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Original paper

On existence and stability results to fuzzy Caputo fractional differential inclusions driven by fuzzy mixed quasivariational inequalities

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Abstract

In this paper, we consider a generalized fuzzy differential system (GFDS) consisting of a fuzzy Caputo fractional differential inclusion combined with a fuzzy mixed quasivariational inequality. The GFDS has been known as a framework of fuzzy fractional differential quasivariational inequalities involving Caputo fractional derivatives. First, we verify the existence of solutions for the fuzzy mixed quasivariational inequality by using the Kakutani-Fan-Glicksberg fixed point theorem. Then, the existence of mild solutions for the GFDS is also obtained under some mild conditions. Finally, the upper semicontinuity of the solution mapping to the GFDS provided in the case of the perturbed parameters is discussed.

Keywords: Fuzzy fractional differential inclusion, fuzzy quasivariational inequality, stability, upper semicontinuity.

1 Introduction

In the fields of economics and finance, engineering, physics and mechanics, chemistry, control science, and microelectronics, differential inclusion (equation) theory has been widely applied (see, e.g. [19] and the references therein). In 1997, Hüllermeier [28] investigated the following differential inclusion in fuzzy environments:

$$\begin{cases} \dot{z}(t) \in \left[H_{(t,z(t))}\right]_a, \quad a \in [0,1], \\ z(0) = [z_0]_a \end{cases} \tag{1}$$

where $\dot{z}(t)$ denotes the time derivative of $z(t)$, $H\colon [0,T] \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n) := \{\mu : \mu\colon \mathbb{R}^n \to [0,1]\}$ is a fuzzy mapping and for a fuzzy set $\mathfrak{B} \in \mathcal{F}(\mathbb{R}^n)$, the set $[\mathfrak{B}]_a = \{u \in \mathbb{R}^n : \mathfrak{B}(u) \geq a\}$ is called the a-level set of $\mathfrak{B}$. The differential inclusion (1) was developed from a family of differential inclusions introduced by Aubin and Cellina [5] and the fuzzy set theory studied by Zadeh [51]. This fuzzy set theory is a powerful tool for simulating uncertain mathematical models that can be utilized to maintain information objectivity. Furthermore, Agarwal et al. [1] proposed the concept of fuzzy differential equation employing a fractional derivative. A large number of works have presented results on existence, numerical algorithms, and applications for various fuzzy fractional differential equations; see e.g., [2, 3, 4, 29] and the references therein.

On the other hand, there have been various problems that can be formulated by variational inequalities in fuzzy environments, such as dynamic traffic networks, image processing and contact mechanics problems. In 1972, Chang and Zadeh [13] introduced a simple fuzzy variational inequality based on the fuzzy mapping notion. Chang et al. [12] introduced and studied a vector quasivariational inequality with fuzzy mappings. Many papers derived results on existence, stability and applications for various fuzzy variational inequalities; see e.g. [27, 30, 31, 33, 45] and the references therein.

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Differential variational inequalities are a type of dynamic systems established by a combination of ordinary differential equations and variational inequalities. It plays a vital role in studying the fields of dynamic Nash equilibrium problem, spatial price equilibrium control problem, dynamic oligopolistic network competition, frictional contact problems, dynamic traffic networks, see e.g., [15, 21, 36, 43]. In 2008, Pang and Stewart [41] firstly introduced and systematically investigated a class of differential variational inequalities in finite dimensional Euclidean spaces. Some sharpest known results about various kinds of differential variational inequalities have been developed recently, see e.g., [7, 10, 38, 44, 52, 53, 54]. Fractional calculus approaches to differential variational inequalities have also been introduced and studied via various fractional differential variational inequalities, see e.g., [32, 39, 40]. Besides, Wang et al. [47] introduced a class of differential fuzzy variational inequalities, which is modeled by a differential equation coupled with a fuzzy variational inequality. Very recently, Wu et al. [49] investigated a class of fuzzy fractional differential inclusions driven by variational inequalities. Wu et al. [50] also studied a new class of fractional differential fuzzy variational inequalities with delay, which consist of a fractional delay differential equation and a fuzzy variational inequality. In their study, they established the existence of Carathéodory weak solution to differential variational inequalities. To the best of our knowledge, until now, there are very few works on the stability analysis for differential variational inequalities. In [24], Gwinner investigated the stability of the solution set to a class of linear differential variational inequalities in the sense of the upper convergence with respect to perturbations in the constraint set and the associated linear mapping. Wang et al. [46] established the upper semicontinuity and continuity results of the solution mapping to a differential mixed variational inequality perturbed by different parameters in the mapping and the constraint set. Recently, [23] studied the stability of the solution mapping for a class of partial differential variational inequalities in infinite dimensional spaces based on ideas introduced in [46].

Motivated by the aforementioned works, in this paper, we consider the following generalized fuzzy differential system (GFDS) described by fuzzy Caputo fractional differential inclusions combined with fuzzy mixed quasivariational inequalities when mappings and constraint sets are perturbed by different parameters:

Problem 1.1. *Find* $x\colon [0,T]\to\mathbb{R}^n$ *and* $u\colon[0,T]\to K$ *such that*

$$\begin{cases} {}_0^C\mathbf{D}_t^q x(t)\in \left[H_{(t,x(t))}\right]_\alpha + B(t,x(t))u(t), & \text{for a.e. } t\in[0,T],\\ u(t)\in \mathbf{SOL}\left(P_\lambda(\cdot),G_{(x(t),\cdot)},\beta,f,A_\gamma+M_\mu(\cdot)\right), & \text{for a.e. } t\in[0,T],\\ x(0)=x_0, \end{cases}$$

where $0<q<1$, $0\le\alpha,\beta\le1$, $x(t)\in\mathbb{R}^n$, $u(t)\in K$, $K\subset\mathbb{R}^m$ *and* $E\subset\mathbb{R}^m$ *are nonempty compact convex sets,* $H\colon[0,T]\times\mathbb{R}^n\to\mathcal{E}^n$ *and* $G\colon\mathbb{R}^n\times\mathbb{R}^m\to\mathcal{E}^m(E)$ *are fuzzy mappings,* $B\colon[0,T]\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$, $P\colon K\times\Lambda\rightrightarrows K$ *is a set-valued mapping,* $f\colon K\times K\to\mathbb{R}$, $A\colon E\times\Sigma\to\mathbb{R}^m$, $M\colon\mathbb{R}^m\times\Xi\to\mathbb{R}^m$ *are given mappings,* Λ,Σ *and* Ξ *are metric spaces.*

Moreover, $\mathbf{SOL}\left(P_\lambda(\cdot),G_{(x(t),\cdot)},\beta,f,A_\gamma+M_\mu(\cdot)\right)$ *stands for the solution set of the following fuzzy mixed quasivariational inequality perturbed by parameters* $\lambda\in\Lambda$, $\gamma\in\Sigma$ *and* $\mu\in\Xi$ *(for short,* $(\mathrm{FMQVI})_x^{\lambda\gamma\mu}$*): for given* $x(t)\in\mathbb{R}^n$*, find* $u\colon[0,T]\to K$ *such that* $u(t)\in P(u(t),\lambda)$ *and there exists* $w\colon[0,T]\to\mathbb{R}^m$ *satisfying* $w(t)\in\left[G_{(x(t),u(t))}\right]_\beta$ *and*

$$\langle A(w(t),\gamma)+M(u(t),\mu),v-u(t)\rangle+f(u(t),v)\ge0,\ \forall v\in P(u(t),\lambda), \tag{2}$$

for a.e. $t\in[0,T]$.

If $P(u,\lambda)=P(u)$, $A(u,\gamma)=A(u)$, $M(u,\mu)=M(u)$ for all $u\in\mathbb{R}^m$, $\lambda\in\Lambda$, $\gamma\in\Sigma$ and $\mu\in\Xi$, then problem $(\mathrm{FMQVI})_x^{\lambda\gamma\mu}$ reduces to the following fuzzy mixed quasivariational inequality (for short, $(\mathrm{FMQVI})_x$): for given $x(t)\in\mathbb{R}^n$, find $u\colon[0,T]\to K$ such that $u(t)\in P(u(t))$ and there exists $w\colon[0,T]\to\mathbb{R}^m$ satisfying $w(t)\in\left[G_{(x(t),u(t))}\right]_\beta$ and

$$\langle A(w(t))+M(u(t)),v-u(t)\rangle+f(u(t),v)\ge0,\ \forall v\in P(u(t)), \tag{3}$$

for a.e. $t\in[0,T]$. Hence, Problem 1.1 perturbed by different parameters reduces to the following fuzzy differential system:

Problem 1.2. *Find* $x\colon [0,T]\to\mathbb{R}^n$ *and* $u\colon[0,T]\to K$ *such that*

$$\begin{cases} {}_0^C\mathbf{D}_t^q x(t)\in \left[H_{(t,x(t))}\right]_\alpha + B(t,x(t))u(t), & \text{for a.e. } t\in[0,T],\\ u(t)\in \mathbf{SOL}\left(P(\cdot),G_{(x(t),\cdot)},\beta,f,A+M(\cdot)\right), & \text{for a.e. } t\in[0,T],\\ x(0)=x_0, \end{cases}$$

where $\mathbf{SOL}\left(P(\cdot),G_{(x(t),\cdot)},\beta,f,A+M(\cdot)\right)$ *stands for the solution set of the problem* $(\mathrm{FMQVI})_x$.

Some special cases of Problem 1.2 are as follows:

(i) If $P(u(t)) = K$, $A(w(t)) = w(t)$, $f = 0$ and $G_{(x(t),u(t))} = \chi_{\{F(t,x(t))\}}$, where $F\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^m$ is a single-valued mapping, and χ_N is the characteristic function of a set N, then Problem 1.2 reduces to a class of fuzzy fractional differential variational inequalities investigated by Wu et al. [49].

(ii) If $P(u(t)) = K$, $A(w(t)) = w(t)$, $f = 0$, $H_{(t,x(t))} = \chi_{Q(t,x(t))}$ and $G_{(x(t),u(t))} = \chi_{\{F(t,x(t))\}}$, where $Q\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ is a set-valued mapping with nonempty convex compact values and $F\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^m$ is a single-valued mapping, then Problem 1.2 reduces to a class of fractional set-valued differential variational inequalities studied by Loi et al. [39].

(iii) If $q = 1$, $P(u(t)) = K$, $A(w(t)) = w(t)$, $f = 0$, $H_{(t,x(t))} = \chi_{\{Q(t,x(t))\}}$ and $G_{(x(t),u(t))} = \chi_{\{F(t,x(t))\}}$, where $Q\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ is a single-valued mapping and $F\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^m$ is a single-valued mapping, then Problem 1.2 reduces to a class of differential variational inequalities introduced by Pang and Stewart [41].

This work has the following specific contributions:

(1) A generalized fuzzy differential system is proposed by using the fuzzy fractional differential inclusion combined with the fuzzy mixed quasivariational inequality involving Caputo fractional derivatives.

(2) By using the Kakutani-Fan-Glicksberg fixed point theorem, the existence of solutions for the fuzzy mixed quasivariational inequality is established and the existence of mild solutions for the GFDS is also obtained under suitable conditions (Section 3).

(3) The upper semicontinuity of the solution mapping to the fuzzy differential system provided in terms of the perturbed parameters is proposed (Section 4).

2 Preliminaries

Throughout the rest of the paper, unless otherwise stated, let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^n$. Let $T > 0$ and $C([0,T];\mathbb{R}^n)$ be the Banach space of all continuous functions $z\colon [0,T]\to\mathbb{R}^n$ with norm $\|z\|_C = \sup_{t\in[0,T]}\|z(t)\|$, and $\mathcal{L}^1([0,T];\mathbb{R}^n)$ be the Banach space of all Lebesgue integrable functions from $[0,T]$ to $\mathbb{R}^n$, i.e.,

$$\mathcal{L}^1([0,T];\mathbb{R}^n) = \left\{ z : \|z\|_{\mathcal{L}^1} = \int_0^T \|z(t)\|dt < \infty \right\}.$$

For our results, we need the following concepts, definitions and lemmas. We first recall the definitions about fractional operators (see [34, 42]).

Definition 2.1. *The Riemann-Liouville fractional integral of order $q > 0$ for a suitable function z, for example $z \in \mathcal{L}^1([t_0,t_1],\mathbb{R}^m)$ is defined by*

$$\mathbf{I}^q_{t_0} z(t) = \frac{1}{\Gamma(q)}\int_{t_0}^t (t-\tau)^{q-1} z(\tau)d\tau, \quad (t > t_0)$$

where $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(q) = \int_0^{+\infty} t^{q-1}e^{-t}dt.$$

It is easy to see that $\Gamma(1) = 1$ and $\Gamma(q+1) = q\Gamma(q)$ for all $q > 0$.

Definition 2.2. *For a suitable function z given on the interval $[t_0,t_1]$, the Riemann-Liouville fractional order derivative of z of order q, is defined as follows:*

$${}_{t_0}\mathbf{D}^q_t z(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_{t_0}^t (t-\tau)^{n-q-1} z(\tau)d\tau,$$

where $n = [q] + 1$ and $[q]$ denotes the integer part of q.

Definition 2.3. *For a suitable function z given on the interval $[t_0, t_1]$, the Caputo fractional order derivative of z of order q, is defined as follows:*

$$ {}_{t_0}^{C}\mathbf{D}_t^q z(t) = {}_{t_0}\mathbf{D}_t^q \left(z(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^k}{k!} z^{(k)}(t_0) \right), $$

which is also expressed by

$$ {}_{t_0}^{C}\mathbf{D}_t^q z(t) = \mathbf{I}_{t_0}^{n-q} z^{(n)}(t) = \begin{cases} z^{(n)}(t) & \text{if } q = n; \\ \dfrac{1}{\Gamma(n-q)} \left(\dfrac{d}{dt}\right)^n \displaystyle\int_{t_0}^{t} (t-\tau)^{n-q-1} z^{(n)}(\tau) d\tau & \text{if } n-1 < q < n. \end{cases} $$

Let the symbol "$\rightharpoonup$" be used to denote the weak convergence. The collection of all nonempty subsets (resp., nonempty compact) of a topological space W is denoted by $\mathbf{Ne}(W)$ (resp., $\mathbf{Cp}(W)$). We now recall some properties of set-valued mappings.

Definition 2.4. [6] *Let X and W be two topological spaces and $G\colon X \to \mathbf{Ne}(W)$ be a set-valued mapping. Then G is said to be*

(a) *convex (resp., closed, bounded, compact, weak compact) valued, if $G(u)$ is convex (resp., closed, bounded, compact, weak compact) for each $u \in X$;*

(b) *lower semicontinuous at $u_0 \in X$ if, $G(u_0) \cap U \neq \emptyset$ for some open set $U \subset W$ implies the existence of a neighborhood V of u_0 such that $G(u) \cap U \neq \emptyset, \forall u \in V$;*

(c) *upper semicontinuous at $u_0 \in X$ if, for each open set U of $G(u_0)$, there is a neighborhood V of u_0 such that $U \supset G(u), \forall u \in V$;*

(d) *continuous at $u_0 \in X$ if, it is both lower and upper semicontinuous at u_0;*

(e) *closed at $u_0 \in X$, if for each $\{u_k\} \subset X$, $u_k \to u_0$ and $\{w_k\} \subset W$, $w_k \to w_0$ such that $w_k \in G(u_k)$, we have $w_0 \in G(u_0)$.*

If G satisfies a certain property at every point of a set $K \subset X$, then G is said to satisfy it on K. If $K \equiv X$, then we omit "on X" in the statement.

Lemma 2.5. [8] *Assume that X and W are two locally convex Hausdorff topological spaces and X is also compact. The set-valued mapping $G\colon X \to \mathbf{Ne}(W)$ is upper semicontinuous with compact values if and only if it is a closed mapping.*

Lemma 2.6. [6] *Let X and W be two Hausdorff topological spaces and $G\colon X \to \mathbf{Ne}(W)$ be a set-valued mapping. Then G is lower semicontinuous if and only if, for any sequence $\{u_k\} \subset X$ with $u_k \to u_0 \in X$ and for any $w_0 \in G(u_0)$, there exists $w_n \in G(u_n)$ such that $w_n \to w_0$.*

Lemma 2.7. [9] *Let W be a Banach space and $\mathcal{K}$ be a nonempty subset of another Banach space. Assume that $G\colon \mathcal{K} \to \mathbf{Ne}(W)$ is a set-valued mapping with weakly compact, convex values. Then G is strongly-weakly upper semicontinuous if and only if, for each sequence $\{u_k\} \subset \mathcal{K}$ which converges to $u_0 \in \mathcal{K}$ and for each sequence $\{w_k\} \subset G(u_k)$, there exists $w_0 \in G(u_0)$ such that $w_k \rightharpoonup w_0$ up to a subsequence.*

We now recall some concepts, definitions and properties of the fuzzy set theory which are needed in the sequel (see e.g., [18, 35]).

Let $\mathcal{Z}$ be a nonempty subset in a base space $\mathcal{X}$. We recall that a function $\theta\colon \mathcal{X} \to [0,1]$ is a fuzzy set of $\mathcal{X}$, and a mapping $H\colon \mathcal{Z} \to \mathcal{F}(\mathcal{X})$ is a fuzzy mapping, where $\mathcal{F}(\mathcal{X})$ is the set of all fuzzy sets. Assume that $H\colon \mathcal{Z} \to \mathcal{F}(\mathcal{X})$ is a fuzzy mapping. Then $H(z)$ (denoted by H_z in the sequel) is a fuzzy set on $\mathcal{Z}$ for each $z \in \mathcal{Z}$ and $H_z(x)$ is the membership grade of the element x in the fuzzy set H_z. Let us denote by $[\theta]_\alpha := \{x \in \mathcal{X} : \theta(x) \geq \alpha\}$, for each $\alpha \in (0,1]$, the α-level set of a fuzzy set θ on $\mathcal{X}$, and the support

$$ [\theta]_0 = \overline{\cup_{\alpha \in (0,1]} [\theta]_\alpha} = \overline{\{x \in \mathcal{X} : \theta(x) > 0\}}, $$

where $\overline{K}$ is the closure of the subset $K \subset \mathcal{X}$. Let $\mathcal{E}^n$ denote the space of all fuzzy subsets θ of $\mathbb{R}^n$ satisfying the following properties:

(i) the set $[\theta]_\alpha = \{x \in \mathbb{R}^n : \theta(x) \geq \alpha\}(\alpha \in (0,1])$ is closed;

(ii) θ is fuzzy convex, i.e., $\theta(\lambda x + (1-\lambda)y) \geq \min\{\theta(x), \theta(y)\}, \quad \forall x, y \in \mathbb{R}^n, \lambda \in [0,1]$;

(iii) θ is normal, i.e., there exists $x_0 \in \mathbb{R}^n$ for which $\theta(x_0) = 1$;

(iv) $[\theta]_0 = \overline{\{x \in \mathcal{X} : \theta(x) > 0\}}$ is compact.

We shall use the symbol $\mathcal{E}^n(K)$ to denote the set $\{\theta \in \mathcal{E}^n : [\theta]_0 \subset K\}$, where K is a nonempty subset of $\mathbb{R}^n$.

Lemma 2.8. [35, Theorem 1.5.1] *Assume that $\theta \in \mathcal{E}^n$. Then*

(i) $\emptyset \neq [\theta]_\alpha \subset \mathbb{R}^n$ *is a convex compact set, for all* $\alpha \in [0,1]$;

(ii) $[\theta]_{\alpha_1} \subset [\theta]_{\alpha_2}$ *for* $0 \leq \alpha_1 \leq \alpha_2 \leq 1$;

(iii) *If a nondecreasing sequence* $\{\alpha_k\}$ *approaches* $\alpha > 0$*, then* $[\theta]_\alpha = \cap_{k\geq 1}[\theta]_{\alpha_k}$.

Conversely, if a family of sets $\{\mathcal{V}_\alpha : \alpha \in [0,1]\} \subset \mathbb{R}^n$ *satisfy (i) − (iii) mentioned above, then there exists a* $\theta \in \mathcal{E}^n$ *such that* $[\theta]_0 = \overline{\cup_{\alpha\in(0,1]}\mathcal{V}_\alpha} \subset \mathcal{V}_0$ *and* $[\theta]_\alpha = \mathcal{V}_\alpha$ $(\alpha \in (0,1])$.

Let $(\mathcal{Y}, d)$ be a metric space. For any $u \in \mathcal{Y}, A_1, A_2 \in \mathbf{Cp}(\mathcal{Y})$, the distance of u from A_1, is defined by $d(u, A_1) = \inf_{a_1\in A_1} d(u, a_1)$, and $d(u, A_1) = +\infty$, if $A_1 = \emptyset$. The Hausdorff distance between A_1 and A_2 is defined by

$$\mathfrak{H}(A_1, A_2) = \max\{\sup_{a_1\in A_1} d(a_1, A_2), \sup_{a_2\in A_2} d(A_1, a_2)\}, \forall A_1, A_2 \in \mathbf{Cp}(\mathcal{Y}).$$

If $(\mathcal{Y}, d)$ is a complete metric space, then $(\mathbf{Cp}(\mathcal{Y}), \mathfrak{H})$ is also a complete metric space (see [26]). The metric on $\mathcal{F}(\mathcal{Y})$ is defined by

$$\mathfrak{H}(u, v) = \sup_{\alpha\in[0,1]} \{\mathfrak{H}([u]_\alpha, [v]_\alpha)\}, \quad \forall u, v \in \mathcal{F}(\mathcal{Y}).$$

Then $(\mathcal{F}(\mathcal{Y}), \mathfrak{H})$ is a complete metric space (see [35]).

Definition 2.9. [11] *Given two topological spaces X, Y, $\alpha \in [0,1]$ and a fuzzy mapping $H : X \to \mathcal{F}(Y)$. H is said to be topologically closed if, for each $u_0 \in X$ and each open set U satisfying $[H_{u_0}(y) \geq \alpha \Longrightarrow v \in U]$ there exists a neighborhood V of u_0 such that $[H_u(v) \geq \alpha \Longrightarrow v \in U, \forall u \in V]$.*

Remark 2.10. *From Definition 2.4(c) and Definition 2.9, it is easily seen that if a fuzzy mapping $H : X \to \mathcal{F}(Y)$ is topologically closed, then the set-valued mapping $\widetilde{H} : X \rightrightarrows Y$ defined by $\widetilde{H}(u) := [H_u]_\alpha$ is upper semicontinuous.*

Let $(\Omega, \mathcal{P})$ be a measurable space and Y be a separable metric space. We know that a set-valued mapping $Q \colon \Omega \rightrightarrows Y$ with nonempty closed values is said to be measurable if $Q^{-1}(V) \in \mathcal{P}$ for every open $V \subset Y$, where $Q^{-1}(V) = \{u \in Y : Q(u) \subset V\}$.

Definition 2.11. [35] *A fuzzy mapping $H \colon [0,T] \to \mathcal{F}(\mathbb{R}^n)$ is called strongly measurable if, for any $\alpha \in [0,1]$, the set-valued mapping $[H_t]_\alpha : [0,T] \rightrightarrows \mathbb{R}^n$ with nonempty compact convex values is (Lebesgue) measurable.*

Lemma 2.12. [16, Proposition 3.4(b)] *Let Y be a Banach space. Then a nonempty strongly measurable map $H \colon [0,T] \rightrightarrows Y$ has a strongly measurable selection.*

Lemma 2.13. [16, Proposition 3.5(a)] *Let Y be a Banach space and $H \colon [0,T] \times Y \rightrightarrows Y$ has nonempty compact values. Assume that $H(t,\cdot)$ is upper semicontinuous and $H(\cdot, x)$ has a strongly measurable selection for all $(t, x) \in [0,T] \times Y$. For $x \in C([0,T]; Y)$, then there exists a strongly measurable selection $h(\cdot) \in H(\cdot, x(\cdot))$.*

Lemma 2.14. [20] *Let P be a nonempty convex compact subset of Hausdorff topological vector space X and $\mathcal{D}$ be a subset of $P \times P$ such that*

(i) *for each at $x \in P$, $(x, x) \notin \mathcal{D}$;*

(ii) *for each at $y \in P$, the set $\{x \in P : (x, y) \in \mathcal{D}\}$ is open on P;*

(iii) *for each at $x \in P$, the set $\{y \in P : (x, y) \in \mathcal{D}\}$ is convex (or empty).*

Then, there exists $x^* \in P$ such that $(x^*, y) \notin \mathcal{D}$ for all $y \in P$.

Lemma 2.15. [25] *Let P be a nonempty convex compact subset of a locally Hausdorff topological vector space X. If the set-valued mapping $F : X \rightrightarrows Y$ is upper semicontinuous and for any $x \in P$, $F(x)$ is nonempty, convex and closed, then there exists an $x^* \in P$ such that $x^* \in F(x^*)$.*

3 Existence results

To establish the existence of solutions for Problem 1.1, we impose the following assumptions on the data of this problem:

($\mathbf{A}_1$): for any $t \in [0,T]$, $x_1, x_2 \in \mathbb{R}^n$, $\mathfrak{H}(H_{(t,x_1)}, H_{(t,x_2)}) \leq \phi(t, \|x_2 - x_1\|)$, where $\phi\colon [0,T] \times [0,+\infty) \to [0,+\infty)$ is single valued function, $\phi(\cdot, s)$ is measurable for each $s \geq 0$, $\phi(t,\cdot)$ is right continuous nondecreasing for each $t \in [0,T]$, and $\phi(t,s) < s$, $\forall s > 0$, $t \in [0,T]$;

($\mathbf{A}_2$): $H_{(\cdot,x)}$ is strongly measurable for every $x \in \mathbb{R}^n$;

($\mathbf{A}_3$): for each $r > 0$, there exists $\eta_r \in \mathcal{L}^1([0,T];\mathbb{R}_+)$ such that $\|H_{(t,x)}\| \leq \eta_r(t)$ for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}^n$ with $\|x\| \leq r$, where $\|H\| = \mathfrak{H}(H, \widetilde{0})$, $\widetilde{0}$ denotes the fuzzy set defined by $\widetilde{0}(y) = 0$ if $y \neq 0$ and $\widetilde{0}(y) = 1$ if $y = 0$;

($\mathbf{A}_4$): the function $B\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is continuous such that there exists $b_B > 0$,

$$\|B(t,x)\| \leq b_B,\ \forall t \in [0,T], x \in \mathbb{R}^n.$$

($\mathbf{A}_5$): for each $\lambda \in \Lambda$, the mapping $P(\cdot,\lambda)\colon K \rightrightarrows K$ is such that

($\mathbf{A}_{5a}$): $P(\cdot,\lambda)$ is continuous with nonempty compact values,

($\mathbf{A}_{5b}$): for all $u_1, u_2 \in K$ and $\kappa \in [0,1]$, we have

$$(1-\kappa)P(u_1,\lambda) + \kappa P(u_2,\lambda) \subset P((1-\kappa)u_1 + \kappa u_2, \lambda); \tag{4}$$

($\mathbf{A}_6$): for each $x \in \mathbb{R}^n$, the fuzzy mapping $G_{(x,\cdot)}\colon \mathbb{R}^m \to \mathcal{E}^m(E)$ is such that

($\mathbf{A}_{6a}$): $G_{(x,\cdot)}$ is topologically closed,

($\mathbf{A}_{6b}$): the mapping $\widetilde{G}(x,\cdot) := [G_{(x,\cdot)}]_\beta$ satisfies

$$(1-\kappa)\widetilde{G}(x,u_1) + \kappa\widetilde{G}(x,u_2) \subset \widetilde{G}(x,(1-\kappa)u_1 + \kappa u_2), \forall u_1,u_2 \in K, \forall \kappa \in [0,1];$$

($\mathbf{A}_7$): for each $\gamma \in \Sigma$, $\mu \in \Xi$, the functions $A(\cdot,\gamma)\colon E \to \mathbb{R}^m$, $M(\cdot,\mu)\colon \mathbb{R}^m \to \mathbb{R}^m$ and $f\colon K \times K \to \mathbb{R}$ are continuos and $f(u,v) + f(v,u) = 0$, for all $u, v \in K$;

($\mathbf{A}_8$): For each $(\lambda,\gamma,\mu) \in \Lambda \times \Sigma \times \Xi$, the set

$$\{v \in P(u,\lambda) : \langle A(w,\gamma) + M(u,\mu), v - u\rangle + f(u,v) < 0,\ \forall (u,w) \in K \times E\},$$

is convex;

($\mathbf{A}_9$): for each $\gamma \in \Sigma$, $\mu \in \Xi$ and $v \in K$, the functions $(u,w) \mapsto \langle A(w,\gamma), v-u\rangle$, $u \mapsto \langle M(u,\mu), v-u\rangle$ and $u \mapsto f(u,v)$ are concave.

Remark 3.1. *(i) We note that assumptions* ($\mathbf{A}_1$)–($\mathbf{A}_4$) *were used by Wu et al.* [48, 50].

(ii) Taking $u_1 = u_2 = u$ *in* (4)*, we have*

$$(1-\kappa)P(u,\lambda) + \kappa P(u,\lambda) \subset P(u,\lambda), \forall u \in K, \forall \kappa \in [0,1],$$

which implies that $P(u,\lambda)$ *is a convex set for all* $(u,\lambda) \in K \times \Lambda$*.*

Using the assumption ($\mathbf{A}_1$) and Lemma 2.8, Wu et al. [49] provided the following result:

Lemma 3.2. [49, Lemma 3.1] *Let* $H\colon [0,T]\times\mathbb{R}^n \to \mathcal{E}^n$ *be a fuzzy mapping and the set-valued mapping* $\widetilde{H}\colon [0,T]\times\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *be defined by*

$$\widetilde{H}(t,x) = \left[H_{(t,x)}\right]_\alpha = \left\{y \in \mathbb{R}^n : H_{(t,x)}(y) \geq \alpha\right\},$$

has nonempty compact convex values. If the hypothesis ($\mathbf{A}_1$) *holds then* $\widetilde{H}(t,\cdot)$ *is upper semicontinuous for all* $t \in [0,T]$*.*

Remark 3.3. *By the hypothesis* ($\mathbf{A}_{6a}$)*, we obtain that* $G_{(x,\cdot)}\colon \mathbb{R}^m \to \mathcal{E}^m(E)$ *is topologically closed. Then it follows from Remark 2.10 and Lemma 3.2 that the set-valued mapping* $\widetilde{G}(x,\cdot) := [G_{(x,\cdot)}]_\beta$ *is upper semicontinuous with nonempty compact convex values for all* $x \in \mathbb{R}^n$*.*

3.1 Results on fuzzy mixed quasivariational inequalities

We now consider the following fuzzy mixed quasivariational inequality perturbed by parameters λ, γ and μ (for short, FMQVI): for given $x \in \mathbb{R}^n$, find $u \in P(u,\lambda)$ such that there exists $w \in \left[G_{(x,u)}\right]_\beta$,

$$\langle A(w,\gamma) + M(u,\mu), v-u\rangle + f(u,v) \geq 0, \ \forall v \in P(u,\lambda). \tag{5}$$

The solution set of FMQVI (5) is denoted by $\mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$.

Theorem 3.4. *Suppose that the conditions* $(\mathbf{A}_5)$*–*$(\mathbf{A}_9)$ *are satisfied. Then* $\mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$ *is nonempty, convex and compact in* $\mathbb{R}^m$.

Proof. (✠) We first verify that $\mathbf{SOL}\,(P_\lambda(\cdot), G_x, \beta, f, A_\gamma + M_\mu)$ is nonempty.

For each $(x,\lambda,\gamma,\mu) \in \mathbb{R}^n \times \Lambda \times \Sigma \times \Xi$ fixed and for all $(u,w) \in K \times E$, we define mapping $\Pi_x^{\lambda\gamma\mu} : K \times E \rightrightarrows K$ by

$$\Pi_x^{\lambda\gamma\mu}(u,w) = \{\hat{u} \in P(u,\lambda) : \langle A(w,\gamma) + M(\hat{u},\mu), v-\hat{u}\rangle + f(\hat{u},v) \geq 0, \ \forall v \in P(u,\lambda)\}.$$

Claim 1: $\Pi_x^{\lambda\gamma\mu}(u,w)$ is nonempty.

For each $(u,w) \in K \times E$, setting

$$\mathcal{D} = \{(\hat{u},v) \in P(u,\lambda) \times P(u,\lambda) : \langle A(w,\gamma) + M(\hat{u},\mu), v-\hat{u}\rangle + f(\hat{u},v) < 0\}.$$

Since $f(\hat{u},v) + f(v,\hat{u}) = 0$, we have $f(\hat{u},\hat{u}) = 0$. This implies that for any $\hat{u} \in P(u,\lambda)$, $(\hat{u},\hat{u}) \notin \mathcal{D}$. It follows from the condition $(\mathbf{A}_8)$ that for any $\hat{u} \in P(u,\lambda)$, the set $\{v \in P(u,\lambda) : (\hat{u},v) \in \mathcal{D}\}$ is convex on K. Moreover, by the condition $(\mathbf{A}_7)$, we obtain that the set $\{(u,w,v) \in K \times E \times K : \langle A(w,\gamma) + M(u,\mu), v-u\rangle + f(u,v) \geq 0\}$ is closed in $K \times E$. Hence, for any $\hat{u} \in P(u,\lambda)$, the set $\{v \in P(u,\lambda) : (\hat{u},v) \in \mathcal{D}\}$ is open in K. Hence, by Lemma 2.14 there exists $\hat{u} \in P(u,\lambda)$ such that $(\hat{u},v) \notin \mathcal{D}$, for all $v \in P(u,\lambda)$, i.e.,

$$\langle A(w,\gamma) + M(\hat{u},\mu), v-\hat{u}\rangle + f(\hat{u},v) \geq 0, \ \forall v \in P(u,\lambda).$$

Thus, $\Pi_x^{\lambda\gamma\mu}(u,v)$ is nonempty.

Claim 2: $\Pi_x^{\lambda\gamma\mu}(u,w)$ is a convex set.

Let $\hat{u}_1, \hat{u}_2 \in \Pi_x^{\lambda\gamma\mu}(u,w)$, $\kappa \in [0,1]$ and put $\hat{u} = (1-\kappa)\hat{u}_1 + \kappa\hat{u}_2$. Since $\hat{u}_1, \hat{u}_2 \in P(u,\lambda)$ and $P(u,\lambda)$ is a convex set (by Remark 3.1(ii)), we have $\hat{u} \in P(u,\lambda)$. It follows from $\hat{u}_1, \hat{u}_2 \in \Pi_x^{\lambda\gamma\mu}(u,w)$ that for any $v \in P(u,\lambda)$, one has

$$\begin{aligned}
&\langle A(w,\gamma) + M(\hat{u}_1,\mu), v-\hat{u}_1\rangle + f(\hat{u}_1,v) \geq 0,\\
&\langle A(w,\gamma) + M(\hat{u}_2,\mu), v-\hat{u}_2\rangle + f(\hat{u}_2,v) \geq 0.
\end{aligned}$$

By the assumption $(\mathbf{A}_9)$, we have

$$\begin{aligned}
0 &\leq (1-\kappa)\left[\langle A(w,\gamma) + M(\hat{u}_1,\mu), v-\hat{u}_1\rangle + f(\hat{u}_1,v)\right] + \kappa\left[\langle A(w,\gamma) + M(\hat{u}_2,\mu), v-\hat{u}_2\rangle + f(\hat{u}_2,v)\right]\\
&= \langle A(w,\gamma), v-\hat{u}\rangle + (1-\kappa)\langle M(\hat{u}_1,\mu), v-\hat{u}_1\rangle + \kappa\langle M(\hat{u}_2,\mu), v-\hat{u}_2\rangle + (1-\kappa)f(\hat{u}_1,v) + \kappa f(\hat{u}_2,v)\\
&\leq \langle A(w,\gamma) + M(\hat{u},\mu), v-\hat{u}\rangle + f(\hat{u},v).
\end{aligned}$$

i.e., $\hat{u} \in \Pi_x^{\lambda\gamma\mu}(u,w)$. Therefore, $\Pi_x^{\lambda\gamma\mu}(u,w)$ is a convex set.

Claim 3: $\Pi_x^{\lambda\gamma\mu}$ is upper semicontinuous on $K \times E$ with closed values.

Since K and E are compact sets, by Lemma 2.5, we need only prove that $\Pi_x^{\lambda\gamma\mu}$ is a closed mapping. Let $\{(u_k,w_k)\} \subset K \times E$ such that $(u_k,w_k) \to (u_0,w_0)$ as $k \to \infty$ and let $\hat{u}_k \in \Pi_x^{\lambda\gamma\mu}(u_k,w_k)$ such that $\hat{u}_k \to \hat{u}_0$ as $k \to \infty$. We now show that $\hat{u}_0 \in \Pi_x^{\lambda\gamma\mu}(u_0,w_0)$. Indeed, since $\hat{u}_k \in P(u_k,\lambda)$ and P is closed on $K \times \Lambda$, we have $\hat{u}_0 \in P(u_0,\lambda)$. On the contrary, we suppose that $\hat{u}_0 \notin \Pi_x^{\lambda\gamma\mu}(u_0,w_0)$. Then, there exists $v_0 \in P(u_0,\lambda)$ such that

$$\langle A(w_0,\gamma) + M(\hat{u}_0,\mu), v_0-\hat{u}_0\rangle + f(\hat{u}_0,v_0) < 0. \tag{6}$$

By the lower semicontinuity of P, there exists a sequence $\{v_k\} \subset \mathbb{R}^m$ such that $v_k \in P(u_k,\lambda)$, $v_k \to v_0$ as $k \to \infty$. Since $\hat{u}_k \in \Pi_x^{\lambda\gamma\mu}(u_k,w_k)$, we have

$$\langle A(w_k,\gamma) + M(\hat{u}_k,\mu), v_k-\hat{u}_k\rangle + f(\hat{u}_k,v_k) \geq 0. \tag{7}$$

By the continuity of A, M and f, it follows from (7) that

$$\langle A(w_0,\gamma) + M(\hat{u}_0,\mu), v_0-\hat{u}_0\rangle + f(\hat{u}_0,v_0) \geq 0. \tag{8}$$

This is a contradiction between (6) and (8). Hence, $\hat{u}_0 \in \Pi_x^{\lambda\gamma\mu}(u_0, w_0)$. Thus, $\Pi_x^{\lambda\gamma\mu}$ is upper semicontinuous on $K \times E$ with closed values.

Claim 4: $\mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$ is nonempty.

Define the set-valued mapping $\Omega : K \times E$ by

$$\Omega(u, w) = (\Pi_x^{\lambda\gamma\mu}(u, w), [G_{(x,u)}]_\beta), \forall (u, w) \in K \times E.$$

Thanks to Remark 3.3, we get that $\widetilde{G}(x, \cdot) = [G_{(x,\cdot)}]_\beta$ is upper semicontinuous on K with nonempty closed convex values. Hence, Ω is upper semicontinuous on $K \times E$ and for all $(u, w) \in K \times E$, $\Omega(u, w)$ is nonempty closed convex subset of $K \times E$. By Lemma 2.15, there exists a point $(u^*, w^*) \in K \times E$ such that $(u^*, w^*) \in \Omega(u^*, w^*)$, i.e., $u^* \in \Pi_x^{\lambda\gamma\mu}(u^*, w^*)$, $w^* \in [G_{(x,u^*)}]_\beta$. This implies that $u^* \in K$, $w^* \in [G_{(x,u^*)}]_\beta$ satisfy

$$u^* \in P(u^*, \lambda) \text{ and } \langle A(w^*, \gamma) + M(u^*, \mu), v - u^* \rangle + f(u^*, v) \geq 0, \forall v \in P(u^*, \lambda).$$

Thus, $\mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu) \neq \emptyset$.

(✠) We show that $\mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu) \subset K \times E$ is a compact set.

Since $K \times E$ is compact, we only prove that $\mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$ is a closed set. In fact, let the sequence $\{u_k\} \subset \mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$ such that $u_k \to u_0$ as $k \to \infty$. By the lower semicontinuity of $P(u_k, \lambda)$, for any $v_0 \in P(u_0, \lambda)$ there exists $v_k \in P(u_k, \lambda)$ such that $v_k \to v_0$ as $k \to \infty$. Since $u_k \in \mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$, we have $u_k \in P(u_k, \lambda)$ and there exists $w_k \in [G_{(x,u_k)}]_\beta$ such that

$$\langle A(w_k, \gamma) + M(u_k, \mu), v_k - u_k \rangle + f(u_k, v_k) \geq 0.$$

By the upper semicontinuity with closed values of $P(\cdot, \lambda)$, we have $u_0 \in P(u_0, \lambda)$. Since $\widetilde{G}(x, \cdot)$ is upper semicontinuous with nonempty closed values, $w_k \to w_0 \in [G_{(x,u_0)}]_\beta$ as $k \to \infty$. By the continuity of A, M and f, we get

$$\langle A(w_0, \gamma) + M(u_0, \mu), v_0 - u_0 \rangle + f(u_0, v_0) \geq 0,$$

i.e., $u_0 \in \mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$. Thus, $\mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$ is closed.

(✠) Finally, we prove that $\mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$ is convex.

Let $u_1^*, u_2^* \in \mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$, $\kappa \in [0, 1]$ and put $u^* = (1 - \kappa)u_1^* + \kappa u_2^*$. Since

$$u_1^*, u_2^* \in \mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu),$$

we get $u_1^* \in P(u_1^*, \lambda)$, $u_2^* \in P(u_2^*, \lambda)$ such that there exist $w_1^* \in [G_{(x,u_1^*)}]_\beta$, $w_2^* \in [G_{(x,u_2^*)}]_\beta$,

$$\langle A(w_1^*, \gamma) + M(u_1^*, \mu), v - u_1^* \rangle + f(u_1^*, v) \geq 0, \ \forall v \in P(u_1^*, \lambda),$$
$$\langle A(w_2^*, \gamma) + M(u_2^*, \mu), v - u_2^* \rangle + f(u_2^*, v) \geq 0, \ \forall v \in P(u_2^*, \lambda).$$

By the conditions $(\mathbf{A}_{5b})$ and $(\mathbf{A}_{6b})$, we obtain

$$u^* = (1 - \kappa)u_1^* + \kappa u_2^* \in P(u^*, \lambda),$$
$$w^* = (1 - \kappa)w_1^* + \kappa w_2^* \in [G_{(x,u^*)}]_\beta \text{ and } v \in P(u^*, \lambda).$$

Moreover, it follows from the assumption $(\mathbf{A}_9)$, that

$$\begin{aligned} 0 &\leq (1 - \kappa)\left[\langle A(w_1^*, \gamma) + M(u_1^*, \mu), v - u_1^* \rangle + f(u_1^*, v)\right] + \kappa\left[\langle A(w_2^*, \gamma) + M(u_2^*, \mu), v - u_2^* \rangle + f(u_2^*, v)\right] \\ &= (1 - \kappa)\langle A(w_1^*, \gamma), v - u_1^* \rangle + \kappa\langle A(w_2^*, \gamma), v - u_2^* \rangle + (1 - \kappa)\langle M(u_1^*, \mu), v - u_1^* \rangle + \kappa\langle M(u_2^*, \mu), v - u_2^* \rangle \\ &\quad + (1 - \kappa)f(u_1^*, v) + \kappa f(u_2^*, v) \\ &\leq \langle A(w^*, \gamma) + M(u^*, \mu), v - u^* \rangle + f(u^*, v), \end{aligned}$$

which implies $u^* \in \mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$. Therefore, we get that $\mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu)$ is a convex set. This completes the proof. □

Remark 3.5. *The problem FMQVI* (5) *is a generalization of fuzzy variational inequalities presented in Chang and Zhu* [14] *and Wu et al.* [50] *to the fuzzy mixed quasivariational inequality under perturbed parameters. Besides, the proof method of the existence result in Theorem 3.4 is different from the proof of Theorem 3.1 in* [14] *and Theorem 3.1 in* [50]. *Indeed, the Kakutani-Fan-Glicksberg fixed point theorem (Lemma 2.15) is used as a main tool for establishing the existence results of solution set in Theorem 3.4, while Chang and Zhu* [14] *and Wu et al.* [50] *investigated existence conditions by using KKM-technique. Furthermore, Theorem 3.1 also proposes the convexity and compactness of the solution set of FMQVI* (5).

For each $x \in C([0,T];\mathbb{R}^n)$ and $(\lambda,\gamma,\mu) \in \Lambda \times \Sigma \times \Xi$, we define a set-valued mapping $\Delta_{\mathbf{U}}^{\lambda\gamma\mu}\colon C([0,T];\mathbb{R}^n) \rightrightarrows \mathcal{L}^1([0,T];\mathbb{R}^m)$ as follows:

$$\Delta_{\mathbf{U}}^{\lambda\gamma\mu}(x) := \left\{u \in \mathcal{L}^1([0,T];\mathbb{R}^m) : u(t) \in \mathbf{U}^{\lambda\gamma\mu}(x(t)) \text{ for a.e. } t \in [0,T]\right\}, \tag{9}$$

where $\mathbf{U}^{\lambda\gamma\mu}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is defined by

$$\mathbf{U}^{\lambda\gamma\mu}(x) := \left\{u \in K : u \in \mathbf{SOL}\left(P_\lambda, G_{(x,\cdot)}, \beta, f, A_\gamma + M_\mu(\cdot)\right)\right\}, \quad \text{for } x \in \mathbb{R}^n.$$

We now provide some properties of solution set of the $\mathrm{FMQVI}_x^{\lambda\gamma\mu}$.

Lemma 3.6. *Assume that all the assumptions* $(\mathbf{A}_5)$*–*$(\mathbf{A}_9)$ *hold. Then* $\Delta_{\mathbf{U}}^{\lambda\gamma\mu}$ *defined by* (9) *is well defined and*

(i) $\Delta_{\mathbf{U}}^{\lambda\gamma\mu}$ *is weakly upper semicontinuous*;

(ii) $\Delta_{\mathbf{U}}^{\lambda\gamma\mu}(x)$ *is a bounded closed and convex set for each* $x \in C([0,T];\mathbb{R}^n)$;

(iii) *there exists a constant* $\mathbf{N}$ *such that*

$$\|\Delta_{\mathbf{U}}^{\lambda\gamma\mu}(x)\| = \sup\left\{\|u\|_{\mathcal{L}^1} : u \in \Delta_{\mathbf{U}}^{\lambda\gamma\mu}(x)\right\} \le \mathbf{N},\ \forall x \in C([0,T];\mathbb{R}^n). \tag{10}$$

Proof. According to Theorem 3.4, $\mathbf{SOL}\left(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu\right)$ is nonempty compact and convex. Therefore, the results of Lemma 3.6 follow from applying the same proof as in [50, Theorem 3.2 and Lemma 4.1]. □

3.2 An existence of mild solutions

For each $(\lambda,\gamma,\mu) \in \Lambda \times \Sigma \times \Xi$, we denote the set of mild solutions to Problem 1.1 by $\Omega(\lambda,\gamma,\mu)$. Inspired from the previous work [37, 49, 50], we now define the solutions to Problem 1.1 in a mild sense as follows:

Definition 3.7. *A pair of functions* $(x,u) \in C([0,T];\mathbb{R}^n) \times \mathcal{L}^1([0,T];\mathbb{R}^m)$ *is said to be a mild solution Problem 1.1 if there exists a function* $h \in \mathcal{Q}^1_{\widetilde{H}}(x)$ *such that*

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[h(s) + B(s,x(s))u(s)\right] ds,$$

for all $t \in [0,T]$*, where* $u(t) \in K$ *solves the problem* $(\mathrm{FMQVI})_x^{\lambda\gamma\mu}$ (2) *for a.e.* $t \in [0,T]$ *and*

$$\mathcal{Q}^1_{\widetilde{H}}(x) = \left\{h \in \mathcal{L}^1([0,T];\mathbb{R}^n) : h(t) \in \widetilde{H}(t,x(t)) = \left[H_{(t,x(t))}\right]_\alpha, \textit{for a.e. } t \in [0,T]\right\}. \tag{11}$$

Lemma 3.8. *Let the assumptions* $(\mathbf{A}_1)$*–*$(\mathbf{A}_3)$ *hold. Then* $\mathcal{Q}^1_{\widetilde{H}}$ *defined by* (11) *is weakly upper semicontinuos with nonempty, bounded, closed and convex values. Furthermore, for any* $r > 0$*,* $x \in C([0,T];\mathbb{R}^n)$ *with* $\|x(t)\| \le r$*, for a.e.* $t \in [0,T]$*, there exists* $\eta_r \in \mathcal{L}^1([0,T];\mathbb{R}_+)$*,*

$$\|h(t)\| \le \eta_r(t), \ \textit{for a.e. } t \in [0,T] \ \textit{ for all } h \in \mathcal{Q}^1_{\widetilde{H}}(x). \tag{12}$$

Proof. By [49, Lemma 3.4], we obtain that $\mathcal{Q}^1_{\widetilde{H}}(x) \neq \emptyset$ for all $x \in C([0,T],\mathbb{R}^n)$. Moreover, by the condition $(\mathbf{A}_3)$, Lemma 2.8(ii) and the definition of $\mathcal{Q}^1_{\widetilde{H}}$ imply that (12) holds. Similar to the proof of [50, Lemma 4.1], by virtue of Lemma 3.2, we conclude that $\mathcal{Q}^1_{\widetilde{H}}$ is weakly upper semicontinuos with bounded, closed and convex values. □

Remark 3.9. *For each* $x \in C([0,T];\mathbb{R}^n)$*, let*

$$\Upsilon^{\lambda\gamma\mu}(t,x(t)) = \left\{B(t,x(t))u(t) : u(t) \in \mathbf{SOL}\left(P_\lambda(\cdot), G_{(x(t),\cdot)}, \beta, f, A_\gamma + M_\mu(\cdot)\right)\right\}. \tag{13}$$

Then it is discovered that the solution of Problem 1.1 can be given by

$$x(t) = x_0 + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left[h(s) + \psi(s)\right] ds,$$

for all $t \in [0,T]$*, where* $h \in \mathcal{Q}^1_{\widetilde{H}}(x)$ *and* $\psi \in \mathcal{Q}^1_{\Upsilon}(x)$ *with* $\mathcal{Q}^1_{\widetilde{H}}(x)$ *defined by* (11) *and* $\mathcal{Q}^1_{\Upsilon}(x)$ *given by*

$$\mathcal{Q}^1_{\Upsilon}(x) = \left\{\psi \in \mathcal{L}^1([0,T];\mathbb{R}^n) : \psi(t) \in \Upsilon^{\lambda\gamma\mu}(t,x(t)), \textit{for a.e. } t \in [0,T]\right\}. \tag{14}$$

Remark 3.10. *Assume that the hypothese* $(\mathbf{A}_4)$*–*$(\mathbf{A}_9)$ *hold. Then* $\mathcal{Q}^1_\Upsilon$ *defined by* (14) *is weakly upper semicontinuos with nonempty, bounded, closed and convex values (see [32, Lemma 3.2],* [49, Remark 3.2 and Lemma 3.4]*). Moerover,*

$$\begin{aligned}\|\Upsilon^{\lambda\gamma\mu}(t,x(t))\| &= \sup\{\|\psi(t)\| : \psi(t) \in \Upsilon^{\lambda\gamma\mu}(t,x(t))\} \\ &\le \|B(t,x(t))\|\|u(t)\| \le b_B\mathbf{N},\end{aligned}$$

where $\mathbf{N}$ *given in* (10)*. Thus, for any* $x \in C([0,T];\mathbb{R}^n)$*, we have* $\|\psi(t)\| \le b_B\mathbf{N}$*, for a.e.* $t \in [0,T]$ *for all* $\psi \in \mathcal{Q}^1_\Upsilon(x)$*.*

Theorem 3.11. *Assume that all the hypothese* $(\mathbf{A}_1)$*–*$(\mathbf{A}_9)$ *are satisfied. If the following condition holds:*

$(\mathbf{A}_{10})$*: there exists a constant* $r > 0$ *such that*

$$r \ge \frac{\|x_0\| + \frac{c_H + b_B\mathbf{N}}{\Gamma(q+1)}T^q}{1 - \frac{T^q}{\Gamma(q+1)}}, \quad \frac{T^q}{\Gamma(q+1)} < 1,$$

where $\mathbf{N}$ *given in* (10) *and* $c_H = \sup_{t\in[0,T]} \|\widetilde{H}(t,0)\|$,

then Problem 1.1 has at least one solution on $[0,T]$*.*

Proof. From Lemma 3.8 and Remark 3.10, using a similar argument as in the proof of [49, Theorem 3.1] based on the Krasnoselskii fixed point theorem, we deduce that Problem 1.1 has at least one solution on $[0,T]$, i.e., $\Omega(\lambda,\gamma,\mu) \ne \emptyset$ for all $(\lambda,\gamma,\mu) \in \Lambda \times \Sigma \times \Xi$. □

To illustrate that all the hypotheses in Theorem 3.11 can be satisfied, we present the following numerical example.

Example 3.12. *Suppose that* $n = 2, m = 1$*,* $q = 0.55$*,* $T = 0.7$*,* $K = \{u \in \mathbb{R} : 0.25 \le u \le 0.85\}$*,* $E = [0, 1.5]$*,* $\alpha, \beta \in [0,1]$*,* $\Lambda = \Sigma = \Xi = \mathbb{R}$*,* $P(u,\lambda) = P(u) = [0.25, u]$*,* $f(u,v) = v^2 - u^2$ *and*

$$\begin{aligned}
&A(w,\gamma) = (1+\cos\gamma^2)w, \quad M(u,\mu) = e^{-\mu} + \sin u, \quad B(t,x) = \begin{pmatrix} -0.15e^{-t} \\ 0.2\sin x_2 \end{pmatrix}, \\
&H_{(t,x)} = \begin{pmatrix} H^{(1)}_{(t,x)} \\ H^{(2)}_{(t,x)} \end{pmatrix} = \begin{pmatrix} e^{-1.2t}\cdot z \\ 1.2\cos x_2 \cdot z \end{pmatrix}, \\
&G_{(x,u)} = \left(0.3\sin\left(\frac{x_1+x_2}{2}\right) + 0.3u^2 + 0.12\right)\cdot\omega,
\end{aligned}$$

for all $(\lambda,\gamma,\mu) \in \Lambda\times\Sigma\times\Xi$*,* $u, v \in \mathbb{R}$*,* $x \in \mathbb{R}^2$*, where* $x = (x_1,x_2)^\top \in \mathbb{R}^2$*,* z *and* ω *are two symmetric triangular fuzzy numbers with level sets as follows:*

$$[z]_\alpha = [0.35(\alpha-1), 0.35(1-\alpha)], [\omega]_\beta = [0.25(1+\beta), 0.25(3-\beta)].$$

That is, we consider the following fuzzy differential system:

$$\begin{cases}
{}^C_0\mathbf{D}^q_t x(t) \in \begin{pmatrix} e^{-1.2t}\cdot[z]_\alpha \\ 1.2\cos x_2(t)\cdot[z]_\alpha \end{pmatrix} + \begin{pmatrix} -0.15e^{-t} \\ 0.2\sin x_2(t) \end{pmatrix} u(t), & a.e.\ t\in[0,T], \\
\langle (1+\cos\gamma^2)w(t) + e^{-\mu} + \sin u(t), v - u(t)\rangle + v^2 - u^2(t) \ge 0,\ \forall v \in [0.25, u(t)], & a.e.\ t\in[0,T], \\
u(t) \in [0.5, u(t)],\ w(t) \in \left(0.3\sin\left(\frac{x_1(t)+x_2(t)}{2}\right) + 0.3u^2(t) + 0.12\right)\cdot[\omega]_\beta & a.e.\ t\in[0,T], \\
x(0) = x_0.
\end{cases} \tag{15}$$

It is easy to check that hypotheses $(\mathbf{A}_2)$*,* $(\mathbf{A}_5)$*–*$(\mathbf{A}_8)$ *and* $(\mathbf{A}_9)$ *are satisfied.*

For any $x^{(j)} = \left(x^{(j)}_1, x^{(j)}_2\right)^\top \in \mathbb{R}^2$ *with* $j = 1,2$*, one has*

$$\mathfrak{H}\left(H^{(1)}_{(t,x^{(1)})}, H^{(1)}_{(t,x^{(2)})}\right) = \sup_{\alpha\in[0,1]} \mathfrak{H}\left(e^{-1.2t}[z]_\alpha, e^{-1.2t}[z]_\alpha\right) = 0,$$

and

$$\begin{aligned}\mathfrak{H}\left(H^{(2)}_{(t,x^{(1)})}, H^{(2)}_{(t,x^{(2)})}\right) &= \sup_{\alpha\in[0,1]} \mathfrak{H}\left(1.2\cos x_2^{(1)}[z]_\alpha, 1.2\cos x_2^{(2)}[z]_\alpha\right)\\ &= 1.2 \sup_{\alpha\in[0,1]} \mathfrak{H}\left(\cos x_2^{(1)}[0.35(\alpha-1), 0.35(1-\alpha)], \cos x_2^{(2)}[0.35(\alpha-1), 0.35(1-\alpha)]\right)\\ &= 0.42(1-\alpha)\left|\left|\cos x_2^{(1)}\right| - \left|\cos x_2^{(2)}\right|\right|\\ &\le 0.42(1-\alpha)\left|\cos x_2^{(1)} - \cos x_2^{(2)}\right|\\ &\le 0.42\left|x_2^{(1)} - x_2^{(2)}\right| \le 0.42\left\|x^{(1)} - x^{(2)}\right\|.\end{aligned}$$

Thus,

$$\begin{aligned}\mathfrak{H}\left(H_{(t,x^{(1)})}, H_{(t,x^{(2)})}\right) &= \mathfrak{H}\left(H^{(1)}_{(t,x^{(1)})}, H^{(1)}_{(t,x^{(2)})}\right) + \mathfrak{H}\left(H^{(2)}_{(t,x^{(1)})}, H^{(2)}_{(t,x^{(2)})}\right)\\ &\le 0.42\left\|x^{(1)} - x^{(2)}\right\| = \phi\left(t, \left\|x^{(1)} - x^{(2)}\right\|\right),\end{aligned}$$

here, $\phi(t,s) = 0.42s$ $(s \ge 0)$. *This verifies that hypothesis* $(\mathbf{A}_1)$ *holds.*

Furthermore, the support $[z]_0 = [-0.35, 0.35]$. *For each* $r > 0$, $x = (x_1, x_2)^\top \in \mathbb{R}^2$, $\|x\| \le r$, *it follows that*

$$\begin{aligned}\|H_{(t,x)}\| &= \mathfrak{H}\left(H^{(1)}_{(t,x)}, \widetilde{0}\right) + \mathfrak{H}\left(H^{(2)}_{(t,x)}, \widetilde{0}\right)\\ &= \sup_{y\in[H^{(1)}_{(t,x)}]_0} \|y\| + \sup_{y\in[H^{(2)}_{(t,x)}]_0} \|y\|\\ &= \sup_{y\in e^{-1.2t}[-0.35,0.35]} \|y\| + \sup_{y\in 1.2\cos x_2[-0.35,0.35]} \|y\|\\ &\le 0.35 + 0.42 = 0.77 = \eta_r(t).\end{aligned}$$

This implies that condition $(\mathbf{A}_3)$ *holds.*

For all $t \in [0,T]$, $x = (x_1, x_2)^\top \in \mathbb{R}^2$, *we have*

$$\|B(t,x)\| = \left\|\begin{pmatrix} -0.15e^{-t} \\ 0.2\sin x_2 \end{pmatrix}\right\| = 0.15e^{-t} + 0.2|\sin x_2| \le 0.35 = b_B,$$

that is, the assumption $(\mathbf{A}_4)$ *holds.*

Finally, we can see that $T^q \approx 0.822 < 0.889 \approx \Gamma(q+1)$. *Since* $\mathbf{SOL}\,(P_\lambda, G_x, \beta, f, A_\gamma + M_\mu) \subset K$, *we have* $\mathbf{N} = 0.85$. *Taking*

$$r \ge \frac{\|x_0\| + \frac{c_H + b_B\mathbf{N}}{\Gamma(q+1)}T^q}{1 - \frac{T^q}{\Gamma(q+1)}},$$

where $c_H = \sup_{t\in[0,T]} \|\widetilde{H}(t,0)\|$, *we can check that the condition* $(\mathbf{A}_{10})$ *is satisfied. As a result, all the assumptions of Theorem 3.11 are satisfied and so the fuzzy differential system* (15) *has at least one solution on* $[0,T]$.

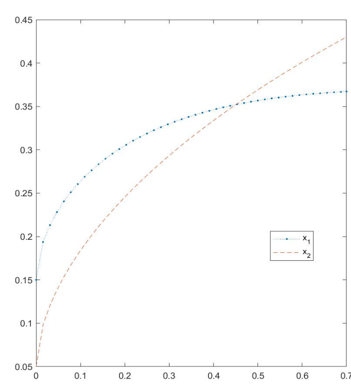

Figure 1: Transient behavior of the mild trajectory $x = (x_1, x_2)^\top$ for the system (16).

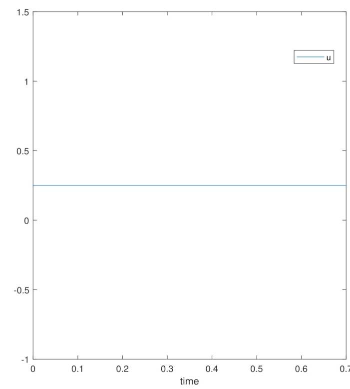

Figure 2: Transient behavior of the variational control trajectory u for the system (16).

In particular, let $[\underline{z}, \overline{z}] = [z]_0 = [-0.35, 0.35]$, $[\underline{\omega}, \overline{\omega}] = [\omega]_0 = [0.25, 0.75]$ *and* $\lambda = \gamma = \mu = 0.5$. *For* $t \in [0, 0.7]$, *we now consider the following case of the fuzzy differential system* (15)*:*

$$\begin{cases} {}_0^C\mathbf{D}_t^q x(t) = \begin{pmatrix} e^{-1.2t}\overline{z} \\ 1.2\cos(x_2(t))\overline{z} \end{pmatrix} + \begin{pmatrix} -0.15e^{-t} \\ 0.2\sin x_2(t) \end{pmatrix} u(t), \\ \langle (1+\cos 0.25)w(t) + e^{-0.5} + \sin u(t), v - u(t) \rangle + v^2 - u^2(t) \geq 0, \ \forall v \in [0.25, u(t)], \\ u(t) \in [0.25, u(t)], \ w(t) \in \left(0.3\sin\left(\dfrac{x_1(t)+x_2(t)}{2} \right) + 0.3u^2(t) + 0.12 \right) \overline{\omega}, \\ x(0) = x_0 = (0.15, 0.05)^\top. \end{cases} \tag{16}$$

In what follows, we provide Fig. 1 and Fig. 2 which show the mild trajectory x *and variational control trajectory* u *of the fuzzy differential system* (16)*, respectively. Here, the Caputo fractional differential equation in the system* (16) *was solved in MATLAB with the code* `fde12.m` *using a modification of the basic Predictor-Corrector method introduced by Garrappa [22]. Step size for* `fde12.m` *was set to* $h = 2^{-6}$.

4 Upper semicontinuity

In this section, the upper semicontinuity of the solution mapping to Problem 1.1 under the perturbed parameters will be discussed. In the sequel, we also need the following assumptions:

($\mathbf{A}'_{5a}$): the mapping $P\colon K \times \Lambda \rightrightarrows K$ is continuous with nonempty convex and compact values;

($\mathbf{A}'_{6b}$): the fuzzy mapping $G\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathcal{E}^m(E)$ is topologically closed;

($\mathbf{A}'_7$): the functions $A\colon E \times \Sigma \to \mathbb{R}^m$, $M\colon \mathbb{R}^m \times \Xi \to \mathbb{R}^m$ and $f\colon K \times K \to \mathbb{R}$ are continuous.

Remark 4.1. *Note that the conditions* ($\mathbf{A}'_{5a}$), ($\mathbf{A}'_{6a}$) *and* ($\mathbf{A}'_7$) *imply the conditions* ($\mathbf{A}_{5a}$), ($\mathbf{A}_{6a}$) *and* ($\mathbf{A}_7$)*, respectively.*

First, we consider a result on the closedness of the solution map to Problem 1.1 with respect to parameters (λ, γ, μ).

Theorem 4.2. *Assume that the hypotheses* ($\mathbf{A}_1$)–($\mathbf{A}_4$), ($\mathbf{A}'_{5a}$), ($\mathbf{A}_{5b}$), ($\mathbf{A}'_{6a}$), ($\mathbf{A}_{6b}$), ($\mathbf{A}'_7$), ($\mathbf{A}_8$)–($\mathbf{A}_{10}$) *hold. Then the set-valued mapping* $(\lambda, \gamma, \mu) \mapsto \Omega(\lambda, \gamma, \mu)$ *is closed.*

Proof. Let $(\lambda, \gamma, \mu) \in \Lambda \times \Sigma \times \Xi$ be fixed. It follows from Theorem 3.11 that the set $\Omega(\lambda, \gamma, \mu)$ is nonempty. We shall verify that the set-valued mapping $(\lambda, \gamma, \mu) \mapsto \Omega(\lambda, \gamma, \mu)$ is closed.

Indeed, for any sequences $\{(\lambda_k, \gamma_k, \mu_k)\} \subset \Lambda \times \Sigma \times \Xi$ and $\{(x_k, u_k)\} \subset C([0,T]; \mathbb{R}^n) \times \mathcal{L}^1([0,T]; \mathbb{R}^m)$ is such that

$$\begin{array}{ll} (\lambda_k, \gamma_k, \mu_k) \to (\lambda^*, \gamma^*, \mu^*) \ \text{ in } \Lambda \times \Sigma \times \Xi & \text{as } k \to \infty, \\ (x_k, u_k) \in \Omega(\lambda_k, \gamma_k, \mu_k) & \text{for each } k \in \mathbb{N}, \\ x_k \to x^* \ \text{ in } C([0,T]; \mathbb{R}^n), u_k \rightharpoonup u^* \ \text{ in } \mathcal{L}^1([0,T]; \mathbb{R}^m) & \text{as } k \to \infty. \end{array}$$

Since $(x_k, u_k) \in \Omega(\lambda_k, \gamma_k, \mu_k)$, there exist $h_k \in \mathcal{Q}^1_{\widetilde{H}}(x_k)$ and $w_k \colon [0,T] \to \mathbb{R}^m$ satisfying

$$x_k(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[h_k(s) + B(s, x_k(s)) u_k(s)\right] ds, \tag{17}$$

for all $t \in [0,T]$ and $u_k(t) \in P(u_k(t), \lambda_k)$, $w_k(t) \in \left[G_{(x_k(t), u_k(t))}\right]_\beta$,

$$\langle A(w_k(t), \gamma_k) + M(u_k(t), \mu_k), v - u_k(t)\rangle + f(u_k(t), v) \geq 0, \tag{18}$$

for all $v \in P(u_k(t), \lambda_k)$, for a.e. $t \in [0,T]$.

In view of Lemma 3.8, we obtain that the sequence $\{h_k\} \subset \mathcal{L}^1([0,T];\mathbb{R}^n)$ is weakly compact. Without loss of generality, assume that $h_k \rightharpoonup h^*$ in $\mathcal{L}^1([0,T];\mathbb{R}^n)$. By the weakly upper semicontinuity of $\mathcal{Q}^1_{\widetilde{H}}$ and $x_k \to x^*$ in $C([0,T];\mathbb{R}^n)$, it follows from Lemma 2.7 that $h^* \in \mathcal{Q}^1_{\widetilde{H}}(x^*)$. Since B is continuous on $[0,T] \times \mathbb{R}^n$, we have

$$h_k(t) + B(t, x_k(t)) u_k(t) \to h^*(t) + B(t, x^*(t)) u^*(t) \text{ for a.e. } t \in [0,T].$$

Moreover, by Lemma 3.8, there exists a function $\varrho \in \mathcal{L}^1([0,T];\mathbb{R}_+)$ such that $\|h_k(t)\| \leq \varrho(t)$ for a.e. $t \in [0,T]$. Let $\widetilde{\varrho} \in \mathcal{L}^1([0,T];\mathbb{R}_+)$ defined by $\widetilde{\varrho}(t) = \varrho(t) + b_B \mathbf{N}$. Then we obtain

$$\begin{aligned} \|h_k(t) + B(t, x_k(t)) u_k(t)\| &\leq \|h_k(t)\| + \|B(t, x_k(t))\| \|u_k(t)\| \\ &\leq \varrho(t) + b_B \mathbf{N} = \widetilde{\varrho}(t) \text{ for a.e. } t \in [0,T]. \end{aligned}$$

Applying the Lebesgue dominated convergence theorem (see [17, Theorem 2.2.19]), one has

$$x_k(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[h_k(s) + B(s, x_k(s)) u_k(s)\right] ds \to x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[h^*(s) + B(s, x^*(s)) u^*(s)\right] ds,$$

for all $t \in [0,T]$. Furthermore, the convergence $x_k \to x^*$ in $C([0,T];\mathbb{R}^n)$ implies

$$x^*(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[h^*(s) + B(s, x^*(s)) u^*(s)\right] ds, \tag{19}$$

for all $t \in [0,T]$.

On the other hand, since P is continuous with compact values, $\lambda_k \to \lambda^*$ and $u_k(t) \to u^*(t)$ for a.e. $t \in [0,T]$, it follows from Lemma 2.5 that $u^*(t) \in P(u^*(t), \lambda^*)$ for a.e. $t \in [0,T]$. Applying Remark 3.3, we get that $\widetilde{G}(\cdot,\cdot) = \left[G_{(\cdot,\cdot)}\right]_\beta$ is upper semicontinuos with nonempty compact values for a.e. $t \in [0,T]$. Without loss of generality, there exists $w^* \colon [0,T] \to \mathbb{R}^m$ such that $w_k(t) \to w^*(t) \in \left[G_{(x^*(t), u^*(t))}\right]_\beta$ for a.e. $t \in [0,T]$.

Moreover, for any $v \in P(u^*(t), \lambda^*)$, it follows from the continuity of P that there exits a sequence $\{v_k\} \subset \mathbb{R}^m$ with $v_k \in P(u_k(t), \lambda_k)$ such that $v_k \to v$ as $k \to \infty$ (by Lemma 2.6). Hence, from (18), we have

$$\langle A(w_k(t), \gamma_k) + M(u_k(t), \mu_k), v_k - u_k(t)\rangle + f(u_k(t), v_k) \geq 0, \tag{20}$$

for a.e. $t \in [0,T]$.

Using the continuity of A, M and f, then (20) implies that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \langle A(w_k(t), \gamma_k) + M(u_k(t), \mu_k), v_k - u_k(t)\rangle + f(u_k(t), v_k) \\ &= \langle A(w^*(t), \gamma^*) + M(u^*(t), \mu^*), v - u^*(t)\rangle + f(u^*(t), v), \end{aligned} \tag{21}$$

for all $v \in P(u^*(t), \lambda^*)$ and a.e. $t \in [0,T]$. Combining (19) and (21), we conclude that $(x^*, u^*) \in \Omega(\lambda^*, \gamma^*, \mu^*)$, i.e., Ω is closed. This completes the proof. □

Next, the following result illustrates the upper semicontinuity of the set-valued operator Ω with respect to parameters.

Theorem 4.3. *Let $(\lambda^*, \gamma^*, \mu^*) \in \Lambda \times \Sigma \times \Xi$. Assume that the hypotheses $(\mathbf{A}_1)$–$(\mathbf{A}_4)$, $(\mathbf{A}'_{5a})$, $(\mathbf{A}_{5b})$, $(\mathbf{A}'_{6a})$, $(\mathbf{A}_{6b})$, $(\mathbf{A}'_7)$, $(\mathbf{A}_8)$–$(\mathbf{A}_{10})$ are satisfied. Then the set-valued mapping $(\lambda, \gamma, \mu) \mapsto \Omega(\lambda, \gamma, \mu)$ is upper semicontinuous at $(\lambda^*, \gamma^*, \mu^*)$.*

Proof. On the contrary, we assume that the set-valued mapping Ω is not upper semicontinuous at $(\lambda^*, \gamma^*, \mu^*) \in \Lambda \times \Sigma \times \Xi$. Then, there exist two sequences $\{(\lambda_p, \gamma_p, \mu_p)\} \subset \Lambda \times \Gamma \times \Xi$ and $\{(x_p, u_p)\} \subset C([0,T]; \mathbb{R}^n) \times \mathcal{L}^1([0,T]; \mathbb{R}^m)$ and an open set $\mathcal{U}$ in $C([0,T]; \mathbb{R}^n) \times \mathcal{L}^1([0,T]; \mathbb{R}^m)$ with

$$\begin{aligned} &(\lambda_p, \gamma_p, \mu_p) \to (\lambda^*, \gamma^*, \mu^*) \ \text{ in } \Lambda \times \Sigma \times \Xi \quad \text{ as } p \to \infty, \\ &(x_p, u_p) \in \Omega\,(\lambda_p, \gamma_p, \mu_p) \quad \text{ for each } p \in \mathbb{N}, \\ &\Omega\,(\lambda^*, \gamma^*, \mu^*) \subset \mathcal{U} \end{aligned}$$

such that $(x_p, u_p) \notin \mathcal{U}$ for each $p \in \mathbb{N}$. Since $(x_p, u_p) \in \Omega\,(\lambda_p, \gamma_p, \mu_p)$, there exist $h_p \in \mathcal{Q}^1_{\widetilde{H}}(x_p)$, $\psi_p \in \mathcal{Q}^1_{\Upsilon}(x_p)$ and $w_p \colon [0,T] \to \mathbb{R}^m$ such that $u_p(t) \in P\,(u_p(t), \lambda_p)\,, w_p(t) \in \left[G_{(u_p(t), x_p(t))}\right]_\beta$,

$$\langle A(w_p(t), \gamma_p) + M(u_p(t), \mu_p), v - u_p(t) \rangle + f(u_p(t), v) \geq 0, \tag{22}$$

for all $v \in P(u_p(t), \lambda_p)$, for a.e. $t \in [0,T]$ and

$$x_p(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[h_p(s) + \psi_p(s)\right] ds, \quad \text{for all } t \in [0,T]. \tag{23}$$

It follows from Lemma 3.8 that the sequence $\{h_p\} \subset \mathcal{Q}^1_{\widetilde{H}}(x_p)$ is bounded in $\mathcal{L}^1([0,T]; \mathbb{R}^n)$. Without loss of generality, we can assume that $h_p \rightharpoonup h^*$ in $\mathcal{L}^1([0,T]; \mathbb{R}^n)$ for some $h^* \in \mathcal{L}^1([0,T]; \mathbb{R}^n)$. By a similar argument, we also obtain that $\psi_p \rightharpoonup \psi^*$ in $\mathcal{L}^1([0,T]; \mathbb{R}^m)$ for some $\psi^* \in \mathcal{L}^1([0,T]; \mathbb{R}^m)$. By Lemma 3.8, there exists a function $\varrho \in \mathcal{L}^1([0,T]; \mathbb{R}_+)$ such that $\|h_p(t)\| \leq \varrho(t)$ for a.e. $t \in [0,T]$ and for all $p \in \mathbb{N}$. Moreover, $\|\psi_p(t)\| \leq b_B \mathbf{N}$ for a.e. $t \in [0,T]$ and for all $p \in \mathbb{N}$ (by Remark 3.10). Thus, applying the Lebesgue dominated convergence theorem, one has

$$x_p(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[h_p(s) + \psi_p(s)\right] ds \to x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[h^*(s) + \psi^*(s)\right] ds,$$

as $p \to \infty$, for all $t \in [0,T]$. Let $x^* \colon [0,T] \to \mathbb{R}^n$ be defined by

$$x^*(t) = x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[h^*(s) + \psi^*(s)\right] ds.$$

It is clear that x^* is continuous on $[0,T]$. Since $x_p \in C([0,T]; \mathbb{R}^n)$ for all $p \in \mathbb{N}$, from (**??**), we deduce that $x_p \to x^*$ in $C([0,T]; \mathbb{R}^n)$. By the weakly upper semicontinuity of $\mathcal{Q}^1_{\widetilde{H}}$ and $\mathcal{Q}^1_{\Upsilon}$, it follows from Lemma 2.7 that $h^* \in \mathcal{Q}^1_{\widetilde{H}}(x^*)$ and $\psi^* \in \mathcal{Q}^1_{\Upsilon}(x^*)$.

Since $\{u_p\} \subset \Delta_{\mathbf{U}}^{\lambda_p \gamma_p \mu_p}(x_p)$, it follows from Lemma 3.6(ii) that $\{u_p\} \subset \mathcal{L}^1([0,T]; \mathbb{R}^n)$ is weakly compact. Hence, there exists a subsequence of $\{u_p\}$, denoted again by $\{u_p\}$, such that $u_p \rightharpoonup u^*$ in $\mathcal{L}^1([0,T]; \mathbb{R}^n)$. Using the assumption that P is continuous with compact values, by Lemma 2.5, we have $u^*(t) \in P\,(u^*(t), \lambda^*)$ for a.e. $t \in [0,T]$.

Next, we prove that $u^*(t) \in \mathbf{SOL}\left(P_{\lambda^*}, G_{(x^*(t),\cdot)}, \beta, f, A_{\gamma^*} + M_{\mu^*}(\cdot)\right)$ for a.e. $t \in [0,T]$. Indeed, analysis similar to that in the proof of Theorem 4.2 shows that there exists $w^* \colon [0,T] \to \mathbb{R}^m$ such that $w^*(t) \in \left[G_{(x^*(t), u^*(t))}\right]_\beta$ and

$$\langle A(w^*(t), \gamma^*) + M(u^*(t), \mu^*), v - u^*(t) \rangle + f(u^*(t), v) \geq 0, \tag{24}$$

for all $v \in P\,(u^*(t), \lambda^*)$ and a.e. $t \in [0,T]$. Thanks to $u^*(t) \in P\,(u^*(t), \lambda^*)$ for a.e. $t \in [0,T]$, we obtain $u^*(t) \in \mathbf{SOL}\left(P_{\lambda^*}, G_{(x^*(t),\cdot)}, \beta, f, A_{\gamma^*} + M_{\mu^*}(\cdot)\right)$ for a.e. $t \in [0,T]$. Furthermore, from Theorem 4.2, we know that the set $\Omega\,(\lambda^*, \gamma^*, \mu^*)$ is closed, hence we get that $(x^*, u^*) \in \Omega\,(\lambda^*, \gamma^*, \mu^*) \subset \mathcal{U}$. However, by assumptions, we know that $(x_p, u_p) \notin \mathcal{U}$ for all $p \in \mathbb{N}$, which is a contradiction. This completes the proof. □

5 Conclusion

In this paper, we studied a generalized fuzzy differential system proposed by the fuzzy fractional differential inclusion combined with the fuzzy mixed quasivariational inequality involving Caputo fractional derivatives, (GFDS). By using the Kakutani-Fan-Glicksberg fixed point theorem, the existence of solutions for the fuzzy mixed quasivariational inequality has been established under suitable conditions (Theorem 3.4). Then we also obtained the existence of mild solutions for the system GFDS (Theorem 3.11). Furthermore, we have given a numerical example to illustrate this main result (Example 3.12). Finally, we discussed the upper semicontinuity of the solution mapping to the system GFDS with the perturbed parameters (Theorem 4.3).

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