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Original paper

Finite-time stability results for fuzzy fractional stochastic delay system under Granular differentiability concept

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Abstract

In present manuscript, we investigate a new type of fuzzy fractional stochastic delay system (FFSDS), in which the derivative is defined by Granular differentiability. We first transform the considered system into an equivalent integral system with the aid of fuzzy Laplace transformation and its inverse involving Mittag-Leffler function. Subsequently, existence and uniqueness results of the solutions for FFSDS are derived by applying Carathéodory approximation, under non-Lipschitz conditions, and contradiction method, respectively. In addition, we establish the finite-time stability of the system by utilizing the generalized Grönwall delay inequality. Finally, the obtained conclusions are expound via an example.

Keywords: Fuzzy differential equation, fractional stochastic differential equation, Carathéodory approximation, existence and uniqueness, finite-time stability.

1 Introduction

Since the groundbreaking work of Zadeh [42], the theory of fuzzy set was established and developed rapidly in the following decades. In real application, the fuzzy set theory now plays a crucial role to describe confusion and uncertainty that appear due to various kinds of errors and imprecisions. In addition, how to assign or determine the initial value of differential system is difficult, and a useful method is to combine fuzzy set theory. Therefore, fuzzy differential system has been investigated extensively by scholars [2, 19, 41].

Fractional calculus also developed rapidly due to its heredity properties and more accurate than integer order. Actually, fractional derivative is a convolution operator with non-locality, and its defined integral term reflects the dependence of system development. This characteristic determines the memorability and long-range correlation of fractional derivatives. Until now, there exist various kinds of fractional derivative, like the Riemann-Liouville type and Caputo type and so on. In [9], Atangana et al. proposed a new kind of Atangana-Baleanu fractional derivative which has a non-singular and nonlocal kernel compared with the derivatives mentioned above. In [17], a new type of conformable fractional derivative was researched. Fractional differential system have been applied to the modeling of various practical problems and a great number of researching results [36] and monographs [18, 46] are available for reading.

Fuzzy fractional differential system involving Hukuhara differentiability has a lot of results. Salahshour et al. is the first one that applying fuzzy Laplace transform to deal with the solutions of fuzzy fractional differential system in [35]. In the sense of Goetschel-Voxman differentiability, Arshad [8] studied the existence and uniqueness results by applying fixed point theorem. Wang et al. [41] studied a kind of Caputo fractional fuzzy differential delay system by utilizing Schauder fixed point theorem and contraction mapping principle under non-Lipschitz condition. Ngo [32] researched two solutions of fuzzy fractional differential and integral system, respectively. Vu et al. [40] investigated

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fractional integral equations involved ψ-kernel with the aid of Picard approximation method. For more detail about fuzzy fractional system, please refer to [4, 10].

In 2018, Mazandarani et al. [28] researched a new kind of fuzzy derivative named Granular derivative. This derivative overcomes the drawbacks of Hukuhara derivative. We know that there exist multiplicity of solutions and unnatural behavior in modelling phenomenon in the sense of Hukuhara differentiability. Therefore, we are going to study fuzzy differential system in the sense of Granular differentiability in present manuscript. In recent years, several results have been established for fuzzy fractional differential system under Granular differentiability, in the papers of [5, 11, 15, 43, 44] and references therein. By applying Elzaki transform homotopy perturbation method, Zhang et al. [44] researched the fractional fuzzy option pricing model based on Granular differentiability. With the aid of Laplace transformation, Dong et al. [11] construct the explicit formula of mild solutions to time-delay fuzzy fractional differential systems under Granular computing, and then the existence, uniqueness and finite-time stability results were established. Hati et al. [15] researched the production inventory optimal control problem under Granular Caputo fractional fuzzy derivative. Under the concept of Granular differentiability and Granular convex fuzzy functions, Zhang et al. [43] researched the fuzzy optimization problem. With the approach of the Granular Caputo fractional derivatives, An et al. [5] established the finite-time stability for a class of non-instantaneous impulsive fuzzy differential equations. Readers can refer [29, 30, 37, 39] for more knowledge.

Fuzzy stochastic differential system has been widely used in the modeling of engineering, finance and population increase and so on. Lupulescu et al. [21] studied two solutions of fractional fuzzy random integral and differential equations by applying Picard successive approximation. In [26, 27], Malinowski et al. researched fuzzy stochastic system by using successive approximation. Under local Lipschitz Condition, Malinowski [23] investigated the local uniqueness theorem of fuzzy stochastic system. Michta [31] applied the theorem of Negoita and Ralescu to establish the existence of a fuzzy solution. Under non-Lipschitz Condition, Fei et al. [13, 14] researched fuzzy stochastic system by applying Picard iteration and the well-known Cauchy-Maruyama approximation, respectively. One can refer papers [24, 25] for more information.

To the best of our knowledge, fuzzy fractional stochastic differential equations have not been studied perfectly, and there are few results up to now. With the aid of Picard iteration method, Arhrrabi et al. [6] and Jafari et al. [16] studied two kinds of fuzzy stochastic system driven by fractional Brownian motion, respectively. Priyadharsini et al. [33] investigated a kind of impulsive fractional fuzzy stochastic system under the sense of Granular differentiability, in which existence and uniqueness results were deduced by using contraction mapping principle. In [34], a seminal work was established, in which solution form involving Mittag-Leffler function was expressed under the concept of Granular differentiability, and the uniqueness of solution was obtained by using contraction mapping principle. In [7], Arhrrabi et al. researched a class of coupled fuzzy fractional Pantograph stochastic system, in which the uniqueness result also obtained by applying contraction mapping principale.

Finite-time stability of fuzzy fractional system is also an important topic. Different from Lyapunov stability, finite-time stability analysis is the state of the system within a finite time interval. Du et al. [12] derived a new type of generalized Grönwall delay inequality, and then established the finite-time stability result of fuzzy fractional cellular neural network. Tyagi et al. [38] investigated a kind of proportional delay fuzzy fractional neural networks, and the finite-time stability result was deduced by differential inequality techniques and several simple algebraic inequalities. In [45], with the aid of Grönwall-Bellman inequality, finite-time stability result of memristor-based fractional fuzzy cellular neural networks was established. In [3], authors derived finite-time stability result by applying Hölder inequality and the generalized Bernoulli inequality. Meanwhile, there exist many results about finite-time stability of fuzzy integer order stochastic system. Based on what we know, there are few results on finite-time stability of fuzzy fractional stochastic system. In this manuscript, we are going to investigate the finite-time stability of a new type of Pantograph fractional system with uncertainties.

From the discussion above, we observe that the existence and uniqueness results can be obtained by applying various fixed point theorems [33, 34, 36, 41], Cauchy Maruyama type approximation [14] and Picard approximation method [21, 40]. To the best of we known, there are few references of applying Carathéodory approximation to deal FFSDS. Inspired by [1, 22, 33, 34] and the discussion above, in present paper, we will investigate a new type of FFSDS, in which the derivative is defined by Granular differentiability. The primary contributions and novelties of present manuscript are as below:

(i) Representation form of solution in terms of Mittag-Leffler function for FFSDS is established in the sense of Granular differentiability, and there exist few literatures that use this technique to solve FFSDS.

(ii) Compared with [34] and Picard approximation method [21, 40], The existence and uniqueness for FFSDS are established by applying Carathéodory approximation under the weaker conditions, and according to all the studies we known that there exist few papers available to handle FFSDS by using this method. Therefore, the analysis method and the derived results are essentially new.

(iii) We point out the definition of finite-time stability of FFSDS, and then a new kind of generalized Grönwall delay inequality is constructed to investigate the finite-time stability, and this is the first time to study this kind stability of FFSDS. In addition, the delay term is well handled by the generalized Grönwall delay inequality.

(iv) In present manuscript, we use the Granular derivative, which overcomes all these six drawbacks of other approaches that are based on fuzzy standard interval arithmetic and then makes it possible for us to solve the system by utilizing the Laplace transformation and its inverse.

The structure of present manuscript is listed as following aspects. In Section 2, some basic concepts, useful lemmas and the focused problem are introduced. In Section 3, we introduce some information about the six drawbacks of other approaches that are based on fuzzy standard interval arithmetic. In Section 4, we first give an equivalent integral expression of FFSDS, and then the existence and uniqueness results for FFSDS are established. Subsequently, finite-time stability of FFSDS is investigated. As an application, an example is shown to expound the correctness of the derived main results in Section 5.

2 Preliminaries and model formulation

In current part, some basic concepts and useful lemmas are presented. In addition, the considered model is given.

2.1 Essential definitions and lemmas

Definition 2.1. [33] *Let $\mathcal{F}\left(\Re^d\right) \subset \Re$ denotes fuzzy subsets, and defined by $\nu : \Re^d \to [0,1]$ meet the following rules:*
(1) $\exists \mathscr{X}_0 \in \Re^d$ *satisfying* $\nu\left(\mathscr{X}_0\right) = 1$.
(2) ν *is convex, i.e* $\min\{\nu(\mathscr{X}) + \nu(\mathscr{Y})\} \leq \nu(\lambda\mathscr{X} + (1-\lambda)\mathscr{Y})$, $\forall \mathscr{X}, \mathscr{Y} \in \Re^d$, *where* $\lambda \in [0,1]$.
(3) $\nu \subset \Re^d$ *is upper semi-continuous.*
(4) *The closure cl* $\left\{\mathscr{X} \in \Re^d : \nu(\mathscr{X}) > 0\right\}$ *is compact.*
The space $\mathcal{F}\left(\Re^d\right)$ *is called fuzzy numbers, and obviously* $\Re^d \subset \mathcal{F}\left(\Re^d\right)$.

Define an embed $\langle\cdot\rangle : \Re^d \to \mathcal{F}\left(\Re^d\right)$ as:

$$\langle r\rangle(b) = \begin{cases} 1, b = r, \\ 0, b \in \Re^d \backslash \{r\}. \end{cases} \qquad \forall r \in \Re^d.$$

For $0 < \alpha \leq 1$, $[\nu]^\alpha = \left\{\mathscr{X} \in \Re^d; \nu(\mathscr{X}) \geq \alpha\right\}$, and $[\nu]^0 = \mathrm{cl}\left\{\mathscr{X} \in \Re^d; \nu(\mathscr{X}) > 0\right\}$. The parametric form $[\nu]^\alpha = [\nu_l^\alpha, \nu_r^\alpha]$ is the α-level set of ν, where ν_l, ν_r are called the left and right end points of $[\nu]^\alpha$, respectively.

Definition 2.2. [33] $\forall \mathscr{X}, \mathscr{Y} \in \mathcal{F}\left(\Re^d\right)$, *the Hausdorff distance is defined by*

$$D(\mathscr{X}, \mathscr{Y}) := \sup_{\alpha \in [0,1]} \max\left\{D\left(\left|\mathscr{X}_l^\alpha - \mathscr{Y}_l^\alpha\right|, \left|\mathscr{X}_r^\alpha - \mathscr{Y}_r^\alpha\right|\right)\right\} = \sup_{\alpha \in [0,1]} \max\left\{D\left([\mathscr{X}]^\alpha, [\mathscr{Y}]^\alpha\right)\right\}.$$

It is easy to verified that $\left(\mathcal{F}\left(\Re^d\right), D\right)$ *is complete and such that the following operations:*
(1) $D(\mathscr{X} + w, \mathscr{Y} + z) = D(\mathscr{X}, \mathscr{Y}) + D(w, z)$, $\forall \mathscr{X}, \mathscr{Y}, z, w \in \mathcal{F}\left(\Re^d\right)$;
(2) $D(k\mathscr{X}, k\mathscr{Y}) = |k| D(\mathscr{X}, \mathscr{Y})$, $\forall \mathscr{X}, \mathscr{Y} \in \mathcal{F}\left(\Re^d\right)$, $k \in \Re$.

Definition 2.3. [28] *Assume that* $\nu : [a,b] \subseteq \mathbb{R} \to [0,1]$ *is a fuzzy number.* $\nu^{\mathscr{G}r} : [0,1] \times [0,1] \to [a,b]$ *is named horizontal membership function and denotes a representation of* $\nu(\mathscr{X})$ *as* $\mathscr{X} = \nu^{\mathscr{G}r}(\alpha, \alpha_\nu)$, *where "$\mathscr{G}r$" stands for the granule of information included in* $\mathscr{X} \in [a,b], \alpha \in [0,1]$ *is the membership degree of* $\mathscr{X}$ *in* $\nu(\mathscr{X})$, $\alpha_\nu \in [0,1]$ *is called relative-distance-measure variable, and* $\nu^{\mathscr{G}r}(\alpha, \alpha_\nu) = \nu_l^\alpha + \left(\nu_r^\alpha - \nu_l^\alpha\right)\alpha_\nu$.

Remark 2.4. *We can also denote the horizontal membership function of* $\nu \in \mathcal{F}\left(\Re^d\right)$ *by* $\mathcal{H}(\nu) \triangleq \nu^{\mathscr{G}\mathrm{r}}(\alpha, \alpha_\nu)$. *In particular, if* $\nu = (a,b,c)$, $a \leq b \leq c$ *is a triangular fuzzy number, then the horizontal membership function of* ν *is* $\mathcal{H}(\nu) = a + (b-a)\alpha + (1-\alpha)(c-a)\alpha_\nu$. *If* $\nu = (a,b,c,d)$, $a \leq b \leq c \leq d$ *is a trapezoidal fuzzy number, then*

$$\mathcal{H}(\nu) = a + (b-a)\alpha + [(d-a) - \alpha(d-a+b-c)]\alpha_\nu.$$

The α-level sets of the vertical membership function of $\nu(\mathscr{X})$ which are the span of the information granule, can be obtained by using

$$\mathcal{H}^{-1}\left(\nu^{\mathscr{G}r}(\alpha,\alpha_\nu)\right)=[\nu]^\alpha=[\inf_{\beta\geq\alpha}\min_{\alpha_\nu}\nu^{\mathscr{G}r}(\beta,\alpha_\nu),\sup_{\beta\geq\alpha}\max_{\alpha_\nu}\nu^{\mathscr{G}r}(\beta,\alpha_\nu)].$$

The horizontal membership function $\mathcal{H}(\nu)$ is similar to a transition from vertical membership functions space to multivariable functions space. Two fuzzy numbers ν and μ is said to be equal if and only if $\mathcal{H}(\nu)=\mathcal{H}(\mu)$ for all $\alpha_\nu=\alpha_\mu\in[0,1]$.

The horizontal membership function is a linear map, i.e., $\mathcal{H}$ satisfies:
(i) $\mathcal{H}(u+v)=\mathcal{H}(u)+\mathcal{H}(v)$;
(ii) $\mathcal{H}(cu)=c\mathcal{H}(u)$, where c is a constant, $u,v\in\mathcal{F}\left(\Re^d\right)$.

Let $f:[a,b]\subseteq\mathbb{R}\to\mathcal{F}\left(\Re^d\right)$ include $n\in\mathbb{N}$ distinct fuzzy numbers $u_1,u_2,\ldots,u_n$. The horizontal membership function of $f(t)$ at the point $t\in[a,b]$ is denoted by $\mathcal{H}(f(t))\triangleq f^{\mathscr{G}r}(t,\mu,\alpha_f)$, and defined as $f^{\mathscr{G}r}:[a,b]\times[0,1]\times[0,1]^n\longrightarrow[c,d]\subseteq\mathbb{R}$ in which $\alpha_f\triangleq(\alpha_{u_1},\alpha_{u_2},\ldots,\alpha_{u_n})$ are the relative-distance-measure variables corresponding to the fuzzy numbers.

Definition 2.5. [28] *For each $\mathscr{X}$, $\mathscr{Y}\in\mathcal{F}\left(\Re^d\right)$ with respective horizontal membership functions denoted by $\mathcal{H}(\mathscr{X})$ and $\mathcal{H}(\mathscr{Y})$, we define*

$$\mathcal{H}(\mathscr{X}\odot\mathscr{Y})\triangleq\mathcal{H}(\mathscr{X})*\mathcal{H}(\mathscr{Y}),$$

where "$\odot$" and "$*$" denote the arithmetic operations in $\mathcal{F}\left(\Re^d\right)$ and $\Re$. It is known that $0\neq\mathscr{Y}^{\mathscr{G}r}(\alpha,\alpha_{\mathscr{Y}})$ when "$\odot$" represents the division operator and the difference, represented by $\ominus^{\mathscr{G}r}$, is called $\mathscr{G}r$-difference.

Definition 2.6. [28] *For each $\mathscr{X}$, $\mathscr{Y}\in\mathcal{F}\left(\Re^d\right)$, the $\mathscr{G}r$-distance is given by*

$${}^{\mathscr{G}r}D(\mathscr{X},\mathscr{Y})=\sup_{\alpha}\max_{\alpha_{\mathscr{X}},\alpha_{\mathscr{Y}}}\left|\mathscr{X}^{\mathscr{G}r}(\alpha,\alpha_{\mathscr{X}})-\mathscr{Y}^{\mathscr{G}r}(\alpha,\alpha_{\mathscr{Y}})\right|.$$

We know that $(\mathcal{F}\left(\Re^d\right),{}^{\mathscr{G}r}D)$ is a complete metric space, according to [28, Theorem 4].

Definition 2.7. [28] *The function $h:[a,b]\subset\Re\to\mathcal{F}\left(\Re^d\right)$ is called $\mathscr{G}_r$-differentiable at a point $\mathscr{X}_0\in[a,b]$, if there exists a fuzzy number $\frac{d^{\mathscr{G}_r}h(\mathscr{X}_0)}{dt}\in\mathcal{F}\left(\Re^d\right)$ satisfying the following limit*

$$\lim_{\delta\to0}\frac{h(\mathscr{X}_0+\delta)\ominus^{\mathscr{G}r}h(\mathscr{X}_0)}{\delta}=\frac{d^{\mathscr{G}_r}h(\mathscr{X}_0)}{dt},$$

exists for δ sufficiently near 0. The limit is taken in $(\mathcal{F}\left(\Re^d\right),{}^{\mathscr{G}r}D)$.

Definition 2.8. [34] *Assume that $F:[a,b]\to\mathcal{F}\left(\Re^d\right)$ is $\mathscr{G}_r$-differentiable. $f(\theta)={}^{\mathscr{G}r}\mathcal{D}F(\theta)$ is continuous on $[a,b]$. Then, integral $\int_a^b f(\theta)d\theta$ is denoted as $\left[\int_a^b f(\theta)d\theta\right]^\alpha=\left[\int_a^b f_l^\alpha(\theta)d\theta,\int_a^b f_r^\alpha(\theta)d\theta\right]$, where $0<\alpha\leq1$, provided that its Lebesgue integrals on the right.*

Lemma 2.9. [34] *Assume that $p\geq1$, and $\mathscr{X},\mathscr{Y}\in L^p\left([0,T],\mathcal{F}\left(\Re^d\right)\right)$, then $\forall t\in[0,T]$, the following hold*

$$\mathbb{E}\sup_{\theta\in[0,t]}D^p\left(\int_0^\theta\mathscr{X}(s)ds,\int_0^\theta\mathscr{Y}(s)ds\right)\leq t^{p-1}\int_0^t\mathbb{E}D^p(\mathscr{X}(s),\mathscr{Y}(s))ds.$$

Lemma 2.10. [16] *If $\mathscr{X},\mathscr{Y}\in L^2\left([0,T],\mathcal{F}\left(\Re^d\right)\right)$, then $\forall t\in[0,T]$, we have*

$$\mathbb{E}\sup_{u\in[0,t]}D^2\left(\left\langle\int_0^u\mathscr{X}(s)dW(s)\right\rangle,\left\langle\int_0^u\mathscr{Y}(s)dW(s)\right\rangle\right)\leq4\mathbb{E}\int_0^t D^2(\langle\mathscr{X}(s)\rangle,\langle\mathscr{Y}(s)\rangle)ds.$$

Lemma 2.11. [6] *Let $\beta>0$, $k(t)$ and $u(t)$ are locally integrable and nonnegative function on $[0,T)$. $g(t)$ is a nondecreasing continuous and nonnegative function on $[0,T)$. If $k(t)$ is nondecreasing and*

$$u(t)\leqslant k(t)+g(t)\int_0^t(t-s)^{\beta-1}u(s)ds.$$

Then, we have

$$u(t)\leqslant k(t)E_\beta\left(g(t)\Gamma(\beta)t^\beta\right),$$

where E_β is defined as $E_\beta(z)=\sum_{k=0}^\infty\frac{z^k}{\Gamma(k\beta+1)}$.

Lemma 2.12. *For $J=[0,T]$, we suppose that $\lambda(t)$, $\xi(t)$, $\zeta(t)\in C(J,\mathbb{R}^+)$ and $\chi(t)\in C([-\tau,0],\mathbb{R}^+)$, in which $\lambda(t)$ and $\chi(t)$ are non-decreasing such that $\lambda(0)=\chi(0)$. If $\alpha(t)\in C([-\tau,T],\mathbb{R}^+)$ and*

$$\begin{cases} \alpha(t)\le\lambda(t)+\int_0^t(t-v)^{2p-2}[\xi(v)\alpha(v)+\zeta(v)\alpha(v-\tau)]dv, t\in J,\\ \alpha(t)\le\chi(t), t\in[-\tau,0],\end{cases}$$

then we have

$$\alpha(t)\le\left\{\Upsilon(t)\exp\left(\int_0^t[\Phi(v)+\Psi(v)]dv\right)\right\}^q, t\in J,$$

where $\Phi(t)=4^{\frac{1}{q}-1}\left(B\left(\frac{2p-1-q}{1-q},\frac{2-2p}{1-q}\right)\right)^{\frac{1-q}{q}}t^{\frac{2p-1-q}{q}}\xi^{\frac{1}{q}}(t)$ and $\Psi(t)=4^{\frac{1}{q}-1}\left(B\left(\frac{2p-1-q}{1-q},\frac{2-2p}{1-q}\right)\right)^{\frac{1-q}{q}}t^{\frac{2p-1-q}{q}}\zeta^{\frac{1}{q}}(t)$, where $B(\cdot,\cdot)$ denotes the well-known Beta function. $\Upsilon(t)=2^{\frac{1}{q}-1}\lambda^{\frac{1}{q}}(t)$. $0<q<2p-1<1$.

Proof. The proof is analogous to [12, Theorem 1]. □

2.2 Model formulation

Notations: Let $\left(\Omega,\mathcal{F}\left(\Re^d\right)\right)=\left(\Omega,\mathcal{A},\{\mathcal{A}_t\}_{t\ge0},\mathcal{F}\left(\Re^d\right)\right)$ denote a complete probability space. $W(\theta)$ is m-dimensional Brownian motion on it. $L_2\left(\Omega,\mathcal{F}\left(\Re^d\right)\right)$ is the set of all strongly measurable square integrable $\mathcal{F}\left(\Re^d\right)$-valued random variable. For $J_1=[-\delta,T]$, let the space $C\left(J_1,L_2\left(\Omega,\mathcal{F}\left(\Re^d\right)\right)\right)$ endowed with the norm

$$\mathbb{E}D^2(\mathscr{X},\mathscr{Y})=\mathbb{E}\sup_{\theta\in J_1}D^2(\mathscr{X}(\theta),\mathscr{Y}(\theta)),$$

where $\mathbb{E}$ denotes the mathematical expectation.

Consider the following FFSDS:

$$\begin{cases} ^{\mathscr{G}_r}\mathcal{D}^p\mathscr{X}(t)=A\mathscr{X}(t)+\sigma(t,\mathscr{X}(t),\mathscr{X}(t-\delta))+\left\langle\int_0^t g(s,\mathscr{X}(s),\mathscr{X}(s-\delta))dW(s)\right\rangle, t\in J:=[0,T],\\ \mathscr{X}(t)=\varphi(t), t\in[-\delta,0],\end{cases}\tag{2.1}$$

where $\frac{1}{2}<p<1$, $\mathscr{X}\in\mathcal{F}\left(\Re^d\right)$ is d-dimensional vector. A is $d\times d$ matrix. $\sigma:J\times\mathcal{F}\left(\Re^d\right)\times\mathcal{F}\left(\Re^d\right)\to\mathcal{F}\left(\Re^d\right)$ and $g:J\times\mathcal{F}\left(\Re^d\right)\times\mathcal{F}\left(\Re^d\right)\to\Re^{d\times m}$ are continuous functions on J. $W(s)$ represents m-dimensional Brownian motion on $\left(\Omega,\mathcal{F}\left(\Re^d\right)\right)$. δ is a positive constant. $\varphi(\cdot)$ is history function satisfying $\mathbb{E}D^2[\varphi(0),\langle0\rangle]<\infty$.

3 Six drawbacks of SGH-derivative, gH-derivative, H-derivative, and g-derivative

These six drawbacks, which can be found in [28, 29, 43], are as follows:

(1) The monotonicity of diameter of fuzzy function: under the mentioned approaches, the diameter of fuzzy function needs to be monotonic or non-decreasing. However, in general cases, we hope that monotonicity is not necessary.

(2) The multiplicity of solutions: in most cases, there are two forms of solutions. However, this fact becomes difficult for higher-order fuzzy differential system. In addition, in all cases there are doubling property.

(3) Existence of the difference: in almost all cases, the corresponding difference does not always exist, such as H-difference and gH-difference.

(4) Unnatural behavior in modeling phenomenon: this phenomenon means that the same element with different structure of the system model may exhibit different behaviors of the system. This phenomenon occurs in all systems defined in terms of the above derivative.

(5) Under the sense of several Hukuhara derivatives, it is unable to obtain the fuzzy solution, due to the zero forms of a solution can not be provided.

(6) About the basic operations: with respect to the derivative mentioned above, we know that in fuzzy system the factorization doesn't work.

In present manuscript, we consider system under the sense of Granular differentiability, which overcomes all these six drawbacks. In fact, there are at least six drawbacks of the mentioned approaches. For example, by applying characterization theorem to find solutions of nonlinear fuzzy system involving unknown functions, it is difficulty to characterize the corresponding level cuts.

4 Main results

In current part, we investigate the existence and uniqueness of solutions for system (2.1) by utilizing Carathéodory approximation, and then finite-time stability analysis is conducted with the aid of the generalized Grönwall delay inequality.

We assume the following conditions of the considered system:

(A1) There is a function $G(t, \mathscr{X}, \mathscr{Y}) : J \times \mathcal{F}\left(\Re^d\right) \times \mathcal{F}\left(\Re^d\right) \to \mathcal{F}\left(\Re^d\right)$ satisfying

(i) $G(\cdot, \mathscr{X}, \mathscr{Y})$ is locally integrable on J, and $G(t, \cdot, \cdot)$ satisfy non-decreasing, continuous, concave, and $G(t, 0, 0) = 0$. For fixed $t \in [0, T]$, and $\mathscr{X}_1, \mathscr{X}_2, \mathscr{Y}_1, \mathscr{Y}_2 \in \mathcal{F}\left(\Re^d\right)$ the following holds:

$$\max\left(D^2[\sigma(t, \mathscr{X}_1, \mathscr{X}_2), \sigma(t, \mathscr{Y}_1, \mathscr{Y}_2)], |g(t, \mathscr{X}_1, \mathscr{X}_2) - g(t, \mathscr{Y}_1, \mathscr{Y}_2)|^2\right) \leq G\big(t, D^2[\mathscr{X}_1, \mathscr{Y}_1], D^2[\mathscr{X}_2, \mathscr{Y}_2]\big).$$

(ii) For $\forall t \in J$, and nonnegative function $L(t)$ satisfying

$$L(t) \leq l \int_0^t G(\theta, L(\theta))\mathrm{d}\theta,$$

where $l > 0$ is a constant and $G(\theta, L(\theta), L(\theta)) = G(\theta, L(\theta))$, we get $L(t) \equiv 0$.

(iii) For $\forall t \in J$, there exists a constant $\bar{M} > 0$ satisfying

$$\max\left(D^2[\sigma(t, \langle 0\rangle, \langle 0\rangle), \langle 0\rangle], |g(t, \langle 0\rangle, \langle 0\rangle)|^2\right) \leq \bar{M}.$$

(A2) For matrix Mittag-Leffler functions E_p and $E_{p,p}$: $\mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$. Since entire functions $E_p(\cdot)$ and $E_{p,p}(\cdot)$ are bounded on $[0, T]$. For simplicity, we consider $\bar{E}_1 := \sup_{0\leq t\leq T} |E_p\left(At^p\right)|$ and $\bar{E}_2 := \sup_{0\leq t\leq T} |E_{p,p}\left(At^p\right)|$.

Lemma 4.1. *System* (2.1) *is equivalent to the following system:*

$$\begin{cases} \mathscr{X}(t) = E_p\left(At^p\right)\varphi(0) + \int_0^t (t-\nu)^{p-1} E_{p,p}\left(A(t-\nu)^p\right)\sigma(\nu, \mathscr{X}(\nu), \mathscr{X}(\nu-\delta))d\nu \\ \qquad + \int_0^t (t-\nu)^{p-1} E_{p,p}\left(A(t-\nu)^p\right)\left\langle \int_0^\nu g(\theta, \mathscr{X}(\theta), \mathscr{X}(\theta-\delta))dW(\theta)\right\rangle d\nu, t \in J, \\ \mathscr{X}(t) = \varphi(t), t \in [-\delta, 0]. \end{cases} \tag{3.1}$$

Proof. Applying the Laplace transform on (2.1), we derive that

$$L[{}^{\mathscr{G}_r}\mathcal{D}^p \mathscr{X}(t)]^\alpha = AL[\mathscr{X}(t)]^\alpha + L[\sigma(t, \mathscr{X}(t), \mathscr{X}(t-\delta))]^\alpha + L[\left\langle \int_0^t g(s, \mathscr{X}(s), \mathscr{X}(s-\delta))dW(s)\right\rangle]^\alpha,$$

and then let $\mathscr{X}_L(s)$, $\sigma_L(s)$ and $g_L(s)$ denote the Laplace transformation of $\mathscr{X}(t)$, $\sigma(t, \mathscr{X}(t), \mathscr{X}(t-\delta))$ and $\left\langle \int_0^t g(s, \mathscr{X}(s), \mathscr{X}(s-\delta))dW(s)\right\rangle$, respectively. We derive the following

$$[s^p \mathscr{X}_L(s) - s^{p-1}\varphi(0)]^\alpha = A[\mathscr{X}_L(s)]^\alpha + [\sigma_L(s)]^\alpha + [g_L(s)]^\alpha,$$

then we obtain

$$[\mathscr{X}_L(s)]^\alpha = \frac{s^{p-1}}{s^p I - A}[\varphi(0)]^\alpha + \frac{1}{s^p I - A}[\sigma_L(s)]^\alpha + \frac{1}{s^p I - A}[g_L(s)]^\alpha,$$

where I denotes the identity matrix. Subsequently, with the aid of inverse transform, we deduce

$$L^{-1}[\mathscr{X}_L(s)]^\alpha = L^{-1}\{\frac{s^{p-1}}{s^p I - A}\}[\varphi(0)]^\alpha + L^{-1}\{\frac{1}{s^p I - A}\} * L^{-1}\{[\sigma_L(s)]^\alpha\} + L^{-1}\{\frac{1}{s^p I - A}\} * L^{-1}\{[g_L(s)]^\alpha\}.$$

Therefore, we derive the following solution involving Mittag-Leffler function

$$[\mathscr{X}(t)]^\alpha = E_p\left(At^p\right)[\varphi(0)]^\alpha + \int_0^t (t-\nu)^{p-1} E_{p,p}\left(A(t-\nu)^p\right)[\sigma(\nu, \mathscr{X}(\nu), \mathscr{X}(\nu-\delta))]^\alpha d\nu$$

$$+\int_0^t (t-\nu)^{p-1}E_{p,p}\left(A(t-\nu)^p\right)\left\langle\int_0^\nu [g(\theta,\mathscr{X}(\theta),\mathscr{X}(\theta-\delta))]^\alpha dW(\theta)\right\rangle d\nu,$$

and then let $\mathscr{X}(t)=[\mathscr{X}(t)]^\alpha$, $\varphi(0)=[\varphi(0)]^\alpha$,

$$\sigma(\nu,\mathscr{X}(\nu),\mathscr{X}(\nu-\delta))=[\sigma(\nu,\mathscr{X}(\nu),\mathscr{X}(\nu-\delta))]^\alpha,$$

and

$$g(\theta,\mathscr{X}(\theta),\mathscr{X}(\theta-\delta))=[g(\theta,\mathscr{X}(\theta),\mathscr{X}(\theta-\delta))]^\alpha,$$

and we obtain (3.1). □

Remark 4.2. *Under the sense of Granular differentiability, Laplace transformation and its inverse involving Mittag-Leffler function can be applied to deduce the form of solution. The method we used is also essentially new compared with literatures* [6, 16, 33, 35].

4.1 Existence and uniqueness of solutions

Theorem 4.3. *Suppose that the hypotheses* (A_1) *and* (A_2) *hold, then system* (2.1) *has a unique solution.*

Proof. Define the Carathéodory approximation by: for any integer $n\geq 1$, define $\mathscr{X}_n(t)=\varphi(0)$, $\forall t\in[-(1+\delta),0]$, and for $\forall t\in J$

$$\begin{aligned}\mathscr{X}_n(t)=&E_p\left(At^p\right)\varphi(0)+\int_0^t (t-\nu)^{p-1}E_{p,p}\left(A(t-\nu)^p\right)\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n}))d\nu\\&+\int_0^t E_{p,p}\left(A(t-\nu)^p\right)(t-\nu)^{p-1}\left\langle\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)\right\rangle d\nu.\end{aligned}\tag{3.2}$$

The proof is divided into four steps below.
Step 1. The boundedness of $\{\mathscr{X}_n(t),n\geq 1\}$.

$$\begin{aligned}&\mathbb{E}\sup_{0\leq u\leq t}D^2[\mathscr{X}_n(u),\langle 0\rangle]\leq 3\mathbb{E}D^2[E_p\left(Au^p\right)\varphi(0),\langle 0\rangle]\\&+3\mathbb{E}\sup_{0\leq u\leq t}D^2[\int_0^u (u-\nu)^{p-1}E_{p,p}\left(A(u-\nu)^p\right)\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n}))d\nu,\langle 0\rangle]\\&+3\mathbb{E}\sup_{0\leq u\leq t}D^2[\int_0^u (u-\nu)^{p-1}E_{p,p}\left(A(u-\nu)^p\right)\left\langle\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)\right\rangle d\nu,\langle 0\rangle]\\:=&I_1+I_2+I_3.\end{aligned}$$

With the aid of the assumptions (A_1), (A_2), Lemma 2.9 and Itô isometry, we get

$$I_1=3\mathbb{E}D^2[E_p\left(Au^p\right)\varphi(0),\langle 0\rangle]\leq 3\bar{E}_1^2\mathbb{E}D^2[\varphi(0),\langle 0\rangle].$$

$$\begin{aligned}I_2\leq&3t\bar{E}_2^2\int_0^t (t-\nu)^{2p-2}\mathbb{E}D^2[\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n})),\langle 0\rangle]d\nu\\\leq&6t\bar{E}_2^2\int_0^t (t-\nu)^{2p-2}\big(\mathbb{E}D^2[\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n})),\sigma(\nu,\langle 0\rangle,\langle 0\rangle)]+\mathbb{E}D^2[\sigma(\nu,\langle 0\rangle,\langle 0\rangle),\langle 0\rangle]\big)d\nu\\\leq&6t\bar{E}_2^2\int_0^t (t-\nu)^{2p-2}\big(G(t,\mathbb{E}D^2[\mathscr{X}_n(\nu-\frac{1}{n}),\langle 0\rangle],\mathbb{E}D^2[\mathscr{X}_n(\nu-\delta-\frac{1}{n}),\langle 0\rangle])+\bar{M}\big)d\nu.\end{aligned}$$

$$\begin{aligned}I_3\leq&3t\bar{E}_2^2\int_0^t (t-\nu)^{2p-2}\mathbb{E}\Big|\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)\Big|^2d\nu\\\leq&6t\bar{E}_2^2\int_0^t (t-\nu)^{2p-2}\int_0^\nu \mathbb{E}\big|g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))-g(0,\langle 0\rangle,\langle 0\rangle)\big|^2+\mathbb{E}\big|g(0,\langle 0\rangle,\langle 0\rangle)\big|^2d\theta d\nu\\\leq&6t\bar{E}_2^2\int_0^t (t-\nu)^{2p-2}\int_0^\nu G(\theta,\mathbb{E}D^2[\mathscr{X}_n(\theta-\frac{1}{n}),\langle 0\rangle],\mathbb{E}D^2[\mathscr{X}_n(\theta-\delta-\frac{1}{n}),\langle 0\rangle])+\bar{M}d\theta d\nu.\end{aligned}$$

Given that $G(t,X,Y)$ is concave, there exist three integrable functions $a(t)$, $b(t)$ and $c(t)$ such that

$$G(t,X,Y)\leq a(t)+b(t)X+c(t)Y,\quad X,Y\geq 0,$$

and suppose that $\sup_{0\leq t\leq T}a(t)=\bar{a}$, $\sup_{0\leq t\leq T}b(t)=\bar{b}$ and $\sup_{0\leq t\leq T}c(t)=\bar{c}$. Then with the aid of mean value theorem of integrals, there exists $\theta_1\in[0,\nu]$ satisfying

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq u\leq t}D^2[\mathscr{X}_n(u),\langle 0\rangle]\leq 3\bar{E}_1^2\mathbb{E}D^2[\varphi(0),\langle 0\rangle]\\
&+6t\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\Big(G(t,\mathbb{E}D^2[\mathscr{X}_n(\nu-\frac{1}{n}),\langle 0\rangle],\mathbb{E}D^2[\mathscr{X}_n(\nu-\delta-\frac{1}{n}),\langle 0\rangle])+\bar{M}\\
&+\int_0^\nu G(\theta,\mathbb{E}D^2[\mathscr{X}_n(\theta-\frac{1}{n}),\langle 0\rangle],\mathbb{E}D^2[\mathscr{X}_n(\theta-\delta-\frac{1}{n}),\langle 0\rangle])+\bar{M}d\theta\Big)d\nu\\
\leq&3\bar{E}_1^2\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+6t\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\Big(G(t,\mathbb{E}D^2[\mathscr{X}_n(\nu-\frac{1}{n}),\langle 0\rangle],\mathbb{E}D^2[\mathscr{X}_n(\nu-\delta-\frac{1}{n}),\langle 0\rangle])\\
&+\bar{M}+\nu\cdot\bar{M}+\nu\cdot G(\theta_1,\mathbb{E}D^2[\mathscr{X}_n(\theta_1-\frac{1}{n}),\langle 0\rangle],\mathbb{E}D^2[\mathscr{X}_n(\theta_1-\delta-\frac{1}{n}),\langle 0\rangle])\Big)d\nu\\
\leq&3\bar{E}_1^2\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+6t\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\Big(a(t)+b(t)\mathbb{E}D^2[\mathscr{X}_n(\nu-\frac{1}{n}),\langle 0\rangle]+c(t)\mathbb{E}D^2[\mathscr{X}_n(\nu-\delta-\frac{1}{n}),\langle 0\rangle])\\
&+\bar{M}+\nu a(\theta_1)+\nu b(\theta_1)\mathbb{E}D^2[\mathscr{X}_n(\theta_1-\frac{1}{n}),\langle 0\rangle]+\nu c(\theta_1)\mathbb{E}D^2[\mathscr{X}_n(\theta_1-\delta-\frac{1}{n}),\langle 0\rangle]+\nu\cdot\bar{M}\Big)d\nu\\
\leq&3\bar{E}_1^2\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+6t\bar{E}_2^2(1+t)\frac{t^{2p-1}}{2p-1}\left(\bar{a}+\bar{M}+(\bar{b}+\bar{c})\mathbb{E}D^2[\varphi(0),\langle 0\rangle]\right)\\
&+6t\bar{E}_2^2(1+t)(\bar{b}+\bar{c})\int_0^t(t-\nu)^{2p-2}\mathbb{E}\sup_{0\leq\nu_1\leq\nu}D^2[\mathscr{X}_n(\nu_1),\langle 0\rangle]d\nu.
\end{aligned}$$

With the aid of Lemma 2.11, we get

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq u\leq t}D^2[\mathscr{X}_n(u),\langle 0\rangle]\leq\Big(3\bar{E}_1^2\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+6t\bar{E}_2^2(1+t)\frac{t^{2p-1}}{2p-1}\left(\bar{a}+\bar{M}+(\bar{b}+\bar{c})\mathbb{E}D^2[\varphi(0),\langle 0\rangle]\right)\Big)\\
&\quad\cdot E_{2p-1}\left(6t\bar{E}_2^2(1+t)(\bar{b}+\bar{c})\Gamma(2p-1)t^{2p-1}\right)\\
\leq&\Big(3\bar{E}_1^2\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+6T\bar{E}_2^2(1+T)\frac{T^{2p-1}}{2p-1}\left(\bar{a}+\bar{M}+(\bar{b}+\bar{c})\mathbb{E}D^2[\varphi(0),\langle 0\rangle]\right)\Big)\\
&\quad\cdot E_{2p-1}\left(6T\bar{E}_2^2(1+T)(\bar{b}+\bar{c})\Gamma(2p-1)T^{2p-1}\right)\\
:=&C_1,
\end{aligned}$$

where C_1 denotes a positive constant. Therefore, the boundedness of sequence be certificated.

Step 2. For $0\leq s<t\leq T$ and any integer $n\geq 1$, and according to (3.2), we have

$$\begin{aligned}
&\mathbb{E}D^2[\mathscr{X}_n(t),\mathscr{X}_n(s)]\leq 2\mathbb{E}D^2[\int_0^t(t-\nu)^{p-1}E_{p,p}\left(A(t-\nu)^p\right)\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n}))d\nu,\\
&\int_0^s(s-\nu)^{p-1}E_{p,p}\left(A(s-\nu)^p\right)\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n}))d\nu]\\
&+2\mathbb{E}D^2[\int_0^t(t-\nu)^{p-1}E_{p,p}\left(A(t-\nu)^p\right)\left\langle\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)\right\rangle d\nu,\\
&\int_0^s(s-\nu)^{p-1}E_{p,p}\left(A(s-\nu)^p\right)\left\langle\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)\right\rangle d\nu]\\
:=&I_4+I_5.
\end{aligned}$$

Applying the assumptions (A_1) and (A_2), we have

$$I_4\leq 4\mathbb{E}D^2[\int_0^s(t-\nu)^{p-1}E_{p,p}\left(A(t-\nu)^p\right)\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n}))d\nu,$$

$$
\begin{aligned}
&\int_0^s (s-\nu)^{p-1}E_{p,p}\left(A(s-\nu)^p\right)\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n}))d\nu]\\
&+4\mathbb{E}D^2[\int_s^t (t-\nu)^{p-1}E_{p,p}\left(A(t-\nu)^p\right)\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n}))d\nu,\langle 0\rangle]\\
\leq&4\int_0^s |(t-\nu)^{2p-2}(E_{p,p}\left(A(t-\nu)^p\right))^2-(s-\nu)^{2p-2}(E_{p,p}\left(A(s-\nu)^p\right))^2|\mathbb{E}D^2[\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n})),\langle 0\rangle]d\nu\\
&+4\bar{E}_2^2\int_s^t (t-\nu)^{2p-2}\mathbb{E}D^2[\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n})),\langle 0\rangle]d\nu\\
\leq&4\bar{E}_2^2(\frac{t^{2p-1}-s^{2p-1}}{2p-1}+\frac{(t-s)^{2p-1}}{2p-1})\mathbb{E}D^2[\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n})),\langle 0\rangle]\\
&+4\bar{E}_2^2\frac{(t-s)^{2p-1}}{2p-1}\mathbb{E}D^2[\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n})),\langle 0\rangle]\\
\leq&12\bar{E}_2^2\frac{(t-s)^{2p-1}}{2p-1}\mathbb{E}D^2[\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n})),\langle 0\rangle]\\
\leq&24\bar{E}_2^2\frac{(t-s)^{2p-1}}{2p-1}\big(G(\nu,\mathbb{E}D^2[\mathscr{X}_n(\nu-\frac{1}{n}),\langle 0\rangle],\mathbb{E}D^2[\mathscr{X}_n(\nu-\delta-\frac{1}{n}),\langle 0\rangle])+\bar{M}\big)\\
\leq&24\bar{E}_2^2\frac{(t-s)^{2p-1}}{2p-1}\big(\bar{a}+\bar{M}+(\bar{b}+\bar{c})\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+(\bar{b}+\bar{c})C_1\big).
\end{aligned}
$$

Applying the assumptions (A_1), (A_2) and Itô isometry, we have

$$
\begin{aligned}
I_5\leq&4\mathbb{E}D^2[\int_0^s (t-\nu)^{p-1}E_{p,p}\left(A(t-\nu)^p\right)\left\langle\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)\right\rangle d\nu,\\
&\quad\int_0^s (s-\nu)^{p-1}E_{p,p}\left(A(s-\nu)^p\right)\left\langle\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)\right\rangle d\nu]\\
&+4\mathbb{E}D^2[\int_s^t (t-\nu)^{p-1}E_{p,p}\left(A(t-\nu)^p\right)\left\langle\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)\right\rangle d\nu,\langle 0\rangle]\\
\leq&4\int_0^s |(t-\nu)^{2p-2}(E_{p,p}\left(A(t-\nu)^p\right))^2-(s-\nu)^{2p-2}(E_{p,p}\left(A(s-\nu)^p\right))^2|\mathbb{E}|\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)|^2d\nu\\
&+4\bar{E}_2^2\int_s^t (t-\nu)^{2p-2}\mathbb{E}|\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)|^2d\nu\\
\leq&4\bar{E}_2^2\int_0^s ((s-\nu)^{2p-2}-(t-\nu)^{2p-2})\mathbb{E}\int_0^\nu |g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))|^2d\theta d\nu\\
&+4\bar{E}_2^2\int_s^t (t-\nu)^{2p-2}\mathbb{E}\int_0^\nu |g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))|^2d\theta d\nu\\
\leq&8\bar{E}_2^2\int_0^s ((s-\nu)^{2p-2}-(t-\nu)^{2p-2})\int_0^\nu G(\theta,\mathbb{E}D^2[\mathscr{X}_n(\theta-\frac{1}{n}),\langle 0\rangle],\mathbb{E}D^2[\mathscr{X}_n(\theta-\delta-\frac{1}{n}),\langle 0\rangle])+\bar{M}d\theta d\nu\\
&+8\bar{E}_2^2\int_s^t (t-\nu)^{2p-2}\int_0^\nu G(\theta,\mathbb{E}D^2[\mathscr{X}_n(\theta-\frac{1}{n}),\langle 0\rangle],\mathbb{E}D^2[\mathscr{X}_n(\theta-\delta-\frac{1}{n}),\langle 0\rangle])+\bar{M}d\theta d\nu\\
\leq&8\bar{E}_2^2\int_0^s ((s-\nu)^{2p-2}-(t-\nu)^{2p-2})t\big(a(\theta_1)+b(\theta_1)\mathbb{E}D^2[\mathscr{X}_n(\theta_1-\frac{1}{n}),\langle 0\rangle]+c(\theta_1)\mathbb{E}D^2[\mathscr{X}_n(\theta_1-\delta-\frac{1}{n})]+\bar{M}\big)d\nu\\
&+8\bar{E}_2^2\int_s^t (t-\nu)^{2p-2}t\big(a(\theta_1)+b(\theta_1)\mathbb{E}D^2[\mathscr{X}_n(\theta_1-\frac{1}{n}),\langle 0\rangle]+c(\theta_1)\mathbb{E}D^2[\mathscr{X}_n(\theta_1-\delta-\frac{1}{n})]+\bar{M}\big)d\nu\\
\leq&8t\bar{E}_2^2(\frac{t^{2p-1}-s^{2p-1}}{2p-1}+\frac{(t-s)^{2p-1}}{2p-1})\big(\bar{a}+\bar{M}+(\bar{b}+\bar{c})\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+(\bar{b}+\bar{c})\mathbb{E}\sup_{0\leq u\leq t}D^2[\mathscr{X}_n(u),\langle 0\rangle]\big)\\
&+8t\bar{E}_2^2\frac{(t-s)^{2p-1}}{2p-1}\big(\bar{a}+\bar{M}+(\bar{b}+\bar{c})\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+(\bar{b}+\bar{c})\mathbb{E}\sup_{0\leq u\leq t}D^2[\mathscr{X}_n(u),\langle 0\rangle]\big)\\
\leq&24T\bar{E}_2^2\frac{(t-s)^{2p-1}}{2p-1}\big(\bar{a}+\bar{M}+(\bar{b}+\bar{c})\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+(\bar{b}+\bar{c})C_1\big).
\end{aligned}
$$

Then, we obtain

$$\begin{aligned}&\mathbb{E}D^2[\mathscr{X}_n(t),\mathscr{X}_n(s)]\le 24\bar{E}_2^2\frac{(t-s)^{2p-1}}{2p-1}(T+1)\big(\bar{a}+\bar{M}+(\bar{b}+\bar{c})\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+(\bar{b}+\bar{c})C_1\big)\\&=C_2(t-s)^{2p-1},\end{aligned}$$

where $C_2=\frac{24\bar{E}_2^2}{2p-1}(T+1)\big(\bar{a}+\bar{M}+(\bar{b}+\bar{c})\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+(\bar{b}+\bar{c})C_1\big)$.

Step 3. We claim that $\{\mathscr{X}_n(t),n\ge 1\}$ is a Cauchy sequence. For integer $m>n\ge 1$, we get from 3.2

$$\begin{aligned}&\mathbb{E}\sup_{0\le u\le t}D^2[\mathscr{X}_m(u),\mathscr{X}_n(u)]\le 2\mathbb{E}\sup_{0\le u\le t}D^2[\int_0^u(u-\nu)^{p-1}E_{p,p}\left(A(u-\nu)^p\right)\sigma(\nu,\mathscr{X}_m(\nu-\frac{1}{m}),\mathscr{X}_m(\nu-\delta-\frac{1}{m}))d\nu,\\&\int_0^u(u-\nu)^{p-1}E_{p,p}\left(A(u-\nu)^p\right)\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n}))d\nu]\\&+2\mathbb{E}\sup_{0\le u\le t}D^2[\int_0^u(u-\nu)^{p-1}E_{p,p}\left(A(u-\nu)^p\right)\left\langle\int_0^\nu g(\theta,\mathscr{X}_m(\theta-\frac{1}{m}),\mathscr{X}_m(\theta-\delta-\frac{1}{m}))dW(\theta)\right\rangle d\nu,\\&\int_0^u(u-\nu)^{p-1}E_{p,p}\left(A(u-\nu)^p\right)\left\langle\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)\right\rangle d\nu]\\&:=I_6+I_7.\end{aligned}$$

Applying the assumptions (A_1), (A_2) and Lemma 2.9 together with the Hölder inequality, we get

$$\begin{aligned}I_6\le&2t\int_0^t(t-\nu)^{2p-2}(E_{p,p}\left(A(t-\nu)^p\right))^2d\nu\int_0^t\mathbb{E}D^2[\sigma(\nu,\mathscr{X}_m(\nu-\frac{1}{m}),\mathscr{X}_m(\nu-\delta-\frac{1}{m})),\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n}))]d\nu\\\le&2t\bar{E}_2^2\frac{t^{2p-1}}{2p-1}\int_0^t\mathbb{E}D^2[\sigma(\nu,\mathscr{X}_m(\nu-\frac{1}{m}),\mathscr{X}_m(\nu-\delta-\frac{1}{m})),\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n}))]d\nu\\\le&4\bar{E}_2^2\frac{t^{2p}}{2p-1}\int_0^t\mathbb{E}D^2[\sigma(\nu,\mathscr{X}_m(\nu-\frac{1}{m}),\mathscr{X}_m(\nu-\delta-\frac{1}{m})),\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{m}),\mathscr{X}_n(\nu-\delta-\frac{1}{m}))]\\&+\mathbb{E}D^2[\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{m}),\mathscr{X}_n(\nu-\delta-\frac{1}{m})),\sigma(\nu,\mathscr{X}_n(\nu-\frac{1}{n}),\mathscr{X}_n(\nu-\delta-\frac{1}{n}))]d\nu\\\le&4\bar{E}_2^2\frac{t^{2p}}{2p-1}\int_0^tG(\nu,\mathbb{E}D^2[\mathscr{X}_m(\nu-\frac{1}{m}),\mathscr{X}_n(\nu-\frac{1}{m})],\mathbb{E}D^2[\mathscr{X}_m(\nu-\delta-\frac{1}{m}),\mathscr{X}_n(\nu-\delta-\frac{1}{m})])d\nu\\&+4\bar{E}_2^2\frac{t^{2p}}{2p-1}\int_0^tG(\nu,\mathbb{E}D^2[\mathscr{X}_n(\nu-\frac{1}{m}),\mathscr{X}_n(\nu-\frac{1}{n})],\mathbb{E}D^2[\mathscr{X}_n(\nu-\delta-\frac{1}{m}),\mathscr{X}_n(\nu-\delta-\frac{1}{n})])d\nu\\\le&4\bar{E}_2^2\frac{t^{2p}}{2p-1}\int_0^tG(\nu,\mathbb{E}\sup_{0\le\nu_1\le\nu}D^2[\mathscr{X}_m(\nu_1),\mathscr{X}_n(\nu_1)],\mathbb{E}\sup_{0\le\nu_1\le\nu}D^2[\mathscr{X}_m(\nu_1),\mathscr{X}_n(\nu_1)])d\nu\\&+4\bar{E}_2^2\frac{t^{2p}}{2p-1}\int_0^tG(\nu,C_2(\frac{1}{n}-\frac{1}{m})^{2p-1},C_2(\frac{1}{n}-\frac{1}{m})^{2p-1})d\nu\\=&4\bar{E}_2^2\frac{t^{2p}}{2p-1}\int_0^tG(\nu,\mathbb{E}\sup_{0\le\nu_1\le\nu}D^2[\mathscr{X}_m(\nu_1),\mathscr{X}_n(\nu_1)])d\nu+4\bar{E}_2^2\frac{t^{2p}}{2p-1}\int_0^tG(\nu,C_2(\frac{1}{n}-\frac{1}{m})^{2p-1})d\nu.\end{aligned}$$

Similar to the proof of I_6, and with the aid of Lemma 2.10 and the mean value theorem of integrals, we get

$$\begin{aligned}I_7\le&2t\int_0^t(t-\nu)^{2p-2}(E_{p,p}\left(A(t-\nu)^p\right))^2d\nu\\&\cdot\int_0^t\mathbb{E}D^2[\left\langle\int_0^\nu g(\theta,\mathscr{X}_m(\theta-\frac{1}{m}),\mathscr{X}_m(\theta-\delta-\frac{1}{m}))dW(\theta)\right\rangle,\left\langle\int_0^\nu g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)\right\rangle]d\nu\\\le&2t\bar{E}_2^2\frac{t^{2p-1}}{2p-1}\int_0^t\mathbb{E}\sup_{0\le\nu_1\le\nu}D^2[\left\langle\int_0^{\nu_1}g(\theta,\mathscr{X}_m(\theta-\frac{1}{m}),\mathscr{X}_m(\theta-\delta-\frac{1}{m}))dW(\theta)\right\rangle,\\&\left\langle\int_0^{\nu_1}g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))dW(\theta)\right\rangle]d\nu\\\le&2t\bar{E}_2^2\frac{t^{2p-1}}{2p-1}\int_0^t4\mathbb{E}\int_0^\nu D^2[\left\langle g(\theta,\mathscr{X}_m(\theta-\frac{1}{m}),\mathscr{X}_m(\theta-\delta-\frac{1}{m}))\right\rangle,\left\langle g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))\right\rangle]d\theta d\nu\\\le&16\bar{E}_2^2\frac{t^{2p}}{2p-1}\int_0^t\mathbb{E}\int_0^\nu|g(\theta,\mathscr{X}_m(\theta-\frac{1}{m}),\mathscr{X}_m(\theta-\delta-\frac{1}{m}))-g(\theta,\mathscr{X}_n(\theta-\frac{1}{m}),\mathscr{X}_n(\theta-\delta-\frac{1}{m}))|^2\end{aligned}$$

$$
\begin{aligned}
&+|g(\theta,\mathscr{X}_n(\theta-\frac{1}{m}),\mathscr{X}_n(\theta-\delta-\frac{1}{m}))-g(\theta,\mathscr{X}_n(\theta-\frac{1}{n}),\mathscr{X}_n(\theta-\delta-\frac{1}{n}))|^2 d\theta d\nu\\
\leq& 16\bar{E}_2^2\frac{t^{2p+1}}{2p-1}\int_0^t G(\theta_1,\mathbb{E}D^2[\mathscr{X}_m(\theta_1-\frac{1}{m}),\mathscr{X}_n(\theta_1-\frac{1}{m})],\mathbb{E}D^2[\mathscr{X}_m(\theta_1-\delta-\frac{1}{m}),\mathscr{X}_n(\theta_1-\delta-\frac{1}{m})])\\
&+G(\theta_1,\mathbb{E}D^2[\mathscr{X}_n(\theta_1-\frac{1}{m}),\mathscr{X}_n(\theta_1-\frac{1}{n})],\mathbb{E}D^2[\mathscr{X}_n(\theta_1-\delta-\frac{1}{m}),\mathscr{X}_n(\theta_1-\delta-\frac{1}{n})])d\nu\\
\leq& 16\bar{E}_2^2\frac{t^{2p+1}}{2p-1}\int_0^t G(\nu,\mathbb{E}\sup_{0\leq\nu_1\leq\nu}D^2[\mathscr{X}_m(\nu_1),\mathscr{X}_n(\nu_1)],\mathbb{E}\sup_{0\leq\nu_1\leq\nu}D^2[\mathscr{X}_m(\nu_1),\mathscr{X}_n(\nu_1)])d\nu\\
&+16\bar{E}_2^2\frac{t^{2p+1}}{2p-1}\int_0^t G(\nu,C_2(\frac{1}{n}-\frac{1}{m})^{2p-1},C_2(\frac{1}{n}-\frac{1}{m})^{2p-1})d\nu.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
&\mathbb{E}\sup_{0\leq u\leq t}D^2[\mathscr{X}_m(u),\mathscr{X}_n(u)]\leq 4\bar{E}_2^2\frac{T^{2p}}{2p-1}(4T+1)\int_0^t G(\nu,\mathbb{E}\sup_{0\leq\nu_1\leq\nu}D^2[\mathscr{X}_m(\nu_1),\mathscr{X}_n(\nu_1)])d\nu\\
&+4\bar{E}_2^2\frac{T^{2p}}{2p-1}(4T+1)\int_0^t G(\nu,C_2(\frac{1}{n}-\frac{1}{m})^{2p-1})d\nu.
\end{aligned}
$$

Let

$$L(t)=\lim_{m,n\to\infty}\mathbb{E}\sup_{0\leq u\leq t}D^2[\mathscr{X}_m(u),\mathscr{X}_n(u)].$$

Then, it yields by Fatous Lemma

$$L(t)\leq 4\bar{E}_2^2\frac{T^{2p}}{2p-1}(4T+1)\int_0^t G(\nu,L(\nu))d\nu.$$

According to the assumption (A_1) (ii), we obtain

$$L(t)=\lim_{m,n\to\infty}\mathbb{E}\sup_{0\leq u\leq t}D^2[\mathscr{X}_m(u),\mathscr{X}_n(u)]=0,$$

which indicate that $\{\mathscr{X}_n(t),n\geq 1\}$ is a Cauchy sequence. Then with the aid of Borel-Cantelli lemma, it deduce that, as $n\to\infty$, $\mathscr{X}_n(t)\to\mathscr{X}(t)$ holds uniformly, for $\forall t\in[0,T]$. Then, as $n\to\infty$ in (3.2), we can conclude that $\mathscr{X}(t)$ is a solution to system 2.1.
Step 4. Let $\mathscr{X}(t)$ and $\mathscr{Y}(t)$ are two solutions of system (2.1), then for $\forall t\in J$, we have by using the same technique as Step 3

$$
\begin{aligned}
\mathbb{E}D^2[\mathscr{X}(t),\mathscr{Y}(t)]\leq& \mathbb{E}\sup_{0\leq u\leq t}D^2[\mathscr{X}(u),\mathscr{Y}(u)]\\
\leq& 2\bar{E}_2^2\frac{T^{2p}}{2p-1}(4T+1)\int_0^t G(v,\mathbb{E}\sup_{0\leq\nu_1\leq\nu}D^2[\mathscr{X}(\nu_1),\mathscr{Y}(\nu_1)])d\nu.
\end{aligned}
$$

Then we conclude that

$$\mathbb{E}D^2[\mathscr{X}(t),\mathscr{Y}(t)]\leq\mathbb{E}\sup_{0\leq u\leq t}D^2[\mathscr{X}(u),\mathscr{Y}(u)]\equiv 0.$$

Therefore, we obtain $\mathbb{E}D^2[\mathscr{X}(t),\mathscr{Y}(t)]=0$ which implies that $\mathscr{X}(t)=\mathscr{Y}(t)$. Meanwhile, for $t\in[-\delta,0]$ the uniqueness is obvious. Then there exists a unique solution to system (2.1). □

Remark 4.4. *It is different from the approximation methods adopted by most researchers to investigate the existence and uniqueness of solution, such as* [14, 21, 22, 40]*. Also, a large number of scholars applied contraction mapping and fixed point theorem to prove it* [6, 16, 33]*. In this paper, we use a new Carathéodory approximation method, and we known that there exist few papers available to handle FFSDS by using this method.*

4.2 Finite-time stability result

Definition 4.5. [20] *System* (2.1) *is finite-time stable with respect to* $\{\eta, \varepsilon, T\}$, $0<\eta<\varepsilon$, *if* $\mathbb{E}D^2[\varphi(0),\langle 0\rangle]\le\eta$ *implies* $\mathbb{E}\sup_{0\le u\le t} D^2[\mathscr{X}(u),\langle 0\rangle]\le\varepsilon$, *for* $t\in[0,T]$.

Theorem 4.6. *Assume that the hypotheses* (A_1), (A_2) *hold and there exist two positive constants* $\eta<\varepsilon$, *and* $\mathbb{E}D^2[\varphi(0),\langle 0\rangle]\le\eta$. *System* (2.1) *is finite-time stable on* $[-\delta,T]$ *w.r.t.* $\{\eta,\varepsilon,T\}$, *if the following hold*

$$3\bar{E}_1^2\eta\le\frac{\varepsilon}{2^{1-q}\exp\{\Xi\}}-6\bar{E}_2^2\frac{T^{2p}}{2p-1}(\bar{a}+\bar{M})(T+1),\tag{3.3}$$

where $\Xi=\frac{q^2}{2p-1}4^{\frac{1}{q}-1}\left(B\left(\frac{2p-1-q}{1-q},\frac{2-2p}{1-q}\right)\right)^{\frac{1-q}{q}}\left((6T\bar{E}_2^2(T+1)\bar{b})^{\frac{1}{q}}+(6T\bar{E}_2^2(T+1)\bar{c})^{\frac{1}{q}}\right)T^{\frac{2p-1}{q}}$.

Proof. For all $t\in J$, it follows from (3.1) that

$$\begin{aligned}
&\mathbb{E}\sup_{0\le u\le t}D^2[\mathscr{X}(u),\langle 0\rangle]\le 3\mathbb{E}D^2[E_p\left(Au^p\right)\varphi(0),\langle 0\rangle]\\
&+3\mathbb{E}\sup_{0\le u\le t}D^2[\int_0^u(u-\nu)^{p-1}E_{p,p}\left(A(u-\nu)^p\right)\sigma(\nu,\mathscr{X}(\nu),\mathscr{X}(\nu-\delta))d\nu,\langle 0\rangle]\\
&+3\mathbb{E}\sup_{0\le u\le t}D^2[\int_0^u(u-\nu)^{p-1}E_{p,p}\left(A(u-\nu)^p\right)\left\langle\int_0^\nu g(\theta,\mathscr{X}(\theta),\mathscr{X}(\theta-\delta))dW(\theta)\right\rangle d\nu,\langle 0\rangle]\\
:=&J_1+J_2+J_3.
\end{aligned}$$

According to the assumptions (A_1), (A_2), Lemma 2.9 and Itô isometry, we get $J_1\le 3\bar{E}_1^2\mathbb{E}D^2[\varphi(0),\langle 0\rangle]$.

$$\begin{aligned}
J_2\le&3t\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\mathbb{E}D^2[\sigma(\nu,\mathscr{X}(\nu),\mathscr{X}(\nu-\delta)),\langle 0\rangle]d\nu\\
\le&6t\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\big(G(\nu,\mathbb{E}D^2[\mathscr{X}(\nu),\langle 0\rangle],\mathbb{E}D^2[\mathscr{X}(\nu-\delta),\langle 0\rangle])+\bar{M}\big)d\nu\\
\le&6t\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\big(a(\nu)+b(\nu)\mathbb{E}D^2[\mathscr{X}(\nu),\langle 0\rangle]+c(\nu)\mathbb{E}D^2[\mathscr{X}(\nu-\delta),\langle 0\rangle])+\bar{M}\big)d\nu\\
\le&6t\bar{E}_2^2\frac{t^{2p-1}}{2p-1}(\bar{a}+\bar{M})+6t\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\big(b(\nu)\mathbb{E}D^2[\mathscr{X}(\nu),\langle 0\rangle]+c(\nu)\mathbb{E}D^2[\mathscr{X}(\nu-\delta),\langle 0\rangle])\big)d\nu.
\end{aligned}$$

$$\begin{aligned}
J_3\le&3t\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\mathbb{E}|\int_0^\nu g(\theta,\mathscr{X}(\theta),\mathscr{X}(\theta-\delta))dW(\theta)|^2d\nu\\
\le&6t\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\int_0^\nu G(\theta,\mathbb{E}D^2[\mathscr{X}(\theta),\langle 0\rangle],\mathbb{E}D^2[\mathscr{X}(\theta-\delta),\langle 0\rangle])+\bar{M}d\theta d\nu\\
\le&6t\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\int_0^\nu a(\theta)+b(\theta)\mathbb{E}D^2[\mathscr{X}(\theta),\langle 0\rangle]+c(\theta)\mathbb{E}D^2[\mathscr{X}(\theta-\delta),\langle 0\rangle]+\bar{M}d\theta d\nu\\
\le&6t^2\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\big(a(\theta_1)+b(\theta_1)\mathbb{E}D^2[\mathscr{X}(\theta_1),\langle 0\rangle]+c(\theta_1)\mathbb{E}D^2[\mathscr{X}(\theta_1-\delta),\langle 0\rangle]+\bar{M}\big)d\nu\\
\le&6t^2\bar{E}_2^2\frac{t^{2p-1}}{2p-1}(\bar{a}+\bar{M})+6t^2\bar{E}_2^2\int_0^t(t-\nu)^{2p-2}\big(b(\theta_1)\mathbb{E}D^2[\mathscr{X}(\theta_1),\langle 0\rangle]+c(\theta_1)\mathbb{E}D^2[\mathscr{X}(\theta_1-\delta),\langle 0\rangle]\big)d\nu.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\mathbb{E}\sup_{0\le u\le t}D^2[\mathscr{X}(u),\langle 0\rangle]\le 3\bar{E}_1^2\mathbb{E}D^2[\varphi(0),\langle 0\rangle]+6t\bar{E}_2^2\frac{t^{2p-1}}{2p-1}(\bar{a}+\bar{M})(t+1)\\
&+6t\bar{E}_2^2(t+1)\int_0^t(t-\nu)^{2p-2}\big(b(\nu)\mathbb{E}\sup_{0\le\nu_1\le\nu}D^2[\mathscr{X}(\nu_1),\langle 0\rangle]+c(\nu)\mathbb{E}\sup_{0\le\nu_1\le\nu}D^2[\mathscr{X}(\nu-\delta),\langle 0\rangle])\big)d\nu\\
\le&\lambda(t)+\int_0^t(t-\nu)^{2p-2}\big(\xi(\nu)\mathbb{E}\sup_{0\le\nu_1\le\nu}D^2[\mathscr{X}(\nu_1),\langle 0\rangle]+\zeta(\nu)\mathbb{E}\sup_{0\le\nu_1\le\nu}D^2[\mathscr{X}(\nu-\delta),\langle 0\rangle])\big)d\nu,
\end{aligned}$$

where $\lambda(t) = 3\bar{E}_1^2\mathbb{E}D^2[\varphi(0),\langle 0\rangle] + 6t\bar{E}_2^2\frac{t^{2p-1}}{2p-1}(\bar{a}+\bar{M})(t+1)$, $\xi(t) = 6T\bar{E}_2^2(T+1)\bar{b}$ and $\zeta(t) = 6T\bar{E}_2^2(T+1)\bar{c}$. Let $\lambda(t) = \chi(t)$, then $\lambda(t)$ and $\chi(t)$ are obvious nondecreasing and $\lambda(0) = \chi(0)$. Then applying the Lemma 2.12, we have

$$\mathbb{E}\sup_{0\le u\le t} D^2[\mathscr{X}(u),\langle 0\rangle] \le \left\{2^{\frac{1}{q}-1}\lambda^{\frac{1}{q}}(t)\exp\left(\int_0^t 4^{\frac{1}{q}-1}\left(B\left(\frac{2p-1-q}{1-q},\frac{2-2p}{1-q}\right)\right)^{\frac{1-q}{q}} s^{\frac{2p-1-q}{q}}(\xi^{\frac{1}{q}}(s)+\zeta^{\frac{1}{q}}(s))ds\right)\right\}^q$$

$$\le 2^{1-q}\big(3\bar{E}_1^2\mathbb{E}D^2[\varphi(0),\langle 0\rangle] + 6\bar{E}_2^2\frac{T^{2p}}{2p-1}(\bar{a}+\bar{M})(T+1)\big)$$

$$\cdot\exp\left(\frac{q^2}{2p-1}4^{\frac{1}{q}-1}\left(B\left(\frac{2p-1-q}{1-q},\frac{2-2p}{1-q}\right)\right)^{\frac{1-q}{q}}\big((6T\bar{E}_2^2(T+1)\bar{b})^{\frac{1}{q}} + (6T\bar{E}_2^2(T+1)\bar{c})^{\frac{1}{q}}\big)T^{\frac{2p-1}{q}}\right).$$

If $\mathbb{E}D^2[\varphi(0),\langle 0\rangle] \le \eta$, then $\mathbb{E}\sup_{0\le u\le t} D^2[\mathscr{X}(u),\langle 0\rangle] \le \varepsilon$ from (3.3). Therefore, system (2.1) is finite-time stable on considered domain. □

Remark 4.7. *Compared with* [3, 12, 38, 45]*, the method used to handel the delay is based on Lemma 2.12. In addition, finite-time stability of FFSDS hasn't been studied very perfectly and the obtained results are essentially new.*

5 Example

In current part, previous results were verified via an example.

Example 5.1. *Investigate the following FFSDS:*

$$\begin{cases} {}^{\mathscr{G}_r}\mathcal{D}^{0.8}\mathscr{X}(t) = A\mathscr{X}(t) + \frac{0.1}{\ln(t+e)}\mathscr{X}(t) + 0.1\sin\mathscr{X}(t-0.05) \\ \qquad + \left\langle \int_0^t 0.1e^{-t}\mathscr{X}(t) + 0.1\cos\mathscr{X}(t-0.05)dW(s)\right\rangle, t\in J := [0,1], \\ \mathscr{X}(t) = (\frac{\pi}{0.05}t, 0, -\frac{\pi}{0.05}t), t\in[-0.05,0], \end{cases} \tag{4.1}$$

where $A = 0.1I$, I is the 2-dimensional identity matrix, $\sigma(t,\mathscr{X}(t),\mathscr{X}(t-\delta)) = \frac{0.1}{\ln(t+e)}\mathscr{X}(t) + 0.1\sin\mathscr{X}(t-0.05)$, $g(s,\mathscr{X}(s),\mathscr{X}(s-\delta)) = 0.1e^{-t}\mathscr{X}(t) + 0.1\cos\mathscr{X}(t-0.05)$ and $\varphi(t) = (\frac{\pi}{0.05}t, 0, -\frac{\pi}{0.05}t)$. Taking $q = 0.4$ and $\bar{M} = 0.02$. By calculation, we get $\bar{E}_1 = 1.115$ and $\bar{E}_2 = 0.9793$. Obviously, assumption (A2) is satisfied. It is easy to verify that there exists a function $G(t,\mathscr{X},\mathscr{Y}) = 0.02\mathscr{X} + 0.02\mathscr{Y}$ satisfying the condition (A1). Then according to Theorem 4.3, we conclude that system (4.1) has a unique solution. By simple calculation, we get the following $\bar{a} = 0$, $\bar{b} = 0.02$, $\bar{c} = 0.02$. Taking $\eta = 0.1$ and $\varepsilon = 3$, we can verify that system (4.1) satisfies all the conditions in Theorem 4.6. According to the Definition 4.5 and Theorem 4.6, we conclude that system (4.1) is finite-time stable on $[-0.05, 1]$.

6 Conclusion

In current article, we study a kind of FFSDS under Granular derivative. By utilizing Laplace transformation and its inverse, we establish the equivalent form of FFSDS. Under some assumed criteria, Carathéodory approximation and contradiction method are applied to investigate the existence and uniqueness of solutions for FFSDS, respectively. Finite-time stability result is deduced by using a new type of generalized Grönwall delay inequality. Additionally, Itô isometry, Hölder inequality, the mean value theorem of integrals and fuzzy stochastic analysis methods are used in the derivation of the main results. In the end, an example is shown to evidence the validity of the derived main conclusions.

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