuzzifying convex matroid. Then $\mathrm{co} : 2^E \to M^E$ satisfies (CL1)–(CL6). From (CL1)–(CL3) we can easily check the following (MH1) and (MH2).

(MH1) $\mathrm{co}_{[a]}(\emptyset) = \emptyset$, $\mathrm{co}_{[a]}(E) = E$.

(MH2) If $A \subseteq B$, then $\mathrm{co}_{[a]}(A) \subseteq \mathrm{co}_{[a]}(B)$.

Next we prove

(MH3) $\mathrm{co}_{[a]}(\mathrm{co}_{[a]}(A)) \subseteq \mathrm{co}_{[a]}(A)$. This can be proved from the following implication.

$$\begin{aligned} x \notin \mathrm{co}_{[a]}(A) &\Rightarrow \mathrm{co}(A)(x) = \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} \mathrm{co}(B)(y) \ngeq a \\ &\Rightarrow \exists B \text{ such that } x \notin B \supseteq A \text{ and } \forall y \notin B, \mathrm{co}(B)(y) \ngeq a \\ &\Rightarrow \exists B \text{ such that } x \notin B \supseteq A \text{ and } \forall y \notin B, y \notin \mathrm{co}(B)_{[a]} \\ &\Rightarrow \exists B \text{ such that } x \notin B = \mathrm{co}_{[a]}(B) \supseteq \mathrm{co}_{[a]}(A) \text{ and } B = \mathrm{co}(B)_{[a]} = \mathrm{co}_{[a]}(B) \\ &\Rightarrow \exists B \text{ such that } x \notin B \supseteq \mathrm{co}_{[a]}(A) \text{ and } B \supseteq \mathrm{co}_{[a]}(\mathrm{co}_{[a]}(A)) \\ &\Rightarrow x \notin \mathrm{co}_{[a]}(\mathrm{co}_{[a]}(A)). \end{aligned}$$

(MH4) $\mathrm{co}_{[a]}(A) = \bigvee\{\mathrm{co}_{[a]}(F) \mid F \in 2^A_{fin}\}$ is true from (CL5).

(MH5) $y \in \mathrm{co}_{[a]}(A \cup \{x\}), y \notin \mathrm{co}(A) \Rightarrow x \in \mathrm{co}_{[a]}(A \cup \{y\})$ holds from (CL6).

This shows that $(E, \mathrm{co}_{[a]})$ is a crisp convex matroid.

($\Leftarrow$) Suppose that for all $a \in J(M)$, $(E, \mathrm{co}_{[a]})$ is a crisp convex matroid. $\forall A \in 2^E$, by

$$\mathrm{co}(A)(x) = \bigvee \{a \in J(M) \mid x \in \mathrm{co}_{[a]}(A)\},$$

we can obtain (CL1)–(CL3).

(CL4) The following inequality is obvious.

$$\mathrm{co}(A)(x) \leq \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} (\mathrm{co}(B))(y).$$

In order to show

$$(\mathrm{co}(A))(x) \geq \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} (\mathrm{co}(B))(y),$$

let $b \in J(M)$ with $\mathrm{co}(A)(x) \ngeq b$. Then there exists $a \prec b$ such that $\mathrm{co}(A)(x) \ngeq a$. By (MH3), we have

$$x \notin (\mathrm{co}(A))_{[a]} = (\mathrm{co}((\mathrm{co}(A))_{[a]}))_{[a]}.$$

Let $D = (\mathrm{co}(A))_{[a]}$. Then it is obvious that $x \notin D \supseteq A$ and $(\mathrm{co}(D))(y) \ngeq a$ are true for every $y \notin D$. This implies $\bigvee_{y \notin D} (\mathrm{co}(D))(y) \ngeq b$ (In fact, if $\bigvee_{y \notin D} (\mathrm{co}(D))(y) \geq b$, then there exists $y \notin D \supseteq A$ such that $(\mathrm{co}(D))(y) \geq a$, which contradicts $(\mathrm{co}(D))(y) \ngeq a$). Further we have

$$\bigwedge_{x \notin D \supseteq A} \bigvee_{y \notin D} (\mathrm{co}(D))(y) \ngeq b.$$

From the arbitrariness of b, we obtain that

$$(\mathrm{co}(A))(x) \geq \bigwedge_{x \notin B \supseteq A} \bigvee_{y \notin B} (\mathrm{co}(B))(y).$$

(CL5) and (CL6) are obvious. This shows (E, co) is an M-fuzzifying convex matroid. □

The following theorem shows that we can induce an M-fuzzy family of dependent sets from M-fuzzifying convex matroid.

Theorem 3.2. *Let (E, co) be an M-fuzzifying convex matroid. Define a map $\mathscr{D} : 2^E \to M$ as follows:*

$$\mathscr{D}(A) = \bigvee_{x \in A} (\mathrm{co}(A - \{x\})(x)).$$

Then $\mathscr{D}$ satisfies the following conditions (FD1)–(FD4). *$\mathscr{D}$ is called the M-fuzzy family of dependent sets of* (E, co).

(FD1) *$\mathscr{D}(\{x\}) = \bot$, $\forall x \in E$;*

(FD2) *For any $A, B \in 2^E$, $A \subseteq B \Rightarrow \mathscr{D}(A) \leq \mathscr{D}(B)$;*

(FD3) $\forall A, B \in 2^E$, $\mathscr{D}(A) \wedge \mathscr{D}(B) \leq \mathscr{D}(A \cap B) \vee \bigwedge_{x \in E} \mathscr{D}(A \cup B - \{x\})$.

(FD4) *For any $A \in 2^E$ and any $a, b \in J(M)$ with $a \geq b$, if there exists $x \in E$ and $B_1 \subseteq A$ such that $a \leq \mathscr{D}(B_1), a \nleq \mathscr{D}(B_1 - \{x\})$, then there exists $B_2 \subseteq A$ such that $b \leq \mathscr{D}(B_2), b \nleq \mathscr{D}(B_2 - \{x\})$.*

Proof. As proved in [27], when $\mathscr{I}(A) = \bigwedge_{x \in A} (\mathrm{co}(A - \{x\})(x))'$, $\mathscr{I}$ is a strong M-fuzzy family of dependent sets on E. Hence we know that $\mathscr{D}$ is an M-fuzzy family of dependence sets on E by $\mathscr{D}(A) = \mathscr{I}(A)'$ for all $A \in 2^E$. This shows that $\mathscr{D}$ satisfies conditions (FD1)–(FD3) (The proof can be found in [17]).

For any $A \in 2^E$ and any $a, b \in J(M)$ with $a \geq b$, if there exists $x \in E$ and $B_1 \subseteq A$ such that $a \leq \mathscr{D}(B_1), a \nleq \mathscr{D}(B_1 - \{x\})$, then

$$x \in \mathrm{co}_{[a]}(B_1 - \{x\}) \subseteq \mathrm{co}_{[a]}(A - \{x\}) \subseteq \mathrm{co}_{[b]}(A - \{x\}).$$

Thus there exists $B_2 \subseteq A - \{x\}$ such that $b \leq \mathscr{D}(B_2), b \nleq \mathscr{D}(B_2 - \{x\})$. This shows that (FD4) holds. □

Corollary 3.3. *Let (E, co) be an M-fuzzifying convex matroid. $\mathscr{D}$ is the M-fuzzy family of dependent sets on (E, co). Then $\forall a \in J(L)$, $\mathscr{D}_{[a]}$ is the family of dependence sets on $(E, \mathrm{co}_{[a]})$.*

Theorem 3.4. *If a map $\mathscr{D} : 2^E \to M$ satisfies* (FD1)–(FD4)*, then there exists an M-fuzzifying convex matroid (E, co) such that $\mathscr{D}$ is M-fuzzy family of dependent sets in (E, co).*

Proof. Suppose that $\mathscr{D} : 2^E \to M$ satisfies (FD1)–(FD4). Then from Corollary 3.3 we know that $\forall a \in J(L)$, $\mathscr{D}_{[a]}$ is a family of dependent sets. Define

$$\mathrm{co}_{[a]}(A) = A \cup \{x \mid \text{ there exists a minimal } B \in \mathscr{D}_{[a]} \text{ such that } x \in B \subseteq A \cup \{x\}\}. \tag{1}$$

By crisp case we know that $\mathrm{co}_{[a]}$ is a hull operator. Define $\mathrm{co} : 2^E \to M^X$ such that

$$\mathrm{co}(A)(x) = \top_A(x) \vee \bigvee \left\{ \mathscr{D}(B) \mid x \in B \subseteq A \cup \{x\}, \mathscr{D}(B) \not\leq \bigvee_{A \subsetneqq B} \mathscr{D}(A) \right\}, \tag{2}$$

where $\top_A$ is the characteristic function of A. Now we prove $\mathrm{co}(A)_{[a]} = \mathrm{co}_{[a]}(A)$. We only need to prove $\mathrm{co}(A)_{[a]} - A = \mathrm{co}_{[a]}(A) - A$

Suppose that $x \in \mathrm{co}_{[a]}(A) - A$. Then there exists a minimal $B \in \mathscr{D}_{[a]}$ such that $x \in B \subseteq A \cup \{x\}$. This implies that there exists a B such that $a \leq \mathscr{D}(B)$, $a \not\leq \bigvee_{C \subsetneqq B} \mathscr{D}(C)$ and $x \in B \subseteq A \cup \{x\}$. Hence we have

$$\begin{aligned} \mathrm{co}(A)(x) &= \top_A(x) \vee \bigvee \left\{ \mathscr{D}(B) \mid x \in B \subseteq A \cup \{x\}, \mathscr{D}(B) \not\leq \bigvee_{A \subsetneqq B} \mathscr{D}(A) \right\} \\ &\geq \bot \vee \bigvee \left\{ \mathscr{D}(B) \mid x \in B \subseteq A \cup \{x\}, \mathscr{D}(B) \not\leq \bigvee_{A \subsetneqq B} \mathscr{D}(A) \right\} \geq a, \end{aligned}$$

i.e., $x \in \mathrm{co}(A)_{[a]}$. This shows $\mathrm{co}(A)_{[a]} - A \supseteq \mathrm{co}_{[a]}(A) - A$.

In order to prove $\mathrm{co}(A)_{[a]} - A \subseteq \mathrm{co}_{[a]}(A) - A$, take $x \in \mathrm{co}(A)_{[a]} - A$. Then $\mathrm{co}(A)(x) \geq a$. Hence there exists a B such that $a \leq \mathscr{D}(B)$, $\mathscr{D}(B) \not\leq \bigvee_{C \subsetneqq B} \mathscr{D}(C)$ and $x \in B \subseteq A \cup \{x\}$ since 2^E is finite set. Let C be a minimal set satisfying $a \leq \mathscr{D}(C)$ and $x \in C \subseteq A \cup \{x\}$. Then C is a circuit in $\mathscr{D}_{[a]}$. Therefore we obtain $x \in \mathrm{co}_{[a]}(A)$. This shows $\mathrm{co}(A)_{[a]} - A \subseteq \mathrm{co}_{[a]}(A) - A$.

From $\mathrm{co}(A)_{[a]} = \mathrm{co}_{[a]}(A)$ we know that (E, co) is an M-fuzzifying convex matroid and $\mathscr{D}$ is M-fuzzy family of dependent set in (E, co). □

Theorem 3.5. *Let (X, co) be an M-fuzzifying convex matroid. Define a map $\mathscr{C} : 2^X \to M$ by*

$$\mathscr{C}(B) = \begin{cases} \mathscr{D}(B), & \mathscr{D}(B) \not\leq \bigvee_{A \subsetneqq B} \mathscr{D}(A), \\ \bot, & \textit{otherwise}. \end{cases}$$

Then $\mathscr{C}$ is called the M-fuzzifying circuit map, it satisfies the following conditions.

(1) *$\forall a \in J(M)$, the family of all minimal elements in $\mathscr{C}_{[a]}$ is the family of circuits of crisp matroid $(E, \mathrm{co}_{[a]})$;*

(2) *If $\mathscr{C}(B) = a$, then B is minimal element in $\mathscr{C}_{[a]}$;*

(3) *For any $A \in 2^E$ and any $a, b \in J(M)$ with $a \geq b$, if there exists $x \in E$ and $B_1 \subseteq A$ such that $a \leq \mathscr{C}(B_1), a \not\leq \mathscr{C}(B_1 - \{x\})$, then there exists $B_2 \subseteq A$ such that $b \leq \mathscr{C}(B_2), b \not\leq \mathscr{C}(B_2 - \{x\})$.*

Proof. (1) and (2) are true from [26] since an M-fuzzifying convex matroid must be an M-fuzzifying matroid in the sense of [15]. (3) holds from Theorem 3.2. □

Theorem 3.6. *Let (E, co) be an M-fuzzifying convex matroid and let $\mathscr{C}, \mathscr{D}$ be respectively the fuzzifying circuit map and the M-fuzzy family of sets. Then*

$$\mathscr{D}(A) = \bigvee_{B \subseteq A} \mathscr{C}(B).$$

Proof. For $a \in J(M)$, let

$$\mathscr{D}(A) = \bigvee_{x \in A} (\mathrm{co}(A - \{x\})(x)) \geq a.$$

Then there exists $x \in A$ such that $\mathrm{co}(A - \{x\})(x) \geq a$, that is,

$$x \in (\mathrm{co}(A - \{x\}))_{[a]} = \mathrm{co}_{[a]}(A - \{x\}).$$

By Theorem 3.1 we know that A is a dependent set in crisp convex matroid $(E, \mathrm{co}_{[a]})$. Hence there exists a circuit B in $(E, \mathrm{co}_{[a]})$ such that $B \subseteq A$. Thus B is a minimal set such that $x \in \mathrm{co}_{[a]}(B - \{x\})$ for some $x \in B$. This implies $\mathscr{C}(B) \geq a$. Therefore we obtain $\mathscr{D}(A) \leq \bigvee_{B \subseteq A} \mathscr{C}(B)$.

Conversely if $a \in J(M)$ and $\bigvee_{B \subseteq A} \mathscr{C}(B) \geq a$, then there exists $B \subseteq A$ such that $\mathscr{C}(B) \geq a$. From the definition of $\mathscr{C}$ we know that $a \leq \mathscr{C}(B) \leq \mathscr{D}(B)$. This implies $B \in (\mathscr{C})_{[a]} \subseteq (\mathscr{D})_{[a]}$. This shows that $A \in (\mathscr{D})_{[a]}$, that is, $\mathscr{D}(A) \geq a$. $\mathscr{D}(A) \geq \bigvee_{B \subseteq A} \mathscr{C}(B)$ is shown. □

From the proof of Theorem 3.4 we can obtain the following corollary.

Corollary 3.7. *Let (E, co) is an M-fuzzifying convex matroid. Then $\forall A \in 2^E$, $\forall x \in E$,*

$$\mathrm{co}(A)(x) = \top_A(x) \vee \bigvee_{x \in B \subseteq A \cup \{x\}} \mathscr{C}(B).$$

4 The hull operators characterized by fuzzy rank functions

In a crisp matroid, the closure operator or the hull operator can be characterized by means of its rank function. But in an M-fuzzifying convex matroid, we don't know how is the M-fuzzifying hull operator characterized. Now we solve this problem.

Theorem 4.1. *For an M-fuzzifying convex matroid (E, co), the following conditions hold.*

(1) $a \leq \mathrm{co}(A)(x) \Leftrightarrow R(A \cup \{x\})_{[b]} \subseteq R(A)_{[b]},\ \forall b \in J(L)$ *with* $b \leq a \in J(M)$.

(2) $a \leq \mathrm{co}(A)(x) \Leftrightarrow a \wedge R(A \cup \{x\})(n) \leq R(A)(n), \forall n \in \mathbb{N}, \forall a \in J(M)$.

(3) $\mathrm{co}(A)(x) = R(A \cup \{x\}) \mapsto R(A) \triangleq \bigwedge_{n \in \mathbb{N}} (R(A \cup \{x\})(n) \mapsto R(A)(n))$.

(4) $a \leq \mathrm{co}(A)(x) \Leftrightarrow R(A \cup \{x\})^{(b)} \subseteq R(A)^{(b)},\ \forall b \in P(M)$ *with* $b \not\geq a$.

(5) $\mathrm{co}(A)(x) = \bigvee\{a \in J(M) \mid R(A \cup \{x\})_{[b]} \subseteq R(A)_{[b]},\ \forall b \in J(M)$ *with* $b \leq a\}$.

(6) $\mathrm{co}(A)(x) = \bigvee\{a \in J(M) \mid a \wedge R(A \cup \{x\})(n) \leq R(A)(n), \forall n \in \mathbb{N}\}$.

(7) $\mathrm{co}(A)(x) = \bigvee\{a \in J(M) \mid R(A \cup \{x\})^{(b)} \subseteq R(A)^{(b)},\ \forall b \in P(M)$ *with* $b \not\geq a\}$.

Proof. (1) can be proved from the following fact.

$$\begin{aligned} \forall b \in J(M),\ b \leq a \leq \mathrm{co}(A)(x) &\Leftrightarrow x \in \mathrm{co}(A)_{[b]} = \mathrm{co}_{[b]}(A),\ \forall b \in J(L) \text{ with } b \leq a \\ &\Leftrightarrow R(A \cup \{x\})_{[b]} \subseteq R(A)_{[b]},\ \forall b \in J(L) \text{ with } b \leq a. \end{aligned}$$

(2) can be obtained from (1) and the following fact.

$$\begin{aligned} & a \leq \mathrm{co}(A)(x) \\ \Leftrightarrow\ & R(A \cup \{x\})_{[b]} \subseteq R(A)_{[b]},\ \forall b \in J(L) \text{ with } b \leq a \\ \Leftrightarrow\ & b \leq a \wedge R(A \cup \{x\})(n) \text{ implies } b \leq R(A)(n), \forall n \in \mathbb{N}, \forall b \in J(M) \\ \Leftrightarrow\ & a \wedge R(A \cup \{x\})(n) \leq R(A)(n), \forall n \in \mathbb{N}. \end{aligned}$$

(3) can be obtained from (2) and the following fact.

$$\begin{aligned} & a \leq \mathrm{co}(A)(x) \\ \Leftrightarrow\ & a \wedge R(A \cup \{x\})(n) \leq R(A)(n), \forall n \in \mathbb{N} \\ \Leftrightarrow\ & a \leq \bigwedge_{n \in \mathbb{N}} (R(A \cup \{x\})(n) \mapsto R(A)(n)). \end{aligned}$$

(4) can be obtained from (2) and the following fact.

$$\begin{aligned} & a \wedge R(A \cup \{x\})(n) \leq R(A)(n), \forall n \in \mathbb{N} \\ \Leftrightarrow \quad & a \wedge R(A \cup \{x\})(n) \not\leq b \text{ implies } R(A)(n) \not\leq b, \forall n \in \mathbb{N}, \forall b \in P(M) \\ \Leftrightarrow \quad & R(A \cup \{x\})^{(b)} \subseteq R(A)^{(b)}, \forall b \in P(M) \text{ with } b \not\geq a. \end{aligned}$$

(5), (6) and (7) can be respectively obtained from (1), (2) and (4). □

5 The relation of different notions of closures

In order to characterize M-fuzzifying matroids, Shi and Wang presented a definition of M-fuzzifying closure operators in [17].

Let $(E, \mathscr{I})$ be an M-fuzzifying matroid. Define a mapping $\mathrm{cl}_{\mathscr{I}} : 2^E \times J(M) \to 2^E$ by

$$\mathrm{cl}_{\mathscr{I}}(A, a) = \{x \in E : R_{[a]}(A) = R_{[a]}(A \cup \{x\})\}, \tag{3}$$

where $R_{[a]}$ is the rank function of $(E, \mathscr{I}_{[a]})$. Then $\mathrm{cl}_{\mathscr{I}}$ is an M-fuzzifying closure operator in the sense of [17].

Now let us replace $(E, \mathscr{I})$ by M-fuzzifying convex matroid (E, co). Then $R_{[a]}$ is the rank function of $(E, \mathrm{co}_{[a]})$, hence $R_{[a]}$ is the rank function of matroid $(E, (\mathscr{I}_{\mathrm{co}})^{(a')})$. In this case, the rank function of $(E, \mathscr{I}_{[a]})$ should be $R^{(a')}$.

Thus by means of Theorem 4.1 (4) we can get the relation between the M-fuzzifying closure operator in [17] and the hull operator of M-fuzzifying convex matroid in Definition 2.9 as follows.

Theorem 5.1. *Let (E, co) be an M-fuzzifying convex matroid. Define a map $\mathrm{cl} : 2^E \times J(M) \to 2^E$ by*

$$\mathrm{cl}(A, a) = \left\{ x \in E \mid R^{(a')}(A) = R^{(a')}(A \cup \{x\}) \right\},$$

where $R^{(a')}$ is the rank function of $(E, (\mathscr{I}_{\mathrm{co}})_{[a]})$. Then $\mathrm{co}(A)_{[b]} = \bigcap_{a \not\leq b'} \mathrm{cl}(A, a)$.

Proof. This can be proved from the following fact.

$$\begin{aligned} x \in \mathrm{co}(A)_{[b]} & \Leftrightarrow \mathrm{co}(A)(x) \geq b \\ & \Leftrightarrow R^{(a')}(A) = R^{(a')}(A \cup \{x\}), \forall a \not\leq b' \\ & \Leftrightarrow x \in \mathrm{cl}(A, a), \forall a \not\leq b' \\ & \Leftrightarrow x \in \bigcap_{a \not\leq b'} \mathrm{cl}(A, a). \end{aligned}$$

□

Conclusions

In this paper, characterizations of an M-fuzzifying convex matroid are presented by means of M-fuzzy family of dependent sets, M-fuzzifying circuit map and M-fuzzifying rank function. These results show the reasonability of the definition of M-fuzzifying convex matroids. Meanwhile they also establish a theoretical foundation for subsequent research.

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