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Methods for obtaining uninorms on some special classes of bounded lattices

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Abstract

In this article, we go on to discuss the structure of uninorms on bounded lattices. We suggest two techniques to yield uninorms with some constraints on the identity element by applying that the t-norms and t-conorms are always present on the considered bounded lattices. These techniques ensure new approaches for getting idempotent uninorms on bounded lattices when regarding infimum t-norm and supremum t-conorm. Furthermore, we display the distinctness between our new construction techniques and the published ones.

Keywords: Bounded lattice, idempotent uninorm, t-norm, t-conorm, uninorm.

1 Introduction

1.1 A brief review on the development of uninorms

As a modification of the logical connectives conjunction and disjunction in the standard two-valued logic, triangular norms (t-norms, in short) and triangular conorms (t-conorms, in short) on the unit interval were introduced by Menger [37] and comprehensively studied by Schweizer and Sklar [44]. They are used more frequently in fuzzy set theory, fuzzy logic, fuzzy systems modeling, approximate reasoning, information aggregation, decision-making, and probabilistic metric spaces [1, 7, 21, 22, 34, 38, 45].

Uninorms on the unit interval $[0,1]$, suggested by Yager and Rybalov [51], are a substantial development of t-norms and t-conorms. The building of uninorms was enlightened, and two well-established families of uninorms, indicated by $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$, were delineated by Fodor et al. [25]. Unlike t-norms and t-conorms that possess the identity element 1 and 0, respectively, uninorms let an identity element e exist anywhere on the unit interval $[0,1]$. Particularly, a uninorm is a t-norm if $e = 1$, while it is a t-conorm if $e = 0$. They are influential operators from a theoretical standpoint and practical application, including fuzzy set theory, fuzzy logic, decision-making, expert systems, neural networks, image processing, and so on [15, 17, 19, 20, 26, 27, 28, 29, 30, 31, 42, 49, 50, 51].

The notion of uninorm on the unit interval was recently extended to the more general setting of bounded lattices by Karaçal and Mesiar [32]. They also classified the smallest and the greatest uninorms on bounded lattices. Hitherto, Bodjanova and Kalina [5] contribute to two remarkable techniques for building uninorms on bounded lattices originating in both t-norms and t-conorms. Çaylı et al. [12] identified two classes of internal and locally internal uninorms on bounded lattices. At the same time, Çaylı [9] worked through idempotent uninorms on bounded lattices and described a method for getting such uninorms with some additional restrictions. Dan et al. [14] suggested two techniques to obtain uninorms emerged from not only t-norms but also t-conorms on specific families of bounded lattices. In the meantime, Dan and Hu [13] proposed a construction way for uninorms on general bounded lattices utilizing only one of a t-norm

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or a t-conorm. Aşıcı and Mesiar [2, 3] introduced some notable techniques for creating uninorms on specific families of bounded lattices. Zhang et al. [53] extended the families $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ of uninorms on the real unit interval to bounded lattices and depicted the construction of their members. The related procedure to get uninorms on bounded lattices is revealed in the literature [6, 8, 10, 33, 35, 46, 48, 52].

1.2 The motivation of our studies

Different families of uninorms on the unit interval that are efficient generalizations of t-norms and t-conorms, particularly $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$, representable uninorms, and idempotent uninorms, were delineated in [15, 16, 18, 25, 39]. Following this, the definition and characterization of a uninorm on bounded lattices become a charming field of inquiry. In line with this consideration, similar to the characterization of uninorms on the unit interval, many authors have attempted to characterize uninorms that possess an identity element with the difference of the top and bottom elements on a bounded lattice. Despite the presentation of various types of uninorms on bounded lattices, their structure has not been completely determined yet. For this reason, from theoretical standpoint, the construction of uninorms on bounded lattices is an outstanding subject.

In order to make clear the structure of uninorms on bounded lattices, Karaçal and Mesiar [32] posed an open problem as follows: "when a bounded lattice $\mathbb{P}$ is a chain (in particular, $\mathbb{P} = [0, 1]$) then, for any pair (T_e, S_e) of a t-norm on $[0_\mathbb{P}, e]^2$ and a t-conorm on $[e, 1_\mathbb{P}]^2$, there are related uninorms. For which bounded lattices a similar claim is valid?"

We should perceive that considering on a bounded chain $\mathbb{C}$ (e.g., real unit interval $[0, 1]$), for any $e \in \mathbb{C}\backslash\{0_\mathbb{P}, 1_\mathbb{P}\}$, for any t-norm T_e acting on $[0_\mathbb{C}, e]$ and any t-conorm S_e acting on $[e, 1_\mathbb{C}]$, (at least two) uninorms extending T_e and S_e exist (see [25]). Even though many authors have concentrated on the generation of uninorms on bounded lattices, most of them have identified the structure of some families of uninorms on bounded lattices originated in only one of the t-norms and t-conorms. Moreover, we note that uninorms constructed by approaches in the literature [3, 5, 6, 10, 13] are not generally idempotent (locally internal).

In this article, motivated by the above observations, we purpose to answer the open problem put by [32]. To be more precise, by applying that the t-norms and t-conorms are always present on a bounded lattice, we develop two techniques for getting uninorms on bounded lattices that possess an identity element, which can be viewed as a generalization of the classes of $\mathcal{U}_{\min}$ or $\mathcal{U}_{\max}$ on the unit interval. As a consequence, we acquire idempotent (locally internal) uninorms on bounded lattices. We stress the differences between our novel construction techniques and the known ones in [2, 3, 8, 10, 13, 14, 32, 48]. Furthermore, we present a characterization of uninorms on bounded lattices based on a t-conorm and a t-subnorm (a t-norm and a t-superconorm) as building blocks. Accordingly, from the theoretical perspective, this study promotes this research subject simultaneously by virtue of t-norms and t-conorms.

The remainder of this article is organized as follows. We think back on some elementary concepts and outcomes concerning bounded lattices and uninorms on them in Section 2. Section 3 is devoted to generation techniques for uninorms on a bounded lattice $\mathbb{P}$ with some restrictions on the element $e \in \mathbb{P}\backslash\{0_\mathbb{P}, 1_\mathbb{P}\}$, playing the role of identity element, with the help of the presence of a t-norm acting on $[0_\mathbb{P}, e]^2$ and a t-conorm on $[e, 1_\mathbb{P}]^2$. This result is well known for the standard real interval $\mathbb{P} = [0, 1]$, but it is a novelty in the framework of lattices. By applying these techniques, we get two novel families of idempotent uninorms on bounded lattices when putting a t-norm $T_e = T_\wedge$ on $[0_\mathbb{P}, e]^2$ and a t-conorm $S_e = S_\vee$ on $[e, 1_\mathbb{P}]^2$. Furthermore, some illustrative examples are presented for a better understanding of the structure of uninorms obtained by our methods.We also exemplify that our constructions are distinct from the present ones. Section 4 is devoted to characterizing the classes of uninorms introduced in Section 3 by virtue of t-subnorms and t-superconorms. Finally, our studies are concluded.

2 Preliminaries

We will review fundamental ideas and terminology for bounded lattices and basic features linked to them in this part.

Letting a set $\mathbb{P} \neq \emptyset$, where $\leqslant$ is a partial order on $\mathbb{P}$, it is said to be a lattice [4] if, for arbitrarily chosen u,$v \in \mathbb{P}$, $\{u, v\}$ has both the greatest lower bound (infimum) and the smallest upper bound (supremum), written as, respectively, $u \wedge v$ and $u \vee v$. Given $l_1, l_2 \in \mathbb{P}$, the symbol $l_1 < l_2$ implies that $l_1 \leqslant l_2$ and $l_1 \neq l_2$. If $l_1 \leqslant l_2$ or $l_2 \leqslant l_1$, they are called comparable elements. In other cases, l_1 and l_2 are incomparable, and, for this statement, we use the symbol $l_1 \parallel l_2$. Given $l \in \mathbb{P}$, $\mathbb{I}_l$ implies the set of all $u \in \mathbb{P}$ such that $u \parallel l$; i.e., $\mathbb{I}_l = \{u \in \mathbb{P} : u \| l\}$. Suppose that $\leqslant^t$ denotes the transpose of $\leqslant$ that is a partial order on a lattice $\mathbb{P}$; namely, $u \leqslant^t v$ iff $v \leqslant u$. Then, we note that it is a partial order on $\mathbb{P}$. Moreover, the infimum and the supremum in regard to $\leqslant^t$ are, respectively, $\vee$ and $\wedge$. That is, $(\mathbb{P}, \leqslant^t, \vee, \wedge)$ is a lattice, which is said to be a dual lattice to $(\mathbb{P}, \leqslant, \wedge, \vee)$.

A lattice $(\mathbb{P}, \leqslant, \wedge, \vee)$ is bounded when two elements, $1_\mathbb{P}$ and $0_\mathbb{P}$, in $\mathbb{P}$ exist such that $0_\mathbb{P} \leqslant u \leqslant 1_\mathbb{P}$ for all $u \in \mathbb{P}$. Unless otherwise noted, $\mathbb{P}$ refers to a bounded lattice $(\mathbb{P}, \leqslant, \wedge, \vee)$ that possesses the top element $1_\mathbb{P}$ and the bottom element $0_{\mathbb{P}}$ in this article.

Letting $l_1, l_2 \in \mathbb{P}$, where $l_1 \leqslant l_2$, we state the subinterval $[l_1, l_2]$ such that

$$[l_1, l_2] = \{u \in \mathbb{P} : l_1 \leqslant u \leqslant l_2\}.$$

The subintervals $[l_1, l_2[$, $]l_1, l_2]$, and $]l_1, l_2[$ can be defined in the same way. $([l_1, l_2], \leqslant, \wedge, \vee)$ is a bounded lattice that possesses the top element l_2 and the bottom element l_1. The sublattice $[l_1, l_2]$ of $(\mathbb{P}, \leqslant^t, \vee, \wedge)$ is identical to the sublattice $[l_2, l_1]$ of $(\mathbb{P}, \leqslant, \wedge, \vee)$.

Definition 2.1. [12, 32, 33] *A function $U : \mathbb{P}^2 \to \mathbb{P}$ is said to be a uninorm on $\mathbb{P}$ when it fulfills the undermentioned properties: for any $u, v, r \in \mathbb{P}$,*
(i) $U(v, u) = U(u, v)$ (commutativity);
(ii) If $v \leqslant u$, then $U(v, r) \leqslant U(u, r)$ (increasingness);
(iii) $U(v, U(u, r)) = U(U(v, u), r)$ (associativity);
(iv) there is an element $e \in \mathbb{P}$, called identity element of U, such that $U(v, e) = v$ (identity element).

We say that a uninorm U is *idempotent* on $\mathbb{P}$ whenever $U(v, v) = v$ for all $v \in \mathbb{P}$ (see [9, 40]).

We say that a uninorm U is conjunctive (resp. disjunctive) on $\mathbb{P}$ whenever $U(0_{\mathbb{P}}, 1_{\mathbb{P}}) = 0_{\mathbb{P}}$ (resp. $U(0_{\mathbb{P}}, 1_{\mathbb{P}}) = 1_{\mathbb{P}}$) (see [8]).

We say that a uninorm U is locally internal on $\mathbb{P}$ whenever $U(u, v) \in \{u, u \wedge v, u \vee v, v\}$ for all $u, v \in \mathbb{P}$ (see [12]).

Precisely, a t-norm T (resp. t-conorm S) on a bounded lattice $\mathbb{P}$ is a uninorm U on $\mathbb{P}$ that possesses the identity element $e = 1_{\mathbb{P}}$ (resp. $e = 0_{\mathbb{P}}$) (see [11, 23, 24, 36, 43, 47]). Notice that S is a t-conorm on the sublattice $[l_1, l_2]$ of $(\mathbb{P}, \leqslant, \wedge, \vee)$ iff it is a t-norm on the sublattice $[l_2, l_1]$ of $(\mathbb{P}, \leqslant^t, \vee, \wedge)$, which implies that they are dual.

Example 2.2. (i) *The largest t-norm on $[l_1, l_2]$ is $T_\wedge$, stated by, for all $v, r \in [l_1, l_2]$, $T_\wedge(v, r) = v \wedge r$, whereas the smallest one on $[l_1, l_2]$ is T_D, putting the output $v \wedge r$ when $l_2 \in \{v, r\}$ and l_1 in other cases. Thus, for any t-norm T on $[l_1, l_2]$, there holds $T_D \leqslant T \leqslant T_\wedge$.*

(ii) *The smallest t-conorm on $[l_1, l_2]$ is $S_\vee$, stated by, for all $v, r \in [l_1, l_2]$, $S_\vee(v, r) = v \vee r$, whereas the largest one on $[l_1, l_2]$ is S_D, putting the output $v \vee r$ when $l_1 \in \{v, r\}$ and l_2 in other cases. Thus, for any t-conorm S on $[l_1, l_2]$, there holds $S_\vee \leqslant S \leqslant S_D$.*

Proposition 2.3. [32] *Suppose that U is a uninorm on $\mathbb{P}$ that possesses the identity element $e \in \mathbb{P} \backslash \{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$. Then, the undermentioned statements are valid:*
(i) $T_e = U|_{[0_{\mathbb{P}}, e]^2} : [0_{\mathbb{P}}, e]^2 \to [0_{\mathbb{P}}, e]$ is a t-norm on $[0_{\mathbb{P}}, e]^2$.
(ii) $S_e = U|_{[e, 1_{\mathbb{P}}]^2} : [e, 1_{\mathbb{P}}]^2 \to [e, 1_{\mathbb{P}}]$ is a t-conorm on $[e, 1_{\mathbb{P}}]^2$.

Definition 2.4. [41, 53] *A binary operation $F : \mathbb{P}^2 \to \mathbb{P}$ is called a t-subnorm (resp. t-superconorm) on $\mathbb{P}$ if it is commutative, increasing, associative, and $F(u, v) \leqslant u \wedge v$ (resp. $u \vee v \leqslant F(u, v)$) for any $u, v \in \mathbb{P}$.*

3 Uninorms with underlying t-norms and t-conorms

This section is devoted to developing two procedures for creating uninorms on a bounded lattice $\mathbb{P}$ that possess an identity element $e \in \mathbb{P} \backslash \{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ with some additional suppositions issued from a t-norm T_e and a t-conorm S_e allowed to act on $[0_{\mathbb{P}}, e]^2$ and $[e, 1_{\mathbb{P}}]^2$, respectively. We note that if the considered t-norm T_e on $[0_{\mathbb{P}}, e]^2$ and t-conorm S_e on $[e, 1_{\mathbb{P}}]^2$ are infimum and supremum, respectively, that is, they are idempotent, then our procedures yield novel types of idempotent uninorms. We also describe some related examples in order to emphasize the distinctness between our creations and the existing ones.

Theorem 3.1. *Suppose that $e \in \mathbb{P} \backslash \{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ and $a < c$ for all $a < e$ and $c \parallel e$. Given a t-norm $T_e : [0_{\mathbb{P}}, e]^2 \to [0_{\mathbb{P}}, e]$ and a t-conorm $S_e : [e, 1_{\mathbb{P}}]^2 \to [e, 1_{\mathbb{P}}]$, then the function $U' : \mathbb{P}^2 \to \mathbb{P}$ expressed by*

$$U'(u, v) = \begin{cases} T_e(u, v) & \text{if } (u, v) \in [0_{\mathbb{P}}, e]^2, \\ S_e(u, v) & \text{if } (u, v) \in [e, 1_{\mathbb{P}}]^2, \\ u \wedge v & \text{if } (u, v) \in [0_{\mathbb{P}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{P}}, e[, \\ v & \text{if } (u, v) \in \{e\} \times \mathbb{I}_e, \\ u & \text{if } (u, v) \in \mathbb{I}_e \times \{e\}, \\ u \vee v & \text{otherwise,} \end{cases} \tag{1}$$

is a uninorm on $\mathbb{P}$ that possesses the identity element e iff $c < b$ and $c \vee d \parallel e$ for all $e < b$ and $c, d \parallel e$.

Proof. Necessity. We note that for any $c, d \parallel e$, either $c \vee d \parallel e$ or $c \vee d > e$ holds. If there are $c, d \parallel e$ such that $c \vee d > e$, then we have

$$U'((c,d), 0_{\mathbb{P}}) = U'(c \vee d, 0_{\mathbb{P}}) = c \vee d \vee 0_{\mathbb{P}} = c \vee d,$$

and

$$U'(c, (d, 0_{\mathbb{P}})) = U'(c, d \wedge 0_{\mathbb{P}}) = U'(c, 0_{\mathbb{P}}) = c \wedge 0_{\mathbb{P}} = 0_{\mathbb{P}},$$

which implies that the associativity of U' is violated. Thus, it is not a uninorm on $\mathbb{P}$. Therefore, to ensure the associativity of U', it must hold that $c \vee d \parallel e$ for all $c, d \parallel e$.

We note that for any $e < b < 1$ and $c \parallel e$, either $b \parallel c$ or $b > c$ holds. If there are $e < b < 1$ and $c \parallel e$ such that $b \parallel c$, then we have

$$U'((b,c), 0_{\mathbb{P}}) = U'(b \vee c, 0_{\mathbb{P}}) = b \vee c \vee 0_{\mathbb{P}} = b \vee c,$$

and

$$U'(b, (c, 0_{\mathbb{P}})) = U'(b, c \wedge 0_{\mathbb{P}}) = U'(b, 0_{\mathbb{P}}) = b \vee 0_{\mathbb{P}} = b,$$

which implies that the associativity of U' is violated. Thus, it is not a uninorm on $\mathbb{P}$. Therefore, to ensure the associativity of U', it must hold that $b > c$ for all $e < b$ and $c \parallel e$.

Sufficiency. By the definition of U', it is explicit that U' is a commutative function that possesses the identity element e. Let us check that the increasingness and the associativity.

Increasingness: Take $u, v, w \in \mathbb{P}$ such that $u \leqslant v$. We prove that U' fulfills the inequality $U'(u,w) \leqslant U'(v,w)$. Whenever $(u,v) \in [0_{\mathbb{P}}, e]^2 \cup]e, 1_{\mathbb{P}}]^2 \cup \mathbb{I}_e \times \mathbb{I}_e$, the increasingness is trivial. Whenever at least one of the elements u, v, w is e, the increasingness is also trivial. We take a note of the undermentioned possibilities.

1. Let $u < e$.
   1.1. $y > e$,
      1.1.1. $w < e$,
      $$U'(u,w) = T_e(u,w) \leqslant v = U'(v,w).$$
      1.1.2. $w > e$,
      $$U'(u,w) = w \leqslant S_e(v,w) = U'(v,w).$$
      1.1.3. $w \parallel e$,
      $$U'(u,w) = u \wedge w \leqslant v \vee w = U'(v,w).$$
   1.2. $v \parallel e$,
      1.2.1. $w < e$,
      $$U'(u,w) = T_e(u,w) \leqslant v \wedge w = U'(v,w).$$
      1.2.2. $w > e$,
      $$U'(u,w) = w \leqslant v \vee w = U'(v,w).$$
      1.2.3. $w \parallel e$,
      $$U'(u,w) = u \wedge w \leqslant v \vee w = U'(v,w).$$
2. Let $u \parallel e$ and $v > e$.
   2.1. $w < e$,
   $$U'(u,w) = u \wedge w \leqslant v = U'(v,w).$$
   2.2. $w > e$,
   $$U'(u,w) = u \vee w \leqslant S_e(v,w) = U'(v,w).$$
   2.3. $w \parallel e$,
   $$U'(u,w) = u \vee w \leqslant v \vee w = U'(v,w).$$

Associativity: Take $u, v, w \in \mathbb{B}$. We prove that U' fulfills the equality $U'(u, U'(v,w)) = U'(U'(u,v), w)$. Whenever at least one of the elements is e, the associativity is trivial. Taking into account that U' is commutative, we take a note of the undermentioned possibilities.

1. Let $u < e$.
   1.1. $v < e$,

1.1.1. $w < e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'\left(u,T_e\left(v,w\right)\right) = T_e\left(u,T_e\left(v,w\right)\right) \\ &= \ T_e\left(T_e\left(u,v\right),w\right) = U'(T_e(u,v),w) \\ &= U'(U'(u,v),w). \end{aligned}$$

1.1.2. $w > e$,

$$\begin{aligned} U'(u,U'(v,w) &= U'\left(u,w\right) = u \vee w = w \\ &= T_e(u,v) \vee w = U'(T_e(u,v),w) \\ &= U'(U'(u,v),w). \end{aligned}$$

1.1.3. $w \parallel e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'\left(u,v \wedge w\right) = U'\left(u,v\right) = T_e\left(u,v\right) \\ &= T_e(u,v) \wedge w = U'(T_e(u,v),w) \\ &= U'(U'(u,v),w). \end{aligned}$$

1.2. $v > e$,

1.2.1. $w < e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'\left(u,v \vee w\right) = U'\left(u,v\right) = u \vee v = v \\ &= v \vee w = U'\left(v,w\right) = U'\left(u \vee v,w\right) \\ &= U'(U'(u,v),w). \end{aligned}$$

1.2.2. $w > e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'\left(u,S_e(v,w)\right) = u \vee S_e(v,w) = S_e(v,w) \\ &= U'\left(v,w\right) = U'\left(u \vee v,w\right) \\ &= U'(U'(u,v),w). \end{aligned}$$

1.2.3. $w \parallel e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'\left(u,v \vee w\right) = u \vee (v \vee w) = v \vee w \\ &= U'\left(v,w\right) = U'\left(u \vee v,w\right) \\ &= U'(U'(u,v),w). \end{aligned}$$

1.3. $v \parallel e$,

1.3.1. $w < e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'\left(u,v \wedge w\right) = U'\left(u,w\right) = T_e\left(u,w\right) \\ &= U'\left(u,w\right) = U'\left(u \wedge v,w\right) \\ &= U'(U'(u,v),w). \end{aligned}$$

1.3.2. $w > e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'\left(u,v \vee w\right) = U'\left(u,w\right) = u \vee w = w \\ &= u \vee w = U'\left(u,w\right) = U'\left(u \wedge v,w\right) \\ &= U'(U'(u,v),w). \end{aligned}$$

1.3.3. $w \parallel e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'\left(u,v \vee w\right) = u \wedge (v \vee w) = u \\ &= u \wedge w = U'\left(u,w\right) = U'\left(u \wedge v,w\right) \\ &= U'(U'(u,v),w). \end{aligned}$$

2. Let $u > e$.

2.1. $v < e$,

2.1.1. $w > e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'\left(u,v \vee w\right) = U'\left(u,w\right) = S_e\left(u,w\right) \\ &= U'\left(u,w\right) = U'\left(u \vee v,w\right) \\ &= U'(U'(u,v),w). \end{aligned}$$

2.1.2. $w \parallel e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'\left(u,v \wedge w\right) = u \vee (v \wedge w) = u \\ &= u \vee w = U'\left(u,w\right) = U'\left(u \vee v,w\right) \\ &= U'(U'(u,v),w). \end{aligned}$$

2.2. $v > e$,

2.2.1. $w > e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'(u, S_e(v,w)) = S_e(u, S_e(v,w)) \\ &= S_e(S_e(u,v), w) = U'(S_e(u,v), w) \\ &= U'(U'(u,v),w). \end{aligned}$$

2.2.2. $w \parallel e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'(u, v \vee w) = U'(u,v) = S_e(u,v) \\ &= S_e(u,v) \vee w = U'(S_e(u,v), w) \\ &= U'(U'(u,v),w). \end{aligned}$$

2.3. $v \parallel e$,

2.3.1. $w > e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'(u, v \vee w) = U'(u,w) = S_e(u,w) \\ &= U'(u,w) = U'(u \vee v, w) \\ &= U'(U'(u,v),w). \end{aligned}$$

2.3.2. $w \parallel e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'(u, v \vee w) = u \vee (v \vee w) = u \\ &= u \vee w = U'(u,w) = U'(u \vee v, w) \\ &= U'(U'(u,v),w). \end{aligned}$$

3. Let $u \parallel e$.

3.1. $v < e$ and $w \parallel e$,

$$\begin{aligned} U'(u,U'(v,w)) &= U'(u, v \wedge w) = u \wedge v \wedge w \\ &= U'(u \wedge v, w) = U'(U'(u,v),w). \end{aligned}$$

3.2. ($v > e$ and $w \parallel e$) or ($v \parallel e$ and $w \parallel e$),

$$\begin{aligned} U'(u,U'(v,w)) &= U'(u, v \vee w) = u \vee v \vee w \\ &= U'(u \vee v, w) = U'(U'(u,v),w). \end{aligned}$$

□

Remark 3.2. *Notice that $U' : \mathbb{P}^2 \to \mathbb{P}$ examined in Theorem 3.1 may also be represented as*

$$U'(u,v) = \begin{cases} T_e(u,v) & \text{if } (u,v) \in [0_{\mathbb{P}}, e]^2, \\ S_e(u,v) & \text{if } (u,v) \in [e, 1_{\mathbb{P}}]^2, \\ u \wedge v & \text{if } (u,v) \in [0_{\mathbb{P}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{P}}, e[\,, \\ u \vee v & \text{if } (u,v) \in]e, 1_{\mathbb{P}}] \times \mathbb{I}_e \cup \mathbb{I}_e \times]e, 1_{\mathbb{P}}] \cup \mathbb{I}_e \times \mathbb{I}_e \\ & \qquad \cup [0_{\mathbb{P}}, e[\times]e, 1_{\mathbb{P}}] \cup]e, 1_{\mathbb{P}}] \times [0_{\mathbb{P}}, e[\,, \\ v & \text{if } (u,v) \in \{e\} \times \mathbb{I}_e, \\ u & \text{if } (u,v) \in \mathbb{I}_e \times \{e\}. \end{cases}$$

Observe that in Theorem 3.1, we cannot generally remove the condition $a < c$ for all $a < e$ and $c \parallel e$. In the following, we give an example of a bounded lattice that violates this condition on which the function U' expressed by (1) in Theorem 3.1 is not a uninorm.

Example 3.3. *The lattice $\mathbb{P}_1$ (Fig. 1) is a negative example, where for an indicated identity element e, the condition $a < c$ for all $a < e$ and $c \parallel e$ in Theorem 3.1 is violated because $x \parallel p$ for $x < e$ and $p \parallel e$. By virtue of the method in Theorem 3.1, we have $U'(x, U'(p,q)) = U'(x, p \vee q) = U'(x,k) = x \wedge k = t$ and $U'(U'(x,p), q) = U'(x \wedge p, q) = U'(t,q) = t \wedge q = 0_{\mathbb{P}_1}$, which implies the associativity of U'. Hence,it is not a uninorm on $\mathbb{P}_1$.*

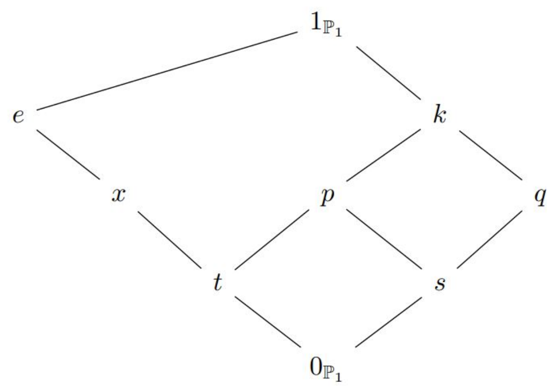

Figure 1: The lattice $\mathbb{P}_1$

By Example 3.3, we see that the condition $a < c$ for all $a < e$ and $c \parallel e$ is a sufficient condition for the function U' expressed by (1) in Theorem 3.1 being a uninorm on $\mathbb{P}$. However, it is not a necessary condition for U' being a uninorm on $\mathbb{P}$. Let us show this argument in the undermentioned example:

Example 3.4. *Consider the lattice $\mathbb{P}_2 = \{0_{\mathbb{P}_2}, x, e, p, 1_{\mathbb{P}_2}\}$ such that $0_{\mathbb{P}_2} < x < e < 1_{\mathbb{P}_2}$, $p \parallel e$ and $x \parallel p$. If we utilize the method in Theorem 3.1 and choose $T_e = T_D$ on $[0_{\mathbb{P}_2}, e]^2$, then the function U' on $\mathbb{P}_2$ is given as in Table 1. Evidently, it is a uninorm on $\mathbb{P}_2$.*

U'	$0_{\mathbb{P}_2}$	x	e	p	$1_{\mathbb{P}_2}$
$0_{\mathbb{P}_2}$	$0_{\mathbb{P}_2}$	$0_{\mathbb{P}_2}$	$0_{\mathbb{P}_2}$	$0_{\mathbb{P}_2}$	$1_{\mathbb{P}_2}$
x	$0_{\mathbb{P}_2}$	$0_{\mathbb{P}_2}$	x	$0_{\mathbb{P}_2}$	$1_{\mathbb{P}_2}$
e	$0_{\mathbb{P}_2}$	x	e	p	$1_{\mathbb{P}_2}$
p	$0_{\mathbb{P}_2}$	$0_{\mathbb{P}_2}$	p	p	$1_{\mathbb{P}_2}$
$1_{\mathbb{P}_2}$	$1_{\mathbb{P}_2}$	$1_{\mathbb{P}_2}$	$1_{\mathbb{P}_2}$	$1_{\mathbb{P}_2}$	$1_{\mathbb{P}_2}$

Table 1: The uninorm U' in $\mathbb{P}_2$

If we suppose in Theorem 3.1 that $T_e = T_\wedge$ on $[0_{\mathbb{P}}, e]^2$ and $S_e = S_\vee$ on $[e, 1_{\mathbb{P}}]^2$, we get the undermentioned Corollary 3.5 that exhibits the presence of a novel family of idempotent uninorms on $\mathbb{P}$ that possesses the identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$.

Corollary 3.5. *Suppose that $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ and $a < c$ for all $a < e$ and $c \parallel e$. Then the function $U'_{\wedge,\vee} : \mathbb{P}^2 \to \mathbb{P}$ expressed by*

$$U'_{\wedge,\vee}(u,v) = \begin{cases} u \wedge v & \text{if } (u,v) \in [0,e]^2 \cup [0,e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0,e[\,, \\ v & \text{if } (u,v) \in \{e\} \times \mathbb{I}_e, \\ u & \text{if } (u,v) \in \mathbb{I}_e \times \{e\}\,, \\ u \vee v & \text{otherwise}, \end{cases}$$

is an idempotent uninorm on $\mathbb{P}$ that possesses the identity element e iff $c < b$ and $c \vee d \parallel e$ for all $e < b$ and $c, d \parallel e$.

Remark 3.6. *Consider the uninorm $U' : \mathbb{P}^2 \to \mathbb{P}$ in Theorem 3.1.*
(i) U' is a disjunctive uninorm, i.e., $U'(0_{\mathbb{P}}, 1_{\mathbb{P}}) = 1_{\mathbb{P}}$.
(ii) It is well acknowledged that the presence of uninorms on a bounded lattice was proved by Karaçal and Mesiar [32]. U' is applied to prove the presence of a novel family of idempotent uninorms on $\mathbb{P}$ with some additional restrictions on the identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ when putting $T_e = T_\wedge$ on $[0_{\mathbb{P}}, e]^2$ and $S_e = S_\vee$ on $[e, 1_{\mathbb{P}}]^2$.
(iii) If $\mathbb{P} = [0,1]$, then $U' = \mathcal{U}_{\max}$.
(iv) U' is equivalent the equation stated as follows:

$$U'(u,v) = \begin{cases} T_e(u,v) & \text{if } (u,v) \in [0_{\mathbb{P}}, e]^2, \\ S_e(u,v) & \text{if } (u,v) \in [e, 1_{\mathbb{P}}]^2, \\ u \wedge v & \text{if } (u,v) \in [0_{\mathbb{P}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{P}}, e[\,, \\ \eta(u) \vee \eta(v) & \text{otherwise}, \end{cases}$$

where a mapping $\eta : \mathbb{P} \to \mathbb{P}$ *is defined such that*

$$\eta(v) = \begin{cases} 0_{\mathbb{P}} & \text{if } v \in [0_{\mathbb{P}}, e], \\ v & \text{otherwise.} \end{cases}$$

In the undermentioned example, we point out that the uninorm $U' : \mathbb{P}^2 \to \mathbb{P}$ in Theorem 3.1 does not need to coincide with the uninorms U defined in the articles [2, 3, 8, 10, 13, 14, 32, 48]. We first present the lattice $\mathbb{P}_3$ which satisfy the constraints of Theorem 3.1. Next, we indicate the distinction between the uninorm U' and the uninorms U in [2, 3, 8, 10, 13, 14, 32, 48].

Example 3.7. *The lattice* $\mathbb{P}_3 = \{0_{\mathbb{P}_3}, e, x, y, t, z, 1_{\mathbb{P}_3}\}$ *drawn in Fig. 2 is a positive example fulfilling the restriction of Theorem 3.1. Through Theorem 3.1, the uninorm* U' *on* $\mathbb{P}_3$ *is given as in Table 2.*

- *The uninorms built by* [2, Theorem 6], [3, Theorem 7], [8, Theorem 7], [10, Theorem 11], [14, Theorem 3.11], [13, Theorem 3.14], [32, Theorem 1 (U_s)], *and* [48, Theorem 3.2] *are conjunctive.*
- *The uninorm* $U : \mathbb{P}_3^2 \to \mathbb{P}_3$ *built by* [2, Theorem 7], [3, Theorem 6], [10, Theorem 10], and [14, Theorem 3.10] *satisfies that* $U(0_{\mathbb{P}_3}, x) = x$.
- *The uninorm* $U : \mathbb{P}_3^2 \to \mathbb{P}_3$ *built by* [8, Theorem 10] *satisfies that* $U(y, x) = t$.
- *The uninorm* $U : \mathbb{P}_3^2 \to \mathbb{P}_3$ *built by* [13, Theorem 3.7] *and* [48, Theorem 3.1] *satisfies that* $U(t, x) = 1_{\mathbb{P}_3}$.
- *The uninorm* $U_t : \mathbb{P}_3^2 \to \mathbb{P}_3$ *built by* [32, Theorem 1] *satisfies that* $U_t(z, z) = 1_{\mathbb{P}_3}$.
- *The uninorm* $U' : \mathbb{P}_3^2 \to \mathbb{P}_3$ *built by Theorem 3.1 is disjunctive and satisfies that* $U'(0_{\mathbb{P}_3}, x) = 0_{\mathbb{P}_3}$, $U'(y, x) = U'(z, z) = z$, $U'(t, x) = x$.

Hence, the uninorm U' *on* $\mathbb{P}_3$ *defined by Theorem 3.1 is different from those in [2, 3, 8, 10, 13, 14, 32, 48].*

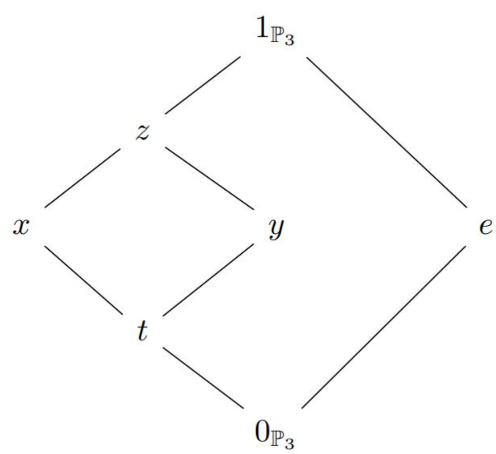

Figure 2: The lattice $\mathbb{P}_3$

U'	0_{P_3}	t	x	y	z	e	$1_{\mathbb{P}_3}$
0_{P_3}	$0_{\mathbb{P}_3}$	0_{P_3}	0_{P_3}	0_{P_3}	0_{P_3}	0_{P_3}	1_{P_3}
t	0_{P_3}	t	x	y	z	t	1_{P_3}
x	0_{P_3}	x	x	z	z	x	1_{P_3}
y	0_{P_3}	y	z	y	z	y	1_{P_3}
z	0_{P_3}	z	z	z	z	z	1_{P_3}
e	0_{P_3}	t	x	y	z	e	$1_{\mathbb{P}_2}$
1_{P_3}	1_{P_3}	1_{P_3}	1_{P_3}	1_{P_3}	1_{P_3}	1_{P_3}	1_{P_3}

Table 2: The uninorm U' in $\mathbb{P}_3$

In the following, we suggest a procedure to generate a uninorm on a bounded lattice $\mathbb{P}$ that possesses an identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ under some additional suppositions emerged from a t-norm T_e and a t-conorm S_e allowed to act on $[0_{\mathbb{P}}, e]^2$ and $[e, 1_{\mathbb{P}}]^2$, respectively.

Theorem 3.8. *Suppose that $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ and $a > c$ for all $a > e$ and $c \parallel e$. Given a t-norm $T_e : [0_{\mathbb{P}}, e]^2 \to [0_{\mathbb{P}}, e]$ and a t-conorm $S_e : [e, 1_{\mathbb{P}}]^2 \to [e, 1_{\mathbb{P}}]$, then the function $U'' : \mathbb{P}^2 \to \mathbb{P}$ expressed by*

$$U''(u,v) = \begin{cases} T_e(u,v) & \text{if } (u,v) \in [0_{\mathbb{P}}, e]^2, \\ S_e(u,v) & \text{if } (u,v) \in [e, 1_{\mathbb{P}}]^2, \\ u \vee v & \text{if } (u,v) \in]e, 1_{\mathbb{P}}] \times \mathbb{I}_e \cup \mathbb{I}_e \times]e, 1_{\mathbb{P}}], \\ v & \text{if } (u,v) \in \{e\} \times \mathbb{I}_e, \\ u & \text{if } (u,v) \in \mathbb{I}_e \times \{e\}, \\ u \wedge v & \text{otherwise}, \end{cases} \tag{2}$$

is a uninorm on $\mathbb{P}$ that possesses the identity element e iff $b < c$ and $c \wedge d \parallel e$ for all $b < e$ and $c, d \parallel e$.

Proof. It is directly related to the proof of Theorem 3.8, and so, it is omitted. This observation springs from the fact that by changing, on the original bounded lattice $\mathbb{P} = (\mathbb{P}, \wedge, \vee, 0_{\mathbb{P}}, 1_{\mathbb{P}})$, $\wedge$ and $\vee$, and $0_{\mathbb{P}}$ and $1_{\mathbb{P}}$, namely, considering the lattice $\widetilde{\mathbb{P}} = (\widetilde{\mathbb{P}}, \widetilde{\wedge}, \widetilde{\vee}, \widetilde{0_{\mathbb{P}}}, \widetilde{1_{\mathbb{P}}})$ such that $\widetilde{\wedge} = \vee$, $\widetilde{\vee} = \wedge$, $\widetilde{0_{\mathbb{P}}} = 1_{\mathbb{P}}$, and $\widetilde{1_{\mathbb{P}}} = 0_{\mathbb{P}}$, we get a dual lattice $\widetilde{\mathbb{P}}$, in this duality, a t-norm (resp. t-conorm) on $\mathbb{P}$ is connected with a t-conorm (resp. t-norm) on $\widetilde{\mathbb{P}}$, in the meantime, the uninorms on $\mathbb{P}$ is connected with to uninorms on $\widetilde{\mathbb{P}}$. □

Remark 3.9. *Notice that $U'' : \mathbb{P}^2 \to \mathbb{P}$ examined in Theorem 3.8 may also be represented as*

$$U''(u,v) = \begin{cases} T_e(u,v) & \text{if } (u,v) \in [0_{\mathbb{P}}, e]^2, \\ S_e(u,v) & \text{if } (u,v) \in [e, 1_{\mathbb{P}}]^2, \\ u \vee v & \text{if } (u,v) \in]e, 1_{\mathbb{P}}] \times \mathbb{I}_e \cup \mathbb{I}_e \times]e, 1_{\mathbb{P}}], \\ u \wedge v & \begin{aligned} &\text{if } (u,v) \in [0_{\mathbb{P}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{P}}, e[\cup \mathbb{I}_e \times \mathbb{I}_e \\ &\qquad \cup [0_{\mathbb{P}}, e[\times]e, 1_{\mathbb{P}}] \cup]e, 1_{\mathbb{P}}] \times [0_{\mathbb{P}}, e[, \end{aligned} \\ v & \text{if } (u,v) \in \{e\} \times \mathbb{I}_e, \\ u & \text{if } (u,v) \in \mathbb{I}_e \times \{e\}, \end{cases}$$

Similar to Examples 3.3 and 3.4, it can be shown that the condition $a > c$ for all $a > e$ and $c \parallel e$ is a sufficient condition for the function U'' expressed by (2) in Theorem 3.8 being a uninorm on $\mathbb{P}$. On the other hand, it is not a necessary condition for U'' being a uninorm on $\mathbb{P}$.

If we suppose in Theorem 3.8 that $T_e = T_\wedge$ on $[0_{\mathbb{P}}, e]^2$ and $S_e = S_\vee$ on $[e, 1_{\mathbb{P}}]^2$, we get the undermentioned Corollary 3.10 that indicates the presence of a novel family of idempotent uninorms on $\mathbb{P}$ that possesses the identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$.

Corollary 3.10. *Suppose that $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ and $a > c$ for all $a > e$ and $c \parallel e$. Then the function $U''_{\wedge,\vee} : \mathbb{P}^2 \to \mathbb{P}$ expressed by*

$$U''_{\wedge,\vee}(u,v) = \begin{cases} u \vee v & \text{if } (u,v) \in [e,1]^2 \cup [e,1[\times \mathbb{I}_e \cup \times \mathbb{I}_e \times [e,1[, \\ v & \text{if } (u,v) \in \{e\} \times \mathbb{I}_e, \\ u & \text{if } (u,v) \in \mathbb{I}_e \times \{e\}, \\ u \wedge v & \text{otherwise}, \end{cases}$$

is an idempotent uninorm on $\mathbb{P}$ that possesses the identity element e iff $b < c$ and $c \wedge d \parallel e$ for all $b < e$ and $c, d \parallel e$.

Remark 3.11. *Consider the uninorm $U'' : \mathbb{P}^2 \to \mathbb{P}$ in Theorem 3.8.*
(i) U'' is a conjunctive uninorm, i.e., $U''(0_{\mathbb{P}}, 1_{\mathbb{P}}) = 0_{\mathbb{P}}$.
(ii) U'' is applied to prove the presence of a novel family of idempotent uninorms on $\mathbb{P}$ with some additional constraints on the identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ when putting $T_e = T_\wedge$ on $[0_{\mathbb{P}}, e]^2$ and $S_e = S_\vee$ on $[e, 1_{\mathbb{P}}]^2$.
(iii) If $\mathbb{P} = [0,1]$, then $U'' = \mathcal{U}_{\min}$.
(iv) U'' is equivalent the equation stated as follows:

$$U''(u,v) = \begin{cases} T_e(u,v) & \text{if } (u,v) \in [0_{\mathbb{P}}, e]^2, \\ S_e(u,v) & \text{if } (u,v) \in [e, 1_{\mathbb{P}}]^2, \\ u \vee v & \text{if } (u,v) \in]e, 1_{\mathbb{P}}] \times \mathbb{I}_e \cup \mathbb{I}_e \times]e, 1_{\mathbb{P}}], \\ \vartheta(u) \wedge \vartheta(v) & \text{otherwise}, \end{cases}$$

where a mapping $\vartheta : \mathbb{P} \to \mathbb{P}$ *is defined such that*

$$\vartheta(v) = \begin{cases} 1_{\mathbb{P}} & \text{if } v \in [e, 1_{\mathbb{P}}], \\ v & \text{otherwise.} \end{cases}$$

In the undermentioned example, we point out that the uninorm $U'' : \mathbb{P}^2 \to \mathbb{P}$ in Theorem 3.1 does not need to coincide with the uninorms U defined in the articles [2, 3, 8, 10, 13, 14, 32, 48]. On the lattice $\mathbb{P}_3$ (Fig. 2), fulfilling the constraints of Theorem 3.8, we demonstrate the distinction between the uninorm U'' and the uninorms U in [2, 3, 8, 10, 13, 14, 32, 48].

Example 3.12. *The lattice* $\mathbb{P}_3$ *drawn in Fig. 2 is also a positive example fulfilling the restriction of Theorem 3.8. Through Theorem 3.8, the uninorm* U'' *on* $\mathbb{P}_3$ *is given as in Table 3.*

- *The uninorms built by* [2, Theorem 7], [3, Theorem 6], [8, Theorem 10], [10, Theorem 10], [14, Theorem 3.10], [13, Theorem 3.7], [32, Theorem 1 (U_t)], *and* [48, Theorem 3.2] *are disjunctive.*
- *The uninorm* $U : \mathbb{P}_3^2 \to \mathbb{P}_3$ *built by* [2, Theorem 6], [3, Theorem 7], [10, Theorem 11], *and* [14, Theorem 3.11] *satisfies that* $U(1_{\mathbb{P}_3}, x) = x$.
- *The uninorm* $U : \mathbb{P}_3^2 \to \mathbb{P}_3$ *built by* [8, Theorem 7] *satisfies that* $U(y, x) = z$.
- *The uninorm* $U : \mathbb{P}_3^2 \to \mathbb{P}_3$ *built by* [13, Theorem 3.14] *and* [48, Theorem 3.1] *satisfies that* $U(y, z) = 0_{\mathbb{P}_3}$.
- *The uninorm* $U_s : \mathbb{P}_3^2 \to \mathbb{P}_3$ *built by* [32, Theorem 1] *satisfies that* $U_s(t, t) = 0_{\mathbb{P}_3}$.
- *The uninorm* $U' : \mathbb{P}_3^2 \to \mathbb{P}_3$ *built by Theorem 3.1 is conjunctive and satisfies that* $U'(1_{\mathbb{P}_3}, x) = 1_{\mathbb{P}_3}$, $U'(y, x) = U'(t, t) = t$, $U'(y, z) = y$.

Hence, the uninorm U'' *on* $\mathbb{P}_3$ *defined by Theorem 3.8 is different from those in [2, 3, 8, 10, 13, 14, 32, 48].*

U''	0_{P_3}	t	x	y	z	e	$1_{\mathbb{P}_3}$
0_{P_3}	$0_{\mathbb{P}_3}$	0_{P_3}	0_{P_3}	0_{P_3}	0_{P_3}	0_{P_3}	0_{P_3}
t	0_{P_3}	t	t	t	t	t	1_{P_3}
x	0_{P_3}	t	x	t	x	x	1_{P_3}
y	0_{P_3}	t	t	y	y	y	1_{P_3}
z	0_{P_3}	t	x	y	z	z	1_{P_3}
e	0_{P_3}	t	x	y	z	e	$1_{\mathbb{P}_2}$
1_{P_3}	0_{P_3}	1_{P_3}	1_{P_3}	1_{P_3}	1_{P_3}	1_{P_3}	1_{P_3}

Table 3: The uninorm U'' in $\mathbb{P}_3$

4 Characterizations of the introduced classes of uninorms

This section is devoted to characterizing the classes of uninorms introduced in Theorems 3.1 and 3.8 with the help of t-subnorms and t-superconorms. In line with this purpose, we characterize in Theorem 4.1 uninorms on bounded lattices derived from a t-conorm and a t-subnorm. Furthermore, Theorem 4.2 presents a characterization of uninorms on bounded lattices based on a t-norm and a t-superconorm.

Theorem 4.1. *Suppose that* $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$, $F : \mathbb{P}^2 \to \mathbb{P}$ *is a t-subnorm,* $S : [e, 1_{\mathbb{P}}]^2 \to [e, 1_{\mathbb{P}}]$ *is a t-conorm and* $U'_F : \mathbb{P}^2 \to \mathbb{P}$ *is a function expressed by*

$$U'_F(u, v) = \begin{cases} S(u, v) & \text{if } (u, v) \in [e, 1_{\mathbb{P}}]^2, \\ F(u, v) & \text{if } (u, v) \in [0_{\mathbb{P}}, e[^2 \cup [0_{\mathbb{P}}, e[\times \mathbb{I}_e \cup \mathbb{I}_e \times [0_{\mathbb{P}}, e[, \\ v & \text{if } (u, v) \in \{e\} \times (\mathbb{I}_e \cup [0, e]), \\ u & \text{if } (u, v) \in ([0, e] \cup \mathbb{I}_e) \times \{e\}, \\ u \vee v & \text{otherwise.} \end{cases}$$

Then U'_F *is a uninorm on* $\mathbb{P}$ *that possesses the identity element* e *iff the following cases hold:*
(i) $c < b$ *and* $c \vee d \parallel e$ *for all* $e < b$ *and* $c, d \parallel e$.
(ii) $F(F(f, c), d) = F(f, c \vee d)$ *for all* $f < e$ *and* $c, d \parallel e$.

Proof. Necessity. Let U'_F be a uninorm on $\mathbb{P}$ that possesses the identity element e. Taking into account Theorem 3.1, the case (i) holds. Hence, we show that the case (ii) holds. From the definition and the associativity of uninorm U'_F, we get that for all $f < e$ and $c, d \parallel e$

$$F\left(F\left(f,c\right),d\right) = U\left(F\left(f,c\right),d\right) = U\left(U\left(f,c\right),d\right) = U\left(f,U\left(c,d\right)\right) = U\left(f,c\vee d\right) = F\left(f,c\vee d\right).$$

Sufficiency. Let the cases (i) and (ii) be satisfied. By the definition of U'_F, it is explicit that U'_F is a commutative function that possesses the identity element e. Let us check that the increasingness and the associativity.
Increasingness: Take $u, v, w \in \mathbb{P}$ such that $u \leqslant v$. We prove that U'_F fulfills the inequality $U'_F(u,w) \leqslant U'_F(v,w)$. In view of Theorem 3.1, we consider only the undermentioned possibilities.

1. Let $u < e$.
   1.1. $v > e$,
      1.1.1. $w < e$,
      $$U'_F(u,w) = F\left(u,w\right) \leqslant v = U'_F(v,w).$$
      1.1.2. $w \parallel e$,
      $$U'_F(u,w) = F\left(u,w\right) \leqslant u \wedge w \leqslant v \vee w = U'_F(v,w).$$
   1.2. $v \parallel e$,
      1.2.1. $w < e$,
      $$U'_F(u,w) = F\left(u,w\right) \leqslant F\left(v,w\right) = U'_F(v,w).$$
      1.2.2. $w \parallel e$,
      $$U'_F(u,w) = F\left(u,w\right) \leqslant u \wedge w \leqslant v \vee w = U'_F(v,w).$$
2. Let $u \parallel e$, $v > e$ and $w < e$,
   $$U'_F(u,w) = F\left(u,w\right) \leqslant u \leqslant v = U'_F(v,w).$$

Associativity: Take $u, v, w \in \mathbb{B}$. We prove that U'_F fulfills the equality $U'_F(u, U'_F(v,w)) = U'_F(U'_F(u,v),w)$. In view of Theorem 3.1, we consider only the undermentioned possibilities.

1. Let $u < e$.
   1.1. $v < e$,
      1.1.1. $w < e$ or $w \parallel e$,
      $$\begin{aligned} U'_F(u,U'_F(v,w)) &= U'_F\left(u,F\left(v,w\right)\right) = F\left(u,F\left(v,w\right)\right) \\ &= F\left(F\left(u,v\right),w\right) = U'_F(F(u,v),w) \\ &= U'_F(U'_F(u,v),w). \end{aligned}$$
      1.1.2. $w > e$,
      $$\begin{aligned} U'_F(u,U'_F(v,w) &= U'_F\left(u,w\right) = u \vee w = w \\ &= F(u,v) \vee w = U'_F(F(u,v),w) \\ &= U'_F(U'_F(u,v),w). \end{aligned}$$
   1.2. $v \parallel e$,
      1.2.1. $w < e$,
      $$\begin{aligned} U'_F(u,U'_F(v,w)) &= U'_F\left(u,F\left(v,w\right)\right) = F\left(u,F\left(v,w\right)\right) \\ &= F\left(F(u,v),w\right) = U'_F\left(F(u,v),w\right) \\ &= U'_F(U'_F(u,v),w). \end{aligned}$$
      1.2.2. $w > e$,
      $$\begin{aligned} U'_F(u,U'_F(v,w)) &= U'_F\left(u,v \vee w\right) = U'_F\left(u,w\right) = u \vee w \\ &= w = F(u,v) \vee w = U'_F\left(F(u,v),w\right) \\ &= U'_F(U'_F(u,v),w). \end{aligned}$$
      1.2.3. $w \parallel e$,
      $$\begin{aligned} U'_F(u,U'_F(v,w)) &= U'_F\left(u,v \vee w\right) = F\left(u,v \vee w\right) \\ &= F\left(F(u,v),w\right) = U'_F\left(F(u,v),w\right) \\ &= U'_F(U'_F(u,v),w). \end{aligned}$$

2. Let $u > e$, $v < e$ and $w \parallel e$,

$$\begin{aligned} U'_F(u, U'_F(v,w)) &= U'_F\left(u, F\left(v,w\right)\right) = u \vee F\left(v,w\right) = u \\ &= u \vee w = U'_F\left(u,w\right) = U'_F\left(u \vee v, w\right) \\ &= U'_F(U'_F(u,v), w). \end{aligned}$$

3. Let $u \parallel e$, $v < e$ and $w \parallel e$,

$$\begin{aligned} U'_F(u, U'_F(v,w)) &= U'_F\left(u, F\left(v,w\right)\right) = F\left(u, F\left(v,w\right)\right) \\ &= F\left(F\left(u,v\right), w\right) = U'_F\left(F\left(u,v\right), w\right) \\ &= U'_F(U'_F(u,v), w). \end{aligned}$$

□

The following theorem describes a construction of uninorms on bounded lattices by virtue of a t-norm and a t-superconorm, where some necessary and sufficient conditions are required.

Theorem 4.2. *Suppose that $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$, $T : [0_{\mathbb{P}}, e]^2 \to [0_{\mathbb{P}}, e]$ is a t-norm, $G : \mathbb{P}^2 \to \mathbb{P}$ is a t-superconorm and $U''_G : \mathbb{P}^2 \to \mathbb{P}$ is a function expressed by*

$$U''_G\left(u,v\right) = \begin{cases} T\left(u,v\right) & \text{if } (u,v) \in [0_{\mathbb{P}}, e]^2, \\ G\left(u,v\right) & \text{if } (u,v) \in \left]e, 1_{\mathbb{P}}\right]^2 \cup \left]e, 1_{\mathbb{P}}\right] \times \mathbb{I}_e \cup \mathbb{I}_e \times \left]e, 1_{\mathbb{P}}\right], \\ v & \text{if } (u,v) \in \{e\} \times \left(\mathbb{I}_e \cup [e, 1_{\mathbb{P}}]\right), \\ u & \text{if } (u,v) \in \left(\mathbb{I}_e \cup [e, 1_{\mathbb{P}}]\right) \times \{e\}, \\ u \wedge v & \text{otherwise}, \end{cases}$$

Then U''_G is a uninorm on $\mathbb{P}$ that possesses the identity element e iff the following cases hold:
(i) $b < c$ and $c \wedge d \parallel e$ for all $b < e$ and $c, d \parallel e$.
(ii) $G\left(G\left(f,c\right), d\right) = G\left(f, c \wedge d\right)$ for all $f > e$ and $c, d \parallel e$.

Proof. Its proof is similar to that of Theorem 4.1. □

5 Concluding remarks

In this article, we have continued to examine uninorms that possess an identity element with the difference of the top and bottom elements on a bounded lattice from theoretical standpoint. To be more precise, we have constructed two families of uninorms on a bounded lattice $\mathbb{P}$ under some restrictions on conceive of $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ as identity element. We have mentioned the main findings in this article as follows:

- In Theorem 3.1, we have presented a novel technique for creating a family of a disjunctive uninorm U' on $\mathbb{P}$, for any fixed identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ and any couple (T_e, S_e) of a t-norm T_e on $[0, e]^2$ and a t-conorm S_e on $[e, 1]^2$, coinciding with T_e on $[0, e]^2$ and S_e on $[e, 1]^2$. Notice that some restrictions are required to that our technique yields a uninorm on $\mathbb{P}$ that possess an identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$. In Examples 3.3 and 3.4, we have exemplified the importance of the role of these restrictions in our techniques.

- As a by-product of Theorem 3.1, we have described in Corollary 3.5 a novel family of idempotent (locally internal) uninorm on $\mathbb{P}$ that possess an identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ when putting $T_e = T_\wedge$ on $[0_{\mathbb{P}}, e]^2$ and $S_e = S_\vee$ on $[e, 1_{\mathbb{P}}]^2$.

- In Example 3.7, we illustrate the distinctness between the procedure in Theorem 3.1 and the present ones in [2, 3, 8, 10, 13, 14, 32, 48].

- In Theorem 3.8, a novel technique to get a family of a conjunctive uninorm U'' on $\mathbb{P}$ that possess an identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ has been described by virtue of the presence of a t-norm T_e on $[0, e]^2$ and a t-conorm S_e on $[e, 1]^2$ under some additional conditions.

- If we consider in Theorem 3.8 that $T_e = T_\wedge$ on $[0_{\mathbb{P}}, e]^2$ and $S_e = S_\vee$ on $[e, 1_{\mathbb{P}}]^2$, a novel family of idempotent (locally internal) uninorm on $\mathbb{P}$ that possess an identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ have been obtained in Corollary 3.10.

- We have displayed an illustrative example in order to demonstrate that the technique in Theorem 3.8 differs from those earlier introduced in [2, 3, 8, 10, 13, 14, 32, 48].

- Note that due to the duality of bounded lattices $(\mathbb{P}, \leqslant, 0_{\mathbb{P}}, 1_{\mathbb{P}})$ and $(\mathbb{P}, \leqslant^t, 1_{\mathbb{P}}, 0_{\mathbb{P}})$, for any construction method for uninorms on $\mathbb{P}$ based on a t-norm, or a t-conorm, or both a t-norm and a t-conorm, we have also a related dual construction method. This fact can be observed in several papers [2, 3, 5, 8, 9, 10, 12, 13, 14, 32, 48, 52], and it is illustrated also in our case, compare Theorems 3.1 and 3.8.

- Theorem 4.1 (resp. Theorem 4.2) presents a characterization of uninorms on bounded lattices based on a t-conorm and a t-subnorm (resp. a t-norm and a t-superconorm) as building blocks.

- We note that the uninorms in Theorems 3.1 and 3.8 belong to the so-called $\mathcal{U}^{\mathrm{r}}_{\min}$ and $\mathcal{U}^{\mathrm{r}}_{\max}$, respectively, introduced in [53], where $\mathcal{U}^{\mathrm{r}}_{\min}$ and $\mathcal{U}^{\mathrm{r}}_{\max}$ denote the classes of all uninorms U on $\mathbb{P}$ with the identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ satisfying the following conditions

$$U(u,v) = u \text{ for all } (u,v) \in \,]e, 1_{\mathbb{P}}] \times \ \mathbb{P}\backslash \left[e, 1_{\mathbb{P}}\right],$$

and

$$U(u,v) = u \text{ for all } (u,v) \in [0_{\mathbb{P}}, e[\times \mathbb{P}\backslash[0_{\mathbb{P}}, e],$$

respectively. The uninorms in Theorems 4.1 and 4.2 also belong to $\mathcal{U}^{\mathrm{r}}_{\min}$ and $\mathcal{U}^{\mathrm{r}}_{\max}$, respectively. Recall that in [53], also some other classes of uniorms on $\mathbb{P}$ with the identity element $e \in \mathbb{P}\backslash\{0_{\mathbb{P}}, 1_{\mathbb{P}}\}$ were introduced and studied, in particular, the classes $\mathcal{U}_{\min}$ and $\mathcal{U}_{\max}$ characterized by the following conditions

$$U(u,v) = v \text{ for all } (u,v) \in \,]e, 1_{\mathbb{P}}] \times \ \mathbb{P}\backslash \left[e, 1_{\mathbb{P}}\right] \text{ for } U \in \mathcal{U}_{\min},$$

and

$$U(u,v) = v \text{ for all } (u,v) \in [0_{\mathbb{P}}, e[\times \mathbb{P}\backslash[0_{\mathbb{P}}, e] \text{ for } U \in \mathcal{U}_{\max}.$$

These classes are completely described in [53, Theorems 4.3 and 4.6]. Concerning the classes $\mathcal{U}^{\mathrm{r}}_{\min}$ and $\mathcal{U}^{\mathrm{r}}_{\max}$, there is only a sufficient condition for $U \in \mathcal{U}^{\mathrm{r}}_{\min}$ and $U \in \mathcal{U}^{\mathrm{r}}_{\max}$, respectively. Hence, a complete description of these two uninorm classes is still an open problem.

- Considering the lattice $\mathbb{P}_3$ drawn in Fig. 2, the disjunctive uninorm U_1 in [53, Theorem 5.7] satisfies that $U_1(y,x) \leqslant t$ and the conjunctive uninorm U_2 in [53, Theorem 5.9] satisfies that $U_2(y,x) \geqslant z$. However, the disjunctive uninorm U' constructed by Theorem 3.1 satisfies that $U'(y,x) = z > t$ and the conjunctive uninorm U'' constructed by Theorem 3.8 satisfies that $U''(y,x) = t < z$. So, uninorms obtained using the construction approaches in Theorems 3.1 and 3.8 are different from those in [53, Theorems 5.7 and 5.9]. Similarly, uninorms constructed by the methods in Theorems 4.1 and 4.2 are also different from those in [53, Theorems 5.7 and 5.9].

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