Iranian Journal of Fuzzy Systems

Volume 20, Number 7, (2023), pp. 145-155

Original paper

A study of BL-algebras by UC-filters

S. Motamed[1], J. Moghaderi[2] and A. Borumand Saeid[3]

[1]*Department of Mathematics, Bandar Abbas Branch, Islamic Azad University, Bandar Abbas, Iran*
[2]*Department of Mathematics, University of Hormozgan, Bandar Abbas, Iran*
[3]*Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran*

somayeh.motamed@iauba.ac.ir, j.moghaderi@hormozgan.ac.ir, arsham@uk.ac.ir

Abstract

In this article, with the aim of further investigating BL-algebras, the concepts of *Unity Co-annihilator*-filters (UC-filters) is introduced and discussed. Also, for the faster study of UC-filters in BL-algebras, some equivalent conditions are obtained, and (with some examples) it is shown that these filters have differences. In addition, we consider several additional conditions imposed on UC-filters and prime filters and establish links between them. So, we get relationships between these types of filters and prime filters in BL-algebras. Finally, the form of all UC-saturated $\vee$-closed subsets of a G-algebras (by the concept of UC-saturated $\vee$-closed subsets) is stated.

Keywords: BL-algebra, (UC, prime)-filter, unity co-annihilator elements.

1 Introduction

Non-classical logic is related to algebraic logic systems. A lot of research has been done to further investigate non-classical logic, which aims to enrich algebraic content. Artificial intelligence is also progressing in this direction, with the aim of simulating humans in dealing with the certainty and uncertainty of information. The foundations of this research are based on logic. Fuzzy logic handles information, such as ambiguity and randomness, with the help of classical logic (many-valued logic). For this reason, non-classical logic is a tool in computer science to deal with uncertain and fuzzy information. For more detailed investigations, these logics were represented as algebras (sets with one or more algebraic operators). Basic logic-algebras (BL-algebras) are one of the most important algebras that have been inspired by logic with this goal in mind. The basic logic, BL for short, was introduced by Hájek to formalize many valued semantics induced by the continuous t-norm on the unit real interval $[0,1]$. This BL is a common piece in three important logics of Łukasiewicz Logic, Gödel Logic, and Product Logic. Similar to MV or Boolean algebras, which were created from classical logic or Łukasiewicz logic, respectively, BL-algebras were also created from logical axioms, as the Lindenbaum algebras. Now, this logic has been developed into fuzzy logic, which is used in various aspects of our world, including quantum mechanical theory, mathematical logic, probability theory, computer science, algebra, soft computing, and many other important aspects. The concept of filters plays an important role in the study of logical algebras because different filters correspond to various sets of provable formulas. The completeness of BL was proved by Hájek with the help of the prime filters in BL-algebras (see [5, 6, 9, 18]).
In the past years, for a more detailed analysis of BL-algebras, different filters were defined and studied by different researchers, such as [1, 2, 7, 10, 12, 17].

This paper aims to analyze the structure of BL-algebras by defining some new types of filters. In fact, this article aims to categorize these logical algebras with the help of new concepts and to generalize the previous concepts including filters. In previous research, the concept of the prime filter has been used to classify BL-algebras in such a way that the researchers showed that a BL-algebra is linear if and only if the filter $\{1\}$ is prime. Linear BL-algebras are of particular

Corresponding Author: A. Borumand Saeid
Received: March 2023; Accepted: Invited paper.
https://doi.org/10.22111/IJFS.2023.7668

importance and many researchers have investigated this type of BL-algebras, including in [13], they have studied the characteristics of some filters in linear BL-algebras. So in this article, we considered the definition of the new filter to be close to the prime filter, but more limited than the definition of the prime filter. In fact, we intend to introduce a weaker filter than the prime filter and, with its help, define a new class of BL-algebras and find more properties of this algebra. So the concept of UC-filters, which is weaker than the prime filter, is introduced:

- a proper filter F of A, is called prime, if $a \vee b \in F$, for $a, b \in A$, implies $a \in F$ or $b \in F$;
- a proper filter F of A, is called a UC-filter, if $a \vee b \in F$, for $a \in U(Co-ann(A))$ and $b \in A$ implies $b \in F$;

Of course, by studying as much as possible the connection of UC-filter with other filters defined in BL-algebras, it is possible to determine the connection with all types of BL-algebras, including Boolean algebra, etc. In fact, like other types of filters in BL-algebras, UC-filters can also be used for the classification of BL-algebras. For this reason, we tried to establish a relationship between UC-filter and the prime filter, which is of particular importance in BL-algebras. In the following, we show that $\{1\}$ is a prime filter of a BL-algebra A if and only if the only UC-filter of A is $\{1\}$. Also, the notions of UC-$\vee$-closed subset, UC-saturated $\vee$-closed subset in BL-algebras are introduced, and some characterizations of them are found.

2 Preliminaries

Definition 2.1. [6] *A BL-algebra is an algebra $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that for all $x, y, z \in A$,:*

(BL_1) $(A, \wedge, \vee, 0, 1)$ *is a bounded lattice,*
(BL_2) $(A, *, 1)$ *is a commutative monoid,*
(BL_3) $*$ *and* $\rightarrow$ *form an adjoint pair i.e.,* $z \leq x \rightarrow y$ *iff* $x * z \leq y$,
(BL_4) $x \wedge y = x * (x \rightarrow y)$,
(BL_5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

• A BL-algebra A is called a Gödel algebra (G-algebra), if $x^2 = x$, for all $x \in A$.

It is easy to prove that if A is a BL-algebra and $x, y, z \in A$, we have the following rules of calculus (for more details see [3, 4, 6, 16]):

(BL_6) $x \leq y$ if and only if $x \rightarrow y = 1$,
(BL_7) $1 \rightarrow x = x$ and $x \leq y \rightarrow x$,
(BL_8) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
(BL_9) If $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$, $z \rightarrow x \leq z \rightarrow y$, $x * z \leq y * z$ and $y^- \leq x^-$, where $x^- = x \rightarrow 0$,
(BL_{10}) $y \leq (y \rightarrow x) \rightarrow x$, $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$,
(BL_{11}) $x \vee (y * z) \geq (x \vee y) * (x \vee z)$, $x * (y \vee z) = (x * y) \vee (x * z)$,
(BL_{12}) $x^m \vee y^n \geqslant (x \vee y)^{mn}$, for all $m, n \in \mathbb{N}$.

Throughout this paper, it is assumed $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra and in short A (unless we write otherwise).

• Let F be a non-empty subset of A and $x, y \in A$. Then the following conditions are equivalent:
(i) $x, y \in F$ implies $x * y \in F$, and if $x \in F$, $x \leq y$, then $y \in F$.
(ii) $1 \in F$, and if $x, x \rightarrow y \in F$ then $y \in F$.

F is called a filter of A if it satisfies in the one of the above conditions. F is proper if $F \neq A$, i.e. $0 \notin F$, [6].

Let $x \in A$ and F, G be filters of A. Recall that
$\langle x \rangle = \{a \in A : a \geq x^n, \text{for some } n \in \mathbb{N}\}$,
$\langle F \cup x \rangle = \{a \in A : a \geq f * x^n, \text{for some } f \in F \text{ and } n \in \mathbb{N}\}$,
$\langle F \cup G \rangle = \{a \in A : a \geq f * g, \text{for some } f \in F \text{ and } g \in G\}$, for see more details, refer to [14].

• Let F be a proper filter of A. Then the following conditions are equivalent:
(i) if for all $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$,
(ii) for all $x, y \in A$, either $x \rightarrow y \in F$ or $y \rightarrow x \in F$.

F is called a prime filter of A if it satisfies one of the above conditions. A proper filter is maximal if it is not contained in any other proper filter, [6].

For non-empty subsets X and Y of A, $X \vee Y = \{x \vee y : x \in X,\ y \in Y\}$ and $a \vee Y = \{a\} \vee Y$, for $a \in A$. For a filter F of A and a non-empty subset X of A, $(F : X) = \{b \in A : X \vee b \subseteq F\}$. It is clear that $(F : X)$ is a filter and $F \subseteq (F : X)$, for any filter F of A and any $X \subseteq A$. Also $(F : a) = (F : \{a\})$, for any $a \in A$, [11].

A filter F of A is called an irreducible filter, if $F = G \cap H$, implies $F = G$ or $F = H$, for any filters G and H. For any $a \in A$; $Co-ann(a) = \{x \in A : a \vee x = 1\}$ is a filter of A. It is clear that $Co-ann(0) = \{1\}$ and $Co-ann(1) = A$, [15].

Definition 2.2. [6] *Let F be a proper filter of a BL-algebra A. The relation $\sim_F$ defined on a BL-algebra A, by $(x, y) \in\sim_F$ if and only if $x \to y \in F$ and $y \to x \in F$, is a congruence relation on A. The quotient algebra $A/\sim_F$ denoted by A/F becomes a BL-algebra in a natural way, with the operations induced from those of A. So, the order relation on A/F is given by $x/F \leq y/F$ if and only if $x \to y \in F$. Hence $x/F = 1/F$ if and only if $x \in F$ and $x/F = 0/F$ if and only if $x^- \in F$.*

3 UC-filters

As we know, filters is a very important concept in BL-algebras, which describes the characteristics of BL-algebras and classifies them. In this section, we introduce the concept of a UC-filter in BL-Algebras and study its properties. Also, under special conditions, we describe the relationship between UC-filters and prime filters.

We begin this section with the notation of $U(Co - ann(A))$, for any BL-algebra A, as follows:

We consider the set of unity co-annihilator elements of A by $U(Co - ann(A)) = \{a \in A : Co - ann(a) = \{1\}\}$. Clearly, for any BL-algebra A, $0 \in U(Co - ann(A))$ and $1 \notin U(Co - ann(A))$.

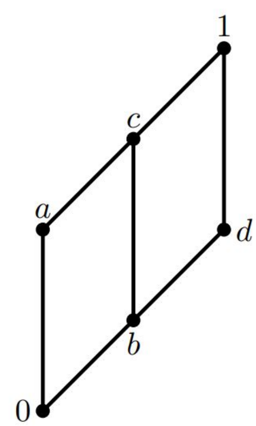
In the following we give examples for the concept of $U(Co - ann(A))$.

Example 3.1. (i) *Let $A = \{0, a, b, c, d, 1\}$, where $0 < a < c < 1$ and $0 < b < c, d < 1$. Define $*$ and $\to$ as follows:*

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

The Hasse diagram of this table is as follows:



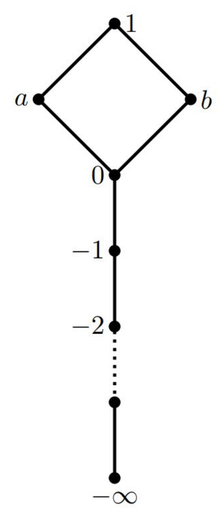
Then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra, [8]*. It is easy to check that $U(Co - ann(A)) = \{0, b\}$.*

(ii) *Let $A = \overline{\mathbb{Z}} \cup \{-\infty\} \cup \{0, a, b, 1\}$, where $\overline{\mathbb{Z}}$ is the set of negative integer numbers and $-\infty < \cdots < -2 < -1 < 0 < a, b < 1$. Operations $*$ and $\to$ are defined as follows:*

$*$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1
$-\infty$	$-\infty$	$\cdots$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	-6	-5	-4	-3	-3	-3	-3
-2	$-\infty$	$\cdots$	-5	-4	-3	-2	-2	-2	-2
-1	$-\infty$	$\cdots$	-4	-3	-2	-1	-1	-1	-1
0	$-\infty$	$\cdots$	-3	-2	-1	0	0	0	0
a	$-\infty$	$\cdots$	-3	-2	-1	0	a	0	a
b	$-\infty$	$\cdots$	-3	-2	-1	0	0	b	b
1	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1

$\rightarrow$	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1
$-\infty$	1	$\cdots$	1	1	1	1	1	1	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	1	1	1	1	1	1	1
-2	$-\infty$	$\cdots$	-1	1	1	1	1	1	1
-1	$-\infty$	$\cdots$	-1	1	1	1	1	1	1
0	$-\infty$	$\cdots$	-3	-2	-1	1	1	1	1
a	$-\infty$	$\cdots$	-3	-2	-1	b	1	b	1
b	$-\infty$	$\cdots$	-3	-2	-1	a	a	1	1
1	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1

The Hasse diagram of this table is as follows:



Then $(A, \wedge, \vee, *, \rightarrow, -\infty, 1)$ *is a BL-algebra,* [8]. *All filters of* A *are* $G_1 = \{1\}$, $G_2 = \{a, 1\}$, $G_3 = \{b, 1\}$, $G_4 = \{0, a, b, 1\}$ *and* $G_5 = \{\ldots, -3, -2, -1, 0, a, b, 1\} \smallsetminus \{-\infty\}$. *It is obvious that* $U(Co - ann(A)) = A \smallsetminus \{a, b, 1\}$.

Definition 3.2. *A proper filter* F *of* A*, is called a Unity* $Co-$*annihilator-filter (or briefly UC-filter), if* $a \vee b \in F$*, for* $a \in U(Co - ann(A))$ *and* $b \in A$ *implies* $b \in F$.

Remark 3.3. *It is clear that* $\{1\}$ *is a UC-filter in any BL-algebra. So if* A *is an MV-algebra, then* $D_S(A) = \{x \in A : x^- = 0\}$ *is a UC-filter.*

Example 3.4. (i) *In Example 3.1(i),* $F_1 = \{1\}$, $F_2 = \{d, 1\}$ *and* $F_3 = \{a, c, 1\}$ *are UC-filters; but* $F_4 = \{c, 1\}$ *and* $F_5 = \{b, c, d, 1\}$ *are not UC-filters. Since* $b \vee a = c$ *and* $b \in U(Co - ann(A))$ *but* $a \notin F_4, F_5$.

(ii) *In Example 3.1(ii), all* G_i *are UC-filters.*

In the following lemma, we show a commonly used property for UC-filters.

Lemma 3.5. *Let F be a UC-filter of A. Then $F \cap U(Co-ann(A)) = \emptyset$.*

Proof. Let $a \in F \cap U(Co-ann(A))$. Then as $0 \vee a = a \in F$ and F is a UC-filter, $0 \in F$; which is a contradiction. □

In the following, by adding a condition, the converse of Lemma 3.5 is proved.

Proposition 3.6. *Let for any $a \in A$, $\{\langle a \rangle : a \in A\}$ be a chain of A. Then every proper filter F of A that $F \cap U(Co-ann(A)) = \emptyset$, is a UC-filter.*

Proof. Let F be a proper filter of A such that $F \cap U(Co-ann(A)) = \emptyset$. Also let $a \vee b \in F$ and $a \in U(Co-ann(A))$, $b \in A$. Then by hypothesis $\langle a \rangle \subseteq \langle b \rangle$ or $\langle b \rangle \subseteq \langle a \rangle$. We know that $\langle a \rangle \cap \langle b \rangle \subseteq \langle a \vee b \rangle \subseteq F$, hence as $\langle a \rangle \subseteq \langle b \rangle$ or $\langle b \rangle \subseteq \langle a \rangle$, so $\langle a \rangle \subseteq F$ or $\langle b \rangle \subseteq F$. If $\langle a \rangle \subseteq F$, then $a \in F \cap U(Co-ann(A))$, which is a contradiction. Hence $\langle b \rangle \subseteq F$, then $b \in F$. Therefore F is a UC-filter. □

In the following, we show the equivalent conditions to check the UC-filters more easily:

Theorem 3.7. *Let F be a proper filter of A such that $F \cap U(Co-ann(A)) = \emptyset$. Then for two non-empty subsets X and Y of A and $a, b \in A$ the following conditions are equivalent:*
(i) F is a UC-filter;
(ii) $F = (F : a)$, for any $a \in U(Co-ann(A))$;
(iii) $X \vee Y \subseteq F$ and $X \cap U(Co-ann(A)) \neq \emptyset$ imply $Y \subseteq F$;
(iv) $a \vee X \subseteq F$ and $a \in U(Co-ann(A))$ imply $X \subseteq F$;
(v) $G \cap H \subseteq F$, for filters G and H of A, implies $G \cap U(Co-ann(A)) = \emptyset$ or $H \subseteq F$;
(vi) $\langle a \rangle \cap \langle b \rangle \subseteq F$ implies $a \notin U(Co-ann(A))$ or $b \in F$;
(vii) $F = G \cap H$, for filters G and H of A such that $G \cap U(Co-ann(A)) \neq \emptyset$, implies $F = H$;
$(viii)$ $(F : X) = A$ or $(F : X)$ is a UC-filter of A.

Proof. $((i) \Rightarrow (ii))$ Assume that $x \in (F : a)$, for $a \in U(Co-ann(A))$. Then $x \vee a \in F$ and so by part (i), $x \in F$. Thus $(F : a) \subseteq F$ and therefore $F = (F : a)$.
$((ii) \Rightarrow (iii))$ Let $X \vee Y \subseteq F$, for non-empty subsets X, Y of A, where $X \cap U(Co-ann(A)) \neq \emptyset$. Consider $x \in X \cap U(Co-ann(A))$ and $y \in Y$. Then $x \vee y \in X \vee Y \subseteq F$ and so $y \in (F : x)$. Hence by part (ii), $y \in F$ and therefore $Y \subseteq F$.
The proofs of $((iii) \Rightarrow (iv))$ and $((v) \Rightarrow (vi))$ are easy.
$((iv) \Rightarrow (v))$ For filters G and H of A, assume that $G \cap H \subseteq F$ and $a \in U(Co-ann(A)) \cap G$. Then $a \vee H \subseteq \langle a \rangle \cap H \subseteq G \cap H \subseteq F$ and so by part (iv), $H \subseteq F$.
$((vi) \Rightarrow (vii))$ Assume that $F = G \cap H$, for filters G and H of A such that $G \cap U(Co-ann(A)) \neq \emptyset$. Consider $a \in G \cap U(Co-ann(A))$ and $x \in H$. Then $\langle a \rangle \cap \langle x \rangle \subseteq G \cap H \subseteq F$. Hence by part (vi), $x \in F$ and therefore $H \subseteq F$.
$((vii) \Rightarrow (viii))$ Assume that for non-empty subset X of A, $(F : X) \neq A$. Let $a \vee b \in (F : X)$, for $a \in U(Co-ann(A))$ and $b \in A$. Then for any $x \in X$, $a \vee b \vee x \in F$. Since $\langle F \cup \{a\} \rangle \cap \langle F \cup \{b \vee x\} \rangle = F$, by part (vii), $F = \langle F \cup \{b \vee x\} \rangle$. Thus $b \vee x \in F$, for any $x \in X$. Hence $b \vee X \subseteq F$ and therefore $b \in (F : X)$. So $(F : X)$ is a UC-filter.
$((viii) \Rightarrow (i))$ As $(F : 0) = F$ is a proper filter, then by part $(viii)$, $(F : 0)$ is a UC-filter. Therefore F is a UC-filter. □

Since $\{1\}$ is a UC-filter, by Theorem 3.7, then we obtain:

Corollary 3.8. *Let F and G be two UC-filters of A, X be a non-empty subset of A and $a \in A \setminus \{1\}$. Then the following statements hold.*
(i) $(\{1\} : X) = A$ or $(\{1\} : X)$ is a UC-filter;
(ii) $Co-ann(a)$ is a UC-filter;
If $X \cap U(Co-ann(A)) \neq \emptyset$ then:
(iii) $X \vee F = X \vee G$, implies $F = G$.

Proof. The proofs of (i) and (ii) are clear.
(iii) Let $X \cap U(Co-ann(A)) \neq \emptyset$ and $b \in F$. So there exists $a \in X \cap U(Co-ann(A))$ and then $a \vee b \in X \vee F = X \vee G$. It is clear that $X \vee G \subseteq G$, hence $a \vee b \in G$. Now as $a \in U(Co-ann(A))$, we get that $b \in G$. Then $F \subseteq G$ and similarly $G \subseteq F$. Therefore, $F = G$. □

Proposition 3.9. *Let F and G be two filters of A.*
(i) If F is a UC-filter, then $a \vee F = F \cap (a \vee A)$, for any $a \in U(Co-ann(A))$.
(ii) If $F \cap U(Co-ann(A)) \neq \emptyset$ and $F \cap G$ is a UC-filter, then G is a UC-filter.

Proof. (i) It is clear that for any $a \in A$, $a \vee F \subseteq F \cap (a \vee A)$. Now let $a \vee x \in F$, for any $x \in A$. Then as F is a UC-filter we get that $x \in F$ and so $a \vee x \in a \vee F$. Therefore $a \vee F = F \cap (a \vee A)$.
(ii) Let $a \in F \cap U(Co - ann(A))$, then $a \vee G \subseteq F \vee G$. We know that $F \vee G \subseteq F \cap G$, hence by Theorem 3.7($(i) \Rightarrow (iv)$), $G \subseteq F \cap G$. Then $G = F \cap G$ and so G is a UC-filter. □

Proposition 3.10. *Let $\{F_i\}_{i \in I}$ be a family of UC-filters of A. Then*
(i) *$\bigcap_{i \in I} F_i$ is a UC-filter;*
(ii) *if $\{F_i\}_{i \in I}$ is a chain, $\bigcup_{i \in I} F_i$ is a UC-filter.*

Let F be a filter of A and S be a non-empty $\vee$-closed subset of A ($x, y \in S$, implies $x \vee y \in S$). Consider $F_S = \{x \in A : x \vee s \in F, \exists s \in S\}$. Then F_S is a filter of A containing F.
In Example 3.1(i), for $S = \{a, b, c\}$, $(\{d, 1\})_S = \{d, 1\}$ and $(\{b, c, d, 1\})_S = A$.

Lemma 3.11. *The set $U(Co - ann(A))$ is a $\vee$-closed subset of A.*

Proof. Assume that $a, b \in U(Co - ann(A))$. Then $Co - ann(a) = Co - ann(b) = \{1\}$. Let $x \in Co - ann(a \vee b)$. Then $x \vee a \vee b = 1$ and so $x \vee a \in Co - ann(b)$. Hence $x \in Co - ann(a)$ and therefore $x = 1$. Thus $a \vee b \in U(Co - ann(A))$. □

Proposition 3.12. *Let F be a filter of A and S be a $\vee$-closed subset of A. Then in each of the following conditions, F_S is a UC-filter.*
(i) *F is a UC-filter and S is contained in $U(Co - ann(A))$.*
(ii) *S contains $U(Co - ann(A))$ such that $F \cap S = \emptyset$.*

Proof. (i) Let F be a UC-filter and $S \subseteq U(Co - ann(A))$. Then by Lemma 3.5, $F \cap U(Co - ann(A)) = \emptyset$ and we get that $F \cap S = \emptyset$. If $0 \in F_s$ then for some $s \in S$, $0 \vee s \in F$ i.e., $F \cap S \neq \emptyset$. Hence $0 \notin F_s$, i.e. F_s is proper. Now assume that $a \vee b \in F_S$, for $a \in U(Co - ann(A))$ and $b \in A$. So there exists $s \in S$ such that $a \vee b \vee s \in F$. By Lemma 3.11, $a \vee s \in U(Co - ann(A))$ and hence $b \in F$, as F is a UC-filter. Thus $b \in F_S$ and therefore F_S is a UC-filter.
(ii) First, we show that F_S is a proper filter of A. If $0 \in F_s$ then for some $s \in S$, $0 \vee s \in F$ i.e., $F \cap S \neq \emptyset$. Hence $0 \notin F_s$, so F_s is proper. Now assume that $a \vee b \in F_S$, for $a \in U(Co - ann(A))$ and $b \in A$. So there exists $s \in S$ such that $a \vee b \vee s \in F$. Then $a \vee s \in S$ and hence $b \in F_S$. Therefore F_S is a UC-filter. □

By using Proposition 3.12(ii):

Corollary 3.13. *Let F be a filter of A such that $F \cap U(Co - ann(A)) = \emptyset$. Then $F_{U(Co-ann(A))}$ is a UC-filter.*

According to Corollaries 3.12 and 3.13, we get that:

Proposition 3.14. *Let F be a filter of A.*
(i) *If S is a $\vee$-closed subset of A which contains $U(Co - ann(A))$ such that $F \cap S = \emptyset$, then there exists a UC-filter G of A contains F such that $G \cap S = \emptyset$.*
(ii) *If $F \cap U(Co - ann(A)) = \emptyset$, then there exists a UC-filter of A containing F.*

Proposition 3.15. *If for $a, b \in A$, $a \wedge b = 0$, then $\langle Co - ann(a) \cup Co - ann(b) \rangle = A$ or $\langle Co - ann(a) \cup Co - ann(b) \rangle$ is a UC-filter.*

Proof. Let $\langle Co - ann(a) \cup Co - ann(b) \rangle \neq A$, for $a, b \in A$, such that $a \wedge b = 0$. Assume that $x \vee y \in \langle Co - ann(a) \cup Co - ann(b) \rangle$, for $x \in U(Co - ann(A))$ and $y \in A$. Then there exist $\alpha \in Co - ann(a)$ and $\beta \in Co - ann(b)$ such that $\alpha * \beta \leq x \vee y$. Thus as $\alpha \vee a = 1$ and $\beta \vee b = 1$, $(\alpha * \beta) \vee (a \vee b) \geq (\alpha \vee (a \vee b)) * (\beta \vee (a \vee b)) = 1$. Therefore $(x \vee y) \vee (a \vee b) = 1$ and hence $y \vee a \vee b = 1$, since $x \in U(Co - ann(A))$ and $\{1\}$ is a UC-filter. Thus $y \vee a \in Co - ann(b)$ and $y \vee b \in Co - ann(a)$. So $y \vee a$, $y \vee b \in \langle Co - ann(a) \cup Co - ann(b) \rangle$. Then $y = y \vee 0 = y \vee (a \wedge b) = (y \vee a) \wedge (y \vee b) \in \langle Co - ann(a) \cup Co - ann(b) \rangle$. Therefore $\langle Co - ann(a) \cup Co - ann(b) \rangle$ is a UC-filter. □

Proposition 3.16. *In A, the following statements are equivalent.*
(i) *Every proper filter is a UC-filter;*
(ii) *Every principal proper filter is a UC-filter.*

Proof. ($(i) \Rightarrow (ii)$) The proof is clear.
($(ii) \Rightarrow (i)$) Let F be a proper filter of A. Assume that $a \vee b \in F$, for $a \in U(Co - ann(A))$ and $b \in A$. If $0 \in \langle a \vee b \rangle$, then $(a \vee b)^m = 0$, for some $m \in \mathbb{N}$. So $0 \in F$, which is a contradiction. Then $\langle a \vee b \rangle$ is a proper filter and since $a \vee b \in \langle a \vee b \rangle$, by hypothesis, $b \in \langle a \vee b \rangle$. Hence $b \in F$, (since $\langle a \vee b \rangle \subseteq F$), and therefore F is a UC-filter. □

Proposition 3.17. *If for any proper filter F of A and for any $a \in U(Co-ann(A))$, $\langle a \vee F \rangle = F$, then every proper filter of A which does not have any intersection with $U(Co-ann(A))$, is a UC-filter.*

Proof. Let F be a proper filter of A, which does not have any intersection with $U(Co-ann(A))$. Assume that $a \vee b \in F$, for $a \in U(Co-ann(A))$ and $b \in A$. If $0 \in \langle b \rangle$, then $b^m = 0$, for some $m \in \mathbb{N}$. Thus as $(a \vee b)^m \leq a \vee b^m = a$, $a \in F \cap U(Co-ann(A))$; which is a contradiction. So $\langle b \rangle$ is a proper filter and thus by assumption, $\langle a \vee \langle b \rangle \rangle = \langle b \rangle$. So $b \in \langle a \vee \langle b \rangle \rangle$ and therefore there exists $n \in \mathbb{N}$ such that $(a \vee b)^n \leq b$, by (BL_{12}). Hence $b \in F$ and then F is a UC-filter. □

Proposition 3.18. *Let F be a proper filter of A such that $F \cap U(Co-ann(A)) = \emptyset$; but F is not a UC-filter. Then there exist two filters G,H of A such that $F \subsetneqq G, H$ where $G \cap U(Co-ann(A)) \neq \emptyset$ and $G \vee H \subseteq F$.*

Proof. As F is not a UC-filter, there exist $a \in U(Co-ann(A))$, $b \in A \setminus F$ such that $a \vee b \in F$. Put $G = (F : b)$. Then $F \subsetneqq G$, since $a \in G \setminus F$. As $a \in G \cap U(Co-ann(A))$, then $G \cap U(Co-ann(A)) \neq \emptyset$. Put $H = (F : G)$. Then $b \in (F : G)$ and so $F \subsetneqq H$, as $b \in H \setminus F$. Also, $G \vee H = G \vee (F : G) \subseteq F$. Therefore, the proof is completed. □

Theorem 3.19. *Let F and F_1 ,..., F_n be filters of A such that $F \subseteq \bigcup_{i=1}^{n} F_i$. If there exists k, $(1 \leq k \leq n)$ such that F_k is a UC-filter, $F_i \cap U(Co-ann(A)) \neq \emptyset$, for all i, $i \neq k$ and $F \nsubseteq \bigcup_{i=1, i\neq k}^{n} F_i$, then $F \subseteq F_k$.*

Proof. Without loss of generality, assume that $k = 1$. So $F \nsubseteq \bigcup_{i=2}^{n} F_i$ and there exists $x \in F \setminus \bigcup_{i=2}^{n} F_i$. Thus $x \in F_1$. Now consider $y \in F \cap F_2 \cap ... \cap F_n$. Then $x * y \in F \setminus \bigcup_{i=2}^{n} F_i$, (since $x \notin \bigcup_{i=2}^{n} F_i$). Hence $x * y \in F_1$ and so $y \in F_1$. Therefore $F \cap F_2 \cap ... \cap F_n \subseteq F_1$. On the other hand, for each i, $(2 \leq i \leq n)$, there exist $a_i \in F_i \cap U(Co-ann(A))$. So for each i, $(2 \leq i \leq n)$, $Co-ann(a_i) = \{1\}$ and hence by Lemma 3.11, $Co-ann(a_2 \vee ... \vee a_n) = \{1\}$. Put $G = \bigcap_{i=2}^{n} F_i$. Then $a_2 \vee ... \vee a_n \in G \cap U(Co-ann(A))$. Thus $G \vee F = (\bigcap_{i=2}^{n} F_i) \vee F \subseteq F \cap F_2 \cap \cdots \cap F_n \subseteq F_1$ and therefore $G \vee F \subseteq F_1$. Now as $G \cap U(Co-ann(A)) \neq \emptyset$ and F_1 is a UC-filter, by Theorem 3.7($(i) \Rightarrow (iii)$), $F \subseteq F_1$. □

Recall that, an algebra A has the $\vee$-irreducible property, if for any $a, b, c \in A$, $a = b \vee c$ implies that $a = b$ or $a = c$.

Proposition 3.20. *Let F and G be two filters of A and $x \in A$. Then the following statements hold.*
(i) If $F \neq \{1\}$, then ${}^{\perp}F = \{a \in A : a \vee b = 1, \ \forall b \in F\}$ is a UC-filter;
(ii) If A is a G-algebra that has $\vee$-irreducible property, then for any $x \in A$, $F_x \cap U(Co-ann(A)) \neq \emptyset$ or F_x is a UC-filter, where $F_x = \{b \in A : x \leq b\}$;
(iii) If F is a UC-filter, then for any filter G, $(G \to F) \cap U(Co-ann(A)) = \emptyset$ or $(G \to F) \to F = F$, where $G \to F = \{a \in A : G \cap \langle a \rangle \subseteq F\}$;
(iv) If F is an irreducible filter of A such that $F \cap U(Co-ann(A)) = \emptyset$, then F is a UC-filter.

Proof. (i) It is clear that ${}^{\perp}F$ is a proper filter of A. Assume that $a \vee b \in {}^{\perp}F$, for $a \in U(Co-ann(A))$ and $b \in A$. Then $a \vee b \vee x = 1$, for any $x \in F$. Thus $b \vee x = 1$, for any $x \in F$ (as $a \in U(Co-ann(A))$). Therefore $b \in {}^{\perp}F$ and so ${}^{\perp}F$ is a UC-filter.
(ii) Let F be a filter of A and $x \in A$ such that $F_x \cap U(Co-ann(A)) = \emptyset$. Assume that $a \vee b \in F_x$, for $a \in U(Co-ann(A))$ and $b \in A$. Then $x \leq a \vee b$. Thus $x = x * x \leq x * (a \vee b) \leq (x * a) \vee (x * b) \leq x \vee x = x$. Thus $x = (x * a) \vee (x * b)$ and therefore by hypothesis $x = x * a$ that implies $a \in F_x$, which is a contradiction (as $F_x \cap U(Co-ann(A)) = \emptyset$) or $x = x * b$ that implies $b \in F_x$. Therefore F_x is a UC-filter.
(iii) Let G be a filter such that $(G \to F) \cap U(Co-ann(A)) \neq \emptyset$. As $(G \to F) \cap ((G \to F) \to F) = F \cap (G \to F) = F$, by Theorem 3.7($(i) \Rightarrow (vii)$), $(G \to F) \to F = F$.
(iv) Let $a \vee b \in F$, for $a \in U(Co-ann(A))$ and $b \in A$. Since $F = \langle F \cup \{a\} \rangle \cap \langle F \cup \{b\} \rangle$, then $F = \langle F \cup \{a\} \rangle$, that implies $a \in F$, which is a contradiction or $F = \langle F \cup \{b\} \rangle$, that implies $b \in F$. Therefore F is a UC-filter. □

Lemma 3.21. *Let F be a proper filter of A. Then*
(i) If $a/F \in U(Co-ann(A/F))$, then $Co-ann(a) \subseteq F$;
(ii) If $a \in U(Co-ann(A))$ and F is a UC-filter, then $a/F \in U(Co-ann(A/F))$.

Proof. (i) Let $b \in Co-ann(a)$. Then $b \vee a = 1$ and so $b/F \vee a/F = 1/F$. Hence $b/F \in Co-ann(a/F) = \{1/F\}$. Thus $b/F = 1/F$ and therefore $b \in F$.
(ii) We must show that $Co-ann(a/F) = \{1/F\}$. Let $b/F \in Co-ann(a/F)$. Then $b/F \vee a/F = 1/F$. Hence $a \vee b \in F$ and so $b \in F$, since $a \in U(Co-ann(A))$ and F is a UC-filter. Thus $b/F = 1/F$ and therefore $a/F \in U(Co-ann(A/F))$. □

Theorem 3.22. *Let F be a proper filter of A. Then F is a UC-filter if and only if $a/F \in U(Co-ann(A/F))$, for any $a \in U(Co-ann(A))$.*

Proof. Let F be a UC-filter. By Lemma 3.21(ii), for $a \in U(Co-ann(A))$, $a/F \in U(Co-ann(A/F))$. Now Assume that $a/F \in U(Co-ann(A/F))$, for any $a \in U(Co-ann(A))$. Let $a \vee b \in F$, for $a \in U(Co-ann(A))$ and $b \in A$. Then $a/F \vee b/F = 1/F$ and so $b/F \in Co-ann(a/F)$. Hence $b/F = 1/F$, since $a/F \in U(Co-ann(A/F))$. Thus $b \in F$ and therefore F is a UC-filter. □

Let A and B be two BL-algebras. It is easy to see that $Co-ann((a,b)) = Co-ann(a) \times Co-ann(b)$, for any $(a,b) \in A \times B$ and so $U(Co-ann(A \times B)) = U(Co-ann(A)) \times U(Co-ann(B))$. Thus the following proposition holds.

Proposition 3.23. *Let A and B be two BL-algebras, F be a proper filter of A and G be a proper filter of B. Then $F \times G$ is a UC-filter of $A \times B$ if and only if F is a UC-filter of A and G is a UC-filter of B.*

Proof. Assume that $F \times G$ is a UC-filter of $A \times B$. Let $a \vee b \in F$, for $a \in U(Co-ann(A))$ and $b \in A$. Then $(a,0) \vee (b,1) \in F \times G$ and $(a,0) \in U(Co-ann(A \times B))$. Hence $(b,1) \in F \times G$ and so $b \in F$. Therefore F is a UC-filter. Similarly G is a UC-filter too. Now assume that F is a UC-filter of A and G is a UC-filter of B. Let $(a,b) \vee (x,y) \in F \times G$, for $(a,b) \in U(Co-ann(A \times B))$ and $(x,y) \in A \times B$. Then $(a,b) \in U(Co-ann(A)) \times U(Co-ann(B))$ and so $(x,y) \in F \times G$. Therefore $F \times G$ is a UC-filter. □

4 The relationship between UC-filters and other types of filters in BL-algebras

In Example 3.1(i), $F_3 = \{a,c,1\}$ is a maximal, prime and UC-filter. $F_2 = \{d,1\}$ is a prime and UC-filter; but it is not a maximal filter. $F_5 = \{b,c,d,1\}$ is a maximal and prime filter; but it is not a UC-filter. $F_1 = \{1\}$ is a UC-filter; but it is not a maximal nor prime filter. $F_4 = \{c,1\}$ is not a maximal, prime nor UC-filter.

Therefore, in general, there is no relationship between prime and maximal filters and UC-filters in BL-algebras.

In the following, we describe the relationship between UC-filters and prime filters under special conditions.

Proposition 4.1. *Let F be a filter of A.*
(i) *If F is a prime filter, then F is a UC-filter if and only if $F \cap U(Co-ann(A)) = \emptyset$;*
(ii) *If F is a UC-filter such that $A \setminus U(Co-ann(A)) \subseteq F$, then F is prime;*
(iii) *Every maximal element by inclusion in the set of UC-filters is prime.*

Proof. (i) Assume that $F \cap U(Co-ann(A)) = \emptyset$ and $a \vee b \in F$, for $a \in U(Co-ann(A))$, $b \in A$. Then as $a \notin F$ and F is a prime filter, $b \in F$. Therefore F is a UC-filter. The converse holds, by Lemma 3.5.
(ii) Let $a \vee b \in F$, for $a, b \in A$. If $a \in U(Co-ann(A))$, then as F is a UC-filter, $b \in F$. If $a \notin U(Co-ann(A))$, then by hypothesis $a \in F$. Therefore F is a prime filter.
(iii) Let F be a maximal UC-filter. Then $F \cap U(Co-ann(A)) = \emptyset$. Now assume that $a \vee b \in F$, for $a, b \in A$ and $a \notin F$. So $F \subseteq (F:a) \neq A$. By Theorem 3.7((i) ⇒ ($viii$)), $(F:a)$ is a UC-filter. Thus by hypothesis, $F = (F:a)$. On other hand, $a \vee b \in F$, implies $b \in (F:a)$. Hence $b \in F$ and therefore F is a prime filter. □

Corollary 4.2. *Let $F = A \setminus U(Co-ann(A))$ be a filter of A. Then F is a prime filter if and only if F is a UC-filter.*

In the following, we will see what relation we get for the UC-filters if $\{1\}$ is a prime filter.

Proposition 4.3. *The following statements are equivalent:*
(i) $A = U(Co-ann(A)) \cup \{1\}$;
(ii) $\{1\}$ is the only UC-filter of A;
(iii) A is a linearly ordered BL-algebra;
(iv) $Co-ann(a \vee b) = Co-ann(a) \cup Co-ann(b)$, for any $a, b \in A$.

Proof. ((i) ⇒ (ii)) We know that $\{1\}$ is a UC-filter. Assume that F is an another UC-filter of A. So there exists $a \in F \setminus \{1\}$. Hence $a \vee 0 = a \in F$ and since F is a UC-filter by part (i), $0 \in F$, which is a contradiction. Therefore $\{1\}$ is the only UC-filter.
((ii) ⇒ (iii)) Let $\{1\}$ be the only UC-filter of A and $a \vee b = 1$, for $a, b \in A$. Hence $b \in Co-ann(a)$. If $a \neq 1$, then as $Co-ann(a)$ is a UC-filter, by hypothesis we get that $Co-ann(a) = \{1\}$. Therefore $b = 1$, hence $\{1\}$ is a prime filter, i.e. A is a linearly ordered BL-algebra.
((iii) ⇒ (iv)) Let $a, b \in A$. It is clear that $Co-ann(a) \cup Co-ann(b) \subseteq Co-ann(a \vee b)$. Assume that $x \in Co-ann(a \vee b)$. Then $x \vee a \vee b = 1$. So $(x \vee a) \vee (x \vee b) = 1$ and by part (iii), $x \vee a = 1$, which implies $x \in Co-ann(a)$ or

$x \vee b = 1$, which implies $x \in Co - ann(b)$. Hence $Co - ann(a \vee b) \subseteq Co - ann(a) \cup Co - ann(b)$. Therefore $Co - ann(a \vee b) = Co - ann(a) \cup Co - ann(b)$.
($(iv) \Rightarrow (i)$) Let $a \in A \setminus \{1\}$. It is enough to show that $Co - ann(a) = \{1\}$. Take $x \in Co - ann(a)$, then $x \vee a = 1$ and by part (iii), $A = Co - ann(1) = Co - ann(x \vee a) = Co - ann(x) \cup Co - ann(a)$. So $0 \in Co - ann(x) \cup Co - ann(a)$. If $0 \in Co - ann(a)$, then $a = 0 \vee a = 1$, which is a contradiction. Hence $0 \in Co - ann(x)$ and then $x = 0 \vee x = 1$. Therefore $Co - ann(a) = \{1\}$ and so $a \in U(Co - ann(A))$. □

Proposition 4.4. *Let $\{F_i\}_{i=1}^n$ be a family of filters of A such that there exists j that F_j is a prime filter and $\bigcap_{i=1, i \neq j}^n F_i \nsubseteq F_j$. If $\bigcap_{i=1}^n F_i$ is a UC-filter, then F_j is a UC-filter.*

Proof. Let $x \in (\bigcap_{i=1, i \neq j}^n F_i) \setminus F_j$ and $a \vee b \in F_j$, for $a \in U(Co - ann(A))$ and $b \in A$. Then $a \vee b \vee x \in \bigcap_{i=1}^n F_i$. Thus $b \vee x \in \bigcap_{i=1}^n F_i$, since $\bigcap_{i=1}^n F_i$ is a UC-filter. Hence $b \vee x \in F_j$ and so $b \in F_j$, since F_j is a prime filter and $x \notin F_j$. Therefore F_j is a UC-filter. □

Proposition 4.5. *Let $\{F_i\}_{i=1}^n$ be a family of prime filters of A such that are not comparable and $\bigcap_{i=1}^n F_i$ is a UC-filter. Then F_i is a UC-filter, for all $1 \leqslant i \leqslant n$.*

Proof. Consider $i \in I = \{1, 2, \ldots, n\}$. Then for any $j \in I$ and $j \neq i$, there exist $x_j \in F_j \setminus F_i$. Hence as F_i is a prime filter then $x_1 \vee \cdots \vee \widehat{x_i} \vee \cdots \vee x_n \in (\bigcap_{j=1, j \neq i}^n F_j) \setminus F_i$. Hence $\bigcap_{j=1, j \neq i}^n F_j \nsubseteq F_i$. Then by Proposition 4.4, F_i is a UC-filter. □

Theorem 4.6. *Let F be a UC-filter of A and G be a filter of A containing F.*
(i) If G/F is a UC-filter, then G is a UC-filter;
(ii) If G is a prime filter such that $G/F \bigcap U(Co - ann(A/F)) = \emptyset$, then G is a UC-filter;
(iii) If G is a UC-filter and $U(Co - ann(A/F)) \subseteq U(Co - ann(A))/F = \{a/F : a \in U(Co - ann(A))\}$, then G/F is a UC-filter.

Proof. (i) It is clear that G is a proper filter of A. Assume that $a \vee b \in G$, for $a \in U(Co - ann(A))$ and $b \in A$. Then $a/F \vee b/F \in G/F$ and by part (ii) of Lemma 3.21, $a/F \in U(Co - ann(A/F))$. Thus $b/F \in G/F$ and so $b \in G$. Therefore G is a UC-filter.
(ii) As G is a prime filter, G/F is a prime filter of A/F. Then by Proposition 4.1, G/F is a UC-filter. Therefore by part (i), G is a UC-filter.
(iii) Let $a/F \vee b/F \in G/F$, for $a/F \in U(Co - ann(A/F))$ and $b/F \in A/F$. Then $a \vee b \in G$ and $a \in U(Co - ann(A))$. Since G is a UC-filter, $b \in G$ and so $b/F \in G/F$. Therefore G/F is a UC-filter. □

Definition 4.7. *A non-empty subset S of A is called UC-$\vee$-closed subset, if $U(Co - ann(A)) \subseteq S$ and for any $a \in U(Co - ann(A))$ and $b \in S$ then $a \vee b \in S$.*

In Example 3.1(i), $S_1 = \{0, b, c\}$ is UC-$\vee$-closed subset; but $S_2 = \{0, a, b\}$ is not. Since $b \in U(Co - ann(A))$ and $a \in S$ while $a \vee b \notin S_2$.

Remark 4.8. *According to Lemma 3.11, always $U(Co - ann(A))$ is a UC-$\vee$-closed subset of A.*

Theorem 4.9. *Let F be a filter of A. Then F is a UC-filter if and only if $A \setminus F$ is a UC-$\vee$-closed subset.*

Proof. Let F be a UC-filter. Then by Lemma 3.5, $F \cap U(Co - ann(A)) = \emptyset$. So $U(Co - ann(A)) \subseteq A \setminus F$. Assume that $a \in U(Co - ann(A))$ and $b \in A \setminus F$. If $a \vee b \in F$, then $b \in F$, since F is a UC-filter, which is a contradiction. Hence $a \vee b \in A \setminus F$ and so $A \setminus F$ is a UC-$\vee$-closed subset. Now assume that $A \setminus F$ is a UC-$\vee$-closed subset and $a \vee b \in F$, for $a \in U(Co - ann(A))$ and $b \in A$. So as $0 \in A \setminus F$ and $a \in U(Co - ann(A))$ then $a \in A \setminus F$. If $b \notin F$, then $b \in A \setminus F$ and as $A \setminus F$ is a UC-$\vee$-closed subset then $a \vee b \in A \setminus F$, which is a contradiction. Therefore $b \in F$ and then F is a UC-filter. □

Proposition 4.10. *Let F be a filter of A and S be a UC-$\vee$-closed subset of A such that $F \cap S = \emptyset$. Then there exists a UC-filter G containing F such that $G \cap S = \emptyset$.*

Proof. Put $\Omega = \{G : G \in F(A),\ F \subseteq G \text{ and } G \cap S = \emptyset\}$. Then $F \in \Omega$ and by Zorn's Lemma, Ω has a maximal member like G. Assume that $a \vee b \in G$, for $a \in U(Co - ann(A))$ and $b \in A$. If $b \notin G$, then $G \subsetneqq (G : a)$. Therefore there exists $x \in (G : a) \cap S$. So $x \vee a \in G$ and $x \in S$. As $a \in U(Co - ann(A))$ then we get that $a \vee x \in S$. Thus $x \vee a \in G \cap S$, which is a contradiction. Therefore $b \in G$ and so G is a UC-filter. □

Definition 4.11. *Let S be UC-$\vee$-closed subset of A. A non-empty subset S^* of A is called S-$\vee$-closed subset (S-$\vee$-closed), if for any $a \in S$ and $b \in S^*$, $a \vee b \in S^*$.*

Example 4.12. *In Example 3.1(i), $S^* = \{d\}$ is a S-$\vee$-closed subset, where $S = \{0, b\}$.*

Proposition 4.13. *Let F be a filter of A and S^* be a S-$\vee$-closed subset of A such that $F \cap S^* = \emptyset$. Then there exists a UC-filter G containing F such that $G \cap S^* = \emptyset$.*

Proof. Put $\Omega = \{G : G \in F(A),\ F \subseteq G \text{ and } G \cap S^* = \emptyset\}$. Then $F \in \Omega$ and by Zorn's Lemma, Ω has a maximal member like G. Assume that $a \vee b \in G$, for $a \in U(Co-ann(A))$ and $b \in A$. If $b \notin G$, then $G \subsetneqq (G : a)$. Therefore there exists $x \in (G : a) \cap S^*$. So $x \vee a \in G$ and $x \in S^*$. Hence as $a \in U(Co-ann(A)) \subseteq S$ and $x \in S^*$, then $a \vee x \in S^*$. Thus $x \vee a \in G \cap S^*$, which is a contradiction. Therefore $b \in G$ and so G is a UC-filter. □

Definition 4.14. *A UC-$\vee$-closed subset S of A is called UC-saturated $\vee$-closed subset, if $x \vee y \in S$ implies $x, y \in S$, for every $x, y \in A$.*

In Example 3.1(i), $S = \{0, b\}$ is a UC-saturated $\vee$-closed subset.

Consider $F_{UC}(A)$, as the set of all UC-filters of A. Then the following proposition holds.

Proposition 4.15. *$A \setminus \bigcup_{F \in F_{UC}(A)} F$ is a UC-saturated $\vee$-closed subset.*

Proof. Put $B = A \setminus \bigcup_{F \in F_{UC}(A)} F$. Then by Lemma 3.5, $U(Co-ann(A)) \cap F = \emptyset$, for any $F \in F_{UC}(A)$ hence $U(Co-ann(A)) \subseteq B$. Assume that $a \vee b \notin B$, for $a \in U(Co-ann(A))$ and $b \in B$. Then $a \vee b \in \bigcup_{F \in F_{UC}(A)} F$ and so $a \vee b \in G$, for some filter $G \in F_{UC}(A)$. Thus $b \in G$, which is a contradiction. Therefore B is a UC-$\vee$-closed subset of A. Now let $x \vee y \in B$, for $x, y \in A$. Then $x \vee y \notin \bigcup_{F \in F_{UC}(A)} F$ and so $x \vee y \notin F$, for every $F \in F_{UC}(A)$. Thus $x, y \notin F$, for any $F \in F_{UC}(A)$ and hence $x, y \notin \bigcup_{F \in F_{UC}(A)} F$. Therefore $x, y \in B$ and so $A \setminus \bigcup_{F \in F_{UC}(A)} F$ is a UC-saturated $\vee$-closed subset. □

The following theorem shows that every UC-saturated $\vee$-closed subset of a G-algebra A is of the latter form.

Theorem 4.16. *Let S be a UC-saturated $\vee$-closed subset of a G-algebra A and $\Omega = \{F : F \in F_{UC}(A);\ F \cap S = \emptyset\}$. Then $S = A \setminus \bigcup_{F \in \Omega} F$.*

Proof. Put $B = A \setminus \bigcup_{F \in \Omega} F$. As for every $F \in \Omega$, $F \cap S = \emptyset$ then $S \subseteq B$. Now assume that $b \in B$. If $b \notin S$, then $\langle b \rangle \cap S = \emptyset$. If $\alpha \in \langle b \rangle \cap S$, then as A is a G-algebra, we get that $b = b^n \leq \alpha$, for some $n \in \mathbb{N}$. Hence $b \vee \alpha = \alpha \in S$. So as S is a UC-saturated $\vee$-closed subset, $b \in S$, which is a contradiction. Thus $\langle b \rangle \cap S = \emptyset$. Thus by Proposition 4.10, there exists a UC-filter G which $G \cap S = \emptyset$ and $\langle b \rangle \subseteq G$. Hence $G \in \Omega$ and $b \in G$, which is a contradiction. Therefore $b \in S$ and so $B \subseteq S$. Then $S = A \setminus \bigcup_{F \in \Omega} F$. □

5 Conclusion

As we know, non-classical logic is used in computer science to deal with uncertain and fuzzy information, which is why this logic has become a tool in computer science. BL-algebras have the most important algebraic structure among all the various logical algebras that have been proposed as the semantic systems of non-classical logical systems. Also, they include some important classes of algebras, like the MV. In this article, we tried to take a step towards a more detailed study of BL-algebras by presenting new concepts. We first defined the concept of UC-filters on BL-algebras and obtained interesting equivalence properties for easier investigation of this type of filter. In a special case, the relationship of UC-filter with some filters, like the prime filters, was studied, and it was shown that $\{1\}$ is a prime filter of a BL-algebra A if and only if the only the UC-filter of A is $\{1\}$. In future works, we can investigate the quotient BL-algebra generated with the UC-filter and determine what kind of the BL-algebra this new filter gives us and it is possible to obtain the connection of UC-filters with other filters in these algebras and after that more connections with other types of BL- algebras. The new concepts defined in this article can be the idea of future research in logical algebras. Some important issues for future work are as follows:

1-Study of the relation between UC-filters and other types of filters in BL-algebras.

2- Providing equivalent logical relationships with these new filters in BL.

3- Introducing a new subclass of BL-algebras.

Since BL-algebras are related to various logical algebraic structures, the results of this article can be studied in other algebraic structures as well. We also hope that this research will be useful for investigating logical algebras as much as possible, and we can classify these algebras more accurately with the help of these new concepts.

Acknowledgement

The authors would like to thank the anonymous reviewers for their constructive suggestion and helpful comments, which enabled us to improve the presentation of our work.

References

[1] A. Borumand Saeid, S. Motamed, *A new filter in BL-algebras*, Journal of Intelligent and Fuzzy Systems, **27** (2014), 2949-2957.

[2] R. A. Borzooei, A. Paad, *Integral filters and integral BL-Algebras*, Italian Journal of Pure and Applied Mathematics, **30** (2013), 303-316.

[3] D. Busneag, D. Piciu, *BL-algebra of fractions relative to an* $\wedge$*-closed system*, Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica, **11**(1) (2003), 31-40.

[4] D. Busneag, D. Piciu, *On the lattice of deductive systems of a BL-algebra*, Central European Journal of Mathematics, **1**(2) (2003), 221-237.

[5] C. C. Chang, *Algebraic analysis of many valued logic*, Transactions of the AMS - American Mathematical Society, **88** (1958), 467-490.

[6] P. Hájek, *Metamathematics of fuzzy logic*, Dordrecht: Kluwer Academic Publishers, 1998.

[7] M. Haveshki, A. Borumand Saeid, E. Eslami, *Some types of filters in BL-algebras*, Soft Computing, **10** (2006), 657-664.

[8] A. Iorgulescu, *Algebras of logic as BCK-algebras*, Bucharest University of Economics Bucharest, Romania, 2008.

[9] L. Leucstean, *Representions of many-valued algebras*, Ph.D. Thesis, University of Bucharest Faculty of Mathematics and Computer Science, 2003.

[10] N. Mohtashamnia, A. Borumand Saeid, *A special type of BL-algebra*, Annals of the University of Craiova, Mathematics and Computer Science Series, **39**(1) (2012), 8-20.

[11] S. Motamed, L. Torkzadeh, *Primary decomposition of filters in BL-algebras*, Afrika Matematika, **24**(4) (2013), 725-737.

[12] F. Najmi Dolat Abadi, J. Moghaderi, *Filters by BL-homomorphisms*, Soft Computing, **23** (2019), 9831-9841.

[13] A. Paad, R. A. Borzooei, *On semi maximal filters in BL-algebras*, Journal of Algebraic Systems, **6**(2) (2019), 101-116.

[14] D. Piciu, *Algebras of fuzzy logic*, Universitaria din Craiova, 2007.

[15] E. Turunen, *BL-algebras of basic fuzzy logic*, Mathware and Soft Computing, **6** (1999), 49-61.

[16] E. Turunen, *Mathematics behind fuzzy logic*, Physica-Verlag, 1999.

[17] E. Turunen, S. Sessa, *Local BL-algebras*, International Journal of Mult-Valued logic, **6** (2001), 229-249.

[18] J. Zhan, B. Davvaz, W. A. Dudek, Y. Bae Jun, H. Sik Kim, *Fuzzy logical algebras and their applications*, The Scientific World Journal, (2015), 1-2. DOI: 10.1155/2015/682648.