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Original paper

Solving fully linear programming problem based on Z-numbers

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Abstract

Generally exploring the exact solution of linear programming problems in which all variables and parameters are Z-numbers, is either not possible or difficult. Therefore, a few numerical methods to find the numerical solutions do act an important role in these problems. In this paper, we concentrate on introducing a new numerical method to solve such problems based on the ranking function. After proving the necessary theories, for more illustrations and the correctness of the topic, some theoretical and practical examples are also provided. Finally, the results obtained from the proposed method have been compared with some existing methods.

Keywords: Z-numbers, linear programming problem (LPP), ranking function.

1 Introduction

The Linear programming (LP) is known as one of the most research techniques in operations. Although the parameters of LP model should be well determined, this is not consistent with reality in the real-world environment. In real problems, there may be uncertainty in terms of parameters. Consequently, the parameters of LP problems may be described as fuzzy numbers. Zimmerman [17] proposed the basic formula for the fuzzy linear programming problem (FLPP). Later, several FLP models were developed to solve them [3, 4, 5, 11, 12, 13]. On the other hand, what is very important is that we do not only consider ambiguity once encountering incomplete information. Another feature of information is reliability. Indeed, estimating interest values whether accurate or soft, is related to trust in the sources of information dealing with knowledge, intuition, assumptions, perception, experience, which generally cannot in accordance with the real-world phenomena. In this matter, Zadeh proposed the Z-number concept as a more appropriate [16], because they have more capability to formulate problems in fuzzy environment, as compared to fuzzy numbers. Yager employed these numbers in a fuzzy environment [14]. In this way, Ezadi and Allahviranloo proposed several approaches for ranking Z-numbers [6, 7]. To read more research in this field, you can refer to references [1, 6, 9, 10, 11, 14]. Note that as Z-numbers is relatively new, their theoretical aspects have not yet been distinguished as well. In terms of solving LPP based on Z-numbers, there have been few research [9]. In the studied linear programming problem such as [9], the confidence part of Z-numbers defined for the variables and parameters of the problem, is expressed in the form of a fuzzy number, not a distribution function. Therefore, the proposed method [9] is not always to be able to solve the linear programming problem based on Z-numbers whose confidence part is a distribution function. In this paper, we try to present a method to solve such problems. This paper is organized as follows: Section 2 brings the preliminaries. In the third section, the mathematical model of the fuzzy problem is introduced. Also, the optimal solution for FZLPP

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is presented using the ranking function. In section 4, two numerical examples and one practical example are presented. Finally, Section 5 brings the conclusion.

2 Preliminaries

The required definitions and theorems are presented as follows.

2.1 Fuzzy set theory

A fuzzy number on the real number is defined as a normal fuzzy set and convex, in which A is described on a universe X as follows:

$$A = \Big\{ \big(x, \mu_A(x)\big) \,\big|\, x \in X \Big\}. \tag{1}$$

So that $\mu_A(x) : X \to [0,1]$ is known as the membership function A. The value $\mu_A(x)$ description the grade of belongingness of $x \in X$ in A [13].

Note that $\tilde{A} = (l, m, u)$ is a triangular fuzzy number if:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & x < l \\ \dfrac{x-l}{m-l} & l \le x < m \\ \dfrac{u-x}{u-m} & m \le x < u \\ 0 & x > u \end{cases} \tag{2}$$

Definition 2.1. *Let $\tilde{A} = (l, m, u)$. Then $\tilde{A}$ is called a non-negative fuzzy number if and only if $l \ge 0$ [12].*

Definition 2.2. *(Fuzzy ranking) Consider $\tilde{A} = (l, m, u)$ and $\tilde{B} = (l', m', u')$ as two triangular fuzzy numbers, such that $\tilde{A} \le \tilde{B}$ if* [12]*: Note that $\tilde{A} = (l, m, u)$ is a triangular fuzzy number if* [16]*:*

$$\begin{aligned} &l \le l' \\ &m \le m' \\ &u \le u' \end{aligned} \tag{3}$$

2.2 Z-number

Zadeh [16] proposed Z-numbers as an unknown variable x including two parts $Z = (A, B)$. A, is a constraint on the values where X is allowed to take a real value. B, is a measure of the reliability (certainty) of the A. Generally, they are in a natural language. In effect, the Z-valuation (X, A, B) may be considered as a constraint (generalized constraint) on X by:

$$Prob\,(X \ is \ A)\, is \ B. \tag{4}$$

Actually, this means that X is A represents a fuzzy event in R if X is a random variable.

$$R(X) : X \ is \ A. \tag{5}$$

p is the probability of this event as follows [16]:

$$p = \int \mu_A(u) p_X(u) du, \tag{6}$$

where p_X refers to the probability density (hidden) of X [16].

2.3 Transforming a Z-number into a fuzzy number

The Z-number should be converted to a regular fuzzy number in order to carry out next calculations. In this regard, Kang et al. [10] provided a useful method to transform a Z-number to a classical fuzzy number [10] as follows:

Consider a Z-number as $Z = (A, B)$, where

$$A = \Big\{ \Big(x, \mu_A(x)\Big) \big| x \in X, \mu_A(x) \in [0,1] \Big\},$$

and we assume that B is as follows

$$B = \Big\{ \Big(x, \mu_B(x)\Big) \big| x \in X, \mu_B(x) \in [0,1] \Big\},$$

where $\mu_A(x)$ is a membership function and $\mu_B(x)$ is a nearest another membership function to $\mu_A(x)$. The following steps are suggested to convert the Z- number into a regular fuzzy number:

(1) Convert the second part (i.e. reliability) into a number as follows.

$$\alpha = \frac{\int x\, \mu_B(x)\, dx}{\int \mu_B(x)\, dx} \tag{7}$$

(2) The weighted Z-number is:

$$Z^{\alpha} = \Big\{ \Big(x, \mu_{A^{\alpha}}(x)\Big) \big|\, \mu_{A^{\alpha}}(x) = \alpha \mu_A\left(x\right), x \in [0,1] \Big\}. \tag{8}$$

α denotes the weight of B of Z-number.

(3) The conversion of irregular fuzzy number can be described as normal fuzzy number:

$$Z^{\alpha} = \Big\{ \Big(x, \mu_{Z^{\alpha}}(x)\Big) \big|\, \mu_{Z^{\alpha}}(x) = \alpha \mu_A(\frac{x}{\sqrt{\alpha}}), x \in [0,1] \Big\}. \tag{9}$$

3 Mathematical model of FZLPP

Consider $F(S)$ as the set of all Z-numbers. The following ZLPP

$$\begin{aligned} &\max(\min)[Z]^Z = \Big([C]^Z \times [X]^Z\Big) \\ &\text{s.t: } [A]^Z \times [X]^Z (\leq = \geq) [B]^Z \\ &\qquad\qquad\qquad\qquad [X]^Z \geq [0]^Z. \end{aligned} \tag{10}$$

Where $[A]^Z, [X]^Z, [B]^Z, [C]^Z \in F(S)$ is called as FZLPP, which $[.]$ indicates valuation with Z-number.

$[A]^Z$ means the coefficients matrix:

$$[A]^Z = \Big(a_{ij}, \mu_{A_{a_{ij}}}, P_{A_{a_{ij}}}\Big). \tag{11}$$

Where $\mu_{A_{a_{ij}}}$ is Gaussian membership function and it is given as

$$\mu_{A_{a_{ij}}} = \mu\left(a_{ij}, \mathcal{C}_{a_{ij}}, \sigma_{a_{ij}}\right) = e^{-\frac{1}{2}\left(\frac{a_{ij} - \mathcal{C}_{a_{ij}}}{\sigma_{a_{ij}}}\right)^2}, \quad -\infty < a_{ij} < \infty. \tag{12}$$

Where $\mathcal{C}_{ij}$ specifies the location of the center of the peak. σ_{ij} (Standard deviation) represents the amount of elongation or expansion of the bell and $i = 1, \ldots, m.,\ j = 1, \ldots, n.$ Also,

$$P_{A_{a_{ij}}} = \mu_{A_{a_{ij}}} . p_{a_{ij}} = \int_R \mu_{A_{a_{ij}}}\left(u\right) . p_{a_{ij}}\left(u\right) du. \tag{13}$$

Where $p_{a_{ij}}$ is probability density function. Here, we get $p_{a_{ij}}$ as the normal density function $N(\mathcal{C}_{ij}, \sigma_a)$. Therefore,

$$p_{a_{ij}} = \frac{1}{\sigma_a \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{a_{ij} - \mathcal{C}_{a_{ij}}}{\sigma_a}\right)^2} \simeq N(\mathcal{C}_{a_{ij}}, \sigma_a). \tag{14}$$

Where $\sigma_a = \max_{i,j} \sigma_{a_{ij}}.$, Then, we obtain from Eq. (12) that

$$P_{A_{a_{ij}}} = \int_{-\infty}^{\infty} \frac{1}{\sigma_a\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{a_{ij}-\mathcal{C}_{a_{ij}}}{\sigma_a}\right)^2} e^{-\frac{1}{2}\left(\frac{a_{ij}-\mathcal{C}_{a_{ij}}}{\sigma_{a_{ij}}}\right)^2} da_{ij} = \frac{2}{\sigma_{a_{ij}}\sqrt{\pi\left(\frac{1}{\sigma_{ij}^2}+\frac{1}{\sigma_a^2}\right)}}. \tag{15}$$

$[X]^Z = [x_{ij}]^Z$ are decision variables that

$$[X]^Z = \left(x_{ij}, \mu_{X_{x_{ij}}}, P_{X_{x_{ij}}}\right). \tag{16}$$

Where $\mu_{X_{x_{ij}}} i$ s membership function as

$$\mu_{X_{x_{ij}}} = \mu\left(x_{ij}, \mathcal{C}_{x_{ij}}, \sigma_{x_{ij}}\right) = e^{-\frac{1}{2}\left(\frac{x_{ij}-\mathcal{C}_{x_{ij}}}{\sigma_{x_{ij}}}\right)^2}, \qquad -\infty < x_{ij} < \infty. \tag{17}$$

Where $i = 1, \ldots, m.,\ j = 1, \ldots, n.$ Also,

$$P_{X_{x_{ij}}} = \mu_{X_{x_{ij}}}.p_{x_{ij}} = \int_R \mu_{X_{x_{ij}}}(u).p_{x_{ij}}(u)\, du. \tag{18}$$

Similarly, we have

$$p_{x_{ij}} = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_{ij}-\mathcal{C}_{x_{ij}}}{\sigma_x}\right)^2} \simeq N(\mathcal{C}_{x_{ij}}, \sigma_x). \tag{19}$$

Where $\sigma_x = \max_{i,j} \sigma_{x_{ij}}.$, Then, we obtain from Eq. (17) that

$$P_{X_{x_{ij}}} = \int_{-\infty}^{\infty} \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_{ij}-\mathcal{C}_{x_{ij}}}{\sigma_x}\right)^2} e^{-\frac{1}{2}\left(\frac{x_{ij}-\mathcal{C}_{x_{ij}}}{\sigma_{x_{ij}}}\right)^2} dx_{ij} = \frac{2}{\sigma_{x_{ij}}\sqrt{\pi\left(\frac{1}{\sigma_{ij}^2}+\frac{1}{\sigma_x^2}\right)}}. \tag{20}$$

$[B]^Z = [b_{ij}]^Z$ denotes vector of the right-side numbers

$$[B]^Z = \left(b_{ij}, \mu_{B_{b_{ij}}}, P_{B_{b_{ij}}}\right). \tag{21}$$

Where $\mu_{B_{b_{ij}}}$ is membership function and given as

$$\mu_{B_{b_{ij}}} = \mu\left(b_{ij}, \mathcal{C}_{b_{ij}}, \sigma_{b_{ij}}\right) = e^{-\frac{1}{2}\left(\frac{b_{ij}-\mathcal{C}_{b_{ij}}}{\sigma_{b_{ij}}}\right)^2}, \qquad -\infty < b_{ij} < \infty. \tag{22}$$

Where $i = 1, \ldots, m., j = 1, \ldots, n.$ Also,

$$P_{B_{b_{ij}}} = \mu_{B_{b_{ij}}}.p_{b_{ij}} = \int_R \mu_{B_{b_{ij}}}(u).p_{b_{ij}}(u)\, du. \tag{23}$$

Therefore,

$$p_{b_{ij}} = \frac{1}{\sigma_b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{b_{ij}-\mathcal{C}_{b_{ij}}}{\sigma_x}\right)^2} \simeq N(\mathcal{C}_{b_{ij}}, \sigma_b). \tag{24}$$

Where $\sigma_b = \max_{i,j} \sigma_{b_{ij}}$, Then, we obtain from Eq. (22) that,

$$P_{B_{b_{ij}}} = \int_{-\infty}^{\infty} \frac{1}{\sigma_b\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{b_{ij}-\mathcal{C}_{b_{ij}}}{\sigma_b}\right)^2} e^{-\frac{1}{2}\left(\frac{b_{ij}-\mathcal{C}_{b_{ij}}}{\sigma_{b_{ij}}}\right)^2} db_{ij} = \frac{2}{\sigma_{b_{ij}}\sqrt{\pi\left(\frac{1}{\sigma_{ij}^2}+\frac{1}{\sigma_b^2}\right)}}. \tag{25}$$

And $[C]^z = [c_i]^z$ is the variables coefficients vector in the objective function that

$$[C]^Z = \Big(c_{ij}, \mu_{C_{c_{ij}}}, P_{C_{c_{ij}}}\Big). \tag{26}$$

Where $\mu_{C_{c_{ij}}}$ is membership function and given as

$$\mu_{C_{c_{ij}}} = \mu\left(c_{ij}, \mathcal{C}_{c_{ij}}, \sigma_{c_{ij}}\right) = e^{-\frac{1}{2}\left(\frac{c_{ij}-\mathcal{C}_{ij}}{\sigma_{c_{ij}}}\right)^2}, \qquad -\infty < c_{ij} < \infty. \tag{27}$$

Where $i = 1, \ldots, m., j = 1, \ldots, n$. Also,

$$P_{C_{c_{ij}}} = \mu_{C_{c_{ij}}}.p_{c_{ij}} = \int_R \mu_{C_{c_{ij}}}(u).p_{c_{ij}}(u)\,du. \tag{28}$$

Therefore,

$$p_{c_{ij}} = \frac{1}{\sigma_c\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{c_{ij}-\mathcal{C}_{c_{ij}}}{\sigma_c}\right)^2} \simeq N(\mathcal{C}_{c_{ij}}, \sigma). \tag{29}$$

Where $\sigma_c = \max_{i,j} \sigma_{c_{ij}}$, Then, we obtain from Eq. (27) that,

$$P_{C_{c_{ij}}} = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{c_{ij}-\mathcal{C}_{c_{ij}}}{\sigma_c}\right)^2} e^{-\frac{1}{2}\left(\frac{c_{ij}-\mathcal{C}_{c_{ij}}}{\sigma_{c_{ij}}}\right)^2} dc_{ij} = \frac{2}{\sigma_{c_{ij}}\sqrt{\pi\left(\frac{1}{\sigma^2_{c_{ij}}} + \frac{1}{\sigma^2_c}\right)}}. \tag{30}$$

Definition 3.1. [9] *Any $[x]^z = \Big([x_1]^z, [x_2]^z, [x_3]^z, \ldots, [x_n]^z\Big) \in F^n(S)$ is said to be a Z feasible solution to (38),where each $[x_i]^z \in F(S)$, satisfies the constraints and non-negativity restrictions of (10).*

3.1 A novel method for ranking Z-numbers

Suppose that $Z = (A, B)$ is a Z-number such that A represents the constraint part for Z, which is a trapezoidal fuzzy number. B denotes the degree of reliability of A as a trapezoidal fuzzy number. The proposed algorithm is as follows: $\omega_{A_i} \in [0, 1]$ and $c = \max_{i,j}(a_{ij}, 1)$ represents the maximum value of the universe of discourse.

First step: for $Z_i = (A_i, B_i)$, if $A_i = \Big(a_{i1}, a_{i2},\ a_{i3}, a_{i4}; \omega_{A_i}\Big)$ converts to a standard fuzzy number $A_i^* = (a^*_{i1}, a^*_{i2}, a^*_{i3}, a^*_{i4}; \omega_{A_i})$ so that $a^*_{ij} = \frac{a_{ij}}{C}$ for $j = 1, 2, 3, 4.$ and $i = 1, \ldots, n$.
$\omega_{A_i} \in [0, 1]$ and $c = \max_{i,j}(a_{ij}, 1)$ represents the maximum value of the universe of discourse.

Second step: B becomes a definite number using Eq. (7) as follows:

$$\alpha = \frac{\int_X x\mu_B\,dx}{\int_X \mu_B\,dx}.$$

Third step: The weight of B is added to the constraint of A respectively using the Eq. (8). Therefore, weighted Z is as follows.

$$Z^\alpha = \Big\{\Big(x, \mu_{A^\alpha}(x)\Big) \big| \,\mu_{A^\alpha}(x) = \alpha\mu_A(x), x \in [0, 1]\Big\}.$$

Fourth step: Calculate the generalized fuzzy number $\hat{A}^*_i$ as follows

$$\hat{A}^*_i = \Big(a^*_{i1}, a^*_{i2}, a^*_{i3}, a^*_{i4}; \omega_{\hat{A}^*_i}\Big) \qquad \omega_{\hat{A}^*_i} = \alpha\omega_{A_i},$$

where $\hat{.}$ is a generalized sense.

Fifth step: Using relation (9), we have

$$\hat{A}^*_i = \Big(\sqrt{\omega_{\hat{A}^*_i}}a^*_{i1}, \sqrt{\omega_{\hat{A}^*_i}}a^*_{i2}, \sqrt{\omega_{\hat{A}^*_i}}a^*_{i3}, \sqrt{\omega_{\hat{A}^*_i}}a^*_{i4}\Big) = (\hat{a}^*_{i1}, \hat{a}^*_{i2}, \hat{a}^*_{i3}, \hat{a}^*_{i4}).$$

Now you can use some ranking methods like [3]. Here we consider the following simple method.

sixth step: Calculate the median $Rank(Z)$ as follows

$$Rank(Z) = \frac{1}{4}(\hat{a}^*_{i1} + \hat{a}^*_{i2} + \hat{a}^*_{i3} + \hat{a}^*_{i4}).$$

Property 3.2. *Let $Z_i = (A_i, B_i)$ if $A_i = (0,0,0,0;0)$ and $B_i = (0,0,0,0;0)$ then $Rank\,(Z) = 0$.*

Proof. Clear that $\alpha = 0$ and $Rank\,(Z) = 0$. □

Property 3.3. *Let $Z_i = (A_i, B_i)$ if $A_i = (1,1,1,1;1)$ and $B_i = (1,1,1,1;1)$ then $Rank\,(Z) = 1$.*

Proof. Because $\alpha = 1$ and $\omega_{\hat{A}_i^*} = 1$, then $Rank\,(Z) = \dfrac{1\,(1+1)}{2} = 1$. □

Property 3.4. *Let $Z_i = (A_i,\ B_i)$, if $A_i = (-1,-1,-1,-1;-1)$ and $B_i = (-1,-1,-1,-1;-1)$, then $Rank\,(Z) = -1$.*

Proof. Because $\alpha = 1$ and $\omega_{\hat{A}_i^*} = 1$, then $Rank\,(Z) = \dfrac{1\,(-1-1)}{2} = -1$. □

We define $Rank\,(Z_i) \in R$ as the function that Rank a Z-number Z_i, such that $\forall i, j \geq 1$
1. $Rank(Z_i) > Rank(Z_j)$ if and only if $Z_i \succ Z_j$.
2. $Rank(Z_i) < Rank(Z_j)$ if and only if $Z_i \prec Z_j$.
3. $Rank(Z_i) = Rank(Z_j)$ if and only if $Z_i \sim Z_j$.

3.2 Optimal solution of the FZLP problem

Let $\mathfrak{R}$ be the concept of linear ranking in section 3.1. The Z-optimal solution of FZLPP (10) will be a $[X]^Z$ if it meets these features:
(i) $[X]^Z = (A_X, B_X), A_X$ is non-negative,
(ii) $\mathfrak{R}\Big([A]^Z \otimes [X]^Z\Big)(\leq=\geq)\mathfrak{R}\Big([B]^Z\Big)$ and its equivalent $\mathfrak{R}\Big([A]^{Z^a} \otimes [X]^{Z^a}\Big)(\leq=\geq)\mathfrak{R}\Big([B]^{Z^a}\Big)$.
(iii) If there is any non-negative $[X^*]^Z$ such

$$\mathfrak{R}\Big([A]^Z \otimes [X^*]^Z\Big)(\leq=\geq)\mathfrak{R}\Big([B]^Z\Big)\mathfrak{R}\Big([A]^Z \otimes [X^*]^Z\Big)(\leq=\geq)\mathfrak{R}\Big([B]^Z\Big),$$

and $\mathfrak{R}$ is the ranking function, then $\mathfrak{R}\Big([C]^Z \otimes [X]^Z\Big) > \mathfrak{R}\Big([C]^Z \otimes [X^*]^Z\Big)$ (about the maxim problem) and

$$\mathfrak{R}\Big([C]^Z \otimes [X]^Z\Big) < \mathfrak{R}\Big([C]^Z \otimes [X^*]^Z\Big),$$

(about the minimization problem) and its equivalent $\mathfrak{R}\Big([C]^{Z^a} \otimes [X]^{Z^a}\Big) < \mathfrak{R}\Big([C]^{Z^a} \otimes [X^*]^{Z^a}\Big)$, (see [9]).

Definition 3.5. *Let Z_i and Z_j are two Z-numbers, and $\mathfrak{R}$ be any concept of linear ranking that Rank a Z-number, Z_i, such that $\forall i, j \geq 1$, which is denoted by $Z_i \prec Z_j$: $\mathfrak{R}\Big(Z_i\Big) < \mathfrak{R}\Big(Z_j\Big)$ if and only if $Z_i \prec Z_j$.*

Remark 3.6. *Considers $[X]^Z$ as Z- optimal solution of FZLPP (10). If there exist a $[Y]^Z$ in which:*
(i) $[Y]^Z = (A_Y, B_Y), A_Y$ is non-negative,
(ii) $\mathfrak{R}\Big([A]^Z \otimes [Y]^Z\Big)(\leq=\geq)\mathfrak{R}\Big([B]^Z\Big)$ and its equivalent $\mathfrak{R}\Big([A]^{Z^a} \otimes [Y]^{Z^a}\Big)(\leq=\geq)\mathfrak{R}\Big([B]^{Z^a}\Big)$.
(iii) $\mathfrak{R}\Big([C]^Z \otimes [X]^Z\Big) = \mathfrak{R}\Big([C]^{Z^a} \otimes [Y]^Z\Big)$, then $[Y]^Z$ said to be an alternative Z-optimal solution of (10), and its equivalent $\mathfrak{R}\Big([C]^{Z^a} \otimes [X]^{Z^a}\Big) = \mathfrak{R}\Big([C]^{Z^a} \otimes [Y]^{Z^a}\Big)$, then $[Y]^{Z^a}$ said to be an alternative Z-optimal solution of (10).

3.3 Solving for FZLPP using the ranking function

Consider FZLPP in (10), According to the contents mentioned in section 3, model can (10) be rewritten to (31):

$$\begin{aligned}
&\max[Z]^z = \sum_{j=1}^{n} \Big(c_{ij}, \mu_{C_{\acute{c}_{ij}}},\ P_{C_{c_{ij}}}\Big) \otimes \Big(x_{ij}, \mu_{X_{x_{ij}}},\ P_{X_{x_{ij}}}\Big) \\
&s.t \quad \sum_{j=1}^{n} \Big(a_{ij}, \mu_{A_{a_{ij}}},\ P_{A_{a_{ij}}}\Big) \otimes \Big(x_{ij}, \mu_{X_{x_{ij}}},\ P_{X_{x_{ij}}}\Big) \leq=\geq \Big(b_{ij}, \mu_{B_{b_{ij}}},\ P_{B_{b_{ij}}}\Big) \\
&x_{ij} \geq 0.
\end{aligned} \tag{31}$$

In this way, a Gaussian fuzzy number approximates to an interval number [8] represented as $< x, \mu(x; c, \sigma) | x \in X >$ where the membership function is:

$$\mu(x; \mathcal{C}, \sigma) = e^{-\frac{1}{2}\left(\frac{x-\mathcal{C}}{\sigma}\right)^2}, \qquad -\infty < x < \infty.$$

An a-cut is defined as set A_α where $A_\alpha = \{x : \mu(x; \mathcal{C}, \sigma) \geq \alpha\}$. So, we have,

$$\mu(x; \mathcal{C}, \sigma) \geq \alpha \Rightarrow e^{-\frac{1}{2}\left(\frac{x-\mathcal{C}}{\sigma}\right)^2} \geq \alpha$$
$$\Rightarrow \mathcal{C} - \sigma\sqrt{2\ln\left(\frac{1}{\alpha}\right)} \leq x \leq \mathcal{C} + \sigma\sqrt{2\ln\left(\frac{1}{\alpha}\right)},$$

and let us consider that $A_{L\alpha} = \mathcal{C} - \sigma\sqrt{2\ln\left(\frac{1}{\alpha}\right)}$ and $A_{R\alpha} = \mathcal{C} + \sigma\sqrt{2\ln\left(\frac{1}{\alpha}\right)}$. Suppose $[a, b]$ be the corresponding interval approximation. Afterwards, $a = \int_0^1 A_{L\alpha} d\alpha = \mathcal{C} - 4\sqrt{2}\sigma$ and $b = \int_0^1 A_{R\alpha} d\alpha = \mathcal{C} + 4\sqrt{2}\sigma$. If $Z = (x, A_x, P_{A_x})$ be Z-number including Gaussian membership function $\mu(x; \mathcal{C}, \sigma)$, interval approximation is created by $\left(x, \left[\mathcal{C} - 4\sqrt{2}\sigma\ , \mathcal{C} + 4\sqrt{2}\sigma\right], P_{A_x}\right)$.

Therefore, we can rewrite relation (31) as follows:

$$\begin{aligned}
\max[Z]^z = \sum_{j=1}^{n} &\left(c_{ij}, \left[\mathcal{C}_{c_{ij}} - 4\sqrt{2}\sigma_{c_{ij}}, \mathcal{C}_{c_{ij}} + 4\sqrt{2}\sigma_{c_{ij}}\right],\ P_{C_{\acute{c}_{ij}}}\right) \\
&\otimes \left(x_{ij}, \left[\mathcal{C}_{x_{ij}} - 4\sqrt{2}\sigma_{x_{ij}}, \mathcal{C}_{x_{ij}} + 4\sqrt{2}\sigma_{x_{ij}}\right],\ P_{X_{x_{ij}}}\right) \\
\sum_{j=1}^{n} &\left(a_{ij}, \left[\mathcal{C}_{a_{ij}} - 4\sqrt{2}\sigma_{a_{ij}}, \mathcal{C}_{a_{ij}} + 4\sqrt{2}\sigma_{a_{ij}}\right], P_{A_{a_{ij}}}\right) \\
&\otimes \left(x_{ij}, \left[\mathcal{C}_{x_{ij}} - 4\sqrt{2}\sigma_{x_{ij}}, \mathcal{C}_{x_{ij}} + 4\sqrt{2}\sigma_{x_{ij}}\right],\ P_{X_{x_{ij}}}\right) \\
&\leq=\geq \left(b_{ij}, \left[\mathcal{C}_{b_{ij}} - 4\sqrt{2}\sigma_{b_{ij}}, \mathcal{C}_{b_{ij}} + 4\sqrt{2}\sigma_{b_{ij}}\right], P_{B_{b_{ij}}}\right) \\
&\qquad x_{ij} \geq 0.
\end{aligned}$$

Assuming that α-cut for the fuzzy interval $\left[\mathcal{C}_{c_{ij}} - 4\sqrt{2}\sigma_{c_{ij}}, \mathcal{C}_{c_{ij}} + 4\sqrt{2}\sigma_{c_{ij}}\right]$ is $4\sqrt{2}\sigma_{c_{ij}}$. Then clear that the fuzzy interval $\left[\mathcal{C}_{c_{ij}} - 4\sqrt{2}\sigma_{c_{ij}}, \mathcal{C}_{c_{ij}} + 4\sqrt{2}\sigma_{c_{ij}}\right]$ can be represented as $\left(\mathcal{C}_{c_{ij}} - 4\sqrt{2}\sigma_{c_{ij}}, \mathcal{C}_{c_{ij}}, \mathcal{C}_{c_{ij}} + 4\sqrt{2}\sigma_{c_{ij}}\right)$.

Similarly, the fuzzy interval $\left[\mathcal{C}_{x_{ij}} - 4\sqrt{2}\sigma_{x_{ij}}, \mathcal{C}_{x_{ij}} + 4\sqrt{2}\sigma_{x_{ij}}\right]$ can be represented as $\left(\mathcal{C}_{x_{ij}} - 4\sqrt{2}\sigma_{x_{ij}}, \mathcal{C}_{x_{ij}}, \mathcal{C}_{x_{ij}} + 4\sqrt{2}\sigma_{x_{ij}}\right)$, and the fuzzy interval $\left[\mathcal{C}_{a_{ij}} - 4\sqrt{2}\sigma_{a_{ij}}, \mathcal{C}_{a_{ij}} + 4\sqrt{2}\sigma_{a_{ij}}\right]$ can be represented as $\left(\mathcal{C}_{a_{ij}} - 4\sqrt{2}\sigma_{a_{ij}}, \mathcal{C}_{a_{ij}}, \mathcal{C}_{a_{ij}} + 4\sqrt{2}\sigma_{a_{ij}}\right)$.

Also, the fuzzy interval $\left[\mathcal{C}_{b_{ij}} - 4\sqrt{2}\sigma_{b_{ij}}, \mathcal{C}_{b_{ij}} + 4\sqrt{2}\sigma_{b_{ij}}\right]$ can be represented as $\left(\mathcal{C}_{b_{ij}} - 4\sqrt{2}\sigma_{b_{ij}}, \mathcal{C}_{b_{ij}}, \mathcal{C}_{b_{ij}} + 4\sqrt{2}\sigma_{b_{ij}}\right)$. So, we have

$$\begin{aligned}
\max[Z]^z = \sum_{j=1}^{n} &\left(\left(\mathcal{C}_{c_{ij}} - 4\sqrt{2}\sigma_{c_{ij}}, \mathcal{C}_{c_{ij}}, \mathcal{C}_{c_{ij}} + 4\sqrt{2}\sigma_{c_{ij}}\right), P_{C_{\acute{c}_{ij}}}\right) \\
&\otimes \left(\left(\mathcal{C}_{x_{ij}} - 4\sqrt{2}\sigma_{x_{ij}}, \mathcal{C}_{x_{ij}}, \mathcal{C}_{x_{ij}} + 4\sqrt{2}\sigma_{x_{ij}}\right),\ P_{X_{x_{ij}}}\right) \\
s.t \sum_{j=1}^{n} &\left(\left(\mathcal{C}_{a_{ij}} - 4\sqrt{2}\sigma_{a_{ij}}, \mathcal{C}_{a_{ij}}, \mathcal{C}_{a_{ij}} + 4\sqrt{2}\sigma_{a_{ij}}\right), P_{A_{a_{ij}}}\right) \\
&\otimes \left(x_{ij}, \left(\mathcal{C}_{x_{ij}} - 4\sqrt{2}\sigma_{x_{ij}}, \mathcal{C}_{x_{ij}}, \mathcal{C}_{x_{ij}} + 4\sqrt{2}\sigma_{x_{ij}}\right), P_{X_{x_{ij}}}\right) \\
&\leq=\geq \left(\left(\mathcal{C}_{b_{ij}} - 4\sqrt{2}\sigma_{b_{ij}}, \mathcal{C}_{b_{ij}}, \mathcal{C}_{b_{ij}} + 4\sqrt{2}\sigma_{b_{ij}}\right), P_{B_{b_{ij}}}\right) \\
&\qquad x_{ij} \geq 0. \qquad (32)
\end{aligned}$$

Let $[X]^Z = \left(A_X, P_{X_{x_{ij}}}\right)$ is Z-number, where $A_X = (a_{1X}\ , a_{2X}, a_{3X})$ is a triangular fuzzy number. Also,
let $[A]^Z = \left(A_A, P_{A_{a_{ij}}}\right)$ is Z-number, where $A_A = (a_{1A}, a_{2A}, a_{3A})$ is a triangular fuzzy number,
let $[B]^Z = \left(A_B, P_{B_{b_{ij}}}\right)$ is Z-number, where $A_B = (a_{1B}, a_{2B}, a_{3B})$ is a triangular fuzzy number, and
let $[C]^Z = \left(A_C, P_{C_{c_{ij}}}\right)$ is Z-number, where $A_C = (a_{1C}, a_{2C}, a_{3C})$ is a triangular fuzzy number.
Optimal solution of FZLPP (31) will be a $[X]^Z = \left(A_X, P_{X_{x_{ij}}}\right)$ if it meets these features:
(i) $[X]^Z = \left(A_X, P_{X_{x_{ij}}}\right)$ is a non-negative,
(ii) If $\mathfrak{R}$ is the ranking function, then $\mathfrak{R}\left(\left(A_A, P_{A_{a_{ij}}}\right) \otimes \left(A_X, P_{X_{x_{ij}}}\right)\right) (\leq=\geq) \mathfrak{R}\left(A_B, P_{B_{b_{ij}}}\right)$,
(iii) If there is any non-negative $[X^*]^Z = \left(A_X, P^*_{X_{x_{ij}}}\right)$ such $\mathfrak{R}\left(\left(A_X, P_{X_{x_{ij}}}\right) \otimes \left(A^*_X, P^*_{X_{x_{ij}}}\right)\right)$
$(\leq=\geq) \mathfrak{R}\left(A_B,\ P_{B_{b_{ij}}}\right)$, then $\mathfrak{R}\left(\left(A_C,\ P_{C_{c_{ij}}}\right) \otimes \left(A_X, P_{X_{x_{ij}}}\right)\right) > \mathfrak{R}\left(\left(A_C, P_{C_{c_{ij}}}\right) \otimes \left(A^*_X, P^*_{X_{x_{ij}}}\right)\right)$
(about the maxim problem). It is equivalent to $\mathfrak{R}\Big([C]^Z \otimes [X]^Z \otimes [X]^Z\Big) > \mathfrak{R}\Big([C]^Z \otimes [X^*]^Z\Big)$, and

$$\mathfrak{R}\left(\left(A_C, P_{C_{c_{ij}}}\right) \otimes \left(A_X, P_{X_{x_{ij}}}\right)\right) < \mathfrak{R}\left(\left(A_C, P_{C_{c_{ij}}}\right) \otimes \left(A^*_X,\ P^*_{X_{x_{ij}}}\right)\right),$$

(about the minimization problem). All it is equivalent to $\mathfrak{R}\Big([C]^Z \otimes [X]^Z\Big) < \mathfrak{R}\Big([C]^Z \otimes [X^*]^Z\Big)$

Theorem 3.7. *If $\left[x^*_{ij}\right]^Z = \left(x^*_{ij}, \mu_{x^*_{ij}}, P_{x^*_{ij}}\right) = \left(\lambda x^*_{ij}, \lambda\ y^*_{ij}, \lambda w^*_{ij}\right)$ be an optimal solution of problems (37) (and hence is a lexicographic optimal solution of problem (33), then it is also an exact optimal solution of problem (10).*

Proof. By contradiction, let $\left[x^*_{ij}\right]^Z = \left(A^*_X, P^*_{X_{x_{ij}}}\right) = \left(\lambda x^*_{ij}, \lambda y^*_{ij}, \lambda w^*_{ij}\right)$ be an optimal solution of (37), but it is not the exact optimal solution of problem (33). Therefore, there exists a feasible solution of problem (10), namel $\left[x^0_{ij}\right]^Z = \left(A^0_X, P^0_{X_{x_{ij}}}\right) = \left(\lambda x^0_{ij}, \lambda y^0_{ij}, \lambda w^0_{ij}\right) \neq \left[x^*_{ij}\right]^Z$ such that $\left(c_{1ij}\lambda x^*_{ij}, c_{2ij}\lambda y^*_{ij}, c_{3ij}\lambda w^*_{ij}\right) < \left(c_{1ij}\lambda x^0_{ij}, c_{2ij}\lambda y^0_{ij}, c_{3ij}\lambda w^0_{ij}\right)$ (in case of minimization problem $\left(c_{1ij}\lambda x^*_{ij}, c_{2ij}\lambda y^*_{ij}, c_{3ij}\lambda w^*_{ij}\right) > \left(c_{1ij}\lambda x^0_{ij}, c_{2ij}\lambda y^0_{ij}, c_{3ij}\lambda w^0_{ij}\right)$) So, with respect to Definition 3.5, we have condition as follows:
If there is any non-negative $[X^*]^Z = \left(A^*_X, P^*_{X_{x_{ij}}}\right) = \left(\lambda x^*_{ij}, \lambda y^*_{ij}, \lambda w^*_{ij}\right)$ such

$$\mathfrak{R}\left(\left(A_c, P_{C_{c_{ij}}}\right) \otimes \left(A^*_X, P^*_{X_{x_{ij}}}\right)\right) (\leq=\geq) \mathfrak{R}\left(A_B, P_{B_{b_{ij}}}\right),$$

It is equivalent to $\mathfrak{R}\left((c_{1ij}\ , c_{2ij}, c_{3ij}) \otimes \left(\lambda x^*_{ij}, \lambda y^*_{ij}, \lambda w^*_{ij}\right)\right) (\leq=\geq) \mathfrak{R}\left(b_{1ij}, b_{2ij}, b_{3ij}\right)$.
Let $\mathfrak{R}\left(\left(A_C, P_{C_{c_{ij}}}\right) \otimes \left(A^0_X, P^0_{X_{x_{ij}}}\right)\right) > \mathfrak{R}\left(\left(A_C, P_{C_{c_{ij}}}\right) \otimes \left(A^*_X, P^*_{X_{x_{ij}}}\right)\right)$, It is equivalent to
$\mathfrak{R}\left((c_{1ij}, c_{2ij}, c_{3ij}) \otimes \left(\lambda x^0_{ij}, \lambda y^0_{ij}, \lambda w^0_{ij}\right)\right) > \mathfrak{R}\left((c_{1ij}, c_{2ij}, c_{3ij}) \otimes \left(\lambda x^*_{ij}, \lambda y^*_{ij}, \lambda w^*_{ij}\right)\right)$ (about the maxim problem).

Therefore, $\left(\lambda x^0_{ij}, \lambda y^0_{ij}, \lambda w^0_{ij}\right)$ is a feasible solution of problem (35) in which the objective value in $\left(\lambda x^0_{ij}, \lambda y^0_{ij}, \lambda w^0_{ij}\right)$ is greater than the objective value in $\left(\lambda x^*_{ij}, \lambda y^*_{ij}, \lambda w^*_{ij}\right)$. But this is a contradiction. Also, let

$$\mathfrak{R}\left(\left(A_C, P_{C_{c_{ij}}}\right) \otimes \left(A_X, P_{X_{x_{ij}}}\right)\right) < \mathfrak{R}\left(\left(A_C, P_{C_{c_{ij}}}\right) \otimes \left(A^*_X, P^*_{X_{x_{ij}}}\right)\right).$$

It is equivalent to $\mathfrak{R}\left((c_{1ij}, c_{2ij}, c_{3ij}) \otimes \left(\lambda x^0_{ij}, \lambda y^0_{ij}, \lambda w^0_{ij}\right)\right) < \mathfrak{R}\left((c_{1ij}, c_{2ij}, c_{3ij}) \otimes \left(\lambda x^*_{ij}, \lambda y^*_{ij}, \lambda w^*_{ij}\right)\right)$ (about the minimization problem). Therefore, $\left(\lambda x^0_{ij}, \lambda y^0_{ij}, \lambda w^0_{ij}\right)$ is a feasible solution of problem (35) in which the objective value in $\left(\lambda x^0_{ij}, \lambda y^0_{ij}, \lambda w^0_{ij}\right)$ is less than the objective value in $\left(\lambda x^*_{ij}, \lambda y^*_{ij}, \lambda w^*_{ij}\right)$. But this is a contradiction. Therefore, $\left[x^*_{ij}\right]^Z = \left(x^*_{ij}, \mu_{x^*_{ij}}, P_{x^*_{ij}}\right) = \left(\lambda x^*_{ij}, \lambda y^*_{ij}, \lambda w^*_{ij}\right)$ is an exact optimal solution of problem (10). □

Remark 3.8. *Considers $[X]^Z =$ as Z- optimal solution of FZLPP (10). If there exist a $[Y]^Z = \left(A_Y,\ P_{Y_{y_{ij}}}\right)$ in which:*
(i) $[Y]^Z = \left(A_Y,\ P_{Y_{y_{ij}}}\right)$ is a non-negative,
(ii) $\mathfrak{R}\left(\left(A_X, P_{X_{x_{ij}}}\right) \otimes \left(A_Y, P_{Y_{y_{ij}}}\right)\right) (\leq=\geq) \mathfrak{R}\left(A_B, P_{B_{b_{ij}}}\right)$. It is equivalent to $\mathfrak{R}\left([A]^Z \otimes [Y]^Z\right) \leq=\geq \mathfrak{R}\left([B]^Z\right)$,

(iii) $\Re\left(\left(A_C, P_{C_{c_{ij}}}\right) \otimes \left(A_X, P_{X_{x_{ij}}}\right)\right) = \Re\left(\left(A_C, P_{C_{c_{ij}}}\right) \otimes \left(A_Y, P_{Y_{y_{ij}}}\right)\right)$.
It is equivalent to $\Re\left([C]^Z \otimes [X]^Z\right) = \Re\left([C]^Z \otimes [Y]^Z\right)$ *then* $[Y]^Z = \left(A_Y, P_{Y_{y_{ij}}}\right)$ *said to be an alternative Z-optimal solution of (10).*

According to the above, the algorithm of the method is as follows:
Step 1. Let $\mathcal{C}_{x_{ij}} - 4\sqrt{2}\sigma_{x_{ij}} = x_{ij}, \mathcal{C}_{x_{ij}} = y_{ij}$ and $\mathcal{C}_{x_{ij}} + 4\sqrt{2}\sigma_{x_{ij}} = w_{ij}$. that's mean $\left(\mathcal{C}_{x_{ij}} - 4\sqrt{2}\sigma_{x_{ij}}, \mathcal{C}_{x_{ij}}, \mathcal{C}_{x_{ij}} + 4\sqrt{2}\sigma_{x_{ij}}\right) = (x_{ij}, y_{ij}, w_{ij})$. So, we have

$$
\begin{aligned}
\max[Z]^z =& \sum_{j=1}^{n}\left(\left(\mathcal{C}_{c_{ij}} - 4\sqrt{2}\sigma_{c_{ij}}, \mathcal{C}_{c_{ij}}, \mathcal{C}_{c_{ij}} + 4\sqrt{2}\sigma_{c_{ij}}\right), P_{C_{\acute{c}_{ij}}}\right) \otimes \left(\left(x_{ij}, y_{ij}, w_{ij}\right), P_{X_{x_{ij}}}\right) \\
& s.t \quad \sum_{j=1}^{n}\left(\left(\mathcal{C}_{a_{ij}} - 4\sqrt{2}\sigma_{a_{ij}}, \mathcal{C}_{a_{ij}}, \mathcal{C}_{a_{ij}} + 4\sqrt{2}\sigma_{a_{ij}}\right), P_{A_{a_{ij}}}\right) \otimes \left(\left(x_{ij}, y_{ij}, w_{ij}\right), P_{X_{x_{ij}}}\right) \\
& \leq=\geq \left(\left(\mathcal{C}_{b_{ij}} - 4\sqrt{2}\sigma_{b_{ij}}, \mathcal{C}_{b_{ij}}, \mathcal{C}_{b_{ij}} + 4\sqrt{2}\sigma_{b_{ij}}\right), P_{B_{b_{ij}}}\right) \\
& x_{ij}, y_{ij}, w_{ij} \geq 0.
\end{aligned} \tag{33}
$$

Step 2. Using, Remark 3.6, FZLPP in Step 1, is converted into the following:

$$
\begin{aligned}
\max[Z]^{Z^{\alpha}} =& \Re\left(\sum_{j=1}^{n}\left(\mathcal{C}_{c_{ij}} - 4\sqrt{2}\sigma_{c_{ij}}\, \mathcal{C}_{c_{ij}}, \mathcal{C}_{c_{ij}} + 4\sqrt{2}\sigma_{c_{ij}}; P_{C_{c_{ij}}}\right) \otimes \left(x_{ij}, y_{ij}, w_{ij}; P_{X_{x_{ij}}}\right)\right) \\
& \sum_{j=1}^{n}\left(\mathcal{C}_{a_{ij}} - 4\sqrt{2}\sigma_{a_{ij}}\, \mathcal{C}_{a_{ij}}, \mathcal{C}_{a_{ij}} + 4\sqrt{2}\sigma_{a_{ij}}; P_{A_{a_{ij}}}\right) \otimes \left(x_{ij}, y_{ij}, w_{ij}; P_{X_{x_{ij}}}\right) \\
& \leq=\geq \left(\mathcal{C}_{b_{ij}} - 4\sqrt{2}\sigma_{b_{ij}}, \mathcal{C}_{b_{ij}}, \mathcal{C}_{b_{ij}} + 4\sqrt{2}\sigma_{b_{ij}}; P_{B_{b_{ij}}}\right) \\
& x_{ij}, y_{ij}, w_{ij} \geq 0.
\end{aligned} \tag{34}
$$

Step 3. Using, Relations (7) and (9). Let

$$
\begin{aligned}
&\sqrt{P_{C_{c_{ij}}}}\left(\mathcal{C}_{c_{ij}} - 4\sqrt{2}\sigma_{c_{ij}}\right) = \hat{c}_1, \quad \sqrt{P_{C_{c_{ij}}}}\mathcal{C}_{c_{ij}} = \hat{c}_2, \quad \sqrt{P_{C_{c_{ij}}}}\left(\mathcal{C}_{c_{ij}} + 4\sqrt{2}\sigma_{c_{ij}}\right) = \hat{c}_3, \\
&\sqrt{P_{A_{a_{ij}}}}\left(\mathcal{C}_{a_{ij}} - 4\sqrt{2}\sigma_{a_{ij}}\right) = \hat{a}_1, \quad \sqrt{P_{A_{a_{ij}}}}\mathcal{C}_{a_{ij}} = \hat{a}_2, \quad \sqrt{P_{A_{a_{ij}}}}\left(\mathcal{C}_{a_{ij}} + 4\sqrt{2}\sigma_{a_{ij}}\right) = \hat{a}_3, \\
&\sqrt{P_{A_{a_{ij}}}}\left(\mathcal{C}_{b_{ij}} - 4\sqrt{2}\sigma_{b_{ij}}\right) = \hat{b}_1, \quad \sqrt{P_{A_{a_{ij}}}}\mathcal{C}_{b_{ij}} = \hat{b}_2, \quad \sqrt{P_{A_{a_{ij}}}}\left(\mathcal{C}_{b_{ij}} + 4\sqrt{2}\sigma_{b_{ij}}\right) = \hat{b}_3.
\end{aligned}
$$

Also, let $\sqrt{P_{X_{x_{ij}}}} = \lambda$, So that λ is non-negative real value. Therefore, Relation (34) rewritten as follows

$$
\begin{aligned}
\max[Z]^Z =& \Re\left(\sum_{j=1}^{n}\left(\hat{c}_1\ , \hat{c}_2,\ \hat{c}_3\ \right) \otimes \left(\lambda x_{ij},\ \lambda y_{ij}, \lambda w_{ij}\right)\right) \\
& \sum_{j=1}^{n}\left(\hat{a}_1\ , \hat{a}_2,\ \hat{a}_3\ \right) \otimes \left(\lambda x_{ij},\ \lambda y_{ij}, \lambda w_{ij}\right) \leq=\geq \left(\hat{b}_1\ , \hat{b}_2,\ \hat{b}_3\ \right) \\
& \lambda x_{ij}, \lambda y_{ij}, \lambda w_{ij} \geq 0.
\end{aligned} \tag{35}
$$

Step 4. Assuming $\left(\hat{a}_1\ , \hat{a}_2,\ \hat{a}_3\ \right) \otimes \left(\lambda x_{ij},\ \lambda y_{ij}, \lambda w_{ij}\right) = \left(\lambda m_{ij}\ , \lambda n_{ij}, \lambda o_{ij}\right)$ the FZLPP, obtained in Step 1, may be written as:

$$\begin{aligned}\max[Z]^Z =&\Re\left(\sum_{j=1}^{n}(\hat{c}_1\ ,\hat{c}_2,\ \hat{c}_3\ )\otimes\left(\lambda x_{ij},\ \lambda y_{ij},\lambda w_{ij}\right)\right)\\ &s.t\quad \sum_{j=1}^{n}(\lambda m_{ij}\ ,\lambda n_{ij},\lambda o_{ij})\le=\ge\left(\hat{b}_1\ ,\hat{b}_2,\ \hat{b}_3\ \right)\\ &\lambda x1_{ij},\ \lambda x2_{ij},\lambda x3_{ij}\ge 0.\end{aligned} \tag{36}$$

Step 5. Calculate the system (36) by Kumar et al method (2011), [11]

$$\begin{aligned}\max[Z]^Z =&\Re\left(\sum_{j=1}^{n}(\hat{c}_1\ ,\hat{c}_2,\ \hat{c}_3\ )\otimes\left(\lambda x_{ij},\ \lambda y_{ij},\lambda w_{ij}\right)\right)\\ &s.t\quad \sum_{j=1}^{n}\lambda m_{ij}\le=\ge\hat{b}_1\\ &\sum_{j=1}^{n}\lambda n_{ij}\le=\ge\hat{b}_2\\ &\sum_{j=1}^{n}\lambda o_{ij}\le=\ge\hat{b}_3\\ &\lambda x2_{ij}-\lambda x1_{ij},\quad \lambda x3_{ij}-\lambda x2_{ij}\ge 0.\end{aligned} \tag{37}$$

Step 6. Get the optimal solution $\lambda x_{ij},\ \lambda y_{ij}, and \lambda w_{ij}$ by solving the linear programming problem (CLPP) obtained in Step 5.
Step 7. Get the value of λ by Step 6.
Step 8. Get the Z- optimal solution by putting the values of $\lambda x_{ij},\ \lambda y_{ij},\lambda w_{ij}$ and λ in $[x_{ij}]^Z=\left(x_{ij},\ y_{ij},w_{ij};\ P_{X_{x_{ij}}}\right)$.
Step 9. Get the Z- optimal value by putting $[x_{ij}]^Z$ in $\sum_{j=1}^{n}[c_{ij}]^Z\otimes[x_{ij}]^Z$.

Theorem 3.9. *If* $\left[x^*_{ij}\right]^Z=\left(x^*_{ij},\mu_{x^*_{ij}},\ P_{x^*_{ij}}\right)=\left(\lambda x^*_{ij},\lambda\ y^*_{ij},\lambda w^*_{ij}\right)$ *be an optimal solution of problems (37) (and hence is a lexicographic optimal solution of problem (33), then it is also an exact optimal solution of problem (10).*

Proof. By contradiction, let $\left[x^*_{ij}\right]^Z=\left(A^*_X,\ P^*_{X_{x_{ij}}}\right)=\left(\lambda x^*_{ij},\lambda\ y^*_{ij},\lambda w^*_{ij}\right)$ be an optimal solution of (37), but it is not the exact optimal solution of problem (33). Therefore, there exists a feasible solution of problem (10), namel

$$\left[x^0_{ij}\right]^Z=\left(A^0_X,\ P^0_{X_{x_{ij}}}\right)=\left(\lambda x^0_{ij},\ \lambda y^0_{ij},\lambda w^0_{ij}\right)\neq\left[x^*_{ij}\right]^Z,$$

such that

$$\left(c_{1ij}\lambda x^*_{ij},c_{2ij}\ \lambda y^*_{ij},c_{3ij}\lambda w^*_{ij}\right)<\left(c_{1ij}\lambda x^0_{ij},c_{2ij}\ \lambda y^0_{ij},c_{3ij}\lambda w^0_{ij}\right),$$

(in case of minimization problem

$$\left(c_{1ij}\lambda x^*_{ij},c_{2ij}\ \lambda y^*_{ij},c_{3ij}\lambda w^*_{ij}\right)>\left(c_{1ij}\lambda x^0_{ij},c_{2ij}\ \lambda y^0_{ij},c_{3ij}\lambda w^0_{ij}\right)).$$

So, with respect to Definition 3.5, we have condition as follows:
If there is any non-negative $[X*]^Z=\left(A^*_X,\ P^*_{X_{x_{ij}}}\right)=\left(\lambda x^*_{ij},\ \lambda y^*_{ij},\lambda w^*_{ij}\right)$ such

$$\Re\left(\left(A_c,\ P_{C_{c_{ij}}}\right)\otimes\left(A^*_X,\ P^*_{X_{x_{ij}}}\right)\right)(\le=\ge)\,\Re\left(A_B,\ P_{B_{b_{ij}}}\right),$$

It is equivalent to $\Re\left((c_{1ij}\ ,c_{2ij},\ c_{3ij})\otimes\left(\lambda x^*_{ij},\ \lambda y^*_{ij},\lambda w^*_{ij}\right)\right)(\le=\ge)\,\Re\left(b_{1ij}\ ,b_{2ij},\ b_{3ij}\right).$

Let $\Re\left(\left(A_C,\ P_{C_{c_{ij}}}\right)\otimes\left(A_X^0,\ P^0_{X_{x_{ij}}}\right)\right)>\Re\left(\left(A_C,\ P_{C_{c_{ij}}}\right)\otimes\left(A_X^*,\ P^*_{X_{x_{ij}}}\right)\right)$. It is equivalent to

$$\Re\left((c_{1ij}\ ,c_{2ij},\ c_{3ij})\otimes\left(\lambda x^0_{ij},\lambda\ y^0_{ij},\lambda w^0_{ij}\right)\right)>\Re\left((c_{1ij}\ ,c_{2ij},\ c_{3ij})\otimes\left(\lambda x^*_{ij},\lambda\ y^*_{ij},\lambda w^*_{ij}\right)\right),$$

(about the maxim problem). Therefore, $\left(\lambda x^0_{ij},\ \lambda y^0_{ij},\lambda w^0_{ij}\right)$ is a feasible solution of problem (35) in which the objective value in $\left(\lambda x^0_{ij},\ \lambda y^0_{ij},\lambda w^0_{ij}\right)$ is greater than the objective value in $\left(\lambda x^*_{ij},\lambda\ y^*_{ij},\lambda w^*_{ij}\right)$.

But this is a contradiction. Moreover, let

$$\Re\left(\left(A_C,\ P_{C_{c_{ij}}}\right)\otimes\left(A_X,\ P_{X_{x_{ij}}}\right)\right)<\Re\left(\left(A_C,\ P_{C_{c_{ij}}}\right)\otimes\left(A_X^*,\ P^*_{X_{x_{ij}}}\right)\right).$$

It is equivalent to $\Re\left((c_{1ij}\ ,c_{2ij},\ c_{3ij})\otimes\left(\lambda x^0_{ij},\lambda\ y^0_{ij},\lambda w^0_{ij}\right)\right)<\Re\left((c_{1ij}\ ,c_{2ij},\ c_{3ij})\otimes\left(\lambda x^*_{ij},\lambda\ y^*_{ij},\lambda w^*_{ij}\right)\right)$ (about the minimization problem).

Therefore, $\left(\lambda x^0_{ij},\ \lambda y^0_{ij},\lambda w^0_{ij}\right)$ is a feasible solution of problem (35) in which the objective value in $\left(\lambda x^0_{ij},\ \lambda y^0_{ij},\lambda w^0_{ij}\right)$ is less than the objective value in $\left(\lambda x^*_{ij},\lambda\ y^*_{ij},\lambda w^*_{ij}\right)$. But this is a contradiction.

Therefore, $\left[x^*_{ij}\right]^Z=\left(x^*_{ij},\mu_{x^*_{ij}},\ P_{x^*_{ij}}\right)=\left(\lambda x^*_{ij},\lambda\ y^*_{ij},\lambda w^*_{ij}\right)$ is an exact optimal solution of problem (10). □

4 Numerical example

Example 4.1.

The following FZLPP is solved using the proposed method

$$\begin{aligned}\max[Z]^Z=&((1.25\ ,7.5,11.25\ ),0.64)\otimes\ [x_1]^Z+((2.5\ ,3.75,10\ ),0.64)\otimes\ [x_2]^Z\\ s.t\quad&((2.5\ ,3.75,5),0.64)\otimes[x_1]^Z+((1.25,2.5\ ,3.75),0.64)\otimes\ [x_2]^Z\\ &=((7.5,20,37.5),0.64)\\ &((-1.25,1.25,2.5),0.64)\otimes[x_1]^Z+((1.25,3.75,5),0.64)\otimes\ [x_2]^Z\\ &=\ ((1.25,21.25,37.5),0.64).\end{aligned}\tag{38}$$

$[x_1]^Z$ and $[x_2]^Z$ are non-negative Z-numbers.

Solution: With the help Step 1, Let $[x_1]^Z=(x_1,\ y_1,w_1;\ P_{X_1})$ and $[x_2]^Z=(x_2,\ y_2,w_2;\ P_{X_2})$ then given FZLPP may be written as:

$$\begin{aligned}\max[Z]^Z=&((1.25\ ,7.5,11.25\ ),0.64)\otimes\ (x_1,\ y_1,w_1;\ P_{X_1})\\ &+((2.5\ ,3.75,10\ ),0.64)\otimes\ (x_2,\ y_2,w_2;\ P_{X_2})\\ s.t\quad&((2.5\ ,3.75,5),0.64)\otimes(x_1,\ y_1,w_1;\ P_{X_1})+((1.25,2.5\ ,3.75),0.64)\\ &\otimes\ (x_2,\ y_2,w_2;\ P_{X_2})=((7.5,20,37.5),0.64)\\ &((-1.25,1.25,2.5),0.64)\otimes(x_1,\ y_1,w_1;\ P_{X_1})+((1.25,3.75,5),0.64)\\ &\otimes\ (x_2,\ y_2,w_2;\ P_{X_2})=\ ((1.25,21.25,37.5),0.64).\end{aligned}\tag{39}$$

Using Relation (6) in Step 3 and Let $[x_1]^{Z^\alpha}=(\lambda x_1,\ \lambda y_1,\lambda w_1)\,and\,[x_2]^{Z^\alpha}=(\lambda x_2,\ \lambda y_2,\lambda w_2)$ where $0<\lambda<1$, then given fully Z-LP problem may be written as:

$$\begin{aligned}\max[Z]^Z=&(1,6,9)\otimes\ (\lambda x_1,\ \lambda y_1,\lambda w_1)+(2,3,8)\otimes\ (\lambda x_2,\ \lambda y_2,\lambda w_2)\\ s.t\quad&(2,3,4)\otimes(\lambda x_1,\ \lambda y_1,\lambda w_1)+(1,2,3)\otimes\ (\lambda x_2,\ \lambda y_2,\lambda w_2)=(6,16,30)\\ &(-1,1,2)\otimes(\lambda x_1,\ \lambda y_1,\lambda w_1)+(1,3,4)\otimes\ (\lambda x_2,\ \lambda y_2,\lambda w_2)=(1,17,30).\end{aligned}\tag{40}$$

$(\lambda x_1,\ \lambda y_1,\lambda w_1)$ and $(\lambda x_2,\ \lambda y_2,\lambda w_3)$ are non-negative triangular fuzzy numbers. With the help Step 4, the above FFLPP may be written as:

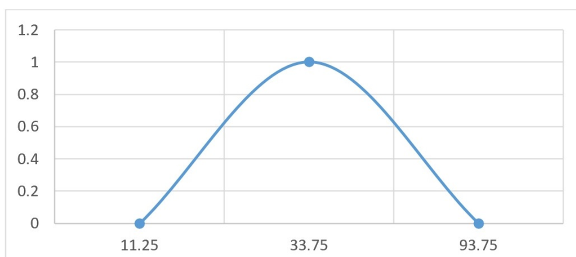
$$\begin{aligned}\max[Z]^Z=&\Re\left(\lambda x_1+2\lambda x_2,6\lambda y_1+3\lambda y_2,9\lambda w_1+8\lambda w_2\right)\\ s.t\quad&(2\lambda x_1+\lambda x_2,3\lambda y_1+2\lambda y_2,4\lambda w_1+3\lambda w_2)=(6,16,30)\\ &(-\lambda x_1+\lambda x_2,\lambda y_1+3\lambda y_2,2\lambda w_1+4\lambda w_2)=\ (1,17,30).\end{aligned}\tag{41}$$

$(\lambda x_1,\ \lambda y_1, \lambda w_1)$ and $(\lambda x_2,\ \lambda y_2, \lambda w_3)$ are non-negative triangular fuzzy numbers.

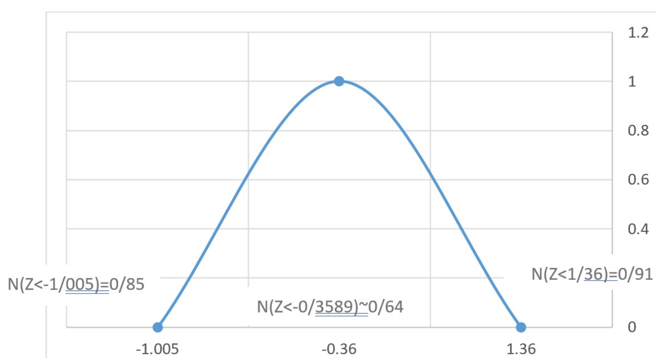
Using Step 5 of the proposed method the above FLPP is converted into the following CLPP

$$\begin{aligned}\max[Z]^Z = &\left(\frac{1}{4}\left(\lambda x_1 + 2\lambda x_2 + 12\lambda y_1 + 6\lambda y_2 + 9\lambda w_1 + 8\lambda w_2\right)\right)\\ s.t\ \ &2\lambda x_1 + \lambda x_2 = 6\\ &-\lambda x_1 + \lambda x_2 = 1\\ &3\lambda y_1 + 2\lambda y_2 = 16\\ &\lambda y_1 + 3\lambda y_2 = 17\\ &4\lambda w_1 + 3\lambda w_2 = 30\\ &2\lambda w_1 + 4\lambda w_2 = 30\\ &\lambda y_1 - \lambda x_1 \geq 0, \lambda w_1 - \lambda y_1 \geq 0,\ \lambda y_2 - \lambda x_2 \geq 0, \lambda w_2 - \lambda y_2 \geq 0. \end{aligned} \tag{42}$$

The optimal solution of the above CLPP is $\lambda x_1 = 1, \lambda y_1 = 2, \lambda w_1 = 3, \lambda x_2 = 4, \lambda y_2 = 5, \lambda w_2 = 6$. So, $[x_1]^{Z^\alpha} = (1,\ 2, 3; 1)$, $[x_2]^{Z^\alpha} = (4,\ 5, 6; 1)$ are regular Z^α-numbers. So, $[x_1]^{Z^\alpha} = \left(\frac{1}{\lambda}, \frac{2}{\lambda}, \frac{3}{\lambda}; \lambda^2\right)$, $[x_2]^{Z^\alpha} = \left(\frac{4}{\lambda}, \frac{5}{\lambda}, \frac{6}{\lambda}; \lambda^2\right)$ are Irregular Z^α-numbers. Assuming that the constraint part of our variable with Z-valuation is a regular fuzzy number, we have $[x_1]^{Z^\alpha} = \left(\frac{1}{\lambda}, \frac{2}{\lambda}, \frac{3}{\lambda}; 1 * \lambda^2\right)$, $[x_2]^{Z^\alpha} = \left(\frac{4}{\lambda}, \frac{5}{\lambda}, \frac{6}{\lambda}; 1 * \lambda^2\right)$. In accordance with relation (7) and relation (9), λ^2 will be $P_{X_{x_{ij}}}$. Using Step 7, the Z-optimal solution is given by $[x_1]^Z = \left(\left(\frac{1}{\lambda}, \frac{2}{\lambda}, \frac{3}{\lambda}\right), \lambda^2\right)$, $[x_2]^Z = \left(\left(\frac{4}{\lambda}, \frac{5}{\lambda}, \frac{6}{\lambda}\right), \lambda^2\right)$. Hence, using Step 8, the Z-optimal value of the given FFLPP is $[Z]^Z = \left(\left(\frac{9}{\lambda}, \frac{27}{\lambda}, \frac{75}{\lambda}\right), \lambda^2\right)$. Let $\lambda = 0.8$, whe have $[x_1]^Z = ((1.25, 2.5, 3.75), 0.64)$, $[x_2]^Z = ((5, 6.25, 7.5), 0.64)$, and $[Z]^Z = ((11.25, 33.75, 93.75), 0.64)$. To demonstrate the correctness of the subject using $z = \frac{x-\mu}{\sigma}$ in the standard normal distribution, we transform the components of $(11.25, 33.75, 93.75)$ into the standard form and have $(-1.00514, -0.35898,\ 1.364121) \cong (-1.005, -0.36, 1.36)$. Using the standard normal distribution table, we have $(Z < -0.36) = 0.64$. For $[Z]^Z$ function, Constraint, Z_A, and reliability (certainty), Z_B, can be seen in Figure 1.



Z_A



Z_B

Figure 1: Z_A: Constraint component of Z function, and Z_B: The reliability (certainty) of Z_A for Example 4.1

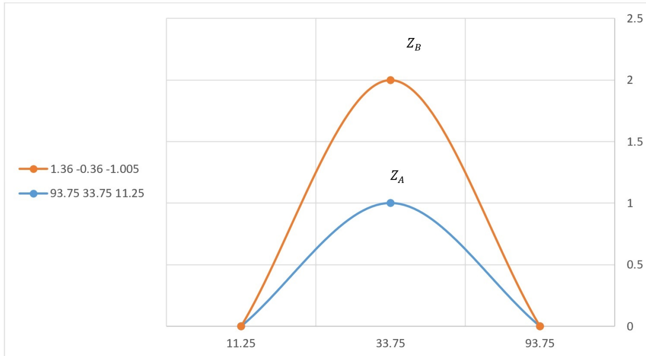


Figure 2: $Z = (Z_A, Z_B)$ for Example 4.1

Example 4.2.

The following FZLPP is solved using the proposed method

$$\begin{aligned}\min[Z]^Z =& ((1.43\ ,2.85,4.28\ ),0.49)\otimes\ [x_1]^Z + ((2.85,4.28,5.715\ ),0.49)\otimes\ [x_2]^Z\\ s.t\quad & ((0\ ,1.43\ ,2.85),0.49)\otimes[x_1]^Z + ((1.43\ ,2.85,4.28),0.49)\otimes\ [x_2]^Z\\ &= ((2.85,14.28,34.28),0.49)\\ & ((1.43\ ,2.85,4.28),0.49)\otimes[x_1]^Z + ((0\ ,1.43\ ,2.85),0.49)\otimes\ [x_2]^Z\\ &= \ ((1.43,11.43,30),0.49)\,.\end{aligned} \tag{43}$$

$[x_1]^Z$ and $[x_2]^Z$ are non-negative Z-numbers.

Solution: With the help Step 1, Let $[x_1]^Z = (x_1,\ y_1, w_1;\ P_{X_1})$ and $[x_2]^Z = (x_2,\ y_2, w_2;\ P_{X_2})$ then given FZLPP may be written as:

$$\begin{aligned}\min[Z]^Z =& ((1.43\ ,2.85,4.28\ ),0.49)\otimes\ (x_1,\ y_1,w_1;\ P_{X_1}) + ((2.85,4.28,5.715\ ),0.49)\otimes\ (x_2,\ y_2,w_2;\ P_{X_2})\\ s.t\quad & ((0\ ,1.43\ ,2.85),0.49)\otimes(x_1,\ y_1,w_1;\ P_{X_1}) + ((1.43\ ,2.85,4.28\ ),0.49)\otimes\ (x_2,\ y_2,w_2;\ P_{X_2})\\ &= ((2.85,14.28,34.28),0.49)\\ & ((1.43\ ,2.85,4.28\ ),0.49)\otimes(x_1,\ y_1,w_1;\ P_{X_1}) + ((0\ ,1.43\ ,2.85),0.49)\otimes\ (x_2,\ y_2,w_2;\ P_{X_2})\\ &= \ ((1.43,11.43,30),0.49)\,.\end{aligned} \tag{44}$$

Using Relation (6) in Step 3 and Let $[x_1]^{Z^\alpha} = (\lambda x_1,\ \lambda y_1, \lambda w_1)$ and $[x_2]^{Z^\alpha} = (\lambda x_2,\ \lambda y_2, \lambda w_2)$ where $0 < \lambda < 1$, then given fully Z-LP problem may be written as:

$$\begin{aligned}\min[Z]^Z =& (1,2,3)\otimes\ (\lambda x_1,\ \lambda y_1,\lambda w_1) + (2,3,4)\otimes\ (\lambda x_2,\ \lambda y_2,\lambda w_2)\\ s.t\quad & (0,1,2)\otimes(\lambda x_1,\ \lambda y_1,\lambda w_1) + (1,2,3)\otimes\ (\lambda x_2,\ \lambda y_2,\lambda w_2) = (2,10,24)\\ & (1,2,3)\otimes(\lambda x_1,\ \lambda y_1,\lambda w_1) + (0,1,2)\otimes\ (\lambda x_2,\ \lambda y_2,\lambda w_2) = \ (1,8,21)\,.\end{aligned} \tag{45}$$

$(\lambda x_1,\ \lambda y_1, \lambda w_1)$ and $(\lambda x_2,\ \lambda y_2, \lambda w_3)$ are non-negative triangular fuzzy numbers. With the help Step 4, the above FFLPP may be written as:

$$\begin{aligned}\min[Z]^Z =& \Re\,(\lambda x_1 + 2\lambda x_2, 2\lambda y_1 + 3\lambda y_2, 3\lambda w_1 + 4\lambda w_2)\\ s.t\quad & (\lambda x_2, \lambda y_1 + 2\lambda y_2, 2\lambda w_1 + 3\lambda w_2) = (6,16,30)\\ & (\lambda x_1, 2\lambda y_1 + \lambda y_2, 3\lambda w_1 + 2\lambda w_2) = \ (1,17,30)\,.\end{aligned} \tag{46}$$

$(\lambda x_1,\ \lambda y_1, \lambda w_1)$ and $(\lambda x_2,\ \lambda y_2, \lambda w_3)$ are non-negative triangular fuzzy numbers. using Step 5 of the proposed method the above FLPP is converted into the following CLPP

$$\begin{aligned}\min[Z]^Z =&\Re\left(\lambda x_1+2\lambda x_2, 2\lambda y_1+3\lambda y_2, 3\lambda w_1+4\lambda w_2\right)\\ &s.t \quad (\lambda x_2, \lambda y_1+2\lambda y_2, 2\lambda w_1+3\lambda w_2)=(6,16,30)\\ &\qquad (\lambda x_1, 2\lambda y_1+\lambda y_2, 3\lambda w_1+2\lambda w_2)= (1,17,30). \end{aligned} \tag{47}$$

$$\begin{aligned}\min[Z]^Z =& \left(\frac{1}{4}\left(\lambda x_1+2\lambda x_2+4\lambda y_1+6\lambda y_2+3\lambda w_1+4\lambda w_2\right)\right)\\ s.t \quad & 0\lambda x_1+\lambda x_2=2\\ & \lambda x_1+0\lambda x_2=1\\ & \lambda y_1+2\lambda y_2=10\\ & \lambda y_1+2\lambda y_2=10\\ & 2\lambda y_1+\lambda y_2=8\\ & 2\lambda w_1+3\lambda w_2=24\\ & 3\lambda w_1+2\lambda w_2=21\\ & \lambda y_1-\lambda x_1\ge 0, \lambda w_1-\lambda y_1\ge 0,\ \lambda y_2-\lambda x_2\ge 0, \lambda w_2-\lambda y_2\ge 0. \end{aligned} \tag{48}$$

The optimal solution of the above CLPP is $\lambda x_1=1, \lambda y_1=2, \lambda w_1=3, \lambda x_2=2, \lambda y_2=4, \lambda w_2=6$. So, $[x_1]^{Z^\alpha}=(1,\ 2,3;1)$, $[x_2]^{Z^\alpha}=(2,\ 4,6;1)$ are regular Z^α-numbers. So,

$$[x_1]^{Z^\alpha}=\left(\frac{1}{\lambda},\frac{2}{\lambda},\frac{3}{\lambda};\lambda^2\right),\quad [x_2]^{Z^\alpha}=\left(\frac{2}{\lambda},\frac{4}{\lambda},\frac{6}{\lambda};\lambda^2\right),$$

are Irregular Z^α-numbers. Assuming that the constraint part of our variable with Z-valuation is a regular fuzzy number, we have $[x_1]^{Z^\alpha}=\left(\frac{1}{\lambda},\frac{2}{\lambda},\frac{3}{\lambda};1*\lambda^2\right)$, $[x_2]^{Z^\alpha}=\left(\frac{2}{\lambda},\frac{4}{\lambda},\frac{6}{\lambda};1*\lambda^2\right)$. In accordance with relation (7) and relation (9), λ^2 will be $P_{X_{x_{ij}}}$. Using Step 7, the Z-optimal solution is given by

$$[x_1]^Z=\left(\left(\frac{1}{\lambda},\frac{2}{\lambda},\frac{3}{\lambda}\right),\lambda^2\right),\ [x_2]^Z=\left(\left(\frac{2}{\lambda},\frac{4}{\lambda},\frac{6}{\lambda}\right),\lambda^2\right).$$

Hence, using Step 8, the Z-optimal value of the given FFLPP is $[Z]^Z=\left(\left(\frac{5}{\lambda},\frac{16}{\lambda},\frac{33}{\lambda}\right),\lambda^2\right)$. Let $\lambda=0.75$, whe have $[x_1]^Z=((1.33,2.66,4),0.56)$, $[x_2]^Z=((2.66,5.33,8),0.56)$, and $[Z]^Z=((6.66,\ 21.33,44),0.56)$.

To demonstrate the correctness of the subject using $z=\frac{x-\mu}{\sigma}$ in the standard normal distribution, we transform the components of $(6.66,\ 21.33,44)$ into the standard form and have $(-1.12868,-0.17361,1.302386)\cong(-1.13,-0.17,1.30)$. Using the standard normal distribution table, we have $(Z<-0.17)=0.56$. For $[Z]^Z$ function, constraint, Z_A, and reliability (certainty),Z_B, can be seen in Figure 3.

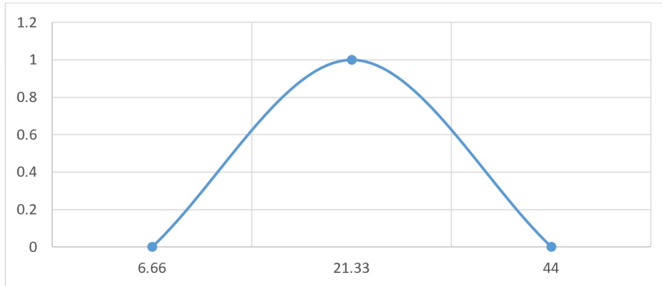
4.1 Results and discussion

It is easy to verify that with the confidence values assumed in the problems, the values $[x_1]^Z$ and $[x_2]^Z$, obtained in Examples 4.1 and 4.2, exactly satisfy all the constraints and also it is not possible to find any non-negative Z numbers $[x_1]^Z$ and $[x_2]^Z$ which will satisfy the following conditions:

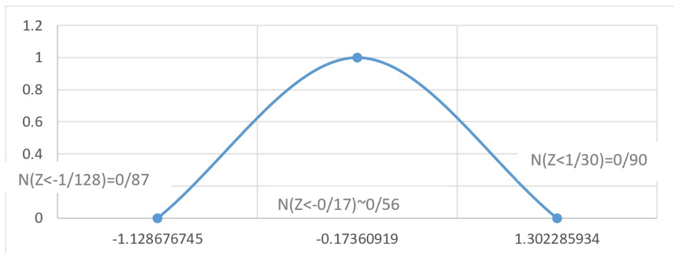
$$\Re\left(((1.25\ ,7.5,11.25\ ),0.64)\otimes\ [x_1]^Z+((2.5\ ,3.75,10\ ),0.64)\right)\otimes\ [x_2]^Z>\Re\left((11.25,33.75,93.75),0.64\right).$$

$$\Re\left(((1.43\ ,2.85,4.28\ ),0.49)\otimes\ [x_1]^Z+((2.85,4.28,5.715\ ),0.49)\otimes\ [x_2]^Z\right)>\Re\left((6.66,\ 21.33,44),0.56\right).$$

Also it is difficult to find the solution by using the existing method [9] as compared to the proposed method



Z_A



Z_B

Figure 3: Z_A: Constraint component of Z function, and Z_B: The reliability (certainty) of Z_A for Example 4.2

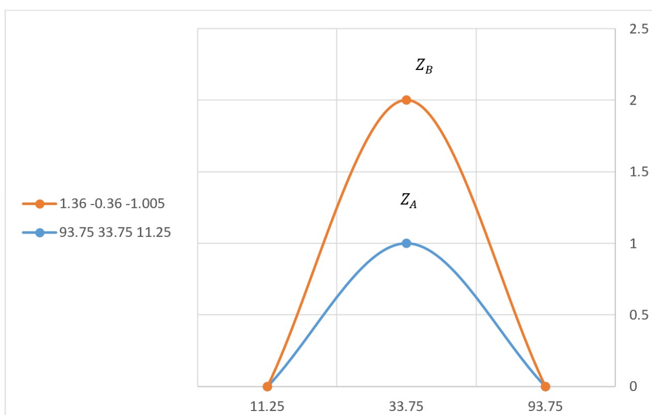


Figure 4: $Z = (Z_A, Z_B)$ for Example 4.2

Example 4.3. (Applied example)

In this section, there is a case of the motor vehicle mixture for travel to illustrate the projected weaving process considering three important criteria (cost, travel time, comfort). The information related to these three criteria, which are described as Z-numbers, can be seen in Table 1 (this information is taken from reference [9]).

After that, the modified decision matrix can be obtained in Table 2 by simplifying the fuzzy data in Table 1.

The information in Table 2 was formulated into a FZLPP and solved by the proposed method. The formulated

Table 1: Linguistic criterion

	Cost (Rupees) ((0.75,1,1),(0.75,1,1))	Travel time (min) ((0.5,0.75,1), (0.75,1,1))	The comfort ((0.25,0.5,0.75),(0.75,1,1))
Train	((10,9,12), (0.75,1,1))	((100,70,120), (0.25,0.5,0.75))	((5,4,6),(0.5,0.75,1))
Auto	((24,20,25),(0.5,0.75,1))	((70,60,100), (0.75,1,1))	((8,7,10), (0.5,0.75,1))
Bus	((15,15,15), (0.5,0.75,1))	((80,70,90), (0.5,0.75,1))	((4,1,7), (0.5,0.75,1))

Table 2: Modified decision matrix with Z-number.

	Cost (Rupees) ((0.75,1,1),(0.75,1,1))	Travel time (min) ((0.5,0.75,1), (0.75,1,1))	The comfort ((0.25,0.5,0.75),(0.75,1,1))
Train	((0.11,0.1,0.14), (0.75,1,1))	((0.23,0. 16,0.27),(0.25,0.5,0.75))	((0.16,0.13,0.19),(0.5,0.75,1))
Auto	((0.27,0.23,0.28), (0.5,0.75,1))	((70,60,100), (0.75,1,1))	((0.25,0.22,0.32), (.5,0.75,1))
Bus	((0.17,0.17,0.17), (0.5,0.75,1))	((0.18,0.16,0.2), (0.5,0.75,1))	((0.13,0.03,0.22), (0.5,0.75,1))

ZLPP is

$$\begin{aligned}
\max[Z]^Z &= ((0.11,0.1,0.14),\ (0.75,1,1)) \otimes [x_1]^Z + ((0.27,0.23,0.28),\ (0.5,0.75,1)) \otimes [x_2]^Z \\
&\quad + ((0.17,0.17,0.17),\ (0.5,0.75,1)) \otimes [x_3]^Z \\
s.t.\ & ((0.23,0.\ 16,0.27),(0.25,0.5,0.75)) \otimes [x_1]^Z + ((0.16,0.14,0.23),\ (0.75,1,1)) \otimes [x_2]^Z \\
&+ ((0.18,0.16,0.2)\,,\ (0.5,0.75,1)) \otimes [x_3]^Z \le ((0.5,0.75,1),(0.75,1,1,1;1)), \\
&((0.16,0.13,0.19),(0.5,0.75,1)) \otimes [x_1]^Z\ ((0.25,0.22,0.32),\ (.5,0.75,1)) \otimes [x_2]^Z \\
&+ ((0.13,0.03,0.22)\,,\ (0.5,0.75,1)) \otimes [x_3]^Z \le ((0.25,0.5,0.75),(0.75,1,1,1;1)).
\end{aligned} \tag{49}$$

$[x_1]^Z, [x_2]^Z$, and $[x_3]^Z$ are non-negative Z-numbers.

Solution: With the help 1, Let $[x_1]^Z = (x_1,\ y_1, w_1;\ P_{X_1}), [x_2]^Z = (x_2,\ y_2, w_2;\ P_{X_2}), [x_3]^Z = (x_3,\ y_3, w_3;\ P_{X_3})$and then given FZLP problem may be written as:

$$\begin{aligned}
\max[Z]^Z =&((0.11,0.1,0.14),\ 0.91) \otimes (x_1,\ y_1,w_1;\ P_{X_1}) + ((0.27,0.23,0.28),\ 0.75) \otimes (x_2,\ y_2,w_2;\ P_{X_2}) \\
&+ ((0.17,0.17,0.17),\ 0.75) \otimes (x_3,\ y_3,w_3;\ P_{X_3}) \\
&s.t.\ ((0.23,0.\ 16,0.27),0.5) \otimes (x_1,\ y_1,w_1;\ P_{X_1}) + ((0.16,0.14,0.23),\ (0.91)) \otimes (x_2,\ y_2,w_2;\ P_{X_2}) \\
&+ ((0.18,0.16,0.2)\,,\ 0.75) \otimes (x_3,\ y_3,w_3;\ P_{X_3}) \le ((0.5,0.75,1),0.93) \\
&((0.16,0.13,0.19),0.75) \otimes (x_1,\ y_1,w_1;\ P_{X_1})\ ((0.25,0.22,0.32),\ 0.75) \otimes (x_2,\ y_2,w_2;\ P_{X_2}) \\
&+ ((0.13,0.03,0.22)\,,0.75) \otimes (x_3,y_3,w_3;P_{X_3}) \le ((0.25,0.5,0.75),0.93).
\end{aligned} \tag{50}$$

With the help Relation (6) in Step 3 and Let $[x_1]^{Z^\alpha} = (\lambda x_1,\ \lambda y_1, \lambda w_1), [x_2]^{Z^\alpha} = (\lambda x_2,\ \lambda y_2, \lambda w_2)$, and $[x_3]^{Z^\alpha} = (\lambda x_3,\ \lambda y_3, \lambda w_3)$ where $0 < \lambda < 1$, then given FZLPP may be written as:

$$\begin{aligned}
\max[Z]^Z =& (0.10,0.09,0.13) \otimes (\lambda x_1,\ \lambda y_1,\lambda w_1) \quad + (0.23,0.19,0.69) \otimes (\lambda x_2,\ \lambda y_2,\lambda w_2) \\
&+\ (0.147,0.147,0.147) \otimes (\lambda x_3,\ \lambda y_3,\lambda w_3) \\
&s.t.\ ((0.16,0.11,0.19) \otimes (\lambda x_1,\ \lambda y_1,\lambda w_1) + (0.15,0.13,0.21) \otimes (\lambda x_2,\ \lambda y_2,\lambda w_2) \\
&+ (0.15,0.13,0.17) \otimes (\lambda x_3,\ \lambda y_3,\lambda w_3) \le (0.48,0.72,0.96), \\
&(0.13,0.11,0.16) \otimes (\lambda x_1,\ \lambda y_1,\lambda w_1)\ (0.21,0.19,0.27) \otimes (\lambda x_2,\ \lambda y_2,\lambda w_2) \\
&+ (0.11,0.02,0.19) \otimes (\lambda x_3,\ \lambda y_3,\lambda w_3) \le (0.24,0.48,0.72).
\end{aligned} \tag{51}$$

$(\lambda x_1,\ \lambda y_1,\lambda w_1), (\lambda x_2,\ \lambda y_2,\lambda w_3)$, and $(\lambda x_3,\ \lambda y_3,\lambda w_3)$ are non-negative fuzzy numbers. using Step 4, the above FFLPP may be written as:

$$\begin{aligned}
\max[Z]^Z =&\Re\,(0.10\lambda x_1 + 0.23\lambda x_2 + 0.14\lambda x_3, 0.09\lambda y_1 + 0.19\lambda y_2 + 0.14\lambda y_3, 0.13\lambda w_1 + 0.69\lambda w_2 + 0.14\lambda w_3) \\
&s.t\,(0.16\lambda x_1 + 0.15\lambda x_2 + 0.15\lambda x_3, 0.11\lambda y_1 + 0.13\lambda y_2 + 0.13\lambda y_3, 0.19\lambda w_1 + 0.21\lambda w_2 + 0.17\lambda w_3) \\
&= (0.48,0.72,0.96) \\
&(0.13\lambda x_1 + 0.21\lambda x_2 + 0.11\lambda x_3, 0.11\lambda y_1 + 0.19\lambda y_2 + 0.02\lambda y_3, 0.16\lambda w_1 + 0.27\lambda w_2 + 0.19\lambda w_3) \\
&=\ (0.24,0.48,0.72)\,.
\end{aligned} \tag{52}$$

With the help 5 of the proposed method the above FLPP is converted into the following CLPP

$$\max[Z]^Z = \left(\frac{1}{4}\left(0.10\lambda x_1 + 0.23\lambda x_2 + 0.14\lambda x_3 + 18\lambda y_1 + 38\lambda y_2 + 28\lambda y_3 + 0.13\lambda w_1 + 0.69\lambda w_2 + 0.14\lambda w_3\right)\right)$$
$$s.t \quad 0.16\lambda x_1 + 0.15\lambda x_2 + 0.15\lambda x_3 = 0.48$$
$$0.13\lambda x_1 + 0.21\lambda x_2 + 0.11\lambda x_3 = 0.24$$
$$0.11\lambda y_1 + 0.13\lambda y_2 + 0.13\lambda y_3 = 0.72$$
$$0.11\lambda y_1 + 0.19\lambda y_2 + 0.02\lambda y_3 = 0.48$$
$$0.19\lambda w_1 + 0.21\lambda w_2 + 0.17\lambda w_3 = 0.96$$
$$0.16\lambda w_1 + 0.27\lambda w_2 + 0.19\lambda w_3 = 0.72$$
$$\lambda y_1 - \lambda x_1 \geq 0, \lambda w_1 - \lambda y_1 \geq 0,\ \lambda y_2 - \lambda x_2 \geq 0, \lambda w_2 - \lambda y_2 \geq 0,\ \lambda y_3 - \lambda x_3 \geq 0, \lambda w_3 - \lambda y_3 \geq 0. \tag{53}$$

The optimal solution of CLPP (53) is $\lambda x_1 = -8.842105263, \lambda y_1 = 0, \lambda w_1 = 0, \lambda x_2 = 0, \lambda y_2 = 0, \lambda w_2 = 10, \quad \lambda x_3 = 12.63157895, \lambda y_3 = 3.366515837, \lambda w_3 = 18. So, [x_1]^{Z^\alpha} = (-8.84, 0, 0; 1), \ [x_2]^{Z^\alpha} = (0,\ 2.17, 10; 1), \ [x_3]^{Z^\alpha} = (12.63, 3.36, 18; 1)$, and $\widetilde{Z}^\alpha = \widetilde{46}$. While the solution obtained in reference [9] is $\widetilde{Z}^\alpha = \widetilde{0.59}$, the answer obtained in reference [10] is 0.65.

With the help Step 7, the Z-optimal solution is given by $[x_1]^Z = \left(\left(\frac{-8.84}{\lambda}, 0, 0\right), \lambda^2\right), \ [x_2]^Z = \left(\left(0, \frac{2.17}{\lambda}, \frac{10}{\lambda}\right), \lambda^2\right)$, and $[x_3]^Z = \left(\left(\frac{12.63}{\lambda}, 3.36/\lambda, 18/\lambda\right), \lambda^2\right)$. Hence, With the help Step 8, the Z-optimal value of the given FFLPP is $[Z]^Z = \left(\left(\frac{0.88}{\lambda}, \frac{176.79}{\lambda}, \frac{9.42}{\lambda}\right), \lambda^2\right)$.

Let $\lambda = 0.99$, whe have, $[x_1]^Z = ((-8.93, 0, 0), 0.9801), \ [x_2]^Z = ((0, 2.19, 10.10), 0.9801)$,
$[x_3]^Z = ((12.75, 3.4, 18.18), 0.9801)$ and $[Z]^Z = ((0.89, 178.58, 9.51), 0.9801)$.

To demonstrate the correctness of the subject using $z = \frac{x-\mu}{\sigma}$ in the standard normal distribution, we transform the components of $(0.89, 178.58, 9.51)$ into the standard form and have $(-0.7591, 1.4112, -0.653) \cong (-0.75, 1.41, -0.65)$. Using the standard normal distribution table, we have $(Z < 1.14) = 0.92$. $[Z]^Z$ function can be seen in Figure 5.

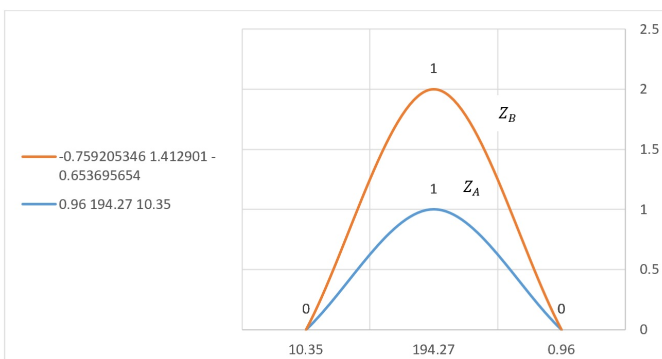


Figure 5: $Z = (Z_A, Z_B)$ for Example 4.3

5 Conclusion

In the few models that have been presented so far to solve linear programming problems with Z-valuation parameters, the final solution of the problem (decision variables) has often been considered real for further simplification or the confidence part of Z- numbers is considered a fuzzy number and not a distribution function. Here, we focus on linear programming problems where all variables and parameters are Z- numbers and the confidence part of Z- numbers is a distribution function. To do so, a method was proposed based on the ranking function. The optimal solution of FZLPP that occurs in real-life situations can be easily achieved using the proposed method. The advantage of the proposed method is that for using the proposed method there is no restriction on the elements of coefficient matrix and the obtained results exactly satisfies all the constraints. Also, the method is that the solution of the problem can be obtained for different λ and it is very easy to apply the proposed method. The examples are solved to demonstrate the proposed method. Of course, validation of the model was possible only by the study of its application in different examples available in the real world. This approach can be developed to solve DEA and supply chain models with Z numbers in the future. therefore, the validity of these approaches will be studied more.

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