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Radical of filters on hoops

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Abstract

In this paper, by using the notion of prime filter, we show representation theorem of hoops and we prove that every nontrivial $\vee$-hoop is a subdirect product of hoop-chains. In the following, by using the concept of maximal filter of hoops, we introduce radical of hoops. Then some equivalence definitions of it and some related properties are investigated. Then by using this notion, we introduce the concepts of r-filters and p-filters on hoops and the relation between them and other filters of hoops are investigated. Finally, by using p-filters on hoops, we define new open sets on hoop that could be used to construct a Zarisky topology.

Keywords: Hoop, maximal filter, prime filter, radical, r-filter, p-filter.

1 Introduction

Non-classical logic (or called alternative logic) are formal systems that differ in a significant way from standard logical systems such as propositional and predicate logic. A hoop is a particular class of algebraic structure that is introduced by Bosbach in [12, 13]. Actually, hoops are naturally ordered commutative residuated integral monoids. The study of hoops is motivated by researchers both in universal algebra and algebraic logic. In recent years, many mathematicians have studied this algebraic structure from different perspectives such as ideals, filters, relationships with other algebraic structures, etc., and good results have been achieved in this regard which can be found in [1, 5, 6, 12, 13]. In [4], the authors defined the concept of the radical of a filter in MTL-algebras and some equivalent definitions and properties are studied. Moreover, they introduced the notion of dense element and proved some properties of it and the relation between dense element and radical filters. In [18], Paad and Borzooei defined a new filter on BL-algebra and called it semi-maximal filter. Then they studied the radical of them and showed that any semi-maximal filter is a primary filter. Moreover, they investigated the relationship among semi-maximal filters and other types of filters. Specially, they showed that any fantastic filter is a semi-maximal filter and any semi-maximal filter is an (n-fold) positive implicative filter in a Gödel algebra. In [16], Motamed presented new properties for the primary filters and defined the notions of prime-like filters and BL-algebras. Then by using this notion, he/she introduced Top-like BL-algebras and by using the concept of prime-like filters, he/she studied a new topology on BL-algebras. In [19], the radical of an implicative filter in Hilbert algebras is studied and some theorems are given on its properties and the authors showed that radical form a closure operator. Also, the radical of the set of all regular elements is characterized. Also, the notion of semi-maximal filter is introduced as a closed set of that closure operator and studied it in detail. In addition, in [17], the notion of the radical of a filter in BL-algebras is defined and several characterizations of the radical of a filter are given. Also, the authors proved that $\frac{A}{F}$ is an MV-algebra if and only if $D_s(A) \subseteq F$. After that the authors defined the notion of semi-maximal filter in BL-algebras and they stated and proved some theorems which determine the relationship between this notion and the other types of filters of a BL-algebra. Moreover, they proved that $\frac{A}{F}$ is a semi-simple BL-algebra

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if and only if F is a semi-maximal filter of A

In this paper, by using the notion of prime filter, we show representation theorem of hoops and we prove that every nontrivial $\vee$-hoop is a subdirect product of hoop-chains. In the following, by using the concept of maximal filter of hoops, we introduce radical of hoops. Then some equivalence definitions of it and some related properties are investigated. Then by using this notion, we introduce the concepts of r-filters and p-filters on hoops and the relation between them and other filters of hoops are investigated. Finally, by using p-filters on hoops, we define new open sets on hoop that could be used to construct a Zarisky topology.

2 Preliminaries

Here, we refer to the basic concepts and features required in the field of hoop that use in the next sections.

An algebraic structure $(H, \odot, \rightarrow, 1)$ is said to be a *hoop*, where $(H, \odot, 1)$ is a commutative monoid and for any $x, y, z \in H$ we have

(Hoop1) $x \rightarrow x = 1$,

(Hoop2) $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$,

(Hoop3) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$.

Define an order $\leq$ on H by $x \leq y$ iff $x \rightarrow y = 1$, which $(H, \leq)$ is a poset. A *bounded hoop*, is a hoop with the least element 0 where for all $x \in H$, $0 \leq x$. Then we can define a unary operation "$'$" where $x' = x \rightarrow 0$, for $x \in H$ and we said that H has (DNP) property if $x'' = x$, for all $x \in H$. Also, we can define a binary operation $\vee$ on a hoop H such that for any $x, y \in H$, $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$. If $\vee$ is a join operation on H, then H is called a $\vee$-*hoop*, and in this case, $(H, \wedge, \vee)$ is a distributive lattice (see [12, 13]).

Note. From now on, we set H as a hoop such as $(H, \odot, \rightarrow, 1)$.

Proposition 2.1. [15] *We have the next properties, for all $x, y, z \in H$:*
(i) $(H, \leq)$ is a meet-semilattice, with $x \wedge y = x \odot (x \rightarrow y)$;
(ii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$;
(iii) $x \odot y \leq x, y$ and $x \leq y \rightarrow x$;
(iv) $x \rightarrow 1 = 1$ and $1 \rightarrow x = x$;
(v) $x \leq y \rightarrow (x \odot y)$;
(vi) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$, $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$ and $x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z)$;
(vii) $x \leq y$ implies $x \odot z \leq y \odot z$, $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$;
(viii) In a bounded hoop, $x \leq x''$, $x \odot x' = 0$ and $x''' = x'$;
(ix) In a $\vee$-hoop, $(x \vee y)^n \rightarrow z = \bigwedge\{(a_1 \odot a_2 \odot \cdots \odot a_n) \rightarrow z \mid a_i \in \{x, y\}\}$, for any $n \in \mathbb{N}$;
(x) In a $\vee$-hoop, $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.
(xi) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.
(xii) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$.

Consider $\emptyset \neq F \subseteq H$ is said to be a *filter* of H if for any $x, y \in H$,
$(F1)$ $x, y \in F$ implies $x \odot y \in F$,
$(F2)$ $x \leq y$ and $x \in F$ imply $y \in F$.
The set of all filters of H is denoted by $\mathcal{F}(H)$. Clearly, $1 \in F$, for each $F \in \mathcal{F}(H)$. In addition, $F \in \mathcal{F}(H)$ is *proper* if $F \neq H$. Obviously, if F is a proper filter of H, then $0 \notin F$, where H is bounded.

Let $F \in \mathcal{F}(H)$. Then for any $x, y \in H$, define the relation $\sim_F$ by

$$x \sim_F y \;\Leftrightarrow\; x \rightarrow y \in F \text{ and } y \rightarrow x \in F.$$

Then we can see that the relation $\sim_F$ is a congruence relation on H and the algebraic structure $(\frac{H}{F}, \otimes, \rightsquigarrow, F)$ is a hoop where

$$[x] \otimes [y] = [x \odot y], \quad [x] \rightsquigarrow [y] = [x \rightarrow y],$$

for any $[x], [y] \in \frac{H}{F}$.

Consider $X \subseteq H$. Then the smallest filter of H containing X is called *the generated filter by X in H* which is denoted by $\langle X\rangle$. Indeed, $\langle X\rangle = \bigcap\limits_{X\subseteq F\in\mathcal{F}(H)} F$.

Proposition 2.2. *Consider $\emptyset \neq X \subseteq H$. Then*

$$\langle X\rangle = \{x \in H \mid \exists\, n \in \mathbb{N} \text{ and } a_1, ..., a_n \in X \text{ such that } a_1 \odot a_2 \odot ... \odot a_n \leq x\}.$$

In particular, for any element $a \in H$, we have $\langle a\rangle = \{x \in H \mid \exists\, n \in \mathbb{N}$ such that $a^n \leq x\}$.
If $F \in \mathcal{F}(H)$ and $a \in H \setminus F$, then $\langle F \cup \{a\}\rangle = \{x \in H \mid \exists\, n \in \mathbb{N}$ such that $a^n \to x \in F\}$.

Corollary 2.3. *If H is a $\vee$-hoop, then for any $x, y \in H$, we have*
(i) $\langle x\rangle \cap \langle y\rangle = \langle x \vee y\rangle$.
(ii) If $F \in \mathcal{F}(H)$, then $\langle F \cup \{x\}\rangle \cap \langle F \cup \{y\}\rangle = \langle F \cup \{x \vee y\}\rangle$.

Definition 2.4. [5] *Let $\emptyset \neq F \subseteq H$. Then for any $x, y, z \in H$, F is called*
(i) an implicative filter, if $1 \in F$ and $x \to ((y \to z) \to y) \in F$ and $x \in F$ imply $y \in F$,
(ii) a positive implicative filter, if $1 \in F$ and $(x \odot y) \to z \in F$ and $x \to y \in F$ imply $x \to z \in F$,
(iii) a fantastic filter if $1 \in F$ and $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$.

Theorem 2.5. [5] *Consider H be a hoop. Then the following statements hold:*
(i) Every implicative filter is a positive implicative filter.
(ii) Every implicative filter is a fantastic filter.
(iii) F is an implicative filter if and only if F is a positive implicative and fantastic filter.

Theorem 2.6. [5] *Let F and G be two filters of H such that $F \subseteq G$. Then*
(i) If F is an implicative filter, so is G.
(ii) If F is a positive implicative filter, so is G.
(iii) If F is a fantastic filter, so is G.

Definition 2.7. [5] (i) *Let H be bounded. A filter U of H is called an ultra filter if $x \in U$ if and only if $x' \notin U$, for any $x \in H$.*
(ii) A proper filter P of H is called a prime filter if for any $x, y \in H$, $x \to y \in P$ or $y \to x \in P$. The set of all prime filters of H is denoted by $\mathcal{S}pec(H)$.
(iii) A maximal filter is a proper filter M of H such that is not included in any other proper filter. The set of all maximal filters of H is denoted by $\mathcal{M}ax(H)$.

Theorem 2.8. [5] *Let H be bounded and $U \subseteq H$. Then U is an ultra filter if and only if U is a maximal filter.*

Proposition 2.9. [5] *Let H be a $\vee$-hoop and P be a proper filter of H. Then P is a prime filter of H if and only if for any two filters F and G of H, if $F \cap G \subseteq P$, then $F \subseteq P$ or $G \subseteq P$.*

Consider H and M be two hoops. A map $\mathfrak{h} : H \to M$ is a *hoop-homomorphism* if it satisfies the following conditions:
$(\mathfrak{h}_1)$ $\mathfrak{h}(x \to y) = \mathfrak{h}(x) \to \mathfrak{h}(y)$,
$(\mathfrak{h}_2)$ $\mathfrak{h}(x \odot y) = \mathfrak{h}(x) \odot \mathfrak{h}(y)$.
A hoop-homomorphism $\mathfrak{h}$ is called a *hoop-isomorphism* if $\mathfrak{h}$ is bijective.

3 Prime and maximal filters of hoops

In this section, we investigate some properties of prime and maximal filters on hoops and we define a concept of $\vee$-prime filter and find the relation between them. Then by using these notions we state the representation theorem of hoops and we prove that every nontrivial $\vee$-hoop is a subdirect product of hoop-chains (see Theorem 3.11).

Proposition 3.1. (i) *For each proper filter F of H there is $M \in \mathcal{M}ax(H)$ such that $F \subseteq M$.*
(ii) If $P \in \mathcal{S}pec(H)$ and $\Lambda = \{F \in \mathcal{F}(H) \mid P \subseteq F$ and F is proper$\}$, then $(\Lambda, \subseteq)$ is a chain.
(iii) If $P \in \mathcal{S}pec(H)$, then a unique maximal filter of H exists such that contains P.
(iv) Assume M is a proper filter of H. Then $M \in \mathcal{M}ax(H)$ iff $\langle M \cup \{x\}\rangle = H$, for any $x \in H \setminus M$.
(v) If H is bounded, then $M \in \mathcal{M}ax(H)$ iff for any $x \in H \setminus M$ iff there is $n \in \mathbb{N}$ such that $(x^n)' \in M$.
(vi) $F \in \mathcal{S}pec(H)$ iff $\dfrac{H}{F}$ is a chain.

(vii) $M \in \mathcal{M}ax(H)$ *iff* $\dfrac{H}{M}$ *is simple.*
(viii) Every maximal filter of H *is a prime filter of* H.
(ix) Consider $P \in \mathcal{S}pec(H)$, *where* H *is bounded and for any* $x, y \in H$.

$$x, y \notin P \text{ implies } x \to y \in P \text{ and } y \to x \in P, \tag{1}$$

Then $P \in \mathcal{M}ax(H)$.

Proof. (i) Zorn's Lemma states that a partially ordered set containing upper bounds for every chain (that is, every totally ordered subset) necessarily contains at least one maximal element. Consider $\Omega = \{P \in \mathcal{F}(H) \mid P \neq H, F \subseteq P\}$. Since $F \in \Omega$, we get $\Omega \neq \emptyset$. Let $\{P_i\}_{i\in I}$ be a chain in partially ordered set $(\Omega, \subseteq)$, where $P_i \subseteq P_{i+1}$, for any $i \in I$. Put $P = \bigcup_{i\in I} P_i$. Clearly, $P \in \mathcal{F}(H)$ and $F \subseteq \bigcup_{i\in I} P_i$. Obviously, $P \neq H$, and so $P \in \Omega$. Hence, P is an upper bound of Ω and by using Zorn's Lemma, there exists a maximal element $M \in \Omega$ which is a maximal filter of H and $F \subseteq M$.
(ii) Let $F, G \in \Lambda$ such that $F \nsubseteq G$ and $G \nsubseteq F$. Then there are $x \in F \setminus G$ and $y \in G \setminus F$. Since $P \in \mathcal{S}pec(H)$, we have $x \to y \in P$ or $y \to x \in P$. If $x \to y \in P \subseteq F$, then since $x \in F$ and $F \in \mathcal{F}(H)$, we obtain $y \in F$, which is a contradiction. Similarly, if $y \to x \in P \subseteq G$, then $x \in G$, which is a contradiction, too. Hence, $F \subseteq G$ or $G \subseteq F$. Therefore, Λ is a chain.
(iii) By (i) and (ii) the proof is clear.
(iv) Suppose $x \in H \setminus M$. Then $M \subsetneq \langle M \cup \{x\}\rangle \subseteq H$. Since $M \in \mathcal{M}ax(H)$, we have $\langle M \cup \{x\}\rangle = H$.
Conversely, consider $M \subseteq Q \subseteq H$. If $M \neq Q$, then there exists $x \in Q \setminus M$. Thus by assumption, $\langle M \cup \{x\}\rangle = H$. Moreover, since $M \cup \{x\} \subseteq Q$, we have $H = \langle M \cup \{x\}\rangle \subseteq Q$. Hence, $Q = H$ and so $M \in \mathcal{M}ax(H)$.
(v) ($\Rightarrow$) Assume $\mathcal{H}$ is bounded and $M \in \mathcal{M}ax(H)$. If $x \in H \setminus M$, then by (iv), $\langle M \cup \{x\}\rangle = H$. Thus $0 \in \langle M \cup \{x\}\rangle$. Hence, by Proposition 2.2, there is $n \in \mathbb{N}$ such that $(x^n)' \in M$.
Suppose for any $x \in M$, we have $(x^n)' \in M$, for some $n \in \mathbb{N}$. Then $0 \in M$, which is a contradiction. Hence, $x \in H \setminus M$.
($\Leftarrow$) Assume $G \in \mathcal{F}(H)$ such that $M \subset G \subseteq H$. Then by assumption $x \in G \setminus M$ implies there exists $n \in \mathbb{N}$ such that $(x^n)' \in M$, and so $(x^n)' \in G$. Hence, $0 \in G$, and so $G = H$. Therefore, $M \in \mathcal{M}ax(H)$.
(vi) Let $F \in \mathcal{S}pec(H)$ and $[x], [y] \in \dfrac{H}{F}$. Then $F \in \mathcal{S}pec(H)$ iff $x \to y \in F$ or $y \to x \in F$ iff $[x] \leq_F [y]$ or $[y] \leq_F [x]$ iff $\dfrac{H}{F}$ is a chain.
(vii) Let $M \in \mathcal{M}ax(H)$ and $\dfrac{G}{M}$ be a proper filter of $\dfrac{H}{M}$. Clearly, $M \subseteq G \subset H$. Since $M \in \mathcal{M}ax(H)$, we have $M = G$ and so $\dfrac{G}{M} = \dfrac{M}{M} = M$. Thus, $\mathcal{F}\left(\dfrac{H}{M}\right) = \{M, \dfrac{H}{M}\}$. Hence, $\dfrac{H}{M}$ is simple.

Conversely, consider $\dfrac{H}{M}$ be simple and there exists $Q \in \mathcal{F}(H)$ such that $M \subseteq Q \subseteq H$. If $M \neq Q$, then $\dfrac{M}{M} \neq \dfrac{Q}{M} \in \mathcal{F}\left(\dfrac{H}{M}\right)$. Thus, $\dfrac{Q}{M} = \dfrac{H}{M}$ and so $Q = H$. Therefore, $M \in \mathcal{M}ax(H)$.
(viii) By (vi) and (vii) the proof is obvious.
(ix) Assume $P \subseteq M \subseteq H$. If $P \neq M$, then there is $x \in M \setminus P$. Since $P \in \mathcal{S}pec(H)$, clearly, $0 \notin P$, and by assumption we get $0 \to x \in P$ and $x \to 0 \in P$ and so $x \to 0 \in M$. Since $M \in \mathcal{F}(H)$ and $x \in M$, we get $0 \in M$ and so $M = H$. Hence, $P \in \mathcal{M}ax(H)$. □

Corollary 3.2. *If* H *is a chain hoop, then all proper filters of* H *are prime.*

Proof. Suppose P is a proper filter of H and $x, y \in H$. By assumption, $x \to y = 1$ or $y \to x = 1$, and so $x \to y \in P$ or $y \to x \in P$. Hence, $P \in Spec(H)$. □

Definition 3.3. *Assume* H *is a* $\vee$*-hoop and* $F \in \mathcal{F}(H)$. *Then* F *is called a* $\vee-$prime filter of H *if* $x \vee y \in F$ *implies* $x \in F$ *or* $y \in F$, *for any* $x, y \in H$.

Example 3.4. *Assume* $(H = \{0, a, b, c, 1\}, \leq)$ *is a poset where* $0 \leq c \leq a, b \leq 1$. *Define the operations* $\odot$ *and* $\to$ *on* H *as follows:*

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Then $(X, \odot, \rightarrow, 0, 1)$ is a hoop. Obviously, $F = \{b, 1\}$ is a $\vee$-prime filter of H but $\{1\}$ is not, since $a \vee b = 1 \in \{1\}$ but $a \notin \{1\}$ and $b \notin \{1\}$.

Proposition 3.5. *Assume H is a $\vee$-hoop and P is a proper filter of H. If $P \in \mathcal{S}pec(H)$, then P is a $\vee$-prime filter.*

Proof. Suppose $x \vee y \in P$. Since $P \in \mathcal{S}pec(H)$, for any $x, y \in H$, $x \rightarrow y \in P$ or $y \rightarrow x \in P$. With out loss of generality, let $x \rightarrow y \in P$. By Proposition 2.1(ix), we have $(x \vee y) \rightarrow y = x \rightarrow y \in P$. Since $P \in \mathcal{F}(H)$ and $x \vee y \in P$, we have $y \in P$. By the similar way, if $y \rightarrow x \in P$, then $x \in P$. □

Corollary 3.6. *If H is a chain, then the notion of prime and $\vee$-prime filters are equivalence.*

Theorem 3.7. *Let H be a $\vee$-hoop and $F \in \mathcal{F}(H)$. Then for each $x \notin F$, there exists a $\vee$-prime filter P containing F such that $x \notin P$.*

Proof. Let $\Omega = \{G \in \mathcal{F}(H) \mid F \subseteq G \neq H,\ x \notin G\}$. Since $F \in \Omega$, we have $\Omega \neq \emptyset$. Since any chain of elements in Ω has an upper bound in Ω, by Zorn's Lemma on $(\Omega, \subseteq)$, there exists a maximal element $P \in \Omega$ such that $F \subseteq P$ and $x \notin P$. Now, we prove that P is a $\vee$-prime filter of H. Let $y \vee z \in P$ and $y, z \notin P$. Then $P \subseteq \langle P \cup \{y\}\rangle \cap \langle P \cup \{z\}\rangle$. Since P is maximal of Ω, we get $\langle P \cup \{y\}\rangle, \langle P \cup \{z\}\rangle \notin \Omega$ and so $x \in \langle P \cup \{y\}\rangle \cap \langle P \cup \{z\}\rangle$. Moreover, by Corollary 2.3(ii), we have $x \in \langle P \cup \{y \vee z\}\rangle = P$, which is a contradiction. Hence, $y \in P$ or $z \in P$. Therefore, there exists a $\vee$-prime filter P containing F such that $x \notin P$. □

Corollary 3.8. *Every $\vee$-hoop has at least one prime filter.*

Proof. By Theorem 3.7, assume $F = \{1\}$. □

Corollary 3.9. *Every filter of H is equal to intersection of all prime filters containing it, where H is a $\vee$-hoop. In particular, $\bigcap_{P \in Spec(H)} P = \{1\}$.*

Proof. Assume F is an arbitrary filter of H. Clearly, $F \subseteq \bigcap_{F \subseteq P, P \in Spec(H)} P$. Conversely, assume $\bigcap_{F \subseteq P, P \in Spec(H)} P \nsubseteq F$. Then there exists $a \in \bigcap_{F \subseteq P, P \in Spec(H)} P \setminus F$. By Theorem 3.7, there is $G \in \mathcal{F}(H)$ such that $F \subseteq G$ and $a \notin G$, which is a contradiction with $a \in G$. So, $\bigcap_{F \subseteq P, P \in Spec(H)} P \subseteq F$. Therefore, $F = \bigcap_{F \subseteq P, P \in Spec(H)} P$. Obviously, since for any $P \in Spec(H)$, $\{1\} \subseteq P$ and $\{1\}$ is filter, we have $\bigcap_{P \in Spec(H)} P = \{1\}$. □

If H is a subdirect product of the family $\{H_i\}_{i \in I}$, then H is isomorphie to the subalgebra $h(H)$ of $\prod_{i \in I} H_i$. In addition, the restriction to $h(H)$ of each projection is a surjective mapping.

Theorem 3.10. *An $\vee$-hoop H is a subdirect product of a family $\{H_i\}_{i \in I}$ of hoops if and only if there is a family $\{F_i\}_{i \in I}$ of filters of H such that*
(i) $H_i \cong \dfrac{H}{F_i}$, for each $i \in I$.
(ii) $\bigcap_{i \in I} F_i = \{1\}$.

Theorem 3.11. *Every nontrivial $\vee$-hoop is a subdirect product of hoop-chains.*

Proof. By Theorem 3.10, a hoop H is a subdirect product of a family of hoop-chains if and only if there is a family $\{P_i\}_{i \in I}$ of prime filters of H such that $\bigcap_{i \in I} P_i = \{1\}$. Now, apply Corollary 3.9 to the filter $\{1\}$. □

4 Radical filter on hoops

In this section, we introduce the concept of radical in hoops and study some properties of it. Then we introduce two filters, related to the radical concept, and examine their relationship with other hoop filters. Finally, by using these, we build a topology on hoops.

Definition 4.1. *Consider F be a proper filter of H. The intersection of all maximal filters of H which contain F is called the radical of F and is denoted by $Rad(F)$.*

Example 4.2. *(i) Let H be a hoop as in Example 3.4. Suppose $F = \{a, 1\}$. Then*

$$Rad(F) = \bigcap_{M \in \mathcal{M}ax(H), F \subseteq M} M = \{c, a, b, 1\}.$$

(ii) Let $(H = \{0, a, b, 1\}, \leq)$ be a poset where $0 \leq a, b \leq 1$. Define two operations $\rightarrow$ and $\odot$ on H as follows:

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Then $(H, \odot, \rightarrow, 0, 1)$ is a bounded hoop. Clearly, $F_1 = \{a, 1\}$, $F_2 = \{b, 1\} \in \mathcal{M}ax(H)$. If $F = \{1\}$, then $Rad(F) = \{1\}$.

Remark 4.3. *(i) According to Example 3.4, by our definition, $Rad(\{1\}) = \bigcap_{M \in \mathcal{M}ax(H), \{1\} \subseteq M} M \neq \{1\}$.*
(ii) Clearly $Rad(F) \in \mathcal{F}(H)$.
(iii) Obviously, $F \subseteq Rad(F)$.
(iv) F is a proper filter of H iff $Rad(F)$ is a proper filter of H.
(v) If $F \in \mathcal{M}ax(H)$, then $Rad(F) = F$.

Corollary 4.4. *Let $P \in \mathcal{S}pec(H)$. Then there exists a unique maximal filter M of H such that $Rad(P) = M$.*

Proof. By Proposition 3.1(iii), the proof is clear. □

Proposition 4.5. *If F is an implicative (positive implicative, fantastic) filter of H, then $Rad(F)$ is, too.*

Proof. Assume F is an implicative filter of H. Since $F \subseteq Rad(F)$, by Remark 4.3(iii) and Theorem 2.6(i), clearly, $Rad(F)$ is an implicative filter of H. The proof of other cases are similar. □

Note. From now on, we let $(H, \odot, \rightarrow, 0, 1)$, or H for short, be a bounded hoop.

Theorem 4.6. *Consider F be a proper filter of H, where H is a chain. Then*

$$Rad(F) = \{x \in H \mid (x^n)' \rightarrow x \in F, \text{ for any } n \in \mathbb{N}\} \subseteq \{x \in H \mid x' \rightarrow x \in F\}.$$

Proof. Suppose

$$\mathcal{C} = \{x \in H \mid (x^n)' \rightarrow x \in F, \text{ for any } n \in \mathbb{N}\},$$

and F is a proper filter of H. Let $x \in Rad(F)$ such that $x \notin \mathcal{C}$. Then there exists $n \in \mathbb{N}$ such that $(x^n)' \rightarrow x \notin F$. Since F is a proper filter of H, by Corollary 3.6 and Theorem 3.7, there exists $P \in \mathcal{S}pec(H)$ such that $F \subseteq P$ and $(x^n)' \rightarrow x \notin P$. Since $P \in \mathcal{S}pec(H)$, by Proposition 3.1(iii), there exists $M \in \mathcal{M}ax(H)$ such that $P \subseteq M$. Moreover, since $P \in \mathcal{S}pec(H)$ and $(x^n)' \rightarrow x \notin P$, we have $x \rightarrow (x^n)' \in P$, and so $x \rightarrow (x^n)' \in M$. If $x \in M$, then $(x^n)' \in M$, and so $0 \in M$, which is a contradiction. If $x \notin M$, then we find a maximal filter of H such that $F \subseteq P \subseteq M$ and $x \notin M$. Thus, $x \notin Rad(F)$, which is a contradiction. Hence, for any $n \in \mathbb{N}$, $(x^n)' \rightarrow x \in F$, and so $x \in \mathcal{C}$.

Conversely, suppose $x \in \mathcal{C}$ and $x \notin Rad(F)$. Since $x \in \mathcal{C}$, for any $n \in \mathbb{N}$ we have $(x^n)' \rightarrow x \in F$. Moreover, $x \notin Rad(F)$, then there exists $M \in \mathcal{M}ax(X)$ such that $F \subseteq M$ and $x \notin M$. Then by Proposition 3.1(v), there exists $n \in \mathbb{N}$ such that $(x^n)' \in M$. Since $(x^n)' \rightarrow x \in F$ and $F \subseteq M$, we get $x \in M$, which is a contradiction. Hence, $x \in Rad(F)$. Therefore,

$$Rad(F) = \{x \in H \mid (x^n)' \rightarrow x \in F, \text{ for any } n \in \mathbb{N}\}.$$

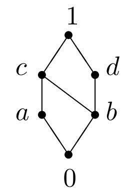
Now, let $x \in Rad(F)$. Then for any $n \in \mathbb{N}$, $(x^n)' \rightarrow x \in F$. By Proposition 2.1(iii), $x \rightarrow 0 \leq x \rightarrow (x \rightarrow 0) = x^2 \rightarrow 0$. By repeating this method $n - 1$ times we have $x \rightarrow 0 \leq (x^n)'$. By Proposition 2.1(vii),

$$(x^n)' \rightarrow x \leq (x \rightarrow 0) \rightarrow x = x' \rightarrow x.$$

Since $F \in \mathcal{F}(H)$ and $(x^n)' \rightarrow x \in F$, we get $x' \rightarrow x \in F$. □

Next example shows $Rad(F)$ is not equal with $\{x \in H \mid x' \rightarrow x \in F\}$, in general.

Example 4.7. *Consider $X = \{0, a, b, c, d, 1\}$ is a lattice with the next Hasse diagram. Define the operations $\odot$ and $\rightarrow$ on E as follows:*



$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	a	1	d	1	d	1
b	0	a	1	1	1	1
c	0	a	d	1	d	1
d	0	a	c	c	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	a	a	a
b	0	a	b	b	b	b
c	0	a	b	c	b	c
d	0	a	b	b	d	d
1	0	a	b	c	d	1

Then $(H, \odot, \rightarrow, 0, 1)$ is a bounded hoop. Let $F = \{c, 1\}$. Clearly, $Rad(F) = \{b, c, d, 1\}$ and $\{x \in H \mid x' \rightarrow x \in F\} = \{a, b, c, d, 1\}$, and so $Rad(F) \subsetneq \{x \in H \mid x' \rightarrow x \in F\}$.

Corollary 4.8. *If H is a chain and F is an implicative filter of H, then*

$$Rad(F) = \{x \in H \mid \ x' \rightarrow x \in F\}.$$

Proof. Assume $\mathcal{B} = \{x \in H \mid \ x' \rightarrow x \in F\}$. By Theorem 4.6, $Rad(F) \subseteq \mathcal{B}$. Now, suppose $x \in \mathcal{B}$. Then

$$1 \rightarrow ((x \rightarrow 0) \rightarrow x) = x' \rightarrow x \in F,$$

and $1 \in F$. Thus, by assumption, F is implicative, and so $x \in F$. By Proposition 2.1(iii), $x \leq (x^n)' \rightarrow x$ and so $(x^n)' \rightarrow x \in F$. Hence, $\mathcal{B} \subseteq Rad(F)$. Therefore, $Rad(F) = \{x \in H \mid \ x' \rightarrow x \in F\}$. □

Proposition 4.9. *Assume $F \in \mathcal{F}(H)$, where H is a chain. If for any $x \in H$, $x \wedge x' = 0$, then*

$$Rad(F) \subseteq \{x \in H \mid x'' \in F\}.$$

Proof. Suppose $x \in Rad(F)$. Then by Theorem 4.6, $x' \rightarrow x \in F$. Since for any $x \in H$, $x \wedge x' = 0$, by Proposition 2.1(xi) we have,

$$x' \rightarrow 0 = x' \rightarrow (x \wedge x') = x' \rightarrow x \in F,$$

and so $x'' \in F$. Hence, $Rad(F) \subseteq \{x \in H \mid x'' \in F\}$. □

Proposition 4.10. *Consider F be an implicative filter of H, where H is a chain. Then*

$$Rad(F) = \{x \in H \mid x'' \in F\}.$$

Proof. Let F be an implicative filter of H and $x \in Rad(F)$. Then by Corollary 4.8, $x' \rightarrow x \in F$. Since F is implicative, by Theorem 2.5(i), F is a positive implicative filter, so we have

$$(x' \rightarrow x) \rightarrow x'' = (x' \rightarrow x) \rightarrow (x' \rightarrow 0) = x' \rightarrow (x \rightarrow 0) = x' \rightarrow x' = 1 \in F.$$

Thus, $(x' \rightarrow x) \rightarrow x'' \in F$. Since $F \in \mathcal{F}(H)$, $(x' \rightarrow x) \rightarrow x'' \in F$ and $x' \rightarrow x \in F$, we get $x'' \in F$. Hence, $Rad(F) \subseteq \{x \in H \mid x'' \in F\}$. Conversely, suppose $x'' \in F$. Since $x'' \leq x' \rightarrow x$ and $F \in \mathcal{F}(H)$, we have $x' \rightarrow x \in F$ and by Corollary 4.8, $x \in Rad(F)$. Therefore, $Rad(F) = \{x \in H \mid x'' \in F\}$. □

As we see in Remark 4.3(i), $Rad(\{1\}) \neq \{1\}$. In the following proposition we investigate that under what condition we have $Rad(\{1\}) = \{1\}$.

Proposition 4.11. *Let H be a chain. Then for any $x \in H$,*

$$Rad(\{1\}) = \{1\} \ \Leftrightarrow \ x = \inf\{(x^n \rightarrow 0) \rightarrow x \mid \ \textit{for any} \ \ n \in \mathbb{N}\}.$$

Proof. Suppose $Rad(\{1\}) = \{1\}$ and for $x \in H$,

$$B = \{(x^n \to 0) \to x \mid \text{ for any } \ n \in \mathbb{N}\}.$$

Since for any $n \in \mathbb{N}$, by Proposition 2.1(iii), $x \leq (x^n \to 0) \to x$, we get x is a lower bound of B. Now, we prove x is the greatest lower bound of B. Suppose there exists $y \in H$ such that for any $n \in \mathbb{N}$, $y \leq (x^n \to 0) \to x$. We prove $y \leq x$. For this, since $y \leq (x^n \to 0) \to x$, we have $y \to ((x^n \to 0) \to x) = 1$. Moreover, by Proposition 2.1(iii) and (vi), $x \leq y \to x$, and so $(y \to x) \to 0 \leq x \to 0$. By Proposition 2.1(vii),

$$(y \to x) \to ((y \to x) \to 0) \leq (y \to x) \to (x \to 0).$$

Also, since $(y \to x) \to 0 \leq x \to 0$, we have $x \to ((y \to x) \to 0) \leq x \to (x \to 0)$. Then

$$\begin{aligned} (y \to x)^2 \to 0 &= (y \to x) \to ((y \to x) \to 0) \\ &\leq (y \to x) \to (x \to 0) \\ &= x \to ((y \to x) \to 0) \\ &\leq x \to (x \to 0) \\ &= x^2 \to 0. \end{aligned}$$

By continuing this method, we get $(y \to x)^n \to 0 \leq x^n \to 0$, for any $n \in \mathbb{N}$. Thus by Proposition 2.1(vii), $(x^n \to 0) \to x \leq ((y \to x)^n \to 0) \to x$, and so by Proposition 2.1(vii),

$$y \to ((x^n \to 0) \to x) \leq y \to (((y \to x)^n \to 0) \to x).$$

From $y \to ((x^n \to 0) \to x) = 1$, we obtain $y \to (((y \to x)^n \to 0) \to x) = 1$, and so $((y \to x)^n \to 0) \to (y \to x) = 1$. Thus $y \to x \in Rad(\{1\}) = \{1\}$. Hence, $y \leq x$. Therefore,

$$x = \inf\{(x^n \to 0) \to x \mid \text{ for any } \ n \in \mathbb{N}\}.$$

Conversely, obviously $\{1\} \subseteq Rad(\{1\})$. Suppose $x \in Rad(\{1\})$. Then for any $n \in \mathbb{N}$, $(x^n \to 0) \to x \in \{1\}$. Thus,

$$1 = (x^n \to 0) \to x \in \inf\{(x^n \to 0) \to x \mid \text{ for any } \ n \in \mathbb{N}\} = x.$$

Hence, $x = 1$. Therefore, $Rad(\{1\}) = \{1\}$. □

Proposition 4.12. *Assume $F, G \in \mathcal{F}(H)$, where H is a chain. Then:*
(i) For any $x \in H \setminus \{0\}$, $x' = 0$ iff $Rad(F) = H \setminus \{0\}$.
(ii) If $F \subseteq G$, then $Rad(F) \subseteq Rad(G)$.
(iii) $Rad(Rad(F)) = Rad(F)$.
(iv) The map $\psi : \mathcal{F}(H) \to H$ such that for any $F \in \mathcal{F}(H)$, $\psi(F) = Rad(F)$ is a closure operator.
(v) $Rad(F) \cap Rad(G) \subseteq Rad(\langle F \cup G \rangle)$.
(vi) $Rad(F \to G) \subseteq Rad(F \to Rad(G))$, where $F \to G = \{x \in H \mid F \cap \langle x \rangle \subseteq G\}$.
(vii) $Rad(F) \to Rad(G) \subseteq F \to Rad(G)$.
(viii) If $f : H \to H$ is a hoop homomorphism, then $Rad(\ker f) = f^{-1}(Rad(\{1\}))$, where $\ker f = \{x \in H \mid f(x) = 1\}$.
(ix) If $\{F_i\}_{i \in I}$ is a family of filters of H, then $Rad(\bigcap_{i \in I} F_i) = \bigcap_{i \in I} Rad(F_i)$.
(x) $Rad([1]) = \dfrac{Rad(F)}{F}$.
(xi) $Rad(\langle F \cup G \rangle) \subseteq Rad\left(\langle Rad(F) \cup Rad(G) \rangle\right)$.
(xii) $[x]_{Rad(F \cap G)} = [x]_{Rad(F)} \cap [x]_{Rad(G)}$.
(xiii) If $G \subseteq F$, then $Rad\left(\dfrac{F}{G}\right) = \dfrac{Rad(F)}{G}$.

Proof. (i) Suppose for any $x \in H \setminus \{0\}$, $x' = 0$. Clearly, $Rad(F) \subseteq H \setminus \{0\}$. If $x \in H \setminus \{0\}$, then by assumption, $x' = 0$

and for any $n \in \mathbb{N}$,

$$\begin{aligned}(x^n \to 0) \to x &= \big(x \to (x \to (\cdots \to (x \to \underbrace{(x \to 0)}_{0})\cdots))\big) \to x \\ &= \big(x \to (x \to (\cdots \to (x \to 0)\cdots))\big) \to x \\ &\vdots \\ &= (x \to 0) \to x \\ &= 0 \to x \\ &= 1 \in F.\end{aligned}$$

Hence, by Theorem 4.6, $x \in Rad(F)$ and so $H \setminus \{0\} \subseteq Rad(F)$. Therefore, $Rad(F) = H \setminus \{0\}$.

Conversely, suppose $Rad(F) = H \setminus \{0\}$. Assume $x \in H \setminus \{0\}$ such that $x' \neq 0$. Then by assumption, $x \in Rad(F)$ and $x \to 0 = x' \in Rad(F)$. Since $Rad(F) \in \mathcal{F}(H)$, we have $0 \in Rad(F)$, a contradiction. Hence, $x' = 0$. Therefore, for any $x \in H \setminus \{0\}$, we have $x' = 0$.

(ii) Consider $x \in Rad(F)$. Then by Theorem 4.6, for any $n \in \mathbb{N}$, $(x^n \to 0) \to x \in F$. Since $F \subseteq G$, we get that for any $n \in \mathbb{N}$, $(x^n \to 0) \to x \in G$, and so $x \in Rad(G)$. Hence, $Rad(F) \subseteq Rad(G)$.

(iii) Since $Rad(F)$ is a proper filter of H, by Remark 4.3(iii) we get $Rad(F) \subseteq Rad(Rad(F))$. Suppose $x \in Rad(Rad(F))$. Then for any $M \in \mathcal{M}ax(H)$ such that $Rad(F) \subseteq M$, we have $x \in M$. Suppose $M_0 \in \mathcal{M}ax(H)$ such that $F \subseteq M_0$. By (ii), $Rad(F) \subseteq Rad(M_0) = M_0$. Then $x \in M_0$. Thus we show that for any $M \in \mathcal{M}ax(H)$ that $F \subseteq M$, we have $x \in M$. Hence, $x \in Rad(F)$, and so $Rad(Rad(F)) \subseteq Rad(F)$. Therefore, $Rad(Rad(F)) = Rad(F)$.

(iv) By (ii), (iii) and Remark 4.3(iii), the proof is clear.

(v) Since $F, G \subseteq \langle F \cup G\rangle$, by (ii), we have $Rad(F), Rad(G) \subseteq Rad(\langle F \cup G\rangle)$. Hence, $Rad(F) \cap Rad(G) \subseteq Rad(\langle F \cup G\rangle)$.

(vi) Assume $x \in Rad(F \to G)$. Then for any $n \in \mathbb{N}$, we have $(x^n \to 0) \to x \in F \to G$. Thus $F \cap \langle (x^n \to 0) \to x\rangle \subseteq G \subseteq Rad(G)$, and so $\langle (x^n \to 0) \to x\rangle \in F \to Rad(G)$. Hence, $x \in Rad(F \to Rad(G))$. Therefore, $Rad(F \to G) \subseteq Rad(F \to Rad(G))$.

(vii) Let $x \in Rad(F) \to Rad(G)$. Then $\langle x\rangle \cap Rad(F) \subseteq Rad(G)$. By Remark 4.3(iii), $F \cap \langle x\rangle \subseteq \langle x\rangle \cap Rad(F) \subseteq Rad(G)$, and so $x \in F \to Rad(G)$. Hence, $Rad(F) \to Rad(G) \subseteq F \to Rad(G)$.

(viii) For any $x \in X$ and for any $n \in \mathbb{N}$, we have

$$\begin{aligned}x \in Rad(\ker f) &\Leftrightarrow (x^n \to 0) \to x \in \ker f \\ &\Leftrightarrow f((x^n \to 0) \to x) = 1 \\ &\Leftrightarrow (f(x)^n \to f(0)) \to f(x) = 1 \\ &\Leftrightarrow f(x) \in Rad(\{1\}) \\ &\Leftrightarrow x \in f^{-1}(Rad(\{1\})).\end{aligned}$$

(ix) Clearly, for any $i \in I$, $\bigcap\limits_{i\in I} F_i \subseteq F_i$. Then by (ii), we have $Rad(\bigcap\limits_{i\in I} F_i) \subseteq \bigcap\limits_{i\in I} Rad(F_i)$. Conversely, suppose $x \in \bigcap\limits_{i\in I} Rad(F_i)$. Then for any $i \in I$, $x \in Rad(F_i)$. Thus for any $i \in I$ and $n \in \mathbb{N}$, $(x^n \to 0) \to x \in F_i$, and so for any $n \in \mathbb{N}$, $(x^n \to 0) \to x \in \bigcap\limits_{i\in I} F_i$. Hence, $x \in Rad(\bigcap\limits_{i\in I} F_i)$. Therefore, $Rad(\bigcap\limits_{i\in I} F_i) = \bigcap\limits_{i\in I} Rad(F_i)$.

(x) For any $x \in X$ and for any $n \in \mathbb{N}$,

$$\begin{aligned}[x] \in Rad([1]) &\Leftrightarrow ([x]^n \to [0]) \to [x] \in [1] \\ &\Leftrightarrow (x^n \to 0) \to x \in F \\ &\Leftrightarrow x \in Rad(F) \\ &\Leftrightarrow [x] \in \frac{Rad(F)}{F}.\end{aligned}$$

(xi) By Remark 4.3(iii), we have $F \subseteq Rad(F)$ and $G \subseteq Rad(G)$. Then $F \cup G \subseteq Rad(F) \cup Rad(G)$, and so $\langle F \cup G\rangle \subseteq \langle Rad(F) \cup Rad(G)\rangle$. Thus by (ii),

$$Rad(\langle F \cup G\rangle) \subseteq Rad\left(\langle Rad(F) \cup Rad(G)\rangle\right).$$

(xii) Since $F \cap G \subseteq F, G$, by (ii) we have $Rad(F \cap G) \subseteq Rad(F), Rad(G)$. Suppose $y \in [x]_{Rad(F\cap G)}$. Then $x \to y, y \to x \in Rad(F \cap G)$. Since $Rad(F \cap G) \subseteq Rad(F)$, we get $x \to y, y \to x \in Rad(F)$ and so $y \in [x]_{Rad(F)}$. By the similar way, we have $y \in [x]_{Rad(G)}$. Hence, $y \in [x]_{Rad(F)} \cap [x]_{Rad(G)}$. Therefore,

$$[x]_{Rad(F\cap G)} \subseteq [x]_{Rad(F)} \cap [x]_{Rad(G)}.$$

Now, suppose $y \in [x]_{Rad(F)} \cap [x]_{Rad(G)}$. Then $x \to y, y \to x \in Rad(F)$ and $x \to y, y \to x \in Rad(G)$. Thus $x \to y, y \to x \in Rad(F) \cap Rad(G)$. By (ix), we have $x \to y, y \to x \in Rad(F \cap G)$, and so $y \in [x]_{Rad(F\cap G)}$. Hence,

$$[x]_{Rad(F)} \cap [x]_{Rad(G)} \subseteq [x]_{Rad(F\cap G)}.$$

Therefore, $[x]_{Rad(F)} \cap [x]_{Rad(G)} = [x]_{Rad(F\cap G)}$.
(xiii) Consider $F, G \in \mathcal{F}(X)$ such that $G \subseteq F$. Then

$$Rad\left(\frac{F}{G}\right) = \bigcap_{\frac{F}{G}\subseteq\frac{M}{G}} \frac{M}{G} = \bigcap_{G\subseteq F\subseteq M} M = \frac{\bigcap_{F\subseteq M} M}{G} = \frac{Rad(F)}{G}.$$

□

Next example shows that the converse of relations in Proposition 4.12(ii) does not hold, in general.

Example 4.13. *According to Example 4.7, let* $F = \{c, 1\}$ *and* $G = \{d, 1\}$. *Clearly,* $Rad(F) = Rad(G) = \{b, c, d, 1\}$. *So, the converse of Proposition 4.12(ii) does not hold.*

The set of dense elements of X is denoted by $D(X) = \{x \in X \mid x' = 0\}$.

Example 4.14. *Suppose* H *is the hoop as in Example 3.4. Clearly* $D(H) = \{a, b, c, 1\}$.

Remark 4.15. *(i) If* $F \in \mathcal{F}(H)$, *then* $D(F) = D(H) \cap F$.
(ii) If H *is a chain and* $F \in \mathcal{F}(H)$, *then* $D(F) \subseteq Rad(F)$. *Since for any* $x \in D(F)$, *we have* $x' = 0$ *and for any* $n \in \mathbb{N}$, $(x^n \to 0) \to x = 0 \to x = 1 \in F$, *and so* $x \in Rad(F)$.

Proposition 4.16. $D(H) \in \mathcal{F}(H)$.

Proof. Since $1' = 0$, clearly $1 \in D(H)$. Suppose $x, x \to y \in D(H)$. Then $x' = (x \to y)' = 0$. By Proposition 2.1(vi), $x \to y \leq (y \to 0) \to (x \to 0)$. Since $x' = 0$ we get $x \to y \leq (y \to 0) \to 0$. Moreover, by Proposition 2.1(xii), we have

$$y \to 0 = ((y \to 0) \to 0) \to 0 \leq (x \to y) \to 0.$$

From $(x \to y)' = 0$, we obtain $y \to 0 = 0$ and so $y \in D(H)$. Therefore, $D(H) \in \mathcal{F}(H)$. □

Proposition 4.17. *Assume* $F \in \mathcal{F}(H)$. *Then:*
(i) If $\frac{H}{F}$ *is a hoop with (DNP), then* $D(H) \subseteq F$.
(ii) If H *is a hoop with (DNP), then* $D(H) = \{1\}$.

Proof. (i) If $\frac{H}{F}$ is a hoop with (DNP), then for any $x \in H$, $[x''] = [x]$. Thus $x'' \to x \in F$ and $x \to x'' \in F$. Suppose $x \in D(H)$. Then $x' = 0$ and so $x'' = 1$. Since for any $x \in H$, $x = 1 \to x = x'' \to x \in F$, we get $x \in F$. Hence, $D(H) \subseteq F$.
(ii) Let H be a hoop with (DNP). Since $\{1\} \in \mathcal{F}(H)$, $\frac{H}{\{1\}}$ is well-defined and $\frac{H}{\{1\}} \cong H$. Thus, $\frac{H}{\{1\}}$ is a hoop with (DNP) and so by (i), $D(H) \subseteq \{1\}$. On the other side, clearly $1' = 0$ and so $\{1\} \subseteq D(H) \subseteq \{1\}$. Hence, $D(H) = \{1\}$. □

Proposition 4.18. *Suppose* $F \in \mathcal{F}(H)$, *where* H *is a chain. Then, we have,*
(i) $D(H) \subseteq Rad(F)$.
(ii) $Rad\left(\frac{F}{D(F)}\right) = \frac{Rad(F)}{D(F)}$.

Proof. (i) Consider $x \in D(H)$. Then $x' = 0$. Thus similar to Remark 4.15(ii), we can see that $x \in Rad(F)$. Hence, $D(H) \subseteq Rad(F)$.
(ii) By Proposition 4.16, since $D(H) \in \mathcal{F}(H)$, $\frac{F}{D(F)}$ is well-defined. For $x \in H$,

$$\begin{aligned}[x] \in Rad\left(\frac{F}{D(F)}\right) &\Leftrightarrow \text{ for any } n \in \mathbb{N},\ ([x]^n \to [0]) \to [x] \in \frac{F}{D(F)} \\ &\Leftrightarrow [(x^n \to 0) \to x] \in \frac{F}{D(F)} \\ &\Leftrightarrow (x^n \to 0) \to x \in F \\ &\Leftrightarrow x \in Rad(F) \\ &\Leftrightarrow [x] \in \frac{Rad(F)}{D(F)}.\end{aligned}$$

□

Example 4.19. *Let H be a hoop as in Example 4.2(ii). If $F = \{a, 1\}$, then $Rad(F) = \{a, 1\}$ and $D(H) = \{1\}$, and clearly, $Rad(F) \nsubseteq D(H)$.*

Proposition 4.20. *Let $f : H \to Y$ be a chain hoop epimorphism such that $F \in \mathcal{F}(H)$, $G \in \mathcal{F}(Y)$, respectively, and $\ker f \subseteq F$. Then $Rad(f(F)) = f(Rad(F))$ and $Rad(f^{-1}(G)) = f^{-1}(Rad(G))$.*

Proof. Since $F \in \mathcal{F}(H)$ and $\ker f \subseteq F$, we get $f(F) \in \mathcal{F}(Y)$. Then

$$\begin{aligned}y \in f(Rad(F)) &\Leftrightarrow \text{ there exists } x \in Rad(F) \text{ such that } y = f(x) \\ &\Leftrightarrow \text{ for any } n \in \mathbb{N},\ (x^n \to 0) \to x \in F \text{ such that } y = f(x) \\ &\Leftrightarrow \text{ for any } n \in \mathbb{N},\ f((x^n \to 0) \to x) \in f(F) \text{ such that } y = f(x) \\ &\Leftrightarrow \text{ for any } n \in \mathbb{N},\ (f(x)^n \to f(0)) \to f(x) \in f(F) \text{ such that } y = f(x) \\ &\Leftrightarrow \text{ for any } n \in \mathbb{N},\ (y^n \to 0) \to y \in f(F) \\ &\Leftrightarrow y \in Rad(f(F)).\end{aligned}$$

Now, we prove $Rad(f^{-1}(G)) = f^{-1}(Rad(G))$. For this,

$$\begin{aligned}x \in f^{-1}(Rad(G)) &\Leftrightarrow f(x) \in Rad(G) \\ &\Leftrightarrow \text{for any } n \in \mathbb{N},\ (f(x)^n \to f(0)) \to f(x) \in G \\ &\Leftrightarrow \text{for any } n \in \mathbb{N},\ (x^n \to 0) \to x \in f^{-1}(G) \\ &\Leftrightarrow x \in Rad(f^{-1}(G)).\end{aligned}$$

□

Definition 4.21. *If $F \in \mathcal{F}(H)$ and $G \in \mathcal{F}(Y)$, respectively, then $F \times G = \{(a, b) \mid a \in F \text{ and } b \in G\} \in \mathcal{F}(H \times Y)$, where for any $(a, b), (c, d) \in H \times Y$ we have*

$$(a, b) \otimes (c, d) = (a \odot_H c, b \odot_Y d),$$
$$(a, b) \dashrightarrow (c, d) = (a \to_H c, b \to_Y d).$$

Proposition 4.22. *If H and Y are chains such that $F \in \mathcal{F}(H)$ and $G \in \mathcal{F}(Y)$, respectively, then $Rad(F) \times Rad(G) = Rad(F \times G)$.*

Proof. According to Definition 4.21, we have

$$\begin{aligned}Rad(F \times G) &= \{(a, b) \in H \times Y \mid \text{ for any } n \in \mathbb{N}, ((a, b)^n \rightsquigarrow (0, 0)) \rightsquigarrow (a, b) \in F \times G\} \\ &= \{(a, b) \in H \times Y \mid \text{ for any } n \in \mathbb{N}, ((a^n \to 0) \to a, (b^n \to 0) \to b)) \in F \times G\} \\ &= \{(a, b) \in H \times Y \mid \text{ for any } n \in \mathbb{N}, (a^n \to 0) \to a \in F \text{ and } (b^n \to 0) \to b \in G\} \\ &= \{(a, b) \in H \times Y \mid a \in Rad(F) \text{ and } b \in Rad(G)\} \\ &= Rad(F) \times Rad(G).\end{aligned}$$

Therefore, $Rad(F) \times Rad(G) = Rad(F \times G)$. □

Corollary 4.23. *If H is a chain and $\{F_i\}_{i\in I}$ is a non-empty family of filters of H, then*

$$Rad(\prod_{i\in I} F_i) = \prod_{i\in I} Rad(F_i).$$

Theorem 4.24. *Assume H and Y are chains such that $F \in \mathcal{F}(H)$ and $G \in \mathcal{F}(Y)$, respectively. Then*

$$\frac{H\times Y}{Rad(F\times G)} \simeq \frac{H}{Rad(F)} \times \frac{Y}{Rad(G)}.$$

Proof. Consider the natural homomorphisms $\pi_H : H \to \dfrac{H}{Rad(F)}$ and $\pi_Y : Y \to \dfrac{Y}{Rad(G)}$ such that for any $x \in H$ and $y \in Y$ we have $\pi_H(x) = [x]$ and $\pi_Y(y) = [y]$. We define $\Theta : H\times Y \to \dfrac{H}{Rad(F)} \times \dfrac{Y}{Rad(G)}$ such that for any $(x, y) \in H\times Y$, $\Theta(x, y) = ([x], [y])$. Then Θ is a well-defined onto homomorphism. Now, we prove $\ker\Theta = Rad(F\times G)$. For this, by Proposition 4.22 we have,

$$\begin{aligned}\ker\Theta &= \{(x, y) \in H\times Y \mid \Theta(x, y) = ([1], [1])\}\\ &= \{(x, y) \in H\times Y \mid \ ([x], [y]) = ([1], [1])\}\\ &= \{(x, y) \in H\times Y \mid \ [x] = [1] \text{ and } [y] = [1]\}\\ &= \{(x, y) \in H\times Y \mid \ x \in Rad(F) \text{ and } y \in Rad(G)\}\\ &= Rad(F)\times Rad(G)\\ &= Rad(F\times G).\end{aligned}$$

Therefore, $\dfrac{H\times Y}{Rad(F\times G)} \simeq \dfrac{H}{Rad(F)} \times \dfrac{Y}{Rad(G)}$. □

Definition 4.25. *Consider $F \in \mathcal{F}(H)$. If $Rad(F) = F$, then F is called an r-filter of H. The set of all r-filters of H is denoted by $\mathcal{RF}(H)$.*

Example 4.26. (i) *According to Example 4.7, let $F = \{b, c, d, 1\}$. Clearly, F is an r-filter of H.*
(ii) *Consider H be a hoop as in Example 3.4. If $F = \{a, 1\}$, then $Rad(F) = \{a, b, c, 1\} \neq F$, and so F is not an r-filter of H.*

In the next example, the relation between different kinds of filters of hoop and r-filter of it is studied.

Example 4.27. (i) *According to Example 4.2(ii), assume $F = \{1\}$. Then $F = Rad(F)$, and so $F \in \mathcal{RF}(H)$ but $F \notin \mathcal{M}ax(H)$. Since $a \vee b = 1$ but $a, b \notin \{1\}$.*
(ii) *According to Example 4.2(ii), assume $F = \{1\}$. Then $F = Rad(F)$, and so $F \in \mathcal{RF}(H)$, but $F \notin \mathcal{S}pec(H)$. Since $a \to b = b \notin F$ and $b \to a = a \notin F$.*
(iii) *Consider $(H = \{0, a, b, 1\}, \leq)$ be a chain where $0 \leq a \leq b \leq 1$. Define two operations $\to$ and $\odot$ on H as follows:*

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $(H, \odot, \to, 0, 1)$ is a bounded hoop. If $F = \{b, 1\}$, then $Rad(F) = F$, and so $F \in \mathcal{RF}(H)$, but F is not a positive implicative filter of H, since $a \to a^2 = a \to 0 = a \notin F$. Also, F is not an implicative filter of H, because $(a' \to a) \to a = a \notin F$.
(iv) *Assume $(H = \{0, a, b, 1\}, \leq)$ is a chain where $0 \leq a \leq b \leq 1$. Define two operations $\to$ and $\odot$ on H as follows:*

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
b	0	a	a	a
c	0	a	a	b
1	0	a	b	1

Then $(H, \odot, \to, 0, 1)$ is a bounded hoop. Clearly, $\{1\}$ is a positive implicative filter of H but $\{1\} \neq \{a, b, 1\} = Rad(\{1\})$.
(v) *According to Example 3.4, $F = \{b, 1\}$ is prime but $F \neq Rad(F)$, and so $F \notin \mathcal{RF}(H)$.*

Proposition 4.28. *Assume $F \in \mathcal{F}(H)$. Then the next statements hold:*
(i) If H is a chain and F is an implicative filter of H, then $F \in \mathcal{R}F(H)$.
(ii) If $F \in \mathcal{M}ax(H)$, then $F \in \mathcal{R}F(H)$.
(iii) If $F \in \mathcal{R}F(H)$ and $\mid \mathcal{M}ax(H) \mid = 1$, then $F \in \mathcal{M}ax(H)$.

Proof. (i) Consider F is an implicative filter of H. By Remark 4.3(iii), $F \subseteq Rad(F)$. Suppose $x \in Rad(F)$. Then by Theorem 4.6, $x' \to x \in F$. Since F is an implicative filter of H, we get $x \in F$. Hence, $Rad(F) \subseteq F$. Therefore, $Rad(F) = F$ and so $F \in \mathcal{R}F(H)$.
(ii) If $F \in \mathcal{M}ax(H)$, clearly $Rad(F) = F$, and so $F \in \mathcal{R}F(H)$.
(iii) Consider $F \in \mathcal{R}F(H)$. Since $F = Rad(F)$, we get F is a proper filter of H. By Proposition 3.1(i), there is $M \in \mathcal{M}ax(H)$ that contains F, and so since $\mid \mathcal{M}ax(H) \mid = 1$, we get $F = Rad(F) = M$. Therefore, $F \in \mathcal{M}ax(H)$. □

Proposition 4.29. *Consider $f : H \to Y$ is a chain hoop homomorphism. Then:*
(i) If $G \in \mathcal{R}F(Y)$, then $f^{-1}(G) \in \mathcal{R}F(H)$.
(ii) If f is onto such that $\ker f \subseteq F$ *and $F \in \mathcal{R}F(H)$, then $f(F) \in \mathcal{R}F(Y)$.*

Proof. (i) Since G is a proper filter of Y, we obtain $f^{-1}(G)$ is a proper filter of H. Moreover, since G is an r-filter, $Rad(G) = G$. Thus

$$\begin{aligned}
Rad(f^{-1}(G)) &= \{x \in H \mid \text{ for any } n \in \mathbb{N}, (x^n \to 0) \to x \in f^{-1}(G)\} \\
&= \{x \in H \mid \text{ for any } n \in \mathbb{N}, (f(x)^n \to f(0)) \to f(x) \in G\} \\
&= \{x \in H \mid f(x) \in Rad(G) = G\} \\
&= \{x \in H \mid x \in f^{-1}(G)\} \\
&= f^{-1}(G).
\end{aligned}$$

Hence, $f^{-1}(G) \in \mathcal{R}F(H)$.
(ii) Since $F \in \mathcal{R}F(H)$, we have $Rad(F) = F$. Clearly, $f(F) \in \mathcal{F}(Y)$ and by Proposition 4.20, $f(F) = f(Rad(F)) = Rad(f(F))$, and so $f(F) \in \mathcal{R}F(Y)$. □

Proposition 4.30. *If H is a chain and $F \in \mathcal{F}(H)$, then:*
(i) $Rad(F)$ is the smallest r-filter of H contains F.
(ii) $\dfrac{Rad(F)}{F} \in \mathcal{R}F\left(\dfrac{H}{F}\right)$.

Proof. (i) By Proposition 4.12(iii), $Rad(Rad(F)) = Rad(F)$. Then $Rad(F) \in \mathcal{R}F(H)$ and by Remark 4.3(iii), $F \subseteq Rad(F)$. Now, suppose $G \in \mathcal{R}F(H)$ such that $F \subseteq G$. Then by Proposition 4.12(ii), $Rad(F) \subseteq Rad(G)$. Since $G \in \mathcal{R}F(H)$, we have $Rad(F) \subseteq G$, and so $Rad(F)$ is the smallest r-filter of H such that $F \subseteq Rad(F)$.
(ii) Clearly, by Remark 4.3(iii), $F \subseteq Rad(F)$, and so $\dfrac{Rad(F)}{F}$ is well-defined. Thus,

$$\frac{Rad(F)}{F} \subseteq Rad\left(\frac{Rad(F)}{F}\right).$$

Assume for $x \in H$, $[x] \in Rad\left(\dfrac{Rad(F)}{F}\right)$. Then for any $n \in \mathbb{N}$, $([x]^n \to [0]) \to [x] \in \dfrac{Rad(F)}{F}$. Thus there exists $y \in Rad(F)$ such that $([x]^n \to [0]) \to [x] = [y]$, and so

$$((x^n \to 0) \to x) \to y \in F \text{ and } y \to ((x^n \to 0) \to x) \in F \subseteq Rad(F).$$

Since $y \in Rad(F)$ and $Rad(F) \in \mathcal{F}(H)$, we get $(x^n \to 0) \to x \in Rad(F)$, and so by Proposition 4.12(iii), $x \in Rad(Rad(F)) = Rad(F)$. Thus, $[x] \in \dfrac{Rad(F)}{F}$. Hence, $\dfrac{Rad(F)}{F} = Rad\left(\dfrac{Rad(F)}{F}\right)$. Therefore, $\dfrac{Rad(F)}{F} \in \mathcal{R}F\left(\dfrac{H}{F}\right)$.
□

Corollary 4.31. *(i) Let $\{F_i\}_{i \in I} \in \mathcal{R}F(H)$. Then $\bigcap_{i \in I} F_i \in \mathcal{R}F(H)$.*
(ii) Consider $F \in \mathcal{F}(H)$. Then $F \in \mathcal{R}F(H)$ iff $[1] \in \mathcal{R}F(H/F)$.

Definition 4.32. *Let H be a $\vee$-hoop. A proper filter F of H is called a p-filter of H if for any $x, y \in H$, $x \vee y \in F$ implies $x \in F$ or $y \in Rad(F)$. The set of all p-filters of H is denoted by $\mathcal{P}F(H)$.*

Example 4.33. *According to Example 4.7, let $F = \{c, 1\}$. Obviously, $Rad(F) = \{b, c, d, 1\}$ and so $F \in \mathcal{PF}(H)$.*

In the next example, we show the relation between different kind of filters of H with p-filters of H.

Example 4.34. (i) *By Example 4.27(iii), $\{b, 1\} \in \mathcal{PF}(H)$ but it is not positive implicative and implicative filters of H.*
(ii) *By Example 4.7, $\{c, 1\} \in \mathcal{PF}(H)$ but $\{c, 1\} \notin \mathcal{M}ax(H) \cap \mathcal{S}pec(H)$.*
(iii) *By Example 4.27(iv), $\{1\} \in \mathcal{PF}(H)$ but $\{1\}$ is not a fantastic filter since $0 \to a = 1$ but*

$$((a \to 0) \to 0) \to a = a \notin \{1\}.$$

Proposition 4.35. *Consider $F, G \in \mathcal{F}(H)$ such that $F \subseteq G$. If $G \in \mathcal{PF}(H)$, then $\frac{G}{F} \in \mathcal{PF}\left(\frac{H}{F}\right)$.*

Proof. Let $\frac{G}{F} \in \mathcal{F}\left(\frac{H}{F}\right)$ such that $[x] \vee [y] \in \frac{G}{F}$. Then $x \vee y \in G$. Since $G \in \mathcal{PF}(H)$, we have $x \in G$ or $y \in Rad(G)$, and so $[x] \in \frac{G}{F}$ or $[y] \in \frac{Rad(G)}{F}$. By Proposition 4.12(xiii), we have $[y] \in Rad\left(\frac{G}{F}\right)$. Therefore, $\frac{G}{F} \in \mathcal{PF}\left(\frac{H}{F}\right)$. □

Proposition 4.36. *Every $\vee$-prime filter of H is a p-filter of H.*

Proof. Suppose F is a $\vee$-prime filter of H and for any $x, y \in H$, $x \vee y \in F$. By assumption we get $x \in F$ or $y \in F$. If $x \in F$, then the proof is complete. If $x \notin F$, then $y \in F$. By Remark 4.3(iii), $F \subseteq Rad(F)$, and so $y \in Rad(F)$. Therefore, $F \in \mathcal{PF}(H)$. □

In the next example, we show that the converse of above proposition may not be true, in general.

Example 4.37. *According to Example 4.7, $F = \{c, 1\} \in \mathcal{PF}(H)$ but is not a $\vee$-prime filter, since $a \vee b = c \in F$ but $a, b \notin F$.*

Proposition 4.38. *Assume F is a proper filter of $\vee$-hoop H such that it satisfies in at least one of the following conditions:*
(i) *$F \in \mathcal{M}ax(H)$.*
(ii) *$F \in \mathcal{RF}(H)$.*
Then $F \in \mathcal{PF}(H)$ iff F is a $\vee$-prime filter of H.

Proof. By Propositions 4.28 and 4.36, the proof is clear. □

Corollary 4.39. (i) *Any proper filter of H can be extended to a p-filter of H.*
(ii) *Every prime filter of H is a p-filter of H.*
(iii) *Every maximal filter of H is a p-filter of H.*

Proof. (i) The proof is clear.
(ii) By Theorem 3.5 and Proposition 4.36, the proof is clear.
(iii) By Proposition 3.1(viii), Theorem 3.5 and Proposition 4.36, the proof is clear. □

Next example shows that the converse of Corollary 4.39(ii) and (iii) do not hold, in general.

Example 4.40. (i) *According to Example 4.7, $F = \{c, 1\} \in \mathcal{PF}(H)$ but $a \to b, b \to a \notin F$.*
(ii) *According to Example 4.7, $F = \{c, 1\} \in \mathcal{PF}(H)$ but $F \notin \mathcal{M}ax(H)$.*

Proposition 4.41. *Suppose H is a chain and F is a proper filter of H. Then $F \in \mathcal{PF}(H)$ iff for any $x, y \in H$, $x \to y \in F$ or $y \to x \in Rad(F)$.*

Proof. Suppose $F \in \mathcal{PF}(H)$ and $x, y \in H$. By assumption, we have $(x \to y) \vee (y \to x) = 1 \in F$. Then $x \to y \in F$ or $y \to x \in Rad(F)$. Conversely, suppose F is a proper filter of H and $x, y \in H$ such that $x \vee y \in F$. Then by Proposition 2.1(ix), $(x \vee y) \to y = x \to y \in F$. Since $(x \vee y) \to y \in F$, $x \vee y \in F$ and $F \in \mathcal{F}(H)$, we get $y \in F$. If $x \to y \notin F$, then $y \to x \in Rad(F)$. By Remark 4.3(iii), $F \subseteq Rad(F)$, then $x \vee y \in Rad(F)$. Again, by Proposition 2.1(ix), $(x \vee y) \to x = y \to x \in Rad(F)$. Since $(x \vee y) \to x \in Rad(F)$, $x \vee y \in Rad(F)$ and $Rad(F) \in \mathcal{F}(H)$, we get $x \in Rad(F)$. Therefore, $F \in \mathcal{PF}(H)$. □

Corollary 4.42. *Assume H is a chain. If $F \in \mathcal{PF}(H)$, then $Rad(F) \in \mathcal{S}pec(H)$.*

Proof. Suppose $F \in \mathcal{P}F(H)$. Then by Proposition 4.41, for any $x, y \in H$, $x \to y \in F$ or $y \to x \in Rad(F)$. By Remark 4.3(iii), $F \subseteq Rad(F)$, thus for any $x, y \in H$, $x \to y \in Rad(F)$ or $y \to x \in Rad(F)$. Hence, $Rad(F) \in \mathcal{S}pec(H)$. □

Theorem 4.43. *Consider H be a chain, $F \in \mathcal{P}F(H)$ and G be a proper filter of H such that $F \subseteq G$. Then $G \in \mathcal{P}F(H)$.*

Proof. Assume G is a proper filter of H such that for $x, y \in H$, $x \vee y \in G$. Since $F \in \mathcal{P}F(H)$, by Proposition 4.41, for any $x, y \in H$, $x \to y \in F$ or $y \to x \in Rad(F)$. If $x \to y \in F$, since $F \subseteq G$, then $x \to y \in G$. Thus by Proposition 2.1(ix), $(x \vee y) \to y = x \to y \in G$. Since $(x \vee y) \to y \in G$, $x \vee y \in G$ and $G \in \mathcal{F}(H)$, we get $y \in G$. If $y \to x \in Rad(F)$, then by Proposition 4.12(ii), since $F \subseteq G$, we have $Rad(F) \subseteq Rad(G)$, and so $y \to x \in Rad(G)$. Thus $(x \vee y) \to x = y \to x \in Rad(G)$. Since $(x \vee y) \to x \in Rad(G)$, $x \vee y \in Rad(G)$ and $Rad(G) \in \mathcal{F}(H)$, we get $x \in Rad(G)$. Therefore, $G \in \mathcal{P}F(H)$. □

Corollary 4.44. *Let H be a chain. Then for any $F \in \mathcal{P}F(H)$, $Rad(F)$ is a $\vee$-prime filter of H.*

Proof. Suppose $F \in \mathcal{P}F(H)$. Since $F \subseteq Rad(F)$, by Proposition 4.43, $Rad(F) \in \mathcal{P}F(H)$. Moreover, by Proposition 4.12(iii), $Rad(Rad(F)) = Rad(F)$, and so $Rad(F) \in \mathcal{R}F(H)$. Then by Proposition 4.38, $Rad(F)$ is a $\vee$-prime filter of H. □

Theorem 4.45. *Let $F \in \mathcal{P}F(H)$. Then for any $a \in H \setminus F$ there exists $G \in \mathcal{P}F(H)$ such that $F \subseteq G$ and $a \notin G$.*

Proof. Define
$$\sum = \{G \in \mathcal{F}(H) \mid \ F \subseteq G \text{ such that } a \notin G\}.$$
Since $F \in \sum$, we get $\sum \neq \emptyset$. Then by Zorn' Lemma, $\sum$ has a maximal element such as G. Now, we prove that $G \in \mathcal{P}F(H)$. For this, suppose $x \vee y \in G$ such that $x \notin G$ and $y \notin Rad(G)$. Obviously, $G \subseteq \langle G \cup \{x\}\rangle$ and $G \subseteq \langle G \cup \{y\}\rangle$. Since G is a maximal element of $\sum$ we get $\langle G \cup \{x\}\rangle, \langle G \cup \{y\}\rangle \notin \sum$. Thus $a \in \langle G \cup \{x\}\rangle \cap \langle G \cup \{y\}\rangle$. Then by Corollary 2.3(ii), $a \in \langle G \cup \{x \vee y\}\rangle$. Since $x \vee y \in G$ and $G \in \mathcal{F}(H)$, we have $\langle G \cup \{x \vee y\}\rangle = G$, and so $a \in G$, which is a contradiction. Hence, $G \in \mathcal{P}F(H)$. □

Proposition 4.46. *Consider $P \in \mathcal{P}F(H)$ and $F, G \in \mathcal{F}(H)$ such that $F \cap G \subseteq P$. Then $F \subseteq P$ or $G \subseteq Rad(P)$.*

Proof. Suppose $F \cap G \subseteq P$ such that $F \nsubseteq P$. Then there exists $x \in F \setminus P$. Thus for any $y \in G$, we have $x \leq x \vee y$ and $y \leq x \vee y$. Since $F, G \in \mathcal{F}(H)$, we get $x \vee y \in F \cap G \subseteq P$. Moreover, $P \in \mathcal{P}F(H)$, then $y \in Rad(P)$. Hence, $G \subseteq Rad(P)$. □

Corollary 4.47. (*i*) *Let $P \in \mathcal{P}F(H)$ and $F, G \in \mathcal{F}(H)$ such that $F \cap G = P$. Then $F = P$ or $Rad(G) = Rad(P)$.*
(*ii*) *Assume $P \in \mathcal{P}F(H)$ and $\{F_i\}_{i \in I}$ is a non-empty family of filters of H such that $\bigcap\limits_{i \in I} F_i \subseteq P$. Then there exists $i \in I$ such that $F_i \subseteq P$ or $Rad(F_i) \subseteq Rad(P)$.*

Proposition 4.48. *Let H be a chain and $F \in \mathcal{P}F(H)$. Then*
$$\mathcal{T} = \{G \in \mathcal{F}(H) \mid \ G \textit{ is proper and } Rad(F) \subseteq G\},$$
is a chain.

Proof. Suppose $G_1, G_2 \in \mathcal{T}$. Then G_1 and G_2 are two proper filters of H such that $Rad(F) \subseteq G_1$ and $Rad(F) \subseteq G_2$. Suppose $G_2 \nsubseteq G_1$ and $G_1 \nsubseteq G_2$. Then there exist $x \in G_1 \setminus G_2$ and $y \in G_2 \setminus G_1$. Since F is a p-filter of H, for any $x, y \in H$, by Proposition 4.41, $x \to y \in F$ or $y \to x \in Rad(F)$. If $x \to y \in F$, since $F \subseteq Rad(F) \subseteq G_1$, then $x \to y \in G_1$. Since $x \in G_1$ and $G \in \mathcal{F}(H)$, we have $y \in G_1$, a contradiction. If $y \to x \in Rad(F)$, since $Rad(F) \subseteq G_2$, then $y \to x \in G_2$. Since $y \in G_2$ and $G_2 \in \mathcal{F}(H)$, we have $x \in G_1$, a contradiction. Hence, $G_2 \subseteq G_1$ or $G_1 \subseteq G_2$. Therefore, $\mathcal{T}$ is a chain. □

Consider $F \in \mathcal{F}(H)$. Define
$$\mathcal{U}(F) = \{P \in \mathcal{F}(H) \mid P \in \mathcal{P}F(H) \text{ such that } F \subseteq P\}.$$

Proposition 4.49. *Assume $F, G, \{F_i\}_{i \in I} \in \mathcal{F}(H)$. We have:*
(*i*) *$\mathcal{U}(H) = \emptyset$ and $\mathcal{U}(\{1\}) = \{P \mid \ P \in \mathcal{P}F(H)\}$.*
(*ii*) *If $F \subseteq G$, then $\mathcal{U}(G) \subseteq \mathcal{U}(F)$.*
(*iii*) *$\bigcap\limits_{i \in I} \mathcal{U}(F_i) = \mathcal{U}(\bigvee\limits_{i \in I} F_i)$, where $\bigvee\limits_{i \in I} F_i = \langle \bigcup\limits_{i \in I} F_i \rangle$.*
(*iv*) *$\mathcal{U}(F) \cup \mathcal{U}(G) \subseteq \mathcal{U}(F \cap G)$.*

Following example shows $\mathcal{U}(F \cap G) \nsubseteq \mathcal{U}(F) \cup \mathcal{U}(G)$.

Example 4.50. *Let H be the hoop as in Example 3.4. Clearly, all filters are p-filters. Suppose $F = \{a, 1\}$ and $G = \{b, 1\}$. Then $\mathcal{U}(F \cap G) = \mathcal{U}(\{1\}) = \{\{1\}, \{a, 1\}, \{b, 1\}, \{a, b, c, 1\}\}$ and $\mathcal{U}(F) \cup \mathcal{U}(G) = \{\{a, 1\}, \{a, b, c, 1\}\} \cup \big\{\{b, 1\}, \{a, b, c, 1\}\big\} = \big\{\{a, 1\}, \{b, 1\}, \{a, b, c, 1\}\big\}$. Therefore, $\mathcal{U}(F \cap G) \nsubseteq \mathcal{U}(F) \cup \mathcal{U}(G)$.*

Proposition 4.51. *Suppose every filter of H is an r-filter of H and $F, G \in \mathcal{F}(H)$. Then $\mathcal{U}(F) \cup \mathcal{U}(G) = \mathcal{U}(F \cap G)$.*

Proof. Suppose $F, G \in \mathcal{F}(H)$. By Proposition 4.49(iv), clearly $\mathcal{U}(F) \cup \mathcal{U}(G) \subseteq \mathcal{U}(F \cap G)$. Suppose $P \in \mathcal{P}F(H)$ such that $P \in \mathcal{U}(F \cap G)$. Then $F \cap G \subseteq P$. Since $P \in \mathcal{P}F(H)$, by Proposition 4.46, $F \subseteq P$ or $G \subseteq Rad(P)$. If $F \subseteq P$, then $P \in \mathcal{U}(F) \subseteq \mathcal{U}(F) \cup \mathcal{U}(G)$, and the proof complete. If $G \subseteq Rad(P)$, then by assumption $P \in \mathcal{R}F(H)$ and so $G \subseteq Rad(P) = P$. Thus $P \in \mathcal{U}(G) \subseteq \mathcal{U}(F) \cup \mathcal{U}(G)$. Hence, $\mathcal{U}(F) \cup \mathcal{U}(G) = \mathcal{U}(F \cap G)$. □

Define $\mathcal{T}(H) = \{\mathcal{U}(F) \mid F \in \mathcal{F}(H)\}$.

Theorem 4.52. *If $\mathcal{T}(H)$ is closed under finite union, then $(H, \mathcal{T}(H))$ is a topological space.*

Corollary 4.53. *If $\mathcal{F}(H) \subseteq \mathcal{R}F(H)$, then $(H, \mathcal{T}(H))$ is a topological space.*

5 Conclusion

Considering the importance of the concept of filters in logical algebras and its application in creating the quotient space and creating logical algebras, one of the aims of this article is to introduce new filters on hoops and examine the relationship between them and the filters that are previously defined. It is shown that the introduced concepts are the expansion of previous filters, such as the prime and maximum filters. On the other hand, by using the notion of prime filter we show representation theorem of hoops and we prove that every nontrivial $\vee$-hoop is a subdirect product of hoop-chains. Also, we used the introduced concepts new filters and define new open sets on hoop that could be used to construct a Zarisky topology. In future, we want to define the notion of primary filter on hoop and we investigate the relation between all kind of filters are introduced on hoop. Also, we try to investigate the notion of flat topology on hoop.

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