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Original paper

The general algebraic solution of fuzzy linear systems based on a block representation of $\{1\}$-inverses

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Abstract

A new method for solving a fuzzy linear system (FLS), $A\tilde{X} = \tilde{Y}$, where the coefficient matrix A is an arbitrary real matrix is obtained. A necessary and sufficient condition for the $\mathcal{R}$-consistency of the associated system of linear equations is obtained, related to its representative solutions. Moreover, the general form of representative solutions of such linear systems is presented. The straightforward method for solving $m \times n$ FLS based on an arbitrary $\{1\}$-inverse of A is introduced. This method is illustrated by interesting examples.

Keywords: Fuzzy linear systems, generalized inverses, singular matrix, general solution.

1 Introduction

Development of science and technology has motivated investigation of methods for solving fuzzy linear systems (FLS), which parameters are rather represented by fuzzy numbers than numbers. Friedman et al. [11] proposed the method for solving a square FLS, $A\tilde{X} = \tilde{Y}$, which coefficient matrix A is a real matrix and $\tilde{X}$ and $\tilde{Y}$ are fuzzy numbers vectors, while $\tilde{X}$ is unknown. This method has some drawbacks indicated and criticized in [2, 3, 14]. It is worth mentioning that the fully fuzzy linear systems with all fuzzy entries, introduced by Buckley and Qu (1991), are related to fuzzy coefficient matrices $\tilde{A}$, however, this type of FLS is not under consideration in this paper (we refer the reader to [8, 9]).

There are numerous of papers inspired by Friedman et al.'s work, regarding to the square form of FLS; an algorithm for obtaining the general algebraic solutions of a square FLS, whose coefficient matrix is non-singular, was proposed by Allahviranlo and Ghanbari [4]. Very recently, a necessary and sufficient condition for the existence of a unique algebraic solution of a dual fuzzy linear system was presented by Ghanbari et al. in [13]. On the other hand, there are papers influenced by [11] which deal with generalized inverses, among others: Assady [6], Allahviranlo and Kermani [5], Abbasbandy et al. [1], Mihailović et al. [15, 16, 18].

Moore (1920) and Penrose (1955)independently came up with the same idea about generalized inverses of a matrix, but from different approaches [19]. The most popular generalized inverses are $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{5\}$ and $\{1^k\}$-inverse. They have been used individually or in the combination with each other. The generalized inverses of matrices are very powerful tool for solving different systems of linear equations. In [15] was shown that the Moore-Penrose inverse is indispensable for solving fuzzy linear systems, $A\tilde{X} = \tilde{Y}$, where the coefficient matrix A is an $m \times n$ real matrix. In the case when the matrix A is full rank by columns (rows), $\{1, 3\}$-inverse ($\{1, 4\}$-inverse) of A has been used for solving fuzzy linear systems. Further, in [16] was also shown that the algebraic (strong) solutions of square FLS, can be represented using the group inverse of A or any of its $\{1\}$-inverses. In [15] the authors presented the first straightforward way for

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obtaining all algebraic (strong) solutions of non-square FLS verified by many examples. In both papers, [15, 16], the proposed algorithms for obtaining algebraic solutions are proper generalizations of the algorithm proposed in [4].

The main goal of this paper is to characterize the general algebraic (strong) solution of FLS, $A\tilde{X} = \tilde{Y}$, where A is an $m \times n$ real matrix, based on an arbitrary $\{1\}$-inverse of its coefficient matrix A. A necessary and sufficient condition for the existence of representative solutions of the associated system of linear equations to FLS is presented. Based on this significant result, a new straightforward way for finding the algebraic (strong) solutions of FLS is presented and illustrated by examples. For the first time in the literature, an algorithm based on $\{1\}$-inverses for obtaining the general (strong) solution of non-square FLS is presented. The proposed algorithm involves the computation of an arbitrary chosen $\{1\}$-inverse of A, proposed in [20], based on its block representation. However, to reduce the computational time, any of such computational methods can be implemented. Therefore, in some cases, the efficiency of the new algorithm can be increased by choosing a particular, less computationally demanding $\{1\}$-inverse of A involved in computational tasks.

The paper is organized as follows. In Section 2, a brief overview of generalized inverses and fuzzy linear systems is recalled. Also, the procedure for calculating $\{1\}$-inverses is presented. In Section 3, the general algebraic solution of FLS whose coefficient matrix is any $m \times n$ real matrix is investigated. In this section, a necessary and sufficient condition for the $\mathcal{R}$-consistency of the associated family of systems of linear equations is obtained and its general representative solution in form of $\{1\}$-inverses is presented. In Section 4, an operative algorithm and numerical examples are given in order to illustrate the proposed method as a straightforward way of solving non-square FLS. In Section 5, some concluding remarks are given.

2 Preliminaries

Recall some basic definitions related to $\{1\}$-inverses of matrices following [7].

Let $\mathcal{M}^{m\times n}$ denotes the class of all $m \times n$ real matrices.

Definition 2.1. *Let $A \in \mathcal{M}^{m\times n}$. Let $X \in \mathcal{M}^{n\times m}$ be a matrix which fulfills the matrix equation $AXA = A$, where $X \in \mathcal{M}^{n\times m}$ is unknown. This matrix X is called a* $\{1\}$-inverse *of A, and it is denoted by $A^{(1)}$.*

Let $A\{1\}$ denote the set of all $\{1\}$-inverses of A. Obviously, if A is invertable, A^{-1} is its only one $\{1\}$-inverse and $A\{1\}$ is a singleton set.

Recall that for any $A \in \mathcal{M}^{m\times n}$ of rank r there exist matrices $Q \in \mathcal{M}^{m\times m}$ and $P \in \mathcal{M}^{n\times n}$ such that:

$$QAP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

where $I_r \in \mathcal{M}^{r\times r}$ is the identity matrix. These matrices P and Q are not uniquely determined (see Theorem 2.7 [20]). Based on the block representation theorem (Theorem 3.3 in [20]), a block representation of any $\{1\}$-inverse of A is presented in the next theorem (for details see [17, 20]).

Theorem 2.2. *Let $A \in \mathcal{M}^{m\times n}$ of rank r. Let Q and P be non-singular, square matrices which fulfill (1). A matrix $X \in \mathcal{M}^{n\times m}$ is a solution of the matrix equation $AXA = A$ if and only if*

$$X = P \cdot \begin{bmatrix} I_r & Z_1 \\ Z_2 & Z_3 \end{bmatrix} \cdot Q, \tag{2}$$

where $I_r \in \mathcal{M}^{r\times r}$ is the identity matrix, and $Z_1 \in \mathcal{M}^{r\times(m-r)}$, $Z_2 \in \mathcal{M}^{(n-r)\times r}$ and $Z_3 \in \mathcal{M}^{(n-r)\times(m-r)}$ are arbitrarily chosen matrices.

The Moore-Penrose inverse of a matrix $A \in \mathcal{M}^{m\times n}$ (denoted by $A^{\dagger}$ or $A^{(1,2,3,4)}$) is the matrix $X \in \mathcal{M}^{n\times m}$ which is the unique solution of the system of four matrix equations $AXA = A$, $XAX = X$, $(AX)^{\mathrm{T}} = AX$ and $(XA)^{\mathrm{T}} = XA$.

The group inverse of a square matrix A with index 1 (denoted by $A^{\sharp}$ or $A^{(1,2,5)}$) is the square matrix X which is the unique solution of the system of three matrix equations $AXA = A$, $XAX = X$ and $AX = XA$.

The block representation of the Moore-Penrose inverse and the group inverse of an appropriate matrix A can be seen in [17].

We present the scheme (written and run in Python) for computing $\{1\}$-inverses of A as follows:

Computation of a {1}-inverse
1. Input: A is $m \times n$ matrix 2. Calculate matrices P and Q using (1) 3. Arbitrary choose matrices Z_1, Z_2 and Z_3 4. Determine the {1}-inverse using (2)

In the sequel of this section, basic notions about fuzzy sets and fuzzy linear systems are recalled. For more details, we refer the reader to [4, 10, 11, 12, 15]

Definition 2.3. *A fuzzy set $\tilde{u}$ with the membership function $\tilde{u} : \mathbb{R} \to [0,1]$ is called a fuzzy number if the following conditions are fulfilled:*

1. *$\tilde{u}$ is upper semi continuous,*
2. *$\tilde{u}(x) = 0$ outside some interval $[c,d]$,*
3. *There are real numbers a and b such that $c \le a \le b \le d$ and*
   - 3.1. *$\tilde{u}(x)$ is monotonic increasing on $[c,a]$,*
   - 3.2. *$\tilde{u}(x)$ is monotonic decreasing on $[b,d]$,*
   - 3.3. *$\tilde{u}(x) = 1$, $a \le x \le b$.*

Recall that a fuzzy number $\tilde{u}$ can be equivalently defined as $(\underline{u}, \overline{u})$, where the upper branch $\overline{u} : [0,1] \to \mathbb{R}$ is non-increasing and left continuous, whereas the lower branch $\underline{u} : [0,1] \to \mathbb{R}$ is non-decreasing and left continuous such that $\underline{u}(\alpha) \le \overline{u}(\alpha)$, for each $\alpha \in [0,1]$. The addition, scalar multiplication and equality of fuzzy numbers are based on the interval arithmetic and the fact that α-cuts of a fuzzy number $\tilde{u}$, given by $[\tilde{u}]_\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$, $\alpha \in [0,1]$, are closed intervals.

Definition 2.4. *For fuzzy numbers $\tilde{u} = (\underline{u}, \overline{u})$ and $\tilde{v} = (\underline{v}, \overline{v})$, and real number c, for each $\alpha \in [0,1]$:*

1. $[\tilde{u} + \tilde{v}]_\alpha = [\underline{u}(\alpha) + \underline{v}(\alpha), \overline{u}(\alpha) + \overline{v}(\alpha)]$,
2. $$[c\tilde{u}]_\alpha = \begin{cases} [c\underline{u}(\alpha), c\overline{u}(\alpha)], & c \ge 0, \\ [c\overline{u}(\alpha), c\underline{u}(\alpha)], & c < 0, \end{cases}$$
3. $\tilde{u} = \tilde{v}$ *iff* $\underline{u}(\alpha) = \underline{v}(\alpha)$ *and* $\overline{u}(\alpha) = \overline{v}(\alpha)$.

The next notation and terminology will be used. Let $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$ denotes a fuzzy number vector, where $\tilde{x}_i \in \mathcal{E}$ ($\mathcal{E}$ represents the set of all fuzzy numbers) for all $i = 1, \ldots, n$. Let $\mathcal{V}$ denotes the class of all fuzzy number vectors. Let $X = (\underline{x}_1, \ldots, \underline{x}_n, -\overline{x}_1, \ldots, -\overline{x}_n)^T$ denotes the associated $2n \times 1$ classical functional vector and let $\mathcal{F}$ denotes a class of all classical functional vectors. For each fuzzy number vector $\tilde{X} \in \mathcal{V}$, the vector $X \in \mathcal{F}$ is called the representative vector for $\tilde{X}$ if all its components are functions on the unit interval. A class of all representative classical functional vectors is marked by $\mathcal{F}^{\mathcal{R}}$. For each $\tilde{X} \in \mathcal{V}$, the associated $n \times 1$ classical functional vectors are $\underline{X} = (\underline{x}_1, \ldots, \underline{x}_n)^T$ and $\overline{X} = (\overline{x}_1, \ldots, \overline{x}_n)^T$. The zero vector is denoted by $\mathsf{O} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$, i.e. $\mathsf{O} = (0, 0, \ldots, 0)^T$.

Definition 2.5. *For the given coefficient matrix $A = [a_{ij}]$, $A \in \mathcal{M}^{m \times n}$ and a fuzzy number vector $\tilde{Y} \in \mathcal{V}$, the $m \times n$ linear system in the matrix form:*

$$A\tilde{X} = \tilde{Y}, \tag{3}$$

with unknown $\tilde{X} \in \mathcal{V}$, is called a fuzzy linear system (FLS).

Obviously, for each $A = [a_{ij}] \in \mathcal{M}^{m \times n}$, it holds $A = A^+ - A^-$ and $|A| = A^+ + A^-$, where $|A| = [|a_{ij}|]$ and $A^+ = [a_{ij}^+]$ and $A^- = [a_{ij}^-]$ are $m \times n$ nonnegative matrices defined by

$$a_{ij}^+ = \begin{cases} a_{ij}, & a_{ij} > 0 \\ 0, & \text{otherwise} \end{cases}, \qquad a_{ij}^- = \begin{cases} |a_{ij}|, & a_{ij} < 0 \\ 0, & \text{otherwise} \end{cases},$$

for all $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Based on the previous definitions, the next definition is obtained.

Definition 2.6. *The solution of (3) is a fuzzy number vector $\tilde{U} \in \mathcal{V}$, $U \in \mathcal{F}^{\mathcal{R}}$, if it holds:*

$$A^+\underline{U} - A^-\overline{U} = \underline{Y},$$

$$A^+\overline{U} - A^-\underline{U} = \overline{Y},$$

for each $\alpha \in [0,1]$.

If $\tilde{X}^0 \in \mathcal{V}$ is a solution of FLS (3), with the coefficient matrix $A \in \mathcal{M}^{m\times n}$, then $X^0 \in \mathcal{F}$ is a solution of the family of classical linear systems [11]

$$S_A X(\alpha) = Y(\alpha), \ \alpha \in [0,1], \tag{4}$$

(if there is no ambiguity, we write $S_A X = Y$), where $S_A \in \mathcal{M}^{2m\times 2n}$ is

$$S_A = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}. \tag{5}$$

The matrix S_A is called the matrix associated to the FLS (3). When $X^0 \in \mathcal{F}$ is a solution vector of (4), it does not mean that it is the representative vector of any fuzzy number vector. Thus, we need a solution vector of (4) such that the obtained functions $\underline{x}_i^0(\alpha)$ and $\overline{x}_i^0(\alpha)$, $\alpha \in [0,1]$, for all $i = 1, 2, \cdots n$, are adequate for the parametric representation of a fuzzy number. Therefore, if $X^0 \in \mathcal{F}^{\mathcal{R}}$ is a solution vector of (4), then $\tilde{X}^0 \in \mathcal{V}$ is a solution of the FLS (3). If a solution exists, we say that a FLS is consistent. In contrary, it is inconsistent. We refer [4, 15, 16, 18] for more details about consistent FLS.

3 The general representative solution of $S_A X = Y$

Consider the family of classical linear systems (4) associated to the FLS (3). Denote the solution set of (4) with $\mathcal{ASS}$. Denote with $\mathcal{ASS}^{\mathcal{R}}$ the subset of $\mathcal{ASS}$ which contains all representative solutions of (4), i.e.

$$\mathcal{ASS}^{\mathcal{R}} = \{X \in \mathcal{F}^{\mathcal{R}} |\ S_A X = Y\} \subseteq \{X \in \mathcal{F} |\ S_A X = Y\} = \mathcal{ASS}.$$

Definition 3.1. *Let $A \in \mathcal{M}^{m\times n}$ be the coefficient matrix of the FLS (3), for given $\tilde{Y} \in \mathcal{V}$. The associated family of classical linear systems (4) for the FLS (3) is said to be $\mathcal{R}$-consistent if there exist at least one representative solution of (4), i.e. if $\mathcal{ASS}^{\mathcal{R}} \neq \emptyset$.*

Let $M \in \mathcal{M}^{n\times m}$, $M = [m_{ij}] \in A\{1\}$ (here M denotes an arbitrary $\{1\}$-inverse of A) and $|M| = [|m_{ij}|]$. Let us consider $S_M \in \mathcal{M}^{2n\times 2m}$ (obtained by (5) replacing A by M):

$$S_M = \begin{bmatrix} M^+ & M^- \\ M^- & M^+ \end{bmatrix}. \tag{6}$$

Let $Y \in \mathcal{F}^{\mathcal{R}}$ be an arbitrary representative vector, and $X^* = S_M Y$. Since S_M, M^+ and M^- are non-negative, it implies that $\tilde{X}^*$ is a fuzzy number vector, even if the FLS (3) has no solutions. In the case of a consistent FLS, by using argumentation similarly to the one presented in the proof of Lemma 2 in [15] (see the proof of Theorem 7 in [16]), we have the next lemma.

Lemma 3.2. *Let $A \in \mathcal{M}^{m\times n}$ be the coefficient matrix of a consistent FLS (3), for given $\tilde{Y} \in \mathcal{V}$. Let $X^* = S_M Y$, where $M \in A\{1\}$ and S_M is given by (6). Then it holds $A(\overline{X}^* + \underline{X}^*) = \overline{Y} + \underline{Y}$.*

In addition, if $|M| \in |A|\{1\}$, i.e. $|M|$ is arbitrary $\{1\}$-inverse of $|A|$, it holds $|A|\left(\overline{X}^* - \underline{X}^*\right) = \overline{Y} - \underline{Y}$ (see the proof of Theorem 7 in [16]), what together with the previous fact ensures that $X^* \in \mathcal{ASS}^{\mathcal{R}}$, hence $\tilde{X}^* \in \mathcal{V}$ is one solution of the consistent FLS (3).

The next theorem presents a necessary and sufficient condition for the $\mathcal{R}$-consistency of (4). Also, the general representative solution of (4) is given, i.e. the elements of $\mathcal{ASS}^{\mathcal{R}}$ are characterised.

Theorem 3.3. *Let $A \in \mathcal{M}^{m\times n}$ be the coefficient matrix of the FLS (3), for given $\tilde{Y} \in \mathcal{V}$. Let $X^* = S_M Y$, where $M \in A\{1\}$ and S_M is given by (6). Let $\mathsf{R} = Y - S_A X^*$.*

The associated family of linear systems (4) for the FLS (3) is $\mathcal{R}$-consistent iff there exist $\mathsf{L} = \begin{bmatrix} L \\ -L \end{bmatrix}$, where $L = (l_1(\alpha), \dots, l_n(\alpha))^T$, such that $S_A \mathsf{L} = \mathsf{O}$, and $\mathsf{V} = \begin{bmatrix} V \\ V \end{bmatrix}$, where $V = (v_1(\alpha), \dots, v_n(\alpha)))^T$, such that $S_A \mathsf{V} = \mathsf{R}$, and $X^* + \frac{1}{2}\mathsf{L} + \mathsf{V} \in \mathcal{F}^R$.

Moreover,

$$\mathcal{ASS}^{\mathcal{R}} = \left\{ X \in \mathcal{F}^R \,|\, X = X^* + \frac{1}{2}\mathsf{L} + \mathsf{V},\ S_A\mathsf{L} = \mathsf{O},\ S_A\mathsf{V} = \mathsf{R} \right\}.$$

Proof. ($\Rightarrow$) Let us suppose $X \in \mathcal{ASS}^R$, i.e. that $X \in \mathcal{F}^R$ is a solution of the family of classical linear equations (4). It is already known (see [15]), that $X \in \mathcal{F}^R$, $X = (\underline{x}_1, \dots, \underline{x}_n, \dots, -\overline{x}_1, \dots, -\overline{x}_n)^T$, is a solution vector of (4) iff the sum vector $\overline{X} + \underline{X}$ is a solution of $A(\overline{X} + \underline{X}) = \overline{Y} + \underline{Y}$ and the difference vector $\overline{X} - \underline{X}$ is a solution of $|A|(\overline{X} - \underline{X}) = \overline{Y} - \underline{Y}$. According to the general solution form in term of {1}-inverses ([7]), there exist two $n \times 1$ vectors K_1 and K_2 that

$$\begin{aligned} \overline{X} + \underline{X} &= A^{(1)}(\overline{Y} + \underline{Y}) + K_1, \\ \overline{X} - \underline{X} &= |A|^{(1)}(\overline{Y} - \underline{Y}) + K_2. \end{aligned}$$

Let us denote $M = A^{(1)}$, $N = |A|^{(1)}$. By adding and subtracting the previous equations, since $X^* = S_M Y$, we obtain, respectively:

$$\begin{aligned} \overline{X} &= \frac{1}{2}\Big(\Big(A^{(1)} + |A|^{(1)}\Big)\overline{Y} - \Big(|A|^{(1)} - A^{(1)}\Big)\underline{Y} + K_1 + K_2 \Big) \\ &= \frac{1}{2}\Big((M + |M|)\overline{Y} - (|M| - M)\underline{Y} + (N - |M|)(\overline{Y} - \underline{Y}) + K_1 + K_2 \Big) \\ &= M^+\overline{Y} - M^-\underline{Y} + \frac{1}{2}(N - |M|)(\overline{Y} - \underline{Y}) + \frac{1}{2}(K_1 + K_2) \\ &= \overline{X}^* + \frac{1}{2}(N - |M|)(\overline{Y} - \underline{Y}) + \frac{1}{2}(K_1 + K_2) \\ &= \overline{X}^* + \frac{1}{2}K_1 - \frac{1}{2}\Big((N - |M|)(\underline{Y} - \overline{Y}) - K_2 \Big), \end{aligned}$$

and similarly,

$$\begin{aligned} \underline{X} &= \underline{X}^* + \frac{1}{2}(N - |M|)(\underline{Y} - \overline{Y}) + \frac{1}{2}(K_1 - K_2) \\ &= \underline{X}^* + \frac{1}{2}K_1 + \frac{1}{2}\Big((N - |M|)(\underline{Y} - \overline{Y}) - K_2 \Big). \end{aligned}$$

Further, let L and V be $n \times 1$ vectors such that

$$L = K_1 \quad \text{and} \quad V = \frac{1}{2}\Big((N - |M|)(\underline{Y} - \overline{Y}) - K_2 \Big). \tag{7}$$

We obtain:

$$\overline{X} = \overline{X}^* + \frac{1}{2}L - V, \tag{8}$$

$$\underline{X} = \underline{X}^* + \frac{1}{2}L + V. \tag{9}$$

By adding (8) and (9), we obtain $\overline{X} + \underline{X} = \overline{X}^* + \underline{X}^* + L$. Since $\overline{X} + \underline{X}$ and $\overline{X}^* + \underline{X}^*$ (by Lemma 3.2) are solutions of $A\left(\overline{X} + \underline{X}\right) = \overline{Y} + \underline{Y}$, it holds

$$\begin{aligned} \overline{Y} + \underline{Y} &= A(\overline{X} + \underline{X}) \\ &= A(\overline{X}^* + \underline{X}^* + L) \\ &= A(\overline{X}^* + \underline{X}^*) + AL \\ &= \overline{Y} + \underline{Y} + AL, \end{aligned}$$

and therefore $AL = \mathsf{O}$. Now, define the $2n \times 1$ functional vector L with: $\mathsf{L} = \begin{bmatrix} L \\ -L \end{bmatrix}$, where $L = (l_1(\alpha), \dots, l_n(\alpha))^T$, $\alpha \in [0,1]$, is given by (7). Further,

$$S_A\mathsf{L} = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix} \begin{bmatrix} L \\ -L \end{bmatrix} = \begin{bmatrix} A^+L - A^-L \\ A^-L - A^+L \end{bmatrix} = \begin{bmatrix} AL \\ -AL \end{bmatrix},$$

since $AL = \mathsf{O}$, we have the claim that $S_A\mathsf{L} = \mathsf{O}$.

The vector R is a $2n \times 1$ functional vector defined by $\mathsf{R} = Y - S_A X^*$. Thus, we have

$$\mathsf{R} = \begin{bmatrix} \underline{Y} \\ -\overline{Y} \end{bmatrix} - \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix} \begin{bmatrix} \underline{X}^* \\ -\overline{X}^* \end{bmatrix} = \begin{bmatrix} \underline{Y} - A^+\underline{X}^* + A^-\overline{X}^* \\ -\overline{Y} + A^+\overline{X}^* - A^-\underline{X}^* \end{bmatrix}.$$

Due to Lemma 3.2, we have

$$\begin{aligned} \overline{Y} + \underline{Y} &= A(\overline{X}^* + \underline{X}^*) \\ &= A\overline{X}^* + A\underline{X}^* \\ &= A^+\overline{X}^* - A^-\overline{X}^* + A^+\underline{X}^* - A^-\underline{X}^*. \end{aligned}$$

Thus,

$$R = \underline{Y} - A^+\underline{X}^* + A^-\overline{X}^* = -\overline{Y} + A^+\overline{X}^* - A^-\underline{X}^*. \tag{10}$$

This implies that $\mathsf{R} = \begin{bmatrix} R \\ R \end{bmatrix}$ and $2R = \underline{Y} - \overline{Y} + |A|(\overline{X}^* - \underline{X}^*)$.

By subtracting (8) and (9), we have $\overline{X} - \underline{X} = \overline{X}^* - \underline{X}^* - 2V$. Since $\overline{X} - \underline{X}$ is one of the solutions of $|A|(\overline{X} - \underline{X}) = \overline{Y} - \underline{Y}$, therefore

$$\begin{aligned} \overline{Y} - \underline{Y} &= |A|(\overline{X} - \underline{X}) \\ &= |A|(\overline{X}^* - \underline{X}^* - 2V) \\ &= |A|(\overline{X}^* - \underline{X}^*) - 2|A|V \\ &= 2R + \overline{Y} - \underline{Y} - 2|A|V, \end{aligned}$$

and we conclude $|A|V = R$. Now, define V as follows: $\mathsf{V} = \begin{bmatrix} V \\ V \end{bmatrix}$, where $V = (v_1(\alpha), \dots, v_n(\alpha)))^T$, $\alpha \in [0,1]$, is given by (7). Further,

$$S_A\mathsf{V} = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix} \begin{bmatrix} V \\ V \end{bmatrix} = \begin{bmatrix} A^+V + A^-V \\ A^+V + A^-V \end{bmatrix} = \begin{bmatrix} |A|V \\ |A|V \end{bmatrix}.$$

Since $|A|V = R$, we conclude that $S_A\mathsf{V} = \mathsf{R}$.
By (8) and (9), we have $X = X^* + \frac{1}{2}\mathsf{L} + \mathsf{V}$.

($\Leftarrow$) Let us suppose $X \in \mathcal{F}^{\mathcal{R}}$ is in the form $X = X^* + \frac{1}{2}\mathsf{L} + \mathsf{V}$, for some L and V such that $S_A\mathsf{L} = \mathsf{O}$ and $S_A\mathsf{V} = \mathsf{R}$. By multiplying the equation $X = X^* + \frac{1}{2}\mathsf{L} + \mathsf{V}$ with S_A from the left-side, we have:

$$\begin{aligned} X &= X^* + \frac{1}{2}\mathsf{L} + \mathsf{V} \\ S_A X &= S_A X^* + \frac{1}{2} S_A \mathsf{L} + S_A\mathsf{V} \\ S_A X &= S_A X^* + Y - S_A X^* \\ S_A X &= Y. \end{aligned}$$

Thus, $X \in \mathcal{A}SS^R$.
We conclude that the general representative solution of the family of linear equations (4) is $X = X^* + \frac{1}{2}\mathsf{L} + \mathsf{V}$, where L and V are solutions of $S_A\mathsf{L} = \mathsf{O}$ and $S_A\mathsf{V} = \mathsf{R}$, respectively, $\mathsf{L} = \begin{bmatrix} L \\ -L \end{bmatrix}$, $\mathsf{V} = \begin{bmatrix} V \\ V \end{bmatrix}$, $L = (l_1(\alpha), \dots, l_n(\alpha))^T$, $V = (v_1(\alpha), \dots, v_n(\alpha)))^T$. □

4 A method for solving FLS

Finally, based on the above result, we present a straightforward method for obtaining the general algebraic solution of an $m \times n$ fuzzy linear system.

Algorithm

Step 1. Compute a starter vector $X^* = S_M Y$, where $M \in A\{1\}$ and S_M given by (6).
Step 2. Compute a $2n \times 1$ functional vector L such that $S_A\mathsf{L} = \mathsf{O}$.
Step 3. Compute a $2n \times 1$ functional vector R such that $\mathsf{R} = Y - S_A X^*$, here X^* is obtained by Step 1.
Step 4. If it exists, compute a $2n \times 1$ functional vector V such that $S_A\mathsf{V} = \mathsf{R}$, where R is obtained by Step 3, else, FLS has no solution.
Step 5. If it exists, compute the representative solution $X = X^* + \frac{1}{2}\mathsf{L} + \mathsf{V}$ of (4), else, FLS has no solution.

The last step means, that from all determined L and V by Step 2 and Step 4, we only choose such that for each $\alpha \in [0,1]$ it holds

$$V \leq \frac{\overline{X}^* - \underline{X}^*}{2},$$

and all components of $\underline{X}^* + \frac{1}{2}L + V$ are non-decreasing and left-continuous function on the unite interval, whereas all components of $\overline{X}^* + \frac{1}{2}L - V$ are non-increasing and left-continuous functions. For such L and V, for all $i = 1, \ldots, n$, the family of intervals $\{[\underline{x}_i^*(\alpha) + \frac{1}{2}l_i(\alpha) + v_i(\alpha), \overline{x}_i^*(\alpha) + \frac{1}{2}l_i(\alpha) - v_i(\alpha)] \,|\, \alpha \in [0,1]\}$ determines α-cuts of a fuzzy number. Then, a fuzzy number vector $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_n)^T$, obtained in a such manner, is a solution of the FLS (3).

Remark 1. *It is not hard to conclude that the efficiency of the algorithm can be increased as follows.*

(*i*) *In Step 2 and Step 4 the computational time can be saved by solving two $m \times n$ classical linear systems $AL = \mathsf{O}$ and $|A|V = R$, where R is obtained by (10), instead of two $2m \times 2n$ systems $S_A\mathsf{L} = \mathsf{O}$ and $S_A\mathsf{V} = \mathsf{R}$, similarly as it was done in [15].*

(*ii*) *If A is a large matrix, in order to reduce the computational cost, by choosing zero matrices for all three arbitrary blocks in (2), a sparse matrix $A^{(1)}$ can be obtained in some cases.*

(*iii*) *If A is a square matrix, by using (i), the solving procedure is exactly the same as the procedure presented in [16], additionally, if $A\{1\}$ is a singleton set, i.e. when the only $\{1\}$-inverse of A is its classical inverse, it reduces to the procedure presented in [4]. In the last case, $S_A\mathsf{L} = \mathsf{O}$ has only the trivial solution.*

Example 4.1. *Consider the next fuzzy linear system:*

$$\begin{array}{rcrcl} \tilde{x}_1 & - & 2\tilde{x}_3 & = & (-1+\alpha, 1-\alpha) \\ -\tilde{x}_1 & + & \tilde{x}_2 & = & (-2+2\alpha, 2-2\alpha) \\ \tilde{x}_2 & - & 2\tilde{x}_3 & = & (-1+\alpha, 1-\alpha) \end{array}.$$

In this example, the coefficient matrix A is singular with index 1, whereas $|A|$ is regular. The group inverse of A is:

$$M = A^\sharp = \frac{1}{3}\begin{bmatrix} 1 & 0 & -2 \\ -1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}.$$

According to the algorithm, Step 1, we have:

$$X^* = S_M Y = \begin{bmatrix} -1+\alpha \\ -1+\alpha \\ -\frac{4}{3}+\frac{4}{3}\alpha \\ -1+\alpha \\ -1+\alpha \\ -\frac{4}{3}+\frac{4}{3}\alpha \end{bmatrix}.$$

According to the algorithm, Step 2, we get a solution vector of $S_A\mathsf{L} = \mathsf{O}$ in the form
$\mathsf{L} = (4f(\alpha), 4f(\alpha), 2f(\alpha), -4f(\alpha), -4f(\alpha), -2f(\alpha))^T$*, where $y = f(\alpha)$, $\alpha \in [0,1]$ is a function on the unite interval.*

Next, according to the algorithm, Step 3, we have:

$$\mathsf{R} = Y - S_A X^* = \begin{bmatrix} \frac{8}{3} - \frac{8}{3}\alpha \\ 0 \\ \frac{8}{3} - \frac{8}{3}\alpha \\ \frac{8}{3} - \frac{8}{3}\alpha \\ 0 \\ \frac{8}{3} - \frac{8}{3}\alpha \end{bmatrix}.$$

Step 4 from the algorithm give us the unique feasible solution V *of the family of classical systems* $S_A\mathsf{V} = \mathsf{R}$*:*

$$\mathsf{V} = \begin{bmatrix} 0 \\ 0 \\ \frac{4}{3} - \frac{4}{3}\alpha \\ 0 \\ 0 \\ \frac{4}{3} - \frac{4}{3}\alpha \end{bmatrix}, \quad \alpha \in [0,1],$$

since the necessary conditions for the vector V *are satisfied:*

$$\begin{aligned} 0 = v_1(\alpha) &\leq 1-\alpha, \\ 0 = v_2(\alpha) &\leq 1-\alpha, \\ \frac{4}{3}(1-\alpha) = v_3(\alpha) &\leq \frac{4}{3}(1-\alpha). \end{aligned}$$

Finally, the general representative solution of the family (4), $X \in \mathcal{ASS}^R$*, is:*

$$\begin{aligned} X &= X^* + \frac{1}{2}\mathsf{L} + \mathsf{V} \\ &= \begin{bmatrix} -1+\alpha \\ -1+\alpha \\ -\frac{4}{3}+\frac{4}{3}\alpha \\ -1+\alpha \\ -1+\alpha \\ -\frac{4}{3}+\frac{4}{3}\alpha \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 4f(\alpha) \\ 4f(\alpha) \\ 2f(\alpha) \\ -4f(\alpha) \\ -4f(\alpha) \\ -2f(\alpha) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{4}{3}-\frac{4}{3}\alpha \\ 0 \\ 0 \\ \frac{4}{3}-\frac{4}{3}\alpha \end{bmatrix} = \begin{bmatrix} -1+\alpha+2f(\alpha) \\ -1+\alpha+2f(\alpha) \\ f(\alpha) \\ -1+\alpha-2f(\alpha) \\ -1+\alpha-2f(\alpha) \\ -f(\alpha) \end{bmatrix}, \end{aligned}$$

where $y = f(\alpha)$*,* $\alpha \in [0,1]$*,* $f \in \mathcal{G}$*, and* $\mathcal{G}$ *denotes the class of functions on the unite interval, such that the adequate functions* $\underline{x}(\alpha)$ *(resp.* $\overline{x}(\alpha)$*) are bounded, non-decreasing (resp. non-increasing) and left-continuous. The general algebraic solution of the considered FLS,* $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^T$*, is obtained from* $X \in \mathcal{ASS}^R$*. To get all solutions with triangular fuzzy number components, it should be satisfied* $f'(\alpha) \geq 0 \geq -\frac{1}{2}$ *and* $f'(\alpha) \leq 0 \leq \frac{1}{2}$*, for all* $\alpha \in (0,1)$*, thus:*

$$\tilde{X} = ((-1+2C+\alpha, 1+2C-\alpha), (-1+2C+\alpha, 1+2C-\alpha), (C,C))^T,\ C \in \mathbb{R}.$$

In order to obtain a particular solution of FLS, e.g., if $f(\alpha) = 3$*,* $\alpha \in [0,1]$*, we get* $\tilde{X}$ *with the next fuzzy number components:*

$$\begin{aligned} \tilde{x}_1 &= (5+\alpha, 7-\alpha), \\ \tilde{x}_2 &= (5+\alpha, 7-\alpha), \\ \tilde{x}_3 &= (3,3). \end{aligned}$$

Example 4.2. *Consider the next fuzzy linear system:*

$$\begin{array}{rcrcl} \tilde{x}_1 & - & 2\tilde{x}_3 & = & (-3+3\alpha, 3-3\alpha) \\ -\tilde{x}_1 & + & \tilde{x}_2 & = & (-1+\alpha, 1-\alpha) \\ \tilde{x}_2 & - & 2\tilde{x}_3 & = & (-2+2\alpha, 2-2\alpha) \end{array}.$$

The one of $\{1\}$*-inverses of the coefficient matrix* A *is in the next form:*

$$M = A^{(1)} = \begin{bmatrix} 3 & 0 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \qquad (11)$$

obtained by (2) with the matrices

$$P = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix},$$

and

$$Z_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad Z_3 = \begin{bmatrix} 1 \end{bmatrix}.$$

Now, using the algorithm, Step 1, we have:

$$X^* = S_M Y = \begin{bmatrix} -13+13\alpha \\ -14+14\alpha \\ -5+5\alpha \\ -13+13\alpha \\ -14+14\alpha \\ -5+5\alpha \end{bmatrix}.$$

According to the algorithm, Step 2, we get a solution vector of $S_A \mathsf{L} = \mathsf{O}$ *in the form* $\mathsf{L} = (4f(\alpha), 4f(\alpha), 2f(\alpha), -4f(\alpha), -4f(\alpha), -2f(\alpha))^T$*, where* $y = f(\alpha)$*,* $\alpha \in [0,1]$ *is a function on the unite interval. Next, according to the algorithm, Step 3, we have:*

$$\mathsf{R} = Y - S_A X^* = \begin{bmatrix} 20-20\alpha \\ 26-26\alpha \\ 22-22\alpha \\ 20-20\alpha \\ 26-26\alpha \\ 22-22\alpha \end{bmatrix}.$$

The Step 4 from the algorithm give us the unique feasible solution V *of the family of classical systems* $S_A \mathsf{V} = \mathsf{R}$*:*

$$\mathsf{V} = \begin{bmatrix} 12-12\alpha \\ 14-14\alpha \\ 4-4\alpha \\ 12-12\alpha \\ 14-14\alpha \\ 4-4\alpha \end{bmatrix}, \quad \alpha \in [0,1],$$

since V *fulfils:*

$$\begin{aligned} 12(1-\alpha) = v_1(\alpha) &\leq 13(1-\alpha), \\ 14(1-\alpha) = v_2(\alpha) &\leq 14(1-\alpha), \\ 4(1-\alpha) = v_3(\alpha) &\leq 5(1-\alpha). \end{aligned}$$

Finally, the general representative solution of the family (4), $X \in \mathcal{A}SS^R$*, is:*

$$\begin{aligned} X &= X^* + \frac{1}{2}\mathsf{L} + \mathsf{V} \\ &= \begin{bmatrix} -13+13\alpha \\ -14+14\alpha \\ -5+5\alpha \\ -13+13\alpha \\ -14+14\alpha \\ -5+5\alpha \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 4f(\alpha) \\ 4f(\alpha) \\ 2f(\alpha) \\ -4f(\alpha) \\ -4f(\alpha) \\ -2f(\alpha) \end{bmatrix} + \begin{bmatrix} 12-12\alpha \\ 14-14\alpha \\ 4-4\alpha \\ 12-12\alpha \\ 14-14\alpha \\ 4-4\alpha \end{bmatrix} = \begin{bmatrix} -1+\alpha+2f(\alpha) \\ 2f(\alpha) \\ -1+\alpha+f(\alpha) \\ -1+\alpha-2f(\alpha) \\ -2f(\alpha) \\ -1+\alpha-f(\alpha) \end{bmatrix}, \end{aligned}$$

where $y = f(\alpha)$*,* $\alpha \in [0,1]$*,* $f \in \mathcal{G}$ *(*f *is a feasible function such that* X *is the representable functional vector for a fuzzy number vector). The general algebraic solution of the considered FLS,* $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^T$*, is obtained using* $X \in \mathcal{A}SS^R$*.*

In order to obtain all solutions with triangular fuzzy number components, we require $2f'(\alpha) \geq 0 \geq -1 \geq -2$ *and* $2f'(\alpha) \leq 0 \leq 1 \leq 2$*, for all* $\alpha \in (0,1)$*, therefore:*

$$\tilde{X} = ((-1+2C+\alpha, 1+2C-\alpha), (2C, 2C), (-1+C+\alpha, 1+C-\alpha))^T,\ C \in \mathbb{R}.$$

E.g. if we take $f(\alpha) = -1$*,* $\alpha \in [0,1]$*, we get* $\tilde{X}$ *as a particular solution:*

$$\begin{aligned} \tilde{x}_1 &= (-3+\alpha, -1-\alpha), \\ \tilde{x}_2 &= (-2, -2), \\ \tilde{x}_3 &= (-2+\alpha, -\alpha). \end{aligned}$$

Example 4.3. *Consider the next fuzzy linear system:*

$$\begin{aligned} \tilde{x}_1 + \tilde{x}_2 - \tilde{x}_3 &= (1+4\alpha, 9-4\alpha) \\ -\tilde{x}_2 + 3\tilde{x}_3 &= (-8+4\alpha, -4\alpha) \end{aligned}.$$

The next $\{1,4\}$*-inverse of the coefficient matrix* A *is chosen from the set of* $\{1\}$*-inverses of* A*,* ($A^{(1,4)} = A^{\dagger} = M$)*:*

$$M = \frac{1}{14}\begin{bmatrix} 10 & 4 \\ 6 & 1 \\ 2 & 5 \end{bmatrix}. \tag{12}$$

Similarly to the previous examples, we have:

$$X^* = \frac{1}{14}\begin{bmatrix} -22+56\alpha \\ -2+28\alpha \\ -38+28\alpha \\ -90+56\alpha \\ -54+28\alpha \\ -18+28\alpha \end{bmatrix},$$

a solution vector of $S_A\mathsf{L} = \mathsf{O}$ *is* $\mathsf{L} = (-4f(\alpha), 6f(\alpha), 2f(\alpha), 4f(\alpha), -6f(\alpha), -2f(\alpha))^T$*,* $\alpha \in [0,1]$*,* f *is a function on the unite interval, and we have:*

$$\mathsf{R} = Y - S_A X^* = \begin{bmatrix} 4-4\alpha \\ 4-4\alpha \\ 4-4\alpha \\ 4-4\alpha \end{bmatrix}.$$

Now, we obtain infinitely many feasible solutions of the family of classical systems $S_A\mathsf{V} = \mathsf{R}$*:*

$$\mathsf{V} = \begin{bmatrix} 2g(\alpha) \\ 4-4\alpha-3g(\alpha) \\ g(\alpha) \\ 2g(\alpha) \\ 4-4\alpha-3g(\alpha) \\ g(\alpha) \end{bmatrix},\ \alpha \in [0,1].$$

The necessary conditions for the vector V *are:*

$$\begin{aligned} 2g(\alpha) = v_1(\alpha) &\leq 4(1-\alpha), \\ 4(1-\alpha) - 3g(\alpha) = v_2(\alpha) &\leq 2(1-\alpha), \\ g(\alpha) = v_3(\alpha) &\leq 2(1-\alpha), \end{aligned}$$

which implies $\frac{2}{3}(1-\alpha) \leq g(\alpha) \leq 2(1-\alpha)$*,* $\alpha \in [0,1]$*, where* g *is a function on the unite interval. Finally, the general* *solution of the family (4), $X \in \mathcal{A}SS^R$, is:*

$$\begin{aligned} X &= X^* + \frac{1}{2}\mathsf{L} + \mathsf{V} \\ &= \frac{1}{14}\begin{bmatrix} -22+56\alpha \\ -2+28\alpha \\ -38+28\alpha \\ -90+56\alpha \\ -54+28\alpha \\ -18+28\alpha \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -4f(\alpha) \\ 6f(\alpha) \\ 2f(\alpha) \\ 4f(\alpha) \\ -6f(\alpha) \\ -2f(\alpha) \end{bmatrix} + \begin{bmatrix} 2g(\alpha) \\ 4(1-\alpha)-3g(\alpha) \\ g(\alpha) \\ 2g(\alpha) \\ 4(1-\alpha)-3g(\alpha) \\ g(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} -\frac{11}{7}+4\alpha-2f(\alpha)+2g(\alpha) \\ -\frac{1}{7}+2\alpha+3f(\alpha)+4(1-\alpha)-3g(\alpha) \\ -\frac{19}{7}+2\alpha+f(\alpha)+g(\alpha) \\ -\frac{45}{7}+4\alpha+2f(\alpha)+2g(\alpha) \\ -\frac{27}{7}+2\alpha-3f(\alpha)+4(1-\alpha)-3g(\alpha) \\ -\frac{9}{7}+2\alpha-f(\alpha)+g(\alpha) \end{bmatrix}, \end{aligned}$$

where f and g are feasible functions, defined on the unite interval. In this example, the analytical consideration is more complex than in the previous examples. In order to obtain all solutions with triangular fuzzy number components, analytical requirements are: $-2 \le f'(\alpha) + g'(\alpha) \le \frac{2}{3}$ and $\frac{2}{3} \le f'(\alpha) - g'(\alpha) \le 2$, for all $\alpha \in (0,1)$, therefore: $f(\alpha) = k_1\alpha + n_1$, $g(\alpha) = k_2\alpha - k_2$, $\alpha \in [0,1]$ where $k_2 \in [-2, -\frac{2}{3}]$, $\frac{2}{3} + k_2 \le k_1 \le 2 + k_2$, $-2 - k_2 \le k_1 \le \frac{2}{3} - k_2$ and $n_1 \in \mathbb{R}$. For example, if we take $f(\alpha) = -\frac{2}{7}$ and $g(\alpha) = 1 - \alpha$, $\alpha \in [0,1]$, we obtain the one of infinitely many solutions of the considered FLS, $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)^T$:

$$\begin{aligned} \tilde{x}_1 &= (1+2\alpha, 5-2\alpha), \\ \tilde{x}_2 &= (\alpha, 2-\alpha), \\ \tilde{x}_3 &= (-2+\alpha, -\alpha). \end{aligned}$$

5 Conclusion

The most important result of this paper is the presented general algebraic solution of FLS. This main result is a generalization of the results obtained in [4, 15, 16]. Remarkably, the obtained result in Theorem 3.3 is a generalization of the known result from [7] for classical linear systems and their general solution form based on {1}-inverses. The proposed method, as a straightforward way for finding the algebraic (strong) solutions of FLS, presents an important brick in the theory of fuzzy linear systems.

A significant contribution of this paper is the introduced algorithm which provides a general approach for solving fuzzy linear systems of Friedman et al.'s type. The presented algorithm is based on the general algebraic (strong) solution form of $m \times n$ FLS, therefore, it provides the possibility to choose any of {1}-inverses of the coefficient matrix A, not necessarily the Moore-Penrose inverse of A. In order to demonstrate the effectiveness of the proposed method, the numerical examples have been given for three FLS with the small coefficient matrices. However, a computation of {1}-inverses is expensive for large sized matrices and solving large classical linear systems of equations traditionally has a high computational cost, what is a disadvantage. The method presented in the author's previous work [16] is limited on a square FLS, whereas the method proposed in [15] is related to general FLS with A of size $m \times n$, however, it is based on the Moore-Penrose inverse of A. The obvious advantage of this new method, in comparison with those from [15], lays in the fact that we can choose a {1}-inverse of A from $A\{1\}$ arbitrarily. The algebraic issues related to the consistency of FLS, their coefficient matrices, strong solutions and their algebraic form have been clarified, however, for undetermined FLS, we need additional analytical methods to analyze the explicit analytical form of solutions, even in the case of triangular fuzzy numbers. In the future work, further investigations on the solution set and the nature of solutions of a consistent FLS will be provided.

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