Iranian Journal of Fuzzy Systems

Volume 20, Number 3, (2023), pp. 95-113

Original paper

A new extension of a triangular norm on a subinterval $[0, \alpha]$ via an interior operator to the underlying entire bounded lattice

B. Wu[1] and H. Zhou[2]

[1,2] *Institute of Mathematics, Shaanxi Normal University, Xi'an, China*

wubg@snnu.edu.cn, hjzhou@snnu.edu.cn

Abstract

As a proper generalization of the ordinal sum t-norm construction on bounded lattices proposed in [E. Aşıcı, R. Mesiar, New constructions of triangular norms and triangular conorms on an arbitrary bounded lattice, International Journal of General Systems, **49**(2) (2020), 143-160], the present paper studies a new extension of a triangular norm on a subinterval $[0, \alpha]$ via an interior operator to the underlying entire bounded lattice, where the necessary and sufficient conditions under which the constructed operation is again a t-norm are given. By comparing the graphic structures of two t-norms on a common bounded lattice which are constructed in different ways, it is shown that the new method in this paper is essentially different from the ones existing in the literature. As an end, this new construction is generalized to construct ordinal sums of finitely many t-norms by recursion on bounded lattices. The dual results for ordinal sum construction of t-conorms via closure operators on bounded lattices are also presented.

Keywords: Triangular norm, ordinal sum, bounded lattice, interior operator.

1 Introduction

Triangular norms (briefly t-norms) are a class of binary aggregation functions. T-norms on the unit interval $[0, 1]$ were studied first by Schweizer and Sklar [36] based on the incipient concept presented by Menger [30] aiming to extend the triangular inequality in metric spaces to probabilistic metric spaces. As many-valued counterparts of Boolean logical conjunctive and disjunctive connectives, t-norms and their dual operations, called triangular conorms (briefly t-conorms), can not only be used to solve functional equations but also play important roles in probability theory, fuzzy set theory, generalized measure and integral theory, information aggregation and other fields [1, 26, 28]. Considering the fact that the unit interval is just a specific algebraic semantics of mathematical fuzzy logic where, other than its lattice structure, there is no more reason to use $[0, 1]$ as the algebraic structure of all truth values of atomic propositions, Goguen demonstrated the lattice-valued algebraic semantics of fuzzy logic [22, 23]. Then the research of t-norms and t-conorms on the unit interval has been extended to bounded lattices [4, 17, 39]. On the other hand, since t-norms and t-conorms are basic building blocks of other more general aggregation functions such as uninorms and nullnorms, they are also playing an important role for constructing and characterizing uninorms, nullnorms and uni-nullnorms on bounded lattices [7, 10, 13, 14, 16, 25, 31, 38, 40].

It is very known that there are several powerful methods to construct and/or represent t-norms (resp. t-conorms) on the real unit interval $[0, 1]$, including the uses of triangular subnorms, additive generators, multiplicative generators, and ordinal sum composition or decomposition. And, among them, the ordinal sum representation is one of the most prominent ways. However, differently from the cases on the unit interval $[0, 1]$, there is no even unified definition for ordinal sum of t-norms and of t-conorms on bounded lattices. It is Saminger [34] who first defined the so-called ordinal sum of t-norms on bounded lattices by imitating the ordinal sum structure of t-norms on the real unit interval. However, Saminger's ordinal sum of t-norms is not necessarily again a t-norm on the given bounded lattice. Saminger presented

Corresponding Author: H. Zhou

Received: May 2022; Revised: December 2022; Accepted: January 2023.

https://doi.org/10.22111/IJFS.2023.7641

then the necessary and sufficient conditions under which the ordinal sum is always a t-norm for any t-norm summands on bounded lattices. Later on, Saminger-Platz, Medina [29, 35] characterized when an ordinal sum of t-norms is a t-norm on bounded lattices. And Ertuğrul et al [19] modified Saminger's ordinal sum such that the resulting ordinal sum is again a t-norm on arbitrary bounded lattices. Çaylı [11, 12] simplified the construction of ordinal sum t-norms on the basis of the results in [19] and proposed then two new constructions. Among the aforementioned methods, the ordinal sum t-norms all work as an extension of a prebehaved t-norm on a subinterval of the form $[\alpha, 1]$ by complementing additionally some appropriate values on the complementary set of $[\alpha, 1]^2$. Ertuğrul, Karaçal and Dan et al studied the possible extension of t-norms on any subintervals $[\alpha, \beta]$ to the whole bounded lattices [15, 20, 24]. Aşıcı, Mesiar and Ertuğrul, Yeşilyurt constructed ordinal sums t-norms based on t-norms on $[0, \alpha]$ and $[\alpha, 1]$ in [8, 21], respectively. In recent years, some constructions of t-norms via interior operators on bounded lattices began to appear in the literature. Dvořák and Holčapk [18] put forward an ordinal sum of finitely many t-norms on bounded lattices, which provided a new direction for ordinal sum t-norm constructions. At the same time, Ouyang et al [32] proposed an ordinal sum of countably many t-norms on bounded lattices with the feature that the obtained ordinal sum is always a t-norm. Sun and Liu [37] came up also new approaches towards constructing ordinal sums of t-norms on bounded lattices by adopting order-preserving functions.

As discussed above, the ordinal sum t-norms given by [11, 12, 19] each are an extension of a t-norm on the subinterval $[\alpha, 1]$. The constructions in [8, 21] can be viewed as extensions of a t-norm on $[\alpha, 1]$, or as extensions of a t-norm on $[0, \alpha]$. But anyway, these methods need to use prefixed t-norms on both subintervals simultaneously. Being different from the above methods, Aşıcı and Mesiar [6] proposed a new construction by replacing a t-norm on $[\alpha, 1]$ with a t-norm on $[0, \alpha]$. Inspired by the idea behind [6] and [18], we will propose a new construction for ordinal sum of t-norms via interior operators on bounded lattices, which is different from the previous methods.

The rest of this paper is organized as follows. We recall some basic definitions concerning lattices, t-norms and so on in Section 2. In Section 3, we list some already-known constructions for ordinal sum of t-norms on bounded lattices to provide a preparation for their comparison with the method in this paper in subsequent sections. In Section 4 we propose a new ordinal sum construction and give some illustrative figures to show its differences from the previous methods. In Section 5 we generalize our method by induction to consider ordinal sums of t-norms on a finite family of intervals whose endpoints form a chain of the underlying lattice. Finally, we summarize the main results and outline further work in Section 6.

2 Preliminaries

In this section, we recall some basic definitions and notations concerning bounded lattices, t-norms, t-conorms, interior operators and closure operators, which are necessary for the sequent sections. For simplicity, we will denote by $\boldsymbol{L}$ a bounded lattice $(L, \wedge, \vee, 0, 1)$ throughout the paper, unless otherwise explicitly stated.

Definition 2.1. [9] *Let $\boldsymbol{L}$ be a bounded lattice and $x, y \in L$. Then x and y are said to be comparable if either $x \leq y$ or $y \leq x$, denoted by $x \nparallel y$. Otherwise, x and y are said to be incomparable, denoted by $x \parallel y$. For $\alpha \in L \backslash \{0, 1\}$, we denote by $I_\alpha = \{x \in L \mid x \parallel \alpha\}$.*

Definition 2.2. [9] *Let $\boldsymbol{L}$ be a bounded lattice and $\alpha, \beta \in L$ with $\alpha \leq \beta$. Then the closed subinterval $[\alpha, \beta]$ is defined as $[\alpha, \beta] = \{x \in L \mid \alpha \leq x \leq \beta\}$. Similarly, the open subinterval (α, β) is defined as $(\alpha, \beta) = \{x \in L \mid \alpha < x < \beta\}$. The half-open subintervals $(\alpha, \beta]$ and $[\alpha, \beta)$ are also defined in the usual way.*

Definition 2.3. [27] *Let $\boldsymbol{L}$ be a bounded lattice. A t-norm on $\boldsymbol{L}$ is a binary operation $T : L^2 \longrightarrow L$ which is commutative, associative, non-decreasing with respect to both variables and has 1 as its neutral element.*

Definition 2.4. [27] *Let $\boldsymbol{L}$ be a bounded lattice. A t-conorm on $\boldsymbol{L}$ is a binary operation $S : L^2 \longrightarrow L$ which is commutative, associative, non-decreasing with respect to both variables and has 0 as its neutral element.*

It is obvious that T_W is the smallest t-norm on L and $T_\wedge$ is the greatest t-norm on L under the pointwise order, where

$$T_W(x, y) = \begin{cases} x \wedge y, & 1 \in \{x, y\}, \\ 0, & \textit{otherwise}. \end{cases}$$

and $T_\wedge(x, y) = x \wedge y$ for all $x, y \in L$, respectively.

Definition 2.5. [18] *Let $\boldsymbol{L}$ be a bounded lattice. An interior operator on $\boldsymbol{L}$ is a unary operation $\gamma : L \longrightarrow L$ which satisfies the following conditions for all $x, y \in L$:*

(i) $\gamma(1) = 1$,

(ii) $\gamma(x) \leq x$,

(iii) $\gamma(\gamma(x)) = \gamma(x)$,

(iv) $\gamma(x \wedge y) = \gamma(x) \wedge \gamma(y)$.

Remark 2.6. *We get $\gamma(x) = 1$ if and only if $x = 1$ by Definition 2.5(i) and (ii). It is worth noting that the interior operators given by* [9] *are just unary self-maps of L which are order-preserving, idempotent and contractive. As done in* [18]*, we use the stronger version of interior operators in this paper, that is, they preserve additionally the meet operations. The reason for this is determined by the proof of the subsequent theorems. Obviously, meet-preservation implies the order-preservation for interior operators, but not vice versa. For a complete characterization of interior operators in the broad sense we refer to* [33].

Dually, we adopt the stronger version of closure operators as used in [18].

Definition 2.7. [18] *Let $\boldsymbol{L}$ be a bounded lattice. A closure operator on $\boldsymbol{L}$ is a unary operation $\delta : L \longrightarrow L$ which satisfies the following conditions for all $x, y \in L$:*

(i) $\delta(0) = 0$,

(ii) $x \leq \delta(x)$,

(iii) $\delta(\delta(x)) = \delta(x)$,

(iv) $\delta(x \vee y) = \delta(x) \vee \delta(y)$.

3 Extant ordinal sums of t-norms on bounded lattices

As reviewed in Introduction, there are many different methods to construct t-norms on the real unit interval $[0, 1]$, among which the ordinal sum construction is the most powerful one in the sense that every continuous t-norm on the real unit interval $[0, 1]$ has a unique ordinal sum representation as the following theorem shows.

Theorem 3.1. [27] *Let $(I, \leq)$ with $I \neq \emptyset$ be a linearly ordered set, $\{T_i\}_{i\in I}$ be a family of t-norms and $(]\alpha_i, \beta_i[)_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0, 1]$. Then the binary operation $T : [0, 1]^2 \longrightarrow [0, 1]$, defined by*

$$T(x, y) = \begin{cases} \alpha_i + (\beta_i - \alpha_i) \cdot T_i(\frac{x-\alpha_i}{\beta_i-\alpha_i}, \frac{y-\alpha_i}{\beta_i-\alpha_i}), & (x, y) \in [\alpha_i, \beta_i]^2, \\ \min(x, y), & otherwise. \end{cases}$$

is a t-norm on $[0, 1]$. In this case, T is called an ordinal sum of $\{T_i\}_{i\in I}$ and denoted by $T = \{\langle\alpha_i, \beta_i, T_i\rangle\}_{i\in I}$. Conversely, every continuous t-norm T on $[0, 1]$ can be uniquely represented as an ordinal sum $T = \{\langle\alpha_i, \beta_i, T_i\rangle\}_{i\in I}$ of an at most countable family of continuous Archimedean t-norms $\{T_i\}_{i\in I}$ on a family of intervals $(]\alpha_i, \beta_i[)_{i\in I}$ with the same index I.

However, the ordinal sums of t-norms on bounded lattices have various representations. In order to illustrate in detail the differences between the method proposed in this paper and the methods in the literatures in next section, we will recall and list related ordinal sums of t-norms on bounded lattices in this section.

Saminger proposed first the ordinal sum on bounded lattices according to Theorem 3.1.

Definition 3.2. [34] *Let $\boldsymbol{L}$ be a bounded lattice, $\{]\alpha_i, \beta_i[\}_{i\in I}$ be a family of non-empty, pairwise disjoint open subintervals of L and $\{T_i\}_{i\in I}$ be a family of t-norms on these subintervals, where I is an index set. Then the binary operation $T : L^2 \longrightarrow L$ is called an ordinal sum of $\{T_i\}_{i\in I}$, denoted still by $T = \{\langle\alpha_i, \beta_i, T_i\rangle\}_{i\in I}$, where*

$$T(x, y) = \begin{cases} T_i(x, y), & (x, y) \in [\alpha_i, \beta_i]^2, \\ x \wedge y, & otherwise. \end{cases}$$

It is easy to see that the ordinal sum of t-norms defined above is not necessarily a t-norm on L. Therefore, Saminger gave the following constraints on the ordinal sum by making it always be a t-norm, which is independent of the t-norm summands.

Proposition 3.3. [34] *Let $\boldsymbol{L}$ be a bounded lattice with $\alpha \in L$, T_1 and T_2 be two t-norms on $[0,\alpha]$ and $[\alpha,1]$ of L, respectively. Then the following are equivalent:*

(*i*) *The binary operation $T_{(1)} : L^2 \longrightarrow L$ is a t-norm for arbitrary T_1 and T_2, where*

$$T_{(1)}(x,y) = \begin{cases} T_1(x,y), & (x,y) \in [0,\alpha]^2, \\ T_2(x,y), & (x,y) \in [\alpha,1]^2, \\ x \wedge y, & otherwise. \end{cases} \tag{1}$$

(*ii*) *For all $x \in L$, it holds that*

(*a*) *if $x \parallel \alpha$, then $x \parallel u$ for all $u \in (0,\alpha]$.*

(*b*) *if $x \parallel \alpha$, then $x \parallel u$ for all $u \in [\alpha,1)$.*

In order to make the ordinal sum construction be applicable to arbitrary bounded lattices, Ertuğrul et al made a modification as follows.

Theorem 3.4. [19] *Let $\boldsymbol{L}$ be a bounded lattice with $\alpha \in L\backslash\{0,1\}$ and T_2 be a t-norm on $[\alpha,1]$. Then the binary operation $T_{(2)} : L^2 \longrightarrow L$ is a t-norm on L, where*

$$T_{(2)}(x,y) = \begin{cases} T_2(x,y), & (x,y) \in [\alpha,1)^2, \\ x \wedge y, & 1 \in \{x,y\}, \\ x \wedge y \wedge \alpha, & otherwise. \end{cases} \tag{2}$$

Çaylı replaced $x \wedge y \wedge \alpha$ in the formula (2) with 0 and proposed a new construction on the basis of Theorem 3.4. And the new construction yields the smallest t-norm on L in the sense that its restriction on $[\alpha,1]$ coincides with the prefixed t-norm T_2 on $[\alpha,1]$, as next theorem shows.

Theorem 3.5. [11] *Let $\boldsymbol{L}$ be a bounded lattice with $\alpha \in L\backslash\{0,1\}$ and T_2 be a t-norm on $[\alpha,1]$. Then the binary operation $T_{(3)} : L^2 \longrightarrow L$ is a t-norm on L, where*

$$T_{(3)}(x,y) = \begin{cases} T_2(x,y), & (x,y) \in [\alpha,1)^2, \\ x \wedge y, & 1 \in \{x,y\}, \\ 0, & otherwise. \end{cases} \tag{3}$$

Çaylı made a further modification by combining Theorems 3.4 and 3.5. The author made a detailed partition of the domain and obtained a new construction as shown in the following theorem.

Theorem 3.6. [12] *Let $\boldsymbol{L}$ be a bounded lattice with $\alpha \in L\backslash\{0,1\}$ and T_2 be a t-norm on $[\alpha,1]$. Then the binary operation $T_{(4)} : L^2 \longrightarrow L$ is a t-norm on L, where*

$$T_{(4)}(x,y) = \begin{cases} T_2(x,y), & (x,y) \in [\alpha,1)^2, \\ 0, & (x,y) \in [0,\alpha)^2 \cup [0,\alpha) \times I_\alpha \cup I_\alpha \times [0,\alpha) \cup I_\alpha \times I_\alpha, \\ x \wedge y, & 1 \in \{x,y\}, \\ x \wedge y \wedge \alpha, & otherwise. \end{cases} \tag{4}$$

By observing Theorems 3.4, 3.5 and 3.6, we find that they are all obtained by reconstructing the function values on the complementary set of $[\alpha,1)^2$ after fixing a t-norm on $[\alpha,1]$. Differently from the previous methods, Aşıcı and Mesiar introduced new constructions based on t-norms on sublattices of the form $[0,\alpha]$.

Theorem 3.7. [6] *Let $\boldsymbol{L}$ be a bounded lattice with $\alpha \in L\backslash\{0,1\}$ satisfying $x \parallel y$ for all $x \in I_\alpha$ and for all $y \in (0,\alpha]$, and T_1 be a t-norm on $[0,\alpha]$. Then the binary operation $T_{(5)} : L^2 \longrightarrow L$ is a t-norm on L, where*

$$T_{(5)}(x,y) = \begin{cases} T_1(x,y), & (x,y) \in [0,\alpha]^2, \\ 0, & (x,y) \in [0,\alpha) \times I_\alpha \cup I_\alpha \times [0,\alpha) \cup I_\alpha \times I_\alpha \\ & \cup [\alpha,1) \times I_\alpha \cup I_\alpha \times [\alpha,1), \\ x \wedge y, & otherwise. \end{cases} \tag{5}$$

Theorem 3.8. [2] *Let* $\mathbf{L}$ *be a bounded lattice with* $\alpha \in L\backslash\{0,1\}$ *satisfying* $x \parallel y$ *for all* $x \in I_\alpha$ *and for all* $y \in (0,\alpha]\cup[\alpha,1)$*, and* T_1 *be a t-norm on* $[0,\alpha]$*. Then the binary operation* $T_{(6)} : L^2 \longrightarrow L$ *is a t-norm on* L*, where*

$$T_{(6)}(x,y) = \begin{cases} T_1(x,y), & (x,y) \in [0,\alpha]^2, \\ 0, & (x,y) \in [0,\alpha) \times I_\alpha \cup I_\alpha \times [0,\alpha) \\ & \cup\, [\alpha,1) \times I_\alpha \cup I_\alpha \times [\alpha,1), \\ \alpha, & (x,y) \in [\alpha,1)^2, \\ x \wedge y, & \textit{otherwise.} \end{cases} \tag{6}$$

Based on Theorem 3.8, Aşıcı replaced α with $x \wedge y$ in the partition of $[\alpha,1)^2$ and proposed another method in [3]. We refer readers to [3] for details.

Theorem 3.9. [5] *Let* $\mathbf{L}$ *be a bounded lattice with* $\alpha \in L\backslash\{0,1\}$ *satisfying* $x \parallel y$ *for all* $x \in I_\alpha$ *and for all* $y \in (0,\alpha]$*,* $x < z$ *for all* $x \in I_\alpha$ *and for all* $z \in (\alpha,1]$*, and* T_1 *be a t-norm on* $[0,\alpha]$*. Then the binary operation* $T_{(7)} : L^2 \longrightarrow L$ *is a t-norm on* L*, where*

$$T_{(7)}(x,y) = \begin{cases} T_1(x,y), & (x,y) \in [0,\alpha]^2, \\ y, & (x,y) \in (\alpha,1] \times I_\alpha, \\ x, & (x,y) \in I_\alpha \times (\alpha,1], \\ 0, & (x,y) \in [0,\alpha] \times I_\alpha \cup I_\alpha \times [0,\alpha] \cup I_\alpha \times I_\alpha, \\ x \wedge y, & \textit{otherwise.} \end{cases} \tag{7}$$

Theorems 3.4, 3.5 and 3.6 only extend a t-norm on $[\alpha,1]$ to L, while Theorems 3.7, 3.8, 3.9 only extend a t-norm on $[0,\alpha]$ to L, respectively. Indeed, we can construct ordinal sum t-norms based on t-norms on these two types of subintervals of L.

Theorem 3.10. [21] *Let* $\mathbf{L}$ *be a bounded lattice with* $\alpha \in L\backslash\{0,1\}$ *and* T_1*,* T_2 *be two t-norms on* $[0,\alpha]$ *and* $[\alpha,1]$ *of* L*, respectively. Then the binary operation* $T_{(8)} : L^2 \longrightarrow L$ *is a t-norm on* L*, where*

$$T_{(8)}(x,y) = \begin{cases} T_1(x,y), & (x,y) \in [0,\alpha)^2, \\ T_2(x,y), & (x,y) \in [\alpha,1)^2, \\ x \wedge y, & (x,y) \in [0,\alpha) \times [\alpha,1) \cup [\alpha,1) \times [0,\alpha) \\ & \cup\, L \times \{1\} \cup \{1\} \times L, \\ T_1(x \wedge \alpha, y \wedge \alpha), & \textit{otherwise.} \end{cases} \tag{8}$$

In a newly published paper [8], Aşıcı and Mesiar proposed an ordinal sum t-norm by adding some restrictions to the underlying lattice in Theorem 3.10, see [8, Theorem 3.6]. An observation shows that it is a special case of Theorem 3.10 when the underlying lattice satisfies these constraints.

The constructions in Theorems 3.4, 3.5, 3.6, 3.7, 3.8, 3.9 and 3.10 all can be generalized by recursion to ordinal sums of finitely many t-norms on bounded lattices (readers can refer to [19, 11, 12, 6, 2, 5, 21]). The following theorem turns to construct ordinal sum t-norms via interior operators which provides a new view for future research.

Theorem 3.11. [18] *Let* L *be a meet semilattice and assume that there exists* $\alpha_1, \cdots, \alpha_n \in L\backslash\{0,1\}$ *such that* $\alpha_1 < \cdots < \alpha_n$ *and* $M = \bigcup_{i=0}^{n}[\alpha_i,\alpha_{i+1}] \subset L$*, where* $\alpha_0 = 0$*,* $\alpha_{n+1} = 1$*. If* $\gamma : L \longrightarrow L$ *is an interior operator on* L *such that* $\gamma(L) \subseteq M$*,* α_i *is a fixed point of* γ *and* T_i *is a t-norm on* $J_{i+} = \gamma(L) \cap [\alpha_i,\alpha_{i+1}]$ *for* $i = 0, \cdots, n$*, then the binary operation* $T_{(9)} : L^2 \longrightarrow L$*, called the* γ*-ordinal sum of t-norms* $T_0, \cdots, T_n$ *and defined by*

$$T_{(9)}(x,y) = \begin{cases} T_i(\gamma(x),\gamma(y)), & (\gamma(x),\gamma(y)) \in J_i^2, \\ \gamma(x) \wedge \gamma(y), & (\gamma(x),\gamma(y)) \in J_i \times J_j, i \neq j, \\ x \wedge y, & \textit{otherwise.} \end{cases} \tag{9}$$

is a t-norm on L*, where* $J_i = \gamma(L) \cap [\alpha_i,\alpha_{i+1})$*.*

Theorem 3.11 provides a construction for the ordinal sum of finitely many t-norms on a meet semilattice. And a construction for ordinal sum of countably many t-norms on a bounded meet semilattice is shown in the following theorem.

Theorem 3.12. [33] *Let* $(L,\leq,\wedge,0,1)$ *be a bounded meet semilattice,* $\{(L_i,\leq,\wedge_i,\alpha_i,\beta_i)\}_{i\in\mathbb{Z}}$ *be a family of bounded meet semilattices with* $\beta_i \leq \alpha_{i+1}$ *and* $\bigcup_{i\in\mathbb{Z}} L_i \subseteq L$*, and* $\{T_i\}_{i\in\mathbb{Z}}$ *be a family of t-norms on* $\{L_i\}_{i\in\mathbb{Z}}$*, where* $\mathbb{Z}$ *is the set of* *all integers. Suppose that* $\tilde{L} = (\bigcap_{i\in\mathbb{Z}}[\beta_i, 1]) \cup (\bigcup_{i\in\mathbb{Z}} L_i) \cup (\bigcap_{i\in\mathbb{Z}}[0, \alpha_i])$ *is an interior range of some associated interior operator* γ*. Then the binary operation* $T_{(10)} = \{\langle L_i, T_i\rangle\}_{i\in\mathbb{Z}} : L^2 \longrightarrow L$*, defined by*

$$T_{(10)}(x,y) = \begin{cases} T_i(\gamma(x),\gamma(y)), & (\gamma(x),\gamma(y)) \in L_i^2 \text{ and } 1 \notin \{x,y\}, \\ x \wedge y, & 1 \in \{x,y\}, \\ \gamma(x) \wedge \gamma(y), & \text{otherwise.} \end{cases} \tag{10}$$

is a t-norm on L*.*

4 A new extension of t-norms on subintervals $[0, \alpha]$ to the entire bounded lattices

In this section, we will stand in the line of Theorem 3.7 to go a further step toward constructing ordinal sum t-norms by means of interior operators. We will show that our new method is completely different from the methods in previous section.

Lemma 4.1. *Let* $\boldsymbol{L}$ *be a bounded lattice with* $\alpha \in L\backslash\{0,1\}$ *and* $\gamma : L \longrightarrow L$ *be an interior operator on* L*. If* $x \parallel y$ *for all* $x \in I_\alpha$ *and for all* $y \in (0, \alpha]$*, then* $\gamma(x) \wedge \gamma(z) \in I_\alpha \cup \{0\}$ *for all* $x \in I_\alpha$ *and for all* $z \in (\alpha, 1)$*.*

Proof. $\forall x \in I_\alpha, \forall z \in (\alpha, 1)$, we have $\gamma(x) \wedge \gamma(z) \leq x \wedge z \leq x$. Then $\gamma(x) \wedge \gamma(z) \notin (0, \alpha] \cup [\alpha, 1]$ from the assumption; otherwise, it will contradict with $x \in I_\alpha$. So $\gamma(x) \wedge \gamma(z) \in I_\alpha \cup \{0\}$ for all $x \in I_\alpha$ and for all $z \in (\alpha, 1)$. □

Theorem 4.2. *Let* $\boldsymbol{L}$ *be a bounded lattice with* $\alpha \in L\backslash\{0,1\}$*,* V *be a t-norm on* $[0, \alpha]$ *and* γ *be an interior operator on* L*. Then the binary operation* $T_{(11)} : L^2 \longrightarrow L$ *defined by*

$$T_{(11)}(x,y) = \begin{cases} V(x,y), & (x,y) \in [0,\alpha]^2, \\ \gamma(x) \wedge \gamma(y), & (x,y) \in (\alpha,1) \times I_\alpha \cup I_\alpha \times (\alpha,1), \\ 0, & (x,y) \in [0,\alpha] \times I_\alpha \cup I_\alpha \times [0,\alpha] \cup I_\alpha \times I_\alpha, \\ x \wedge y, & \text{otherwise.} \end{cases} \tag{11}$$

is a t-norm on L *if and only if* $x \parallel y$ *for all* $x \in I_\alpha$ *and for all* $y \in (0, \alpha]$*.*

Proof. ($\Longrightarrow$) Suppose that $T_{(11)}$ is a t-norm on L, we need to prove $x \parallel y$ for all $x \in I_\alpha$ and for all $y \in (0, \alpha]$.

If $y = \alpha$, then it is trivial that $x \parallel y$ for all $x \in I_\alpha$. So we only need to show that $x \parallel y$ for all $x \in I_\alpha$ and for all $y \in (0, \alpha)$. Suppose to the contrary that there exist $x_0 \in I_\alpha$ and $y_0 \in (0, \alpha)$ satisfying $x_0 \nparallel y_0$. If it was this case, then there would be $x_0 > y_0$ because of $x_0 \in I_\alpha$. According to the definition of $T_{(11)}$, we have $T_{(11)}(x_0, z) = 0 < y_0 = V(y_0, z) = T_{(11)}(y_0, z)$ for $z = \alpha$. This is in contradiction to the monotonicity of $T_{(11)}$. Therefore, $x \parallel y$ for all $x \in I_\alpha$ and for all $y \in (0, \alpha]$.

($\Longleftarrow$) Assume that $x \parallel y$ for all $x \in I_\alpha$ and for all $y \in (0, \alpha]$, we need to prove that $T_{(11)}$ is a t-norm on L. It is obvious that $T_{(11)}$ is commutative and has the neutral element 1 by (11), then it is left to prove the monotonicity and the associativity.

(i) Monotonicity: Take any $x, y, z \in L$ such that $x \leq y$ and we need to prove $T_{(11)}(x, z) \leq T_{(11)}(y, z)$. If $x = 1$, then $y = 1$, and consequently, $T_{(11)}(x, z) = z = T_{(11)}(y, z)$; If $y = 1$, then $T_{(11)}(x, z) \leq x \wedge z \leq z = T_{(11)}(y, z)$; If $z = 1$, then $T_{(11)}(x, z) = x \leq y = T_{(11)}(y, z)$. In addition, $T_{(11)}(x, z) \leq T_{(11)}(y, z)$ is always true if x, y belong to the same range $[0, \alpha]$, $(\alpha, 1)$ or I_α. Therefore, it suffices to consider the cases where $1 \notin \{x, y, z\}$ and x, y belong to *different* ranges. We proceed the proof by distinguishing the following possible cases.

1. $x \in [0, \alpha]$, then it follows from the assumption that $y \in (\alpha, 1)$,

   1.1. $z \in [0, \alpha]$,
   $T_{(11)}(x, z) = V(x, z) \leq x \wedge z \leq z = y \wedge z = T_{(11)}(y, z)$.

   1.2. $z \in (\alpha, 1)$,
   $T_{(11)}(x, z) = x \wedge z \leq y \wedge z = T_{(11)}(y, z)$.

   1.3. $z \in I_\alpha$,
   $T_{(11)}(x, z) = 0 \leq \gamma(y) \wedge \gamma(z) = T_{(11)}(y, z)$.

2. $x \in (\alpha, 1)$, then it must be $y \in (\alpha, 1]$, which has been considered as above.
3. $x \in I_\alpha$, then in this case, it is only possible that $y \in (\alpha, 1)$,
   - 3.1. $z \in [0, \alpha]$,
     $T_{(11)}(x, z) = 0 \leq z = y \wedge z = T_{(11)}(y, z)$.
   - 3.2. $z \in (\alpha, 1)$,
     $T_{(11)}(x, z) = \gamma(x) \wedge \gamma(z) \leq x \wedge z \leq y \wedge z = T_{(11)}(y, z)$.
   - 3.3. $z \in I_\alpha$,
     $T_{(11)}(x, z) = 0 \leq \gamma(y) \wedge \gamma(z) = T_{(11)}(y, z)$.

(ii) Associativity: Lemma 4.1 plays an important role in proving the associativity and $\gamma(x \wedge \alpha) = 0$ for all $x \in I_\alpha$. In addition, the $\gamma(x \wedge \alpha) = 0$ for all $x \in I_\alpha$ will be used in Cases 2.2.3 and 3.2.2 below.

Take any $x, y, z \in L$ and we need to prove $T_{(11)}(x, T_{(11)}(y, z)) = T_{(11)}(T_{(11)}(x, y), z)$. If $1 \in \{x, y, z\}$, it is obvious that $T_{(11)}(x, T_{(11)}(y, z)) = T_{(11)}(T_{(11)}(x, y), z)$. Therefore, it suffices to consider the cases $1 \notin \{x, y, z\}$ below. We proceed with the proof by considering all possible cases according to the order relationships of x, y, z and α.

1. $x \in [0, \alpha]$,
   - 1.1. $y \in [0, \alpha]$,
     - 1.1.1. $z \in [0, \alpha]$,
       $T_{(11)}(x, T_{(11)}(y, z)) = V(x, V(y, z)) = V(V(x, y), z) = T_{(11)}(T_{(11)}(x, y), z)$.
     - 1.1.2. $z \in (\alpha, 1)$,
       $T_{(11)}(x, T_{(11)}(y, z)) = V(x, y) = T_{(11)}(T_{(11)}(x, y), z)$.
     - 1.1.3. $z \in I_\alpha$,
       $T_{(11)}(x, T_{(11)}(y, z)) = 0 = T_{(11)}(T_{(11)}(x, y), z)$.
   - 1.2. $y \in (\alpha, 1)$,
     - 1.2.1. $z \in [0, \alpha]$,
       $T_{(11)}(x, T_{(11)}(y, z)) = V(x, z) = T_{(11)}(T_{(11)}(x, y), z)$.
     - 1.2.2. $z \in (\alpha, 1)$,
       $T_{(11)}(x, T_{(11)}(y, z)) = x = T_{(11)}(T_{(11)}(x, y), z)$.
     - 1.2.3. $z \in I_\alpha$,
       $$T_{(11)}(x, T_{(11)}(y, z)) = \begin{cases} 0, & \gamma(y) \wedge \gamma(z) \in I_\alpha \\ T_{(11)}(x, 0), & \gamma(y) \wedge \gamma(z) = 0 \end{cases} = 0 = T_{(11)}(T_{(11)}(x, y), z).$$
   - 1.3. $y \in I_\alpha, z \in L \backslash \{1\}$,
     $T_{(11)}(x, T_{(11)}(y, z)) = 0 = T_{(11)}(T_{(11)}(x, y), z)$.
2. $x \in (\alpha, 1)$,
   - 2.1. $y \in [0, \alpha]$,
     - 2.1.1. $z \in [0, \alpha]$,
       $T_{(11)}(x, T_{(11)}(y, z)) = V(y, z) = T_{(11)}(T_{(11)}(x, y), z)$.
     - 2.1.2. $z \in (\alpha, 1)$,
       $T_{(11)}(x, T_{(11)}(y, z)) = y = T_{(11)}(T_{(11)}(x, y), z)$.
     - 2.1.3. $z \in I_\alpha$,
       $T_{(11)}(x, T_{(11)}(y, z)) = 0 = T_{(11)}(T_{(11)}(x, y), z)$.
   - 2.2. $y \in (\alpha, 1)$,
     - 2.2.1. $z \in [0, \alpha]$,
       $T_{(11)}(x, T_{(11)}(y, z)) = z = T_{(11)}(T_{(11)}(x, y), z)$.
     - 2.2.2. $z \in (\alpha, 1)$,
       $T_{(11)}(x, T_{(11)}(y, z)) = x \wedge y \wedge z = T_{(11)}(T_{(11)}(x, y), z)$.
     - 2.2.3. $z \in I_\alpha$,
       $$T_{(11)}(x, T_{(11)}(y, z)) = \begin{cases} \gamma(x) \wedge \gamma(y) \wedge \gamma(z), & \gamma(y) \wedge \gamma(z) \in I_\alpha \\ 0, & \gamma(y) \wedge \gamma(z) = 0 \end{cases} = \gamma(x) \wedge \gamma(y) \wedge \gamma(z).$$
       $$T_{(11)}(T_{(11)}(x, y), z) = \begin{cases} \gamma(x) \wedge \gamma(y) \wedge \gamma(z), & x \wedge y \in (\alpha, 1) \\ 0, & x \wedge y = \alpha \end{cases} = \gamma(x) \wedge \gamma(y) \wedge \gamma(z).$$

2.3. $y \in I_\alpha$,

2.3.1. $z \in [0,\alpha]$,

$$T_{(11)}(x, T_{(11)}(y,z)) = 0 = \begin{cases} 0, & \gamma(x) \wedge \gamma(y) \in I_\alpha \\ T_{(11)}(0,z), & \gamma(x) \wedge \gamma(y) = 0 \end{cases} = T_{(11)}(T_{(11)}(x,y), z).$$

2.3.2. $z \in (\alpha, 1)$,

$$T_{(11)}(x, T_{(11)}(y,z)) = \begin{cases} \gamma(x) \wedge \gamma(y) \wedge \gamma(z), & \gamma(y) \wedge \gamma(z) \in I_\alpha \\ 0, & \gamma(y) \wedge \gamma(z) = 0 \end{cases} = \gamma(x) \wedge \gamma(y) \wedge \gamma(z).$$

$$T_{(11)}(T_{(11)}(x,y), z) = \begin{cases} \gamma(x) \wedge \gamma(y) \wedge \gamma(z), & \gamma(x) \wedge \gamma(y) \in I_\alpha \\ 0, & \gamma(x) \wedge \gamma(y) = 0 \end{cases} = \gamma(x) \wedge \gamma(y) \wedge \gamma(z).$$

2.3.3. $z \in I_\alpha$,

$$T_{(11)}(x, T_{(11)}(y,z)) = 0 = \begin{cases} 0, & \gamma(x) \wedge \gamma(y) \in I_\alpha \\ T_{(11)}(0,z), & \gamma(x) \wedge \gamma(y) = 0 \end{cases} = T_{(11)}(T_{(11)}(x,y), z).$$

3. $x \in I_\alpha$,

3.1. $y \in [0,\alpha], z \in L\backslash\{1\}$,
$T_{(11)}(x, T_{(11)}(y,z)) = 0 = T_{(11)}(T_{(11)}(x,y), z)$.

3.2. $y \in (\alpha, 1)$,

3.2.1. $z \in [0,\alpha]$,

$$T_{(11)}(x, T_{(11)}(y,z)) = 0 = \begin{cases} 0, & \gamma(x) \wedge \gamma(y) \in I_\alpha \\ T_{(11)}(0,z), & \gamma(x) \wedge \gamma(y) = 0 \end{cases} = T_{(11)}(T_{(11)}(x,y), z).$$

3.2.2. $z \in (\alpha, 1)$,

$$T_{(11)}(x, T_{(11)}(y,z)) = \begin{cases} \gamma(x) \wedge \gamma(y) \wedge \gamma(z), & y \wedge z \in (\alpha, 1) \\ 0, & y \wedge z = \alpha \end{cases} = \gamma(x) \wedge \gamma(y) \wedge \gamma(z).$$

$$T_{(11)}(T_{(11)}(x,y), z) = \begin{cases} \gamma(x) \wedge \gamma(y) \wedge \gamma(z), & \gamma(x) \wedge \gamma(y) \in I_\alpha \\ 0, & \gamma(x) \wedge \gamma(y) = 0 \end{cases} = \gamma(x) \wedge \gamma(y) \wedge \gamma(z).$$

3.2.3. $z \in I_\alpha$,

$$T_{(11)}(x, T_{(11)}(y,z)) = \begin{cases} 0, & \gamma(y) \wedge \gamma(z) \in I_\alpha \\ T_{(11)}(x,0), & \gamma(y) \wedge \gamma(z) = 0 \end{cases} = 0.$$

$$T_{(11)}(T_{(11)}(x,y), z) = \begin{cases} 0, & \gamma(x) \wedge \gamma(y) \in I_\alpha \\ T_{(11)}(0,z), & \gamma(x) \wedge \gamma(y) = 0 \end{cases} = 0.$$

3.3. $y \in I_\alpha, z \in L\backslash\{1\}$,
$T_{(11)}(x, T_{(11)}(y,z)) = 0 = T_{(11)}(T_{(11)}(x,y), z)$.

So, we have the fact that $T_{(11)}$ is a t-norm on L. □

By duality, we have the following results for t-conorms.

Lemma 4.3. *Let $\boldsymbol{L}$ be a bounded lattice with $\alpha \in L\backslash\{0,1\}$ and $\delta : L \longrightarrow L$ be a closure operator on L. If $x \parallel y$ for all $x \in I_\alpha$ and for all $y \in [\alpha, 1)$, then $\delta(x) \vee \delta(z) \in I_\alpha \cup \{1\}$ for all $x \in I_\alpha$ and for all $z \in (0,\alpha)$.*

Theorem 4.4. *Let $\boldsymbol{L}$ be a bounded lattice with $\alpha \in L\backslash\{0,1\}$, W be a t-conorm on $[\alpha, 1]$ and δ be a closure operator on L. Then the binary operation $S : L^2 \longrightarrow L$ defined by*

$$S(x,y) = \begin{cases} W(x,y), & (x,y) \in [\alpha,1]^2, \\ \delta(x) \vee \delta(y), & (x,y) \in (0,\alpha) \times I_\alpha \cup I_\alpha \times (0,\alpha), \\ 1, & (x,y) \in [\alpha,1] \times I_\alpha \cup I_\alpha \times [\alpha,1] \cup I_\alpha \times I_\alpha, \\ x \vee y, & \textit{otherwise}. \end{cases} \tag{12}$$

is a t-conorm on L if and only if $x \parallel y$ for all $x \in I_\alpha$ and for all $y \in [\alpha, 1)$.

Corollary 4.5.

(*i*) *Take the interior operator* γ *defined by*

$$\gamma(x) = \begin{cases} 1, & x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then the t-norm $T_{(11)}$ *given by Theorem 4.2 reduces to* $T_{(5)}$ *in Theorem 3.7. Hence, Theorem 4.2 extends Theorem 3.7.*

(*ii*) *Dually, take the closure operator* δ *be*

$$\delta(x) = \begin{cases} 0, & x = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then the t-conorm S *in Theorem 4.4 reduces to the t-conorm given in* [6, Theorem 3.8]*. So Theorem 4.4 is a generalization of* [6, Theorem 3.8].

We will illustrate that Theorem 4.2 is fundamentally different from those ordinal sum t-norm constructions in Section 3 by comparing their corresponding structures of t-norms. We need to require the underlying bounded lattices to satisfy simultaneously the conditions of theorems under consideration and assume $I_\alpha \neq \emptyset$ when drawing the structures.

Remark 4.6.

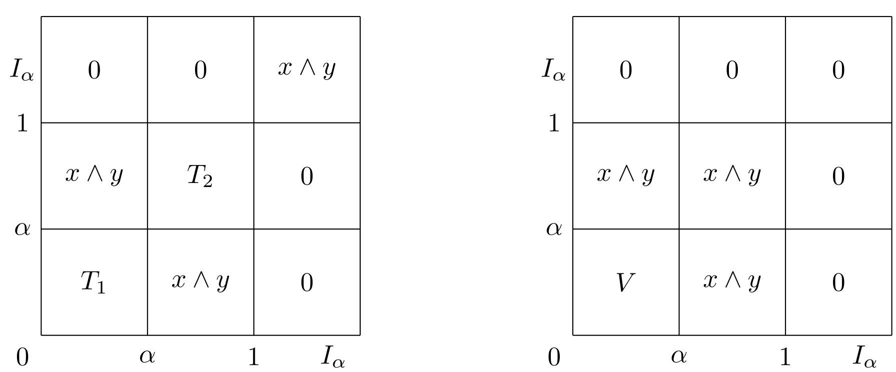
(*i*) *The comparison of Theorem 4.2 and Proposition 3.3:*
Let $\boldsymbol{L}$ *be a bounded lattice satisfying the condition of Proposition 3.3 and the condition of Theorem 4.2 simultaneously, which are copied as follows:*

(*a*) T_1 *and* T_2 *are two respective t-norms on* $[0, \alpha]$ *and* $[\alpha, 1]$ *of* L *in Proposition 3.3, and* V *is a t-norm on* $[0, \alpha]$ *of* L *in Theorem 4.2.*

(*b*) $\forall x \in L$*, if* $x \parallel \alpha$*, then* $x \parallel y$ *for all* $y \in (0, \alpha]$.

(*c*) $\forall x \in L$*, if* $x \parallel \alpha$*, then* $x \parallel y$ *for all* $y \in [\alpha, 1)$.

From (b), (c) and $\gamma(x) \leq x$*, we find that* $x \wedge y = 0$ *and* $\gamma(x) \wedge \gamma(y) = 0$ *for all* $(x, y) \in I_\alpha \times ((0, \alpha] \cup [\alpha, 1)) \cup ((0, \alpha] \cup [\alpha, 1)) \times I_\alpha$*. Then we get the structures of* $T_{(1)}$ *and* $T_{(11)}$ *as shown in Figure 1.*



Proposition 3.3 $T_{(1)}$ Theorem 4.2 $T_{(11)}$

Figure 1: The structures of $T_{(1)}$ and $T_{(11)}$

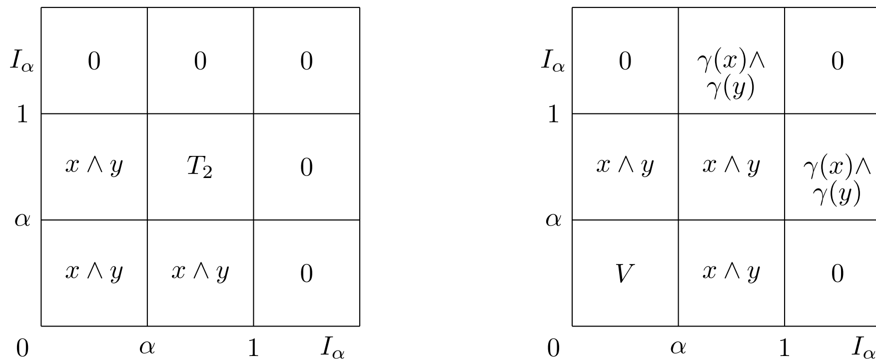
It is clear that $\forall (x, x) \in I_\alpha \times I_\alpha$*,* $T_{(1)}(x, x) = x \neq 0 = T_{(11)}(x, x)$*. Therefore, Theorem 4.2 is essentially different from Proposition 3.3.*

(*ii*) *The comparison of Theorem 4.2 and Theorem 3.4:*
Let $\boldsymbol{L}$ *be a bounded lattice satisfying the condition of Theorem 3.4 and the condition of Theorem 4.2 simultaneously, which are copied as follows:*

(*a*) T_2 *is a t-norm on* $[\alpha, 1]$ *of* L *in Theorem 3.4 and* V *is a t-norm on* $[0, \alpha]$ *of* L *in Theorem 4.2.*

(b) $\forall x \in L$, *if* $x \parallel \alpha$, *then* $x \parallel y$ *for all* $y \in (0, \alpha]$.

We find that $x \wedge y \wedge \alpha = 0$ *for all* $(x, y) \in I_\alpha \times L \backslash \{1\} \cup L \backslash \{1\} \times I_\alpha$ *from* (b). *Then we get the structures of* $T_{(2)}$ *and* $T_{(11)}$ *as shown in Figure 2.*



Theorem 3.4 $T_{(2)}$ Theorem 4.2 $T_{(11)}$

Figure 2: The structures of $T_{(2)}$ and $T_{(11)}$

We discover that it may happen for some $(x, y) \in I_\alpha \times (\alpha, 1)$, $T_{(2)}(x, y) = 0 \neq \gamma(x) \wedge \gamma(y) = T_{(11)}(x, y)$. *On the other hand, it holds that* $T_{(2)} = T_{(11)}$ *if* $T_2 = V = T_\wedge$ *and*

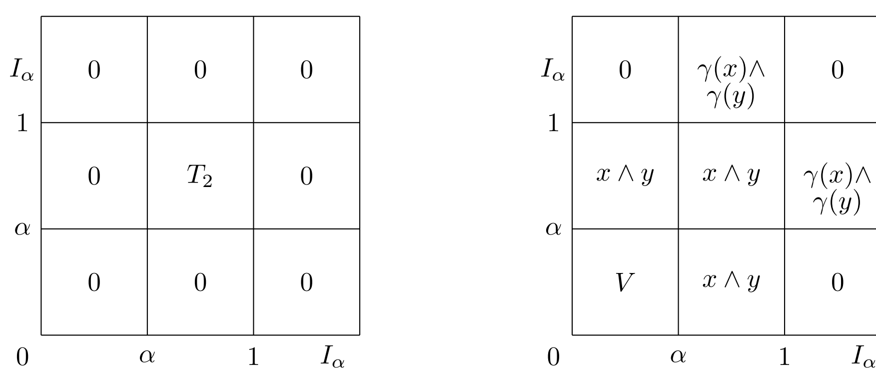
$$\gamma(x) = \begin{cases} 1, & x = 1, \\ x \wedge \alpha, & \textit{otherwise}. \end{cases}$$

Therefore, Theorem 4.2 is essentially different from Theorem 3.4.

(iii) *The comparison of Theorem 4.2 and Theorem 3.5:*
Let $\boldsymbol{L}$ *be a bounded lattice satisfying the condition of Theorem 3.5 and the condition of Theorem 4.2 simultaneously, which are copied as follows:*

(a) T_2 *is a t-norm on* $[\alpha, 1]$ *of* L *in Theorem 3.5 and* V *is a t-norm on* $[0, \alpha]$ *of* L *in Theorem 4.2.*

(b) $\forall x \in L$, *if* $x \parallel \alpha$, *then* $x \parallel y$ *for all* $y \in (0, \alpha]$.



Theorem 3.5 $T_{(3)}$ Theorem 4.2 $T_{(11)}$

Figure 3: The structures of $T_{(3)}$ and $T_{(11)}$

It is obvious that $\forall (x, y) \in (0, \alpha) \times [\alpha, 1)$, $T_{(3)}(x, y) = 0 \neq x = T_{(11)}(x, y)$. *An important thing to note here is that* $T_{(3)} = T_{(11)}$ *if the interval* $[0, \alpha)$ *only has one element 0,* $T_2 = T_\wedge$ *and*

$$\gamma(x) = \begin{cases} 1, & x = 1, \\ x \wedge \alpha, & \textit{otherwise}. \end{cases}$$

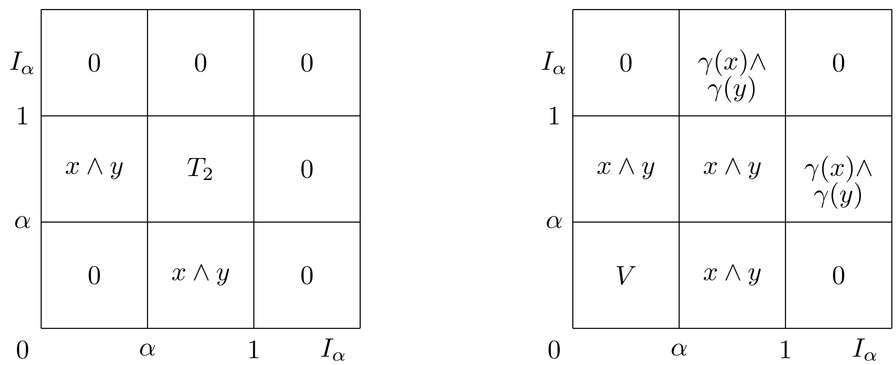
Therefore, Theorem 4.2 is essentially different from Theorem 3.5.

(*iv*) *The comparison of Theorem 4.2 and Theorem 3.6:*
Let $\boldsymbol{L}$ *be a bounded lattice satisfying the condition of Theorem 3.6 and the condition of Theorem 4.2 simultaneously, which are copied as follows:*

(*a*) T_2 *is a t-norm on* $[\alpha, 1]$ *of* L *in Theorem 3.6 and* V *is a t-norm on* $[0, \alpha]$ *of* L *in Theorem 4.2.*

(*b*) $\forall x \in L$, *if* $x \parallel \alpha$, *then* $x \parallel y$ *for all* $y \in (0, \alpha]$.

We find that $x \wedge y \wedge \alpha = 0$ *for all* $(x, y) \in I_\alpha \times [\alpha, 1) \cup [\alpha, 1) \times I_\alpha$ *from* (*b*). *Then we get the structures of* $T_{(4)}$ *and* $T_{(11)}$ *as shown in Figure 4.*



Theorem 3.6 $T_{(4)}$ Theorem 4.2 $T_{(11)}$

Figure 4: The structures of $T_{(4)}$ and $T_{(11)}$

We discover that it may happen for some $(x, y) \in I_\alpha \times (\alpha, 1)$, $T_{(4)}(x, y) = 0 \neq \gamma(x) \wedge \gamma(y) = T_{(11)}(x, y)$. *Significantly, it holds that* $T_{(4)} = T_{(11)}$ *if* $T_2 = T_\wedge$, $V = T_W$ *and*

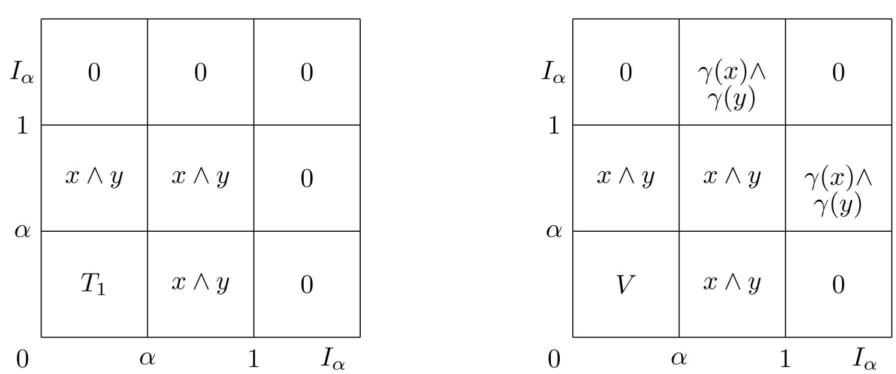
$$\gamma(x) = \begin{cases} 1, & x = 1, \\ x \wedge \alpha, & \textit{otherwise}. \end{cases}$$

Therefore, Theorem 4.2 is essentially different from Theorem 3.6.

(*v*) *The comparison of Theorem 4.2 and Theorem 3.7:*
Let $\boldsymbol{L}$ *be a bounded lattice satisfying the condition of Theorem 3.7 and the condition of Theorem 4.2 simultaneously, which are copied as follows:*

(*a*) T_1 *is a t-norm on* $[0, \alpha]$ *of* L *in Theorem 3.7 and* V *is a t-norm on* $[0, \alpha]$ *of* L *in Theorem 4.2.*

(*b*) $\forall x \in L$, *if* $x \parallel \alpha$, *then* $x \parallel y$ *for all* $y \in (0, \alpha]$.



Theorem 3.7 $T_{(5)}$ Theorem 4.2 $T_{(11)}$

Figure 5: The structures of $T_{(5)}$ and $T_{(11)}$

We find that it may happen for some $(x,y) \in I_\alpha \times (\alpha,1)$*,* $T_{(5)}(x,y) = 0 \neq \gamma(x) \wedge \gamma(y) = T_{(11)}(x,y)$*. On the other hand, it holds that* $T_{(5)} = T_{(11)}$ *if* $T_1 = V$ *and*

$$\gamma(x) = \begin{cases} 1, & x = 1, \\ x \wedge \alpha, & otherwise. \end{cases}$$

Therefore, Theorem 4.2 is essentially different from Theorem 3.7.

(*vi*) *The comparison of Theorem 4.2 and Theorem 3.8:*
Let $\boldsymbol{L}$ *be a bounded lattice satisfying the condition of Theorem 3.8 and the condition of Theorem 4.2 simultaneously, which are copied as follows:*

(*a*) T_1 *is a t-norm on* $[0,\alpha]$ *of* L *in Theorem 3.8 and* V *is a t-norm on* $[0,\alpha]$ *of* L *in Theorem 4.2.*

(*b*) $\forall x \in L$*, if* $x \parallel \alpha$*, then* $x \parallel y$ *for all* $y \in (0,\alpha]$*.*

(*c*) $\forall x \in L$*, if* $x \parallel \alpha$*, then* $x \parallel y$ *for all* $y \in [\alpha,1)$*.*

From (b), (c) and $\gamma(x) \leq x$*, we find that* $x \wedge y = 0$ *and* $\gamma(x) \wedge \gamma(y) = 0$ *for all* $(x,y) \in I_\alpha \times ((0,\alpha] \cup [\alpha,1)) \cup ((0,\alpha] \cup [\alpha,1)) \times I_\alpha$*. Then we get the structures of* $T_{(6)}$ *and* $T_{(11)}$ *as shown in Figure 6.*

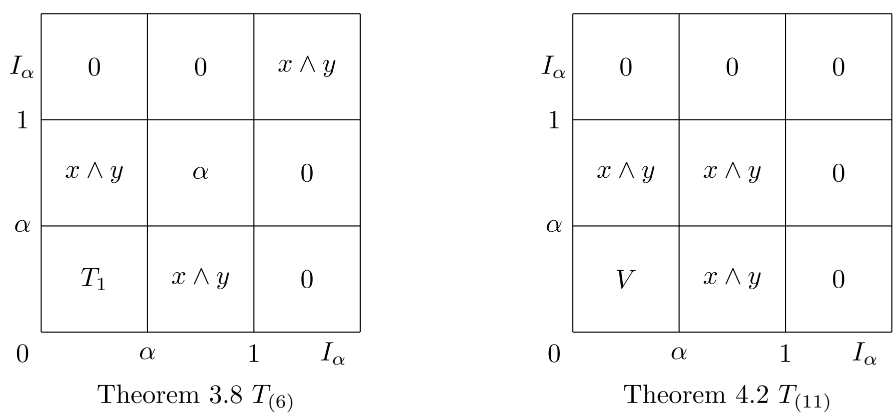


Figure 6: The structures of $T_{(6)}$ and $T_{(11)}$

It is obvious that $\forall (x,x) \in I_\alpha \times I_\alpha$*,* $T_{(6)}(x,x) = x \neq 0 = T_{(11)}(x,x)$*. Therefore, Theorem 4.2 is essentially different from Theorem 3.8. Because of the relationship between Theorem 3.8 and* [3, Theorem 2]*, Theorem 4.2 is also fundamentally different from the construction in* [3]*.*

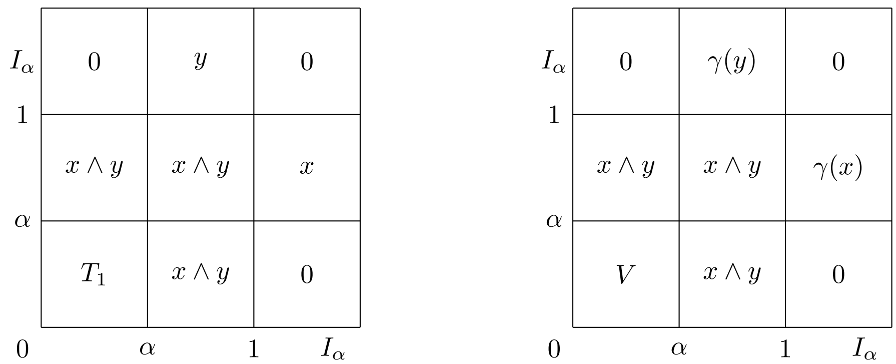
(*vii*) *The comparison of Theorem 4.2 and Theorem 3.9:*
Let $\boldsymbol{L}$ *be a bounded lattice satisfying the condition of Theorem 3.9 and the condition of Theorem 4.2 simultaneously, which are copied as follows:*

(*a*) T_1 *is a t-norm on* $[0,\alpha]$ *of* L *in Theorem 3.9 and* V *is a t-norm on* $[0,\alpha]$ *of* L *in Theorem 4.2.*

(*b*) $\forall x \in L$*, if* $x \parallel \alpha$*, then* $x \parallel y$ *for all* $y \in (0,\alpha]$*.*

(*c*) $\forall x \in L$*, if* $x \parallel \alpha$*, then* $x < z$ *for all* $z \in (\alpha,1]$*.*

We find that $T_{(11)}(x,y) = \gamma(x)$ *for all* $(x,y) \in I_\alpha \times (\alpha,1)$ $(T_{(11)}(x,y) = \gamma(y)$ *for all* $(x,y) \in (\alpha,1) \times I_\alpha)$ *from (c). Then we get the structures of* $T_{(7)}$ *and* $T_{(11)}$ *as shown in Figure 7.*



Theorem 3.9 $T_{(7)}$ Theorem 4.2 $T_{(11)}$

Figure 7: The structures of $T_{(7)}$ and $T_{(11)}$

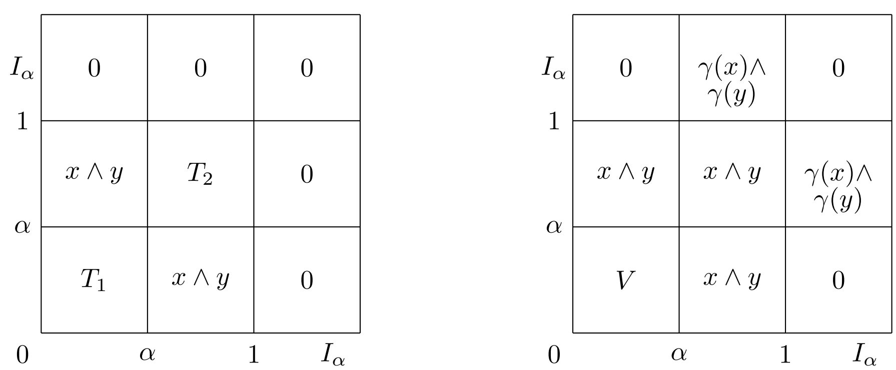
We find that it may happen for some $(x, y) \in I_\alpha \times (\alpha, 1)$, $T_{(7)}(x, y) = x \neq \gamma(x) = T_{(11)}(x, y)$. *On the other hand, it holds that* $T_{(7)} = T_{(11)}$ *if* $T_1 = V$ *and* $\gamma = id_L$.
Therefore, Theorem 4.2 is essentially different from Theorem 3.9.

(*viii*) *The comparison of Theorem 4.2 and Theorem 3.10:*
Let $\boldsymbol{L}$ *be a bounded lattice satisfying the condition of Theorem 3.10 and the condition of Theorem 4.2 simultaneously, which are copied as follows:*

(*a*) T_1 *and* T_2 *are two t-norms on* $[0, \alpha]$ *and* $[\alpha, 1]$ *of* L *in Theorem 3.10 and* V *is a t-norm on* $[0, \alpha]$ *of* L *in Theorem 4.2.*

(*b*) $\forall x \in L$, *if* $x \parallel \alpha$, *then* $x \parallel y$ *for all* $y \in (0, \alpha]$.

We find that $x \wedge \alpha = 0$ *for all* $x \in I_\alpha$ *and* $T_1(x \wedge \alpha, y \wedge \alpha) = 0$ *for all* $(x, y) \in I_\alpha \times L \backslash \{1\} \cup L \backslash \{1\} \times I_\alpha$ *from (b). Then we get the structures of* $T_{(8)}$ *and* $T_{(11)}$ *as shown in Figure 8.*



Theorem 3.10 $T_{(8)}$ Theorem 4.2 $T_{(11)}$

Figure 8: The structures of $T_{(8)}$ and $T_{(11)}$

We discover that it may happen for some $(x, y) \in I_\alpha \times (\alpha, 1)$, $T_{(8)}(x, y) = 0 \neq \gamma(x) \wedge \gamma(y) = T_{(11)}(x, y)$. *On the other hand, it holds that* $T_{(8)} = T_{(11)}$ *if* $T_1 = V$, $T_2 = T_\wedge$ *and*

$$\gamma(x) = \begin{cases} 1, & x = 1, \\ x \wedge \alpha, & \textit{otherwise}. \end{cases}$$

Therefore, Theorem 4.2 is essentially different from Theorem 3.10. Similarly, Theorem 4.2 is also fundamentally different from the construction in [8] *since* [8, Theorem 3.6] *is a special case of Theorem 3.10.*

Remark 4.7. *In Theorem 3.11 (the case of $n = 1$) and Theorem 4.2, let $\boldsymbol{L}$ be a bounded lattice with $\alpha \in L \setminus \{0,1\}$ such that $0 < \alpha < 1$, $M = [0,\alpha] \cup [\alpha,1] \subset L$, and $x \parallel y$ for all $x \in I_\alpha$ and for all $y \in (0,\alpha]$. If γ_1, γ_2 are two interior operators on L satisfying $\gamma_1(0) = 0$, $\gamma_1(\alpha) = \alpha$, $\gamma_1(1) = 1$ and $\gamma_1(L) \subseteq M$. Meanwhile, T_0 and T_1 are t-norms on $J_{0+} = \gamma_1(L) \cap [0,\alpha]$, $J_{1+} = \gamma_1(L) \cap [\alpha,1]$, respectively. Then we get that $0 \leq \gamma_1(x) \leq x < \alpha$ for all $x \in [0,\alpha)$, and $\alpha = \gamma_1(\alpha) \leq \gamma_1(y) \leq y < 1$ for all $y \in [\alpha,1)$, i.e., $\gamma_1(x) \in \gamma_1(L) \cap [0,\alpha)$, $\gamma_1(y) \in \gamma_1(L) \cap [\alpha,1)$. Then we get $T'_{(9)}$ and $T_{(11)}$ as follows:*

$$T'_{(9)}(x,y) = \begin{cases} T_0(\gamma_1(x),\gamma_1(y)), & (\gamma_1(x),\gamma_1(y)) \in J_0^2, \\ T_1(\gamma_1(x),\gamma_1(y)), & (\gamma_1(x),\gamma_1(y)) \in J_1^2, \\ \gamma_1(x) \wedge \gamma_1(y), & (\gamma_1(x),\gamma_1(y)) \in J_0 \times J_1 \cup J_1 \times J_0, \\ x \wedge y, & \text{otherwise.} \end{cases}$$

where $J_0 = \gamma_1(L) \cap [0,\alpha)$, $J_1 = \gamma_1(L) \cap [\alpha,1)$,

$$T_{(11)}(x,y) = \begin{cases} V(x,y), & (x,y) \in [0,\alpha]^2, \\ \gamma_2(x) \wedge \gamma_2(y), & (x,y) \in (\alpha,1) \times I_\alpha \cup I_\alpha \times (\alpha,1), \\ 0, & (x,y) \in [0,\alpha] \times I_\alpha \cup I_\alpha \times [0,\alpha] \cup I_\alpha \times I_\alpha, \\ x \wedge y, & \text{otherwise.} \end{cases}$$

Therefore, it may happen for some $(x,y) \in (0,\alpha) \times (\alpha,1)$, $T'_{(9)}(x,y) = \gamma_1(x) \wedge \gamma_1(y) = \gamma_1(x) \neq x = x \wedge y = T_{(11)}(x,y)$. On the other hand, it holds that $T'_{(9)} = T_{(11)}$ unless $T_0 = V$, $T_1 = T_\wedge$ and

$$\gamma_1(x) = \begin{cases} x, & x \notin I_\alpha, \\ x \wedge \alpha, & x \in I_\alpha. \end{cases}, \qquad \gamma_2(x) = \begin{cases} 1, & x = 1, \\ x \wedge \alpha, & \text{otherwise.} \end{cases}$$

Therefore, Theorem 4.2 is essentially different from Theorem 3.11.

Remark 4.8. *In Theorem 3.12 (the case of $i = 2$) and Theorem 4.2, let $\boldsymbol{L}$ be a bounded lattice with $\alpha \in L\setminus\{0,1\}$ such that $0 < \alpha < 1$, and $x \parallel y$ for all $x \in I_\alpha$ and for all $y \in (0,\alpha]$. If $(L_1, \leq, \wedge_1, 0, \alpha)$, $(L_2, \leq, \wedge_2, \alpha, 1)$ are bounded meet semilattices satisfying $L_1 \cup L_2 \subseteq L$, and T_1, T_2 are t-norms on L_1 and L_2, respectively. Meanwhile, γ_1, γ_2 are two interior operators on L satisfying $\gamma_1(L) = \tilde{L} = \{0,1\} \cup L_1 \cup L_2 = L_1 \cup L_2$. Then we get $\gamma_1(x) = 0$ for all $x \in I_\alpha$ and $\gamma_1(y) \in L_1 \cup (\alpha,y] \subseteq L_1 \cup L_2$ for all $y \in (\alpha,1)$. So we get $T'_{(10)}$ and $T_{(11)}$ as follows:*

$$T'_{(10)}(x,y) = \begin{cases} T_1(\gamma_1(x),\gamma_1(y)), & (\gamma_1(x),\gamma_1(y)) \in L_1^2 \text{ and } 1 \notin \{x,y\}, \\ T_2(\gamma_1(x),\gamma_1(y)), & (\gamma_1(x),\gamma_1(y)) \in L_2^2 \text{ and } 1 \notin \{x,y\}, \\ x \wedge y, & 1 \in \{x,y\}, \\ \gamma_1(x) \wedge \gamma_1(y), & \text{otherwise.} \end{cases}$$

$$T_{(11)}(x,y) = \begin{cases} V(x,y), & (x,y) \in [0,\alpha]^2, \\ \gamma_2(x) \wedge \gamma_2(y), & (x,y) \in (\alpha,1) \times I_\alpha \cup I_\alpha \times (\alpha,1), \\ 0, & (x,y) \in [0,\alpha] \times I_\alpha \cup I_\alpha \times [0,\alpha] \cup I_\alpha \times I_\alpha, \\ x \wedge y, & \text{otherwise.} \end{cases}$$

Therefore, it may happen that for some $(x,y) \in I_\alpha \times (\alpha,1)$, i.e., $(\gamma_1(x),\gamma_1(y)) \in \{0\} \times (L_1 \cup (\alpha,y])$,

$$T'_{(10)}(x,y) = \begin{cases} T_1(0,\gamma_1(y)), & \gamma_1(y) \in L_1 \\ 0 \wedge \gamma_1(y), & \gamma_1(y) \in (\alpha,y] \end{cases} = 0 \neq \gamma_2(x) \wedge \gamma_2(y) = T_{(11)}(x,y).$$

On the other hand, it holds that $T'_{(10)} = T_{(11)}$ if $T_1 = V$, $T_2 = T_\wedge$ and

$$\gamma_1(x) = \begin{cases} x, & x \notin I_\alpha, \\ x \wedge \alpha, & x \in I_\alpha. \end{cases}, \qquad \gamma_2(x) = \begin{cases} 1, & x = 1, \\ x \wedge \alpha, & \text{otherwise.} \end{cases}$$

Therefore, Theorem 4.2 is essentially different from Theorem 3.12.

Remark 4.9.

(i) *As seen from Remark 4.6, 4.7 and 4.8, we have that Theorem 4.2 is different from any of the methods in the third section.*

(ii) *If $I_\alpha = \emptyset$, then $T_{(11)} = T_{(5)}$.*

5 Recursive construction for ordinal sum t-norms and t-conorms on bounded lattices

In the previous section, we divide $\boldsymbol{L}$ into two subintervals to construct t-norms and t-conorms. In this section, we will generalize the number of subintervals to any positive integer n and construct ordinal sum of finitely many t-norms on bounded lattices.

Theorem 5.1. *Let $\boldsymbol{L}$ be a bounded lattice, $\{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ be a finite chain in L such that $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$, V be a t-norm on $[0, \alpha_1]$ and γ_i be an interior operator on $[0, \alpha_i]$. Then the binary operation $T_i : [0, \alpha_i]^2 \longrightarrow [0, \alpha_i]$ defined by*

$$T_i(x,y) = \begin{cases} T_{i-1}(x,y), & (x,y) \in [0,\alpha_{i-1}]^2, \\ \gamma_i(x) \wedge \gamma_i(y), & (x,y) \in (\alpha_{i-1},\alpha_i) \times I_{\alpha_{i-1}} \cup I_{\alpha_{i-1}} \times (\alpha_{i-1},\alpha_i), \\ 0, & (x,y) \in [0,\alpha_{i-1}] \times I_{\alpha_{i-1}} \cup I_{\alpha_{i-1}} \times [0,\alpha_{i-1}] \cup I_{\alpha_{i-1}} \times I_{\alpha_{i-1}}, \\ x \wedge y, & \text{otherwise}. \end{cases} \tag{13}$$

is a t-norm on $[0, \alpha_i]$ if and only if $x \parallel y$ for all $x \in I_{\alpha_{i-1}}$ and for all $y \in (0, \alpha_{i-1}]$, where $T_1 = V$ and $i \in \{2, \cdots, n\}$.

In order to understand Theorem 5.1, we will give an illustrative example as follows.

Example 5.2. *Let $(L'_1, \wedge, \vee, 0, 1)$ be a bounded lattice and $\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a finite chain in L'_1 such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 = 1$, which is shown in Figure 9.*

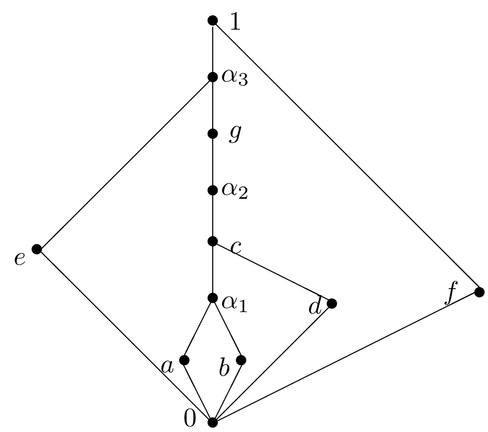


Figure 9: The lattice L'_1

Let $V = T_1 = T_\wedge$ be a t-norm on $[0, \alpha_1]$, γ_2, γ_3 and γ_4 interior operators on $[0, \alpha_2], [0, \alpha_3]$ and L'_1 defined by

$$\gamma_2(x) = \begin{cases} x, & c \leq x, \\ x \wedge c, & \text{otherwise}. \end{cases}$$

$$\gamma_3(x) = \begin{cases} x, & g \leq x, \\ x \wedge g, & \text{otherwise}. \end{cases}$$

$$\gamma_4(x) = \begin{cases} x, & \alpha_3 \leq x, \\ x \wedge \alpha_3, & \text{otherwise}. \end{cases}$$

By Theorem 5.1, we get the t-norms $T_2 : [0, \alpha_2]^2 \longrightarrow [0, \alpha_2]$, $T_3 : [0, \alpha_3]^2 \longrightarrow [0, \alpha_3]$ and $T_4 : {L'_1}^2 \longrightarrow L'_1$, which are shown in Table 1 – Table 3, respectively.

Table 1: The t-norm T_2 on $[0,\alpha_2]$.

T_2	0	a	b	α_1	c	d	α_2
0	0	0	0	0	0	0	0
a	0	a	0	a	a	0	a
b	0	0	b	b	b	0	b
α_1	0	a	b	α_1	α_1	0	α_1
c	0	a	b	α_1	c	d	c
d	0	0	0	0	d	0	d
α_2	0	a	b	α_1	c	d	α_2

Table 2: The t-norm T_3 on $[0,\alpha_3]$.

T_3	0	a	b	α_1	c	d	α_2	e	g	α_3
0	0	0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	0	a	0	a	a
b	0	0	b	b	b	0	b	0	b	b
α_1	0	a	b	α_1	α_1	0	α_1	0	α_1	α_1
c	0	a	b	α_1	c	d	c	0	c	c
d	0	0	0	0	d	0	d	0	d	d
α_2	0	a	b	α_1	c	d	α_2	0	α_2	α_2
e	0	0	0	0	0	0	0	0	0	e
g	0	a	b	α_1	c	d	α_2	0	g	g
α_3	0	a	b	α_1	c	d	α_2	e	g	α_3

Table 3: The t-norm T_4 on L_1'.

T_4	0	a	b	α_1	c	d	α_2	e	g	α_3	f	1
0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	a	0	a	a	0	a	0	a	a	0	a
b	0	0	b	b	0	b	b	0	b	b	0	b
α_1	0	a	b	α_1	α_1	0	α_1	0	α_1	α_1	0	α_1
c	0	a	b	α_1	c	d	c	0	c	c	0	c
d	0	0	0	0	d	0	d	0	d	d	0	d
α_2	0	a	b	α_1	c	d	α_2	0	α_2	α_2	0	α_2
e	0	0	0	0	0	0	0	0	0	e	0	e
g	0	a	b	α_1	c	d	α_2	0	g	g	0	g
α_3	0	a	b	α_1	c	d	α_2	e	g	α_3	0	α_3
f	0	0	0	0	0	0	0	0	0	0	0	f
1	0	a	b	α_1	c	d	α_2	e	g	α_3	f	1

In fact, though Theorem 5.1 is a generalization of Theorem 4.2, they are different from each other. We will give a concrete example to explain it.

Example 5.3. *Let $(L_2',\wedge,\vee,0,1)$ be a bounded lattice and $\{\alpha_0,\alpha_1,\alpha_2,\alpha_3\}$ be a finite chain in L_2' such that $0=\alpha_0<\alpha_1<\alpha_2<\alpha_3=1$, which is shown in Figure 10.*

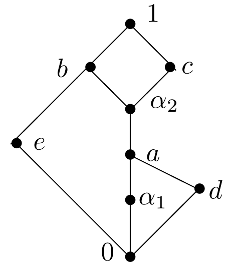


Figure 10: The lattice L_2'

In Theorem 4.2, let $\gamma(x)$ be an interior operator on L_2', which is defined by

$$\gamma(x)=\begin{cases} x, & \alpha_1\le x,\\ x\wedge\alpha_1, & \text{otherwise.}\end{cases}$$

In Theorem 5.1(the case of $n=3$), γ_2, γ_3 are interior operators on $[0,\alpha_2]$ and $[0,\alpha_3]$, respectively. According to (11) and (13), we get

$$T_{(11)}(x,y)=\begin{cases} V(x,y), & (x,y)\in[0,\alpha_1]^2,\\ \gamma(x)\wedge\gamma(y), & (x,y)\in(\alpha_1,1)\times I_{\alpha_1}\cup I_{\alpha_1}\times(\alpha_1,1),\\ 0, & (x,y)\in[0,\alpha_1]\times I_{\alpha_1}\cup I_{\alpha_1}\times[0,\alpha_1]\cup I_{\alpha_1}\times I_{\alpha_1},\\ x\wedge y, & \text{otherwise.}\end{cases}$$

$$T_3(x,y) = \begin{cases} T_2(x,y), & (x,y) \in [0,\alpha_2]^2, \\ \gamma_3(x) \wedge \gamma_3(y), & (x,y) \in (\alpha_2,\alpha_3) \times I_{\alpha_2} \cup I_{\alpha_2} \times (\alpha_2,\alpha_3), \\ 0, & (x,y) \in [0,\alpha_2] \times I_{\alpha_2} \cup I_{\alpha_2} \times [0,\alpha_2] \cup I_{\alpha_2} \times I_{\alpha_2}, \\ x \wedge y, & otherwise. \end{cases}$$

and $T_{(11)}(b,d) = \gamma(b) \wedge \gamma(d) = 0 \neq d = b \wedge d = T_3(b,d)$.

Theorem 5.4. *Let* $\boldsymbol{L}$ *be a bounded lattice,* $\{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ *be a finite chain in* L *such that* $1 = \alpha_0 > \alpha_1 > \cdots > \alpha_n = 0$*,* W *be a t-conorm on* $[\alpha_1, 1]$ *and* δ_i *be a closure operator on* $[\alpha_i, 1]$*. Then the binary operation* $S_i : [\alpha_i, 1]^2 \longrightarrow [\alpha_i, 1]$ *defined by*

$$S_i(x,y) = \begin{cases} S_{i-1}(x,y), & (x,y) \in [\alpha_{i-1},1]^2, \\ \delta_i(x) \vee \delta_i(y), & (x,y) \in (\alpha_i,\alpha_{i-1}) \times I_{\alpha_{i-1}} \cup I_{\alpha_{i-1}} \times (\alpha_i,\alpha_{i-1}), \\ 1, & (x,y) \in [\alpha_{i-1},1] \times I_{\alpha_{i-1}} \cup I_{\alpha_{i-1}} \times [\alpha_{i-1},1] \cup I_{\alpha_{i-1}} \times I_{\alpha_{i-1}} \\ x \vee y, & otherwise. \end{cases} \tag{14}$$

is a t-conorm on $[\alpha_i, 1]$ *if and only if* $x \parallel y$ *for all* $x \in I_{\alpha_{i-1}}$ *and for all* $y \in [\alpha_{i-1}, 1)$*, where* $S_1 = W$ *and* $i \in \{2, \cdots, n\}$*.*

6 Conclusions

In this paper, we proposed a new method for constructing the ordinal sum of t-norms on bounded lattices by combining interior operators with a given t-norm on a subinterval $[0, \alpha]$ of L. And we illustrate that such an ordinal sum construction is different from the constructions listed in Section 3. Finally, we further generalized the result to be ordinal sums of finitely many t-norm summands. For future studies, we hope to figure out how to weaken or transform the constraint condition in Theorem 4.2 to make it more universal. Secondly, we will focus on possible characterization of some classes of ordinal sum t-norms on bounded lattices. Moreover, we will consider the possible extension to the setting of more general aggregation functions such as uninorms on bounded lattices.

Acknowledgement

This work is partially supported by the National Nature Science Foundation of China (Grant No. 12171292) and the Fundamental Research Funds For the Central Universities (Grant No. GK202101009).

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