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Original paper

Characterizing three classes of idempotent uninorms on a bounded lattice

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Abstract

This study presents characterizations of three classes of idempotent uninorms on a bounded lattice by the orders of their associated meet-semilattices. The first one is the class of internal uninorms, the second one is the class of idempotent uninorms defined on a lattice in which all elements are comparable with the corresponding neutral element and the third one is the class of idempotent uninorms defined on a lattice in which a single point is incomparable with the corresponding neutral element.

Keywords: Bounded lattice, internal uninorm, idempotent uninorm, partial order.

1 Introduction

Let $(L, \leq, 0, 1)$ be a bounded lattice. A uninorm [16] is a binary operation $U : L^2 \to L$ such that (L, U) is an abelian, ordered semigroup with a neutral element $e \in L$. A pointed bounded commutative residuated lattice [13] is an algebra: $\mathcal{S} = \langle L, \wedge, \vee, *, \to, e, f, \perp, \top\rangle$ with universe L, binary operations $\wedge$, $\vee$, $*$, $\to$ and constants e, f, $\perp$, $\top$ such that

1. $\langle L, \wedge, \vee, \perp, \top\rangle$ is a bounded lattice with top element $\top$ and bottom element $\perp$.
2. $(L, *, e)$ is a commutative monoid.
3. $z \leq x \to y$ iff $x * z \leq y$, for all $x, y, z \in L$.

Note that in a pointed bounded commutative residuated lattice $\mathcal{S}$, $*$ is a uninrom on a bounded lattice L with neutral element e and $\to$ is the residuum of $*$ defined by $x \to y = \sup\{z \mid x * z \leq y\}$ for all $x, y \in L$.

A **UL**-algebra is a pointed bounded commutative residuated lattice satisfying the prelinearity condition:

$$e \leq ((x \to y) \wedge e) \vee ((y \to x) \wedge e),$$

for all $x, y \in L$. The corresponding algebraic structures for Uninorm Logic are **UL**-algebras. A **UML**-algebra is a **UL**-algebra satisfying: $x * x = x$, for all $x \in L$. The corresponding algebraic structures for Uninorm Mingle Logic are **UML**-algebras. Refer to [13] for details on algebraic semantics for uninorm based logics.

A great deal of attention is being paid to the structure of idempotent uninorms, for example,

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- In [2], Çaylı and Drygaś discussed some properties of idempotent uninorms on a special class of bounded lattices, and characterized the class of idempotent uninorms with neutral element e on a finite bounded lattice L with $x \nparallel e$ for all $x \in L$ in a way similar to the framework of the unit interval $[0,1]$.
- Çaylı, Karaçal and Mesiar [3] demonstrated the existence of idempotent uninorms with an arbitrary neutral element on an arbitrary bounded lattice.
- Çaylı, Karaçal and Mesiar [4] introduced internal uninorms on a bounded lattice L, showing that there are bounded lattices on which no internal uninorm with the neutral element $e \notin \{0,1\}$ exists, and presented two construction methods for internal uninorms on a bounded lattice L with the fixed neutral element $e \in L\backslash\{0,1\}$ under some additional assumptions on L. Different from the $[0,1]$-case, there exist non-internal idempotent uninorms in the framework of bounded lattices [4].
- In [6], De Baets fully described left- or right-continuous idempotent uninorms defined on the unit interval $[0,1]$ in terms of the characterizing functions g.
- Martín et al. [9] presented a complete characterization of locally internal, associative and increasing functions on on the unit interval $[0,1]$; an improvement of this last result restricted to idempotent uninorms was done in [15].
- Mesiarová-Zemánková [10] showed that each idempotent uninorm on the unit interval $[0,1]$ can be characterized as an ordinal sum of singleton semigroups and introduced necessary and sufficient conditions for an ordinal sum of singleton semigroups to be an idempotent uninorm in the unit interval.
- In [14], Ouyang et al., characterized idempotent uninorms on a complete chain in terms of decreasing unary functions with a symmetry-related property.

Observe that, for any idempotent uninorm U on a bounded lattice $(L,\leq,0,1)$, (L,U) is a semilattice, i.e., a system with a single binary, idempotent, commutative and associative operation [1]. A semilattice is viewed as an algebraic structure (L,U). Alternatively, a semilattice is also viewed as a poset of a special type: every semilattice (L,U) is the *meet-semilattice* associated with the order $\preceq^U$ defined by

$$x \preceq^U y \text{ if and only if } U(x,y)=x. \tag{1}$$

That is, the meet operation $\curlywedge^U$ of $\preceq^U$ is U. The order $\preceq^U$ is the meet-semilattice order associated with U. Therefore, the class of idempotent uninorms is in one-to-one correspondence with the class of orders of meet-semilattices of a special type. Plainly, the meet operation $\curlywedge$ of a meet-semilattice with order $\preceq$ is idempotent, commutative and associative; and e is the greatest element of a meet-semilattice $(L,\preceq)$ if and only if the meet operation $\curlywedge$ of $\preceq$ has a neutral element e. Thus, the study on idempotent uninorms on a bounded lattice $(L,\leq)$ can be transformed into that on the partial orders $\preceq^U$ on L which are compatible with the partial order $\leq$ on L, i.e., such that the meet operation $\curlywedge^U$ is $\leq$-preserving. Devillet and Teheux [7] have discussed the case where $(L,\leq)$ is a chain, and a general case, i.e., the case where $(L,\leq)$ is a poset, was presented by them as an open question, that is, "Find characterizations for semilattice operations that are $\leq$-preserving on a poset $(L,\leq)$." In this paper, we partially answer this question and present the characterizations for such partial orders: (a) when $(L,\leq)$ is a lattice, and $\curlywedge^U$ is internal and has a neutral element, (b) when $\curlywedge^U$ has a neutral element e and $(L,\leq)$ is a lattice such that all $x \in L$ are comparable with e, and (c) when $\curlywedge^U$ has a neutral element e and $(L,\leq)$ is a lattice such that a single point is incomparable with e. As a consequence, we can obtain the characterizations of three classes of idempotent uninorms on a bounded lattice: the first one is the class of internal uninorms (Section 3), the second one is the class of idempotent uninorms defined on a lattice in which all elements are comparable with the corresponding neutral element and the third one is the class of idempotent uninorms defined on a lattice in which a single point is incomparable with the corresponding neutral element (Section 4).

2 Preliminaries

In this section, we recall some basic notions and results concerning bounded lattices (for more information, see [1]), and associative functions (especially uninorms) on a bounded lattice.

A *poset* $(P,\leq)$ is a nonempty set P equipped with an order relation $\leq$ (i.e., a reflexive, antisymmetric and transitive binary relation). If $a \leq b$ and $a \neq b$, we write $a < b$. For $a,b \in P$ with $a \leq b$, the subintervals $[a,b]$, $]a,b]$, $[a,b[$ and $]a,b[$ of P are defined as usual. A poset $(P,\leq)$ is called a *chain* if a and b are comparable, for all $a,b \in P$, i.e., either $a \leq b$ or $b \leq a$, for all $a,b \in P$. The notation $a\|b$ means that a and b are incomparable. The *least* element of any subset H of a poset P is an element $a \in H$ such that $a \leq h$, for all $h \in H$. The *greatest* element of H is an element $a \in H$ such that $h \leq a$ for all $h \in H$. An *upper bound* of H is an element $a \in P$ such that $h \leq a$, for all $h \in H$; the *least upper bound* of H or *supremum* of H, is the least element a of all upper bounds of H and we write $a = \sup H$ or $a = \vee H$. The concepts of *lower bound* and *greatest lower bound* or *infimum* are similarly defined; the latter is denoted by $\inf H$ or $\wedge H$. A poset $(L, \leq)$ is a *lattice* if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist, for all $a, b \in L$. A *bounded lattice* $(L, \leq)$ is a lattice which has the greatest element and the least element, written as 1 and 0, respectively. A poset $(L, \leq)$ is a *meet-semilattice* (or *lower semilattice*) if $\inf\{a, b\}$ exists, for all $a, b \in L$.

Next, we recall several notions and results related to uninorms on a bounded lattice.

Definition 2.1. ([4, Definition 3.2], [8, Definition 4]) *Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U : L^2 \to L$ is called*

(i) *a uninorm on L if it is commutative, associative, increasing with respect to both variables and has a neutral element $e \in L$.*

(ii) *internal if $U(x, y) \in \{x, y\}$, for all $x, y \in L$.*

(iii) *idempotent if $U(x, x) = x$, for all $x \in L$.*

It is easy to observe that each internal uninorm is idempotent, however, the converse it not true, in general. Let $I_e = \{x \in L \mid x \parallel e\}$.

Proposition 2.2. ([3, Corollary 1], [2, Proposition 6]) *Given a bounded lattice $(L, \leq, 0, 1)$ and $e \in L \backslash \{0, 1\}$, if U is an idempotent uninorm with neutral element e on L, then*

(i) *$U |_{[0,e]^2} (x, y) = x \wedge y$, for all $x, y \in [0, e]$.*

(ii) *$U |_{[e,1]^2} (x, y) = x \vee y$, for all $x, y \in [e, 1]$.*

(iii) *$U(x, y) \in \{x \wedge y, x \vee y\}$ or $U(x, y) \in I_e$, for all $x, y \in L$.*

Proposition 2.3. ([4, Propositions 3.5 and 3.6]) *Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L \backslash \{0, 1\}$. If U is an internal uninorm with neutral element e on L, then the subintervals $([0, e], \leq)$ and $([e, 1], \leq)$ are chains.*

3 Internal uninorms on bounded lattices

In this section we assume that $(L, \leq, 0, 1)$ is a bounded lattice. It is easy to see that *if U is an internal uninorm on L with the neutral element $e \in L \backslash \{0, 1\}$, then $(L, \preceq^U)$ is a chain and e is the greatest element of $(L, \preceq^U)$, where $\preceq^U$ is defined by* (1). The converse of the previous proposition is not true as is shown in the following example.

Example 3.1. *Consider a bounded lattice $(L, \leq) = (\{0, x, s, m, e, y, 1\}, \leq)$ with Hasse diagram shown in Figure 1. For the linear order $\preceq$ on L defined by $0 \prec x \prec s \prec m \prec 1 \prec y \prec e$, the meet operation $\curlywedge$ of $\preceq$ is given by the following table*

$\curlywedge$	0	x	s	m	e	y	1
0	0	0	0	0	0	0	0
x	0	x	x	x	x	x	x
s	0	x	s	s	s	s	s
m	0	x	s	m	m	m	m
e	0	x	s	m	e	y	1
y	0	x	s	m	y	y	1
1	0	x	s	m	1	1	1

Plainly, $\curlywedge$ is commutative, associative and internal, and it is obvious that it has neutral element e. However, $\curlywedge$ is not increasing with respect to $\leq$ since $x < y$ but $s \curlywedge x = x \| s = s \curlywedge y$. [10, Proposition 3] *manifested that idempotent uninorms on the unit interval $[0, 1]$ are in one-to-one correspondence with linear orders $\preceq$ on $[0, 1]$ such that the linear order $\preceq$ coincides with the order $\leq$ on $[0, e]$ and is the dual of the order $\leq$ on $[e, 1]$. This example shows that this one-to-one correspondence does not hold for the case of idempotent uninorms on bounded lattices.*

The following result shows necessary and sufficient conditions for compatibility of the lattice order $\leq$ and the linear order $\preceq$ which ensure that the meet operation $\curlywedge$ of $\preceq$ yields an internal uninorm.

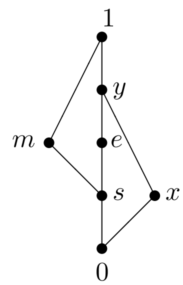

Figure 1: The lattice L from Example 3.1.

Theorem 3.2. *Let $e \in L\backslash\{0,1\}$ and let $\preceq$ be a linear order on L. Then the meet operation $\curlywedge$ of $\preceq$ is an internal uninorm with the neutral element e on $(L,\leq)$ if and only if for any $y,z \in L$ we have the following:*

(i) *If $y < z$, $y \prec z$ and there exists some $x \in L$ such that $y \prec x \prec z$, then $y < x$.*

(ii) *If $y < z$, $z \prec y$ and there exists some $x \in L$ such that $z \prec x \prec y$, then $x < z$.*

(iii) *$y \preceq e$.*

Proof. Suppose that $\curlywedge$ is an internal uninorm with neutral element e. If $y < z$, $y \prec z$ and there exists some $x \in L$ such that $y \prec x \prec z$, then $x \curlywedge y = y$ and $x \curlywedge z = x$, which together with the monotonicity of $\curlywedge$ implies $y < x$ and thus item (i) holds. Similarly we can verify that item (ii) holds. For the linear order $\preceq$ we can easily observe that $y \preceq e$, i.e., that item (iii) holds.

Conversely, suppose that items (i)-(iii) hold. Since $(L,\preceq)$ is a chain, we then have that $\curlywedge$ is internal, commutative and associative. By (iii) we know that $(L,\preceq)$ has the top element e and $x \curlywedge e = x$ for all $x \in L$ implies that e is the neutral element of $\curlywedge$. Thus we only need to show the monotonicity of $\curlywedge$. Consider $x,y,z \in L$ with $y < z$. The rest of the proof is divided into the following cases:

- If $x = y$, then $x \curlywedge y = y$ and $x \curlywedge z \in \{y,z\}$ since U is internal. Hence, $x \curlywedge y \leq x \curlywedge z$.
- If $x = z$, then similarly as above $x \curlywedge y \in \{y,z\}$ and $x \curlywedge z = z$. Hence, $x \curlywedge y \leq x \curlywedge z$.
- If $x \notin \{y,z\}$, i.e., x,y,z are mutually different, then there are the following possibilities:
  - If $y \prec x \prec z$, then $x \curlywedge y = y$ and $x \curlywedge z = x$. Hence, $x \curlywedge y < x \curlywedge z$ by (i).
  - If $z \prec y \prec x$ or $y \prec z \prec x$, then $x \curlywedge y = y < z = x \curlywedge z$.
  - If $x \prec z \prec y$ or $x \prec y \prec z$, then $x \curlywedge y = x = x \curlywedge z$
  - If $z \prec x \prec y$, then $x \curlywedge y = x$ and $x \curlywedge z = z$. Hence, $x \curlywedge y < x \curlywedge z$ by (ii).

Combining the facts above, we know that $\curlywedge$ is an internal uninorm with neutral element e. □

The following corollary specifies the relation of the lattice order $\leq$ and the linear order $\preceq$.

Corollary 3.3. *Let $e \in L\backslash\{0,1\}$ and let $\preceq$ be a linear order on L. If the meet operation $\curlywedge$ of $\preceq$ is an internal uninorm with the neutral element e on $(L,\leq)$ then for any $x_1,x_2 \in L$*

(i) *$x_1 \prec x_2$ if $x_1 < x_2 \leq e$.*

(ii) *$x_1 \prec x_2$ if $x_1 > x_2 \geq e$.*

(iii) *$x_1 < x_2$ if $x_2 \prec x_1$ and $x_2 > e$.*

(iv) *$x_1 > x_2$ if $x_2 \prec x_1$ and $x_2 < e$.*

(v) *$x_1 \prec x_2$ if $x_2 \leq e$ and $x_1 \parallel x_2$.*

(vi) *$x_1 \prec x_2$ if $x_2 \geq e$ and $x_1 \parallel x_2$.*

(vii) *The least element of* $(L, \preceq)$ *is* $0 \curlywedge 1 \in \{0, 1\}$.

Proof. (i) Assume that $x_2 \prec x_1$, then we have $x_2 \prec x_1 \preceq e$ from Theorem 3.2(iii). Thus the monotonicity of $\curlywedge$ yields that $x_2 = x_1 \curlywedge x_2 \leq x_1 \curlywedge e = x_1$, which contradicts with $x_1 < x_2 \leq e$.

(ii) The proof is similar to that of (i).

(iii) If $x_2 \prec x_1$ and $x_2 > e$, then $x_1 \curlywedge x_2 = x_2$ and the monotonicity of $\curlywedge$ yields $x_1 = x_1 \curlywedge e \leq x_1 \curlywedge x_2 = x_2$. Since $x_2 \prec x_1$ we have $x_1 < x_2$.

(iv) The proof is similar to that of (iii).

(v) From the monotonicity of $\curlywedge$ it follows $x_1 \curlywedge x_2 \leq x_1 \curlywedge e = x_1$, which together with $x_1 \parallel x_2$, gives $x_1 \curlywedge x_2 = x_1$ and hence $x_1 \prec x_2$.

(vi) The proof is similar to that of (v).

(vii) [11, Lemma 4.2] proved that $0 \curlywedge 1$ is the least element of $(L, \preceq)$. Since $\curlywedge$ is internal, we have $0 \curlywedge 1 \in \{0, 1\}$, implying (vii). □

Example 3.4. *Consider a bounded lattice* $L = \{0, a, e, b, 1\}$ *with Hasse diagram shown in Figure 2. Next, we show all*

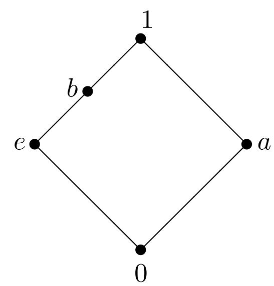

Figure 2: The lattice L from Example 3.4.

possible linear orders $\preceq$ *on* L *which induce internal uninorms on* L *with neutral element* e *by* $U(x, y) = x \curlywedge y$.

By Theorem 3.2 and Corollary 3.3 (ii), (vi) and (vii) we obtain that $\top = e$ *is the greatest element of* $(L, \preceq)$ *and*

$$1 \prec b, \quad a \prec b, \quad \bot \in \{0, 1\}, \tag{2}$$

where $\bot$ *denotes the least element of* $(L, \preceq)$.

1. *There are two possible linear orders* $\preceq$ *on* L *with property* (2)*, the least element* $\bot = 0$ *and the greatest element* $\top = e$*:*

   1) $0 \prec a \prec 1 \prec b \prec e$. *In this case the corresponding function* U_1 *is given by*

U_1	0	a	e	b	1
0	0	0	0	0	0
a	0	a	a	a	a
e	0	a	e	b	1
b	0	a	b	b	1
1	0	a	1	1	1

   Clearly, U_1 *is increasing and hence* U_1 *is an internal uninorm on* L.

   2) $0 \prec 1 \prec a \prec b \prec e$. *In this case the corresponding function* U_2 *is given by*

U_2	0	a	e	b	1
0	0	0	0	0	0
a	0	a	a	a	1
e	0	a	e	b	1
b	0	a	b	b	1
1	0	1	1	1	1

   Plainly, U_2 *is increasing and hence* U_2 *is an internal uninorm on* L.

2. *There are three possible linear orders* $\preceq$ *on* L *with property* (2)*, the least element* $\bot = 1$ *and the greatest element* $\top = e$*:*

   3) $1 \prec a \prec b \prec 0 \prec e$. *In this case the corresponding function* U_3 *is given by*

U_3	0	a	e	b	1
0	0	a	0	b	1
a	a	a	a	a	1
e	0	a	e	b	1
b	b	a	b	b	1
1	1	1	1	1	1

   Since $U_3(b, 0) = b \parallel a = U_3(b, a)$, *we have that* U_3 *is not an internal uninorm on* L.

   4) $1 \prec a \prec 0 \prec b \prec e$. *In this case the corresponding function* U_4 *is given by*

U_4	0	a	e	b	1
0	0	a	0	0	1
a	a	a	a	a	1
e	0	a	e	b	1
b	0	a	b	b	1
1	1	1	1	1	1

   Clearly, U_4 *is increasing and hence* U_4 *is an internal uninorm on* L.

   5) $1 \prec 0 \prec a \prec b \prec e$. *In this case the corresponding function* U_5 *is given by*

U_5	0	a	e	b	1
0	0	0	0	0	1
a	0	a	a	a	1
e	0	a	e	b	1
b	0	a	b	b	1
1	1	1	1	1	1

   Plainly, U_5 *is increasing and hence* U_5 *is an internal uninorm on* L.

Summarizing, there are only four internal uninorms on L *with neutral element* e*, namely,* U_1*,* U_2*,* U_4 *and* U_5.

By Proposition 2.3, we can see that there exist bounded lattices on which internal uninorms cannot be defined. Below, we will discuss the structure of the underlying bounded lattices which allow (or do not allow) the definition of internal uninorms.

For any bounded lattice $(L, \leq, 0, 1)$ and $e \in L\backslash\{0, 1\}$, $L = [0, e] \cup [e, 1] \cup I_e$. Suppose U is an internal uninorm with neutral element e on L. Then $([0, e] \cup [e, 1], \leq)$ is a chain. Hence, $U \mid_{([0,e]\cup[e,1])^2}$ is an internal uninorm on the chain $([0, e] \cup [e, 1], \leq)$. Therefore, we start with the simplest case, i.e., $(L, \leq)$ is a chain.

Theorem 3.5. *Let* $(L, \leq)$ *be a chain,* $e \in L\backslash\{0, 1\}$ *and let* $\preceq$ *be a linear order on* L. *Then the meet operation* $\curlywedge$ *of* $\preceq$ *is an internal uninorm with the neutral element* e *on* $(L, \leq)$ *if and only if, for all* $x, y \in L$,

(i) $x \prec y$ *if* $x < y \leq e$.

(ii) $x \prec y$ *if* $e \leq y < x$.

Proof. The necessity follows from Corollary 3.3(iii) and (iv).

Conversely, suppose that items (i) and (ii) hold. Consider $x, y, z \in L$.

- Suppose $y < z$ and $y \prec x \prec z$. Then by (ii) there is $y < e$ because $(L, \leq)$ is a chain and $e \leq y$ implies $z \prec y$, which is a contradiction. If $e \leq x$, then $y < x$. If $x < e$, then $y < x$ by $y \prec x$, (i) and the fact that $(L, \leq)$ is a chain.

- Suppose $y < z$ and $z \prec x \prec y$. Then $e < z$ for (i) and $(L, \leq)$ is a chain. If $x \leq e$, then $x < z$. If $e < x$, then $x < z$ by $z \prec x$, (ii) and the fact that $(L, \leq)$ is a chain.

- Since $(L, \leq)$ is a chain we have either $x < e$, or $x = e$, or $x > e$, which together with (i) and (ii), produces $x \preceq e$.

The result now follows from Theorem 3.2. □

From the previous theorem, we can see that on any chain, we can always define an internal uninorm. For $L = [0, 1]$, the previous result retrieves the already known result [10, Proposition 3].

Finally, we present a class of bounded lattices on which an internal uninorm cannot be defined.

Theorem 3.6. *Given a bounded lattice $(L, \leq, 0, 1)$ and $e \in L\backslash\{0, 1\}$, if there are elements $x, y, s, m \in L$ such that $s \in]0, e]$, $y \in [e, 1[$, $x, m \in I_e$, $s < m$, $m \| y$, $x < y$ and $x \| s$, there is no internal uninorm U on L with neutral element e.*

Proof. Suppose that there exists an internal uninorm U on L with neutral element e. Then $U(x, y) = x \curlywedge^U y$ by (1). By $m \| y$ and $y \geq e$, Corollary 3.3(viii) yields $m \prec^U y$ and hence $U(m, y) = m$. From $x \| s$, $s \leq e$ and Corollary 3.3(vii), it follows $x \prec^U s$. If $s \prec^U y$, then $x \prec^U s \prec^U y$, which, with $x < y$ and Theorem 3.2(i), yields $x < s$, contradicting $x \| s$; if $y \prec^U s$, then $U(s, y) = y \| m = U(m, y)$, contradicting the monotonicity of U. Hence, there is no internal uninorm U on L with neutral element e. □

Remark 3.7. *Theorem 3.6 clearly generalizes* [4, Theorem 3.11], *from which we deduce that if there exists some internal uninorm U on a bounded lattice L with the neutral element $e \in L\backslash\{0, 1\}$, then L has no sublattice with Hasse diagram shown in Figure 1.*

4 Idempotent uninorms on bounded lattices

In this section we again assume that $(L, \leq, 0, 1)$ is a bounded lattice. When the underlying lattice is a chain, idempotent and internal uninorms coincide. However, in general it is not true when the underlying lattice is not a chain. In this section we will focus on a case when $(L, \leq)$ is not a chain and we will generalize results from the previous section for idempotent uninorms.

Similarly as Corollary 3.3 we can show the following result that show the relation of the lattice order $\leq$ and the linear order $\preceq$.

Proposition 4.1. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L\backslash\{0, 1\}$ and let $(L, \preceq)$ be a meet-semilattice. If the meet operation $\curlywedge$ of $\preceq$ is an idempotent uninorm with the neutral element e on $(L, \leq)$ then for any $x_1, x_2 \in L$,*

(i) $x_1 \preceq e$.

(ii) $x_1 \prec x_2$ *if* $x_1 < x_2 \leq e$.

(iii) $x_1 \prec x_2$ *if* $x_1 > x_2 \geq e$.

(iv) $x_1 < x_2$ *if* $x_2 \prec x_1$ *and* $x_2 > e$.

(v) $x_1 > x_2$ *if* $x_2 \prec x_1$ *and* $x_2 < e$.

(vi) $x_1 \prec x_2$ *or* $x_1 \|_{\preceq} x_2$ *if* $x_2 \leq e$ *and* $x_1 \| x_2$.

(vii) $x_1 \prec x_2$ *or* $x_1 \|_{\preceq} x_2$ *if* $x_2 \geq e$ *and* $x_1 \| x_2$.

(viii) *The least element of* $(L, \preceq)$ *is* $0 \curlywedge 1$.

Proof. The proof is similar to that of Corollary 3.3. □

Note that in the case of idempotent uninorms $0 \curlywedge 1 \in \{0, 1\}$ does not hold, in general. We can observe this fact in the following example which is based on [11, Example 5.5].

Example 4.2. *Consider a bounded lattice $L = \{0, a, e, 1\}$ with Hasse diagram shown in Figure 3. Then for an idempotent uninorm U given by*

U	0	a	e	1
0	0	a	0	a
a	a	a	a	a
e	0	a	e	1
1	a	a	1	1

The corresponding partial order $\preceq^U$ on L is given by Hasse diagram shown in Figure 4.

In this case a is the annihilator of U and e is the neutral element of U. Observe that U is both an idempotent uninorm with the neutral element e and an idempotent nullnorm with the annihilator a.

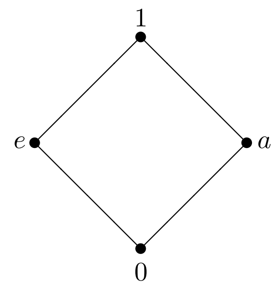

Figure 3: The lattice L from Example 4.2.

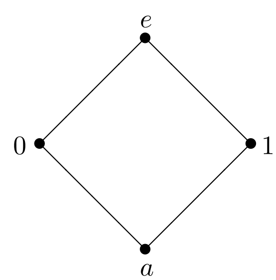

Figure 4: The partial order $\preceq^U$ on L from Example 4.2.

4.1 The case $I_e = \emptyset$

Let $(P, \leq_1)$ and $(Q, \leq_2)$ be two disjoint posets. Then the *linear sum* $P \oplus Q$ [5] is defined by taking the following relation on $P \cup Q$:

$$x \leq_3 y \text{ if and only if } \begin{cases} \text{either } x, y \in P \text{ and } x \leq_1 y, \\ \text{or } x, y \in Q \text{ and } x \leq_2 y, \\ \text{or } x \in P \text{ and } y \in Q. \end{cases} \tag{3}$$

Let $(L, \leq, 0, 1)$ be a bounded lattice and $e \in L \backslash \{0, 1\}$. If $I_e = \emptyset$, then $L = P \oplus Q$, where $P = ([0, e], \leq)$ and $Q = (]e, 1], \leq)$. Finally, we present a characterization of idempotent uninorms on such a special bounded lattice.

Proposition 4.3. *Let $(L, \leq, 0, 1)$ be a bounded lattice, $e \in L \backslash \{0, 1\}$ be an element such that $I_e = \emptyset$ and let $(L, \preceq)$ be a meet-semilattice. Then the meet operation $\curlywedge$ of $\preceq$ is an idempotent uninorm with the neutral element e on $(L, \leq)$ if and only if, for all $x, y \in L$,*

(i) *$x \prec y$ if $x < y \leq e$.*

(ii) *$x \prec y$ if $e \leq y < x$.*

(iii) *x is comparable with y in $\leq$ if and only if x is comparable with y in $\preceq$.*

Proof. Let U be an idempotent uninorm. Then items (i) and (ii) follows from Proposition 4.1(ii) and (iii), respectively. If $x, y \in [0, e]$ or $x, y \in [e, 1]$ then item (iii) follows from items (i) and (ii). [2, Proposition 7] proved that $x \curlywedge y \in \{x \wedge y, x \vee y\}$ for all $x, y \in L$. If $(x, y) \in [0, e] \times [e, 1]$ (and similarly for $(x, y) \in [e, 1] \times [0, e]$) then $x \leq y$ and $x \curlywedge y \in \{x, y\}$ which means that x and y are comparable in $\preceq$. Thus, item (iii) holds.

Conversely, suppose that items (i)–(iii) hold. Then $\curlywedge$ is commutative, associative and idempotent. From (i) and (ii), we know that $\curlywedge|_{[0,e]^2} = \wedge$, $\curlywedge|_{[e,1]^2} = \vee$ and e is the greatest element of $(L, \preceq)$, implying that e is the neutral element of $\curlywedge$. In order to show the monotonicity of $\curlywedge$, we only have to show the monotonicity on $[0, e] \times [e, 1] \cup [e, 1] \times [0, e]$. We focus on $[0, e] \times [e, 1]$ as the other case is similar. It is enough to show that for $x \in [0, e]$ and $y_1, y_2 \in [e, 1]$ with $y_1 < y_2$ there is $x \curlywedge y_1 \leq x \curlywedge y_2$. From (ii), we then have $y_2 \prec y_1$. Item (iii) yields that x is comparable in $\preceq$ with y_1 and with y_2.

- If $y_2 \prec y_1 \prec x$, then $x \curlywedge y_1 = y_1 < y_2 = x \curlywedge y_2$.
- If $y_2 \prec x \prec y_1$, then $x \curlywedge y_1 = x < y_2 = x \curlywedge y_2$.
- If $x \prec y_2 \prec y_1$, then $x \curlywedge y_1 = x = x \curlywedge y_2$.

Thus in all cases $x \curlywedge y_1 \leq x \curlywedge y_2$ and therefore $\curlywedge$ is increasing. Summarizing, $\curlywedge$ is an idempotent uninorm. □

4.2 The case $I_e = \{w\}$

We will denote by $\mathcal{L}_e^w$ the set of all bounded lattices L such that for some $e, w \in L \setminus \{0, 1\}$ there is $I_e = \{w\}$.

Lemma 4.4. [12, Lemma 12] *Let $L \in \mathcal{L}_e^w$ and let $U : L^2 \to L$ be an idempotent uninorm with the neutral element e.*

(i) *If $U(x, w) = w = U(y, w)$ for some $x, y \in L$ with $x < w < y$, then $U(x, y) = w$.*

(ii) *If $U(x, y) = w$ for some $x, y \in L \setminus \{w\}$, then either $x < w < y$ or $x > w > y$.*

(iii) *If $U(x, y) = e$ for some $x, y \in L$, then $x = y = e$.*

Proof. Assume any $x, y \in L$. Due to the monotonicity and the idempotency of U we get $x \wedge y = U(x \wedge y, x \wedge y) \leq U(x, y) \leq U(x \vee y, x \vee y) = x \vee y$.

(i) If $U(x, w) = w = U(y, w)$ for some $x, y \in L$ with $x < w < y$ then the monotonicity of U implies $w = U(x, w) \leq U(x, y) \leq U(w, y) = w$, i.e., $U(x, y) = w$.

(ii) If $U(x, y) = w$ for some $x, y \in L \setminus \{w\}$, then $e \notin \{x, y\}$ and $(x, y) \in L^2 \setminus ([0, e]^2 \cup [e, 1]^2)$. If $x < e < y$, then $x \leq w \leq y$, i.e., $x < w < y$. If $x > e > y$, then $x \geq w \geq y$, i.e., $x > w > y$.

(iii) If $U(x, y) = e$, then $e = U(x, y) = U(U(x, x), y) = U(x, U(x, y)) = U(x, e) = x$, i.e., $x = e$ and $e = U(e, y) = y$ implies $y = e$.

□

Proposition 4.5. *$L \in \mathcal{L}_e^w$ and let $(L, \preceq^*)$ be a meet semi-lattice. Then the meet operation $\curlywedge^*$ of $\preceq^*$ is an idempotent uninorm with the neutral element e on $(L, \leq)$ if and only if, for all $x, y \in L$,*

(i) *$x \prec^* y$ if $x < y \leq e$.*

(ii) *$x \prec^* y$ if $e \leq y < x$.*

(iii) *If $x \curlywedge^* y \neq w$, then x and y are comparable in $\leq$ if and only if x and y are comparable in $\preceq^*$.*

(iv) *There is $y \curlywedge^* w \leq w$ for $y \leq e$ and $y \curlywedge^* w \geq w$ for $y \geq e$.*

(v) *If $w \preceq^* x \curlywedge^* y$, $w \notin \{x, y\}$ and $e \in]x \wedge y, x \vee y[$, then $x \notin I_w$ implies $x \curlywedge^* y \neq y$ and $x \in I_w$ implies $x \curlywedge^* y \neq w$.*

Proof. Let U be an idempotent uninorm.

(i) Follows from Proposition 2.2.

(ii) Follows from Proposition 2.2.

(iii) Due to Proposition 2.2 there is $x \curlywedge^* y = U(x, y) \in \{x \wedge y, x \vee y, w\}$ for all $x, y \in L$. If x is comparable with y in $\leq$ then $x \curlywedge^* y \in \{x, y, w\}$, i.e., either x and y are comparable in $\preceq^*$ or $x \curlywedge^* y = w$. If x is comparable with y in $\preceq^*$, i.e., $U(x, y) \in \{x, y\}$ and $x \| y$, then $x \wedge y \notin \{x, y\}$, $x \vee y \notin \{x, y\}$ and therefore, $x \curlywedge^* y = U(x, y) = w \in \{x, y\}$.

(iv) The monotonicity of U gives us $U(w, y) \leq U(w, e) = w$ for $y \leq e$ and $U(w, y) \geq U(w, e) = w$ for $y \geq e$.

(v) If $w \notin \{x, y\}$ and $e \in]x \wedge y, x \vee y[$, then either $x < e < y$ or $y < e < x$. We will suppose $x < e < y$ since the proof for $y < e < x$ is analogous. Now $w \preceq^* x \curlywedge^* y$ gives us $U(w, U(x, y)) = w$ and by the associativity and the idempotency of U we get $U(x, w) = w$ and $U(y, w) = w$. Then $x \notin I_w$ implies $x < w$ and due to the monotonicity of U there is $U(x, y) \leq U(w, y) = w$ and thus $U(x, y) = y$ gives us $e < y = U(x, y) \leq w$, which is a contradiction. If $x \in I_w$, then Lemma 4.4(ii) implies $U(x, y) \neq w$.

Conversely, suppose that items (i)–(v) hold. Then $\curlywedge^*$ is commutative, associative and idempotent. From (i) and (ii), we know that $\curlywedge^*|_{[0,e]^2} = \wedge$, $\curlywedge^*|_{[e,1]^2} = \vee$ and item (iv) shows that $w \curlywedge^* e \geq w$ and $w \curlywedge^* e \leq w$, i.e., $w \curlywedge^* e = w$ and $w \prec^* e$. Therefore e is the greatest element of $(L, \preceq^*)$, implying that e is the neutral element of $\curlywedge^*$. Now we will show that $x \curlywedge^* y \in \{x \wedge y, x \vee y, w\}$ for all $x, y \in L$.

1. If $x, y \in [0, e]$, then $x \curlywedge^* y = x \wedge y$.
2. If $x, y \in [e, 1]$, then $x \curlywedge^* y = x \vee y$.
3. If $x < e < y$ or $y < e < x$, then x is comparable with y in $\leq$ and item (iv) implies $x \curlywedge^* y \in \{x, y, w\}$.
4. If $x = y = w$, then the claim evidently holds.
5. If $e \in \{x, y\}$, then the claim is obvious.
6. If $x = w$ and $y < w$, then x and y are comparable, i.e., by item (iii) $x \curlywedge^* y \in \{y, w\}$.
7. If $x = w$ and $y \in I_w \cap [0, e[$, then if y is comparable with w in $\preceq^*$, then item (iii) implies $x \curlywedge^* y = w$. If $y \|^* w$, then item (i) implies $y \wedge w \prec^* y$ and since $y \wedge w$ is comparable with w in $\leq$ the Case 6. gives us $(y \wedge w) \curlywedge^* w \in \{w, y \wedge w\}$.
   - If $(y \wedge w) \curlywedge^* w = w$, then $w \prec^* y \wedge w \prec^* y$ and $x \curlywedge^* y = w$.
   - If $(y \wedge w) \curlywedge^* w = y \wedge w$, then assume that $y \curlywedge^* w = z \neq w$. Here item (iv) implies that $z < w$, i.e., $z < e$ and since $z \curlywedge^* y = z$, while $\curlywedge^*|_{[0,e]^2} = \wedge$, we get $z < y$. Therefore, $z \leq w \wedge y$ and then item (i) implies $z \preceq^* y \wedge w \preceq^* y \curlywedge^* w = z$, i.e., $x \curlywedge^* y = y \wedge w$.
8. If $x = w$ and $y > e$ we can analogously as in 6. and 7. obtain that $x \curlywedge^* y \in \{y \vee w, w\}$.

Further, if $x \curlywedge^* y = w$ for some $x \leq y$ with $w \notin \{x, y\}$, then $\curlywedge^*|_{[0,e]^2} = \wedge$ and $\curlywedge^*|_{[e,1]^2} = \vee$ imply that $x < e < y$. If $x \in I_w$ or $y \in I_w$, then item (v) implies $x \curlywedge^* y \neq w$, which is a contradiction. Therefore $x < w < y$.

What remains is to show the monotonicity. Assume $x, y, z \in L$, $y < z$. If $x = e$, then the claim evidently holds. If $x, y, z \in [0, e]$ or $x, y, z \in [e, 1]$, then the monotonicity follows from the monotonicity of $\wedge$ and $\vee$. Otherwise we have the following possibilities.

1. If $x < e$ and $z = w$, then $y < w$ implies $y < e$ and we get $x \curlywedge^* y = x \wedge y$ and $x \curlywedge^* z \in \{x \wedge w, w\}$ and $x \wedge y \leq x \wedge w < w$ ensures that the monotonicity holds.
2. If $x < e$,$z > e$ and $y > e$, then $x \curlywedge^* y \in \{x, y, w\}$ and $x \curlywedge^* z \in \{x, z, w\}$ and item (ii) implies $z \prec^* y$. Then $x \curlywedge^* z \preceq^* x \curlywedge^* y$.
   (a) If $x \curlywedge^* y = x$, then the monotonicity can be violated only if $x \curlywedge^* z = w$ in which case item (v) implies $x < w$, i.e., $x \curlywedge^* y = x < w = x \curlywedge^* z$.
   (b) If $x \curlywedge^* y = y \neq x$, then $z \prec^* y \prec^* x$ implies $x \curlywedge^* z = z$ and the monotonicity holds.
   (c) If $x \curlywedge^* y = w$, then item (v) implies $x < w < y < z$ and the monotonicity can be violated only if $x \curlywedge^* z = x$. Here, however, $x \preceq^* z \prec^* y$ implies $x \curlywedge^* y = x$, which is a contradiction.
3. If $x < e$ and $w = y < z$, then $x \curlywedge^* y \in \{x \wedge w, w\}$ and $x \curlywedge^* z \in \{x, z, w\}$
   (a) If $x \curlywedge^* y = x \wedge w$, then the monotonicity obviously holds.
   (b) If $x \curlywedge^* y = w$, then the monotonicity can be violated only if $x \curlywedge^* z = x$. Here, however, $y = w \prec^* x = x \curlywedge^* z$ and $z \notin I_w$, $e < z$ and therefore, item (v) implies $x \curlywedge^* z \neq x$, which is a contradiction.
4. If $x < e$,$z > e$ and $y \leq e$, then $x \curlywedge^* y = x \wedge y$, $x \curlywedge^* z \in \{x, z, w\}$ and the monotonicity can be violated only if $x \curlywedge^* z = w$ which implies $x < w$, i.e., $x \curlywedge^* y = x \wedge y \leq w = x \curlywedge^* z$.
5. If $x = w$
   (a) If $y, z \in [0, e]$, then $x \curlywedge^* y \in \{w, y \wedge w\}$ and $x \curlywedge^* z \in \{w, z \wedge w\}$ and the monotonicity can be violated only if $x \curlywedge^* y = w$ and $x \curlywedge^* z = z \wedge w$. Here, however item (i) implies $y \prec^* z$ and thus $w \prec^* y \prec^* z$ gives us $x \curlywedge^* z = w$, which is a contradiction.
   (b) If $y, z \in [e, 1]$, then $x \curlywedge^* y \in \{w, y \vee w\}$ and $x \curlywedge^* z \in \{w, z \vee w\}$ and the monotonicity can be violated only if $x \curlywedge^* y = y \vee w$ and $x \curlywedge^* z = w$. Here, however item (ii) implies $z \prec^* y$ and thus $w \prec^* z \prec^* y$ gives us $x \curlywedge^* y = w$, which is a contradiction.

(c) If $y < e < z$, then $x \curlywedge^* y \in \{w, y \wedge w\}$ and $x \curlywedge^* z \in \{w, z \vee w\}$ and the monotonicity clearly holds.

(d) If $w = y < z$, then $x \curlywedge^* y = w$ and $x \curlywedge^* z \in \{w, z \vee w\}$, i.e., the monotonicity holds.

(e) If $y < z = w$, then $x \curlywedge^* z = w$ and $x \curlywedge^* y \in \{w, y \wedge w\}$, i.e., the monotonicity holds.

6. If $x > e$, then the monotonicity can be shown similarly as in items 1., 2., 3. and 4.

□

Remark 4.6. *In the case that for a bounded lattice L and $e \in L \setminus \{0, 1\}$ the cardinality of the set I_e is finite the meet operation $\curlywedge$ related to an idempotent uninorm on L can be characterized similarly as in the case when $|I_e| = 1$. However, with the growing number of elements in I_e there are more and more conditions that ensure the monotonicity of $\curlywedge$. For example, in the case when $I_e = \{x_1, x_2, \cdots, x_n\}$ we have $U(x, y) \in \{x \wedge y, x \vee y, x_1, x_2, \cdots, x_n\}$ by Proposition 2.2 and in order to describe the meet-semilattice order $\preceq^U$ associated with an idempotent uninorm U, it is necessary to consider different orders among x_i, $i = 1, 2, \cdots, n$.*

5 Conclusions and future work

In this paper, we have presented characterizations of three classes of idempotent uninorms on a bounded lattice by the orders of their associated meet-semilattices. The first one is the class of internal uninorms, the second one is the class of idempotent uninorms defined on a lattice in which all elements are comparable with the corresponding neutral element and the third one is the class of idempotent uninorms defined on a lattice in which a single point is incomparable with the corresponding neutral element. The results obtained in this paper partially answer this open question presented by Devillet and Teheux [7], that is, "Find characterizations for semilattice operations that are $\leq$-preserving on a poset $(L, \leq)$."

Theorem 3.5 shows that internal uninorms on a chain L are in one-to-one correspondence with linear orders $\preceq$ on L with properties (i) and (ii). We denote by $\mathcal{L}_e$ the set of all linear orders on L that fulfill these conditions. If $\preceq \in \mathcal{L}_e$ then condition (i) completely determines $\preceq$ on $[0, e]^2$ and condition (ii) completely determines $\preceq$ on $[e, 1]^2$. However, on $[0, e] \times [e, 1] \cup [e, 1] \times [0, e]$ linear order $\preceq$ is not specified. In the future, we will focus on a complete characterization of the corresponding linear order $\preceq \in \mathcal{L}_e$.

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