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Original paper

Ranking intuitionistic fuzzy numbers by relative preference relation

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Abstract

Ranking fuzzy numbers(FNs) was a critical issue in fuzzy computing field. Generally, triangular FNs, trapezoidal FNs, and even interval-valued FNs(IVFNs) were often expressed in ranking. However, ranking intuitionistic FNs(IFNs) were less mentioned due to the complicated components in membership functions. Herein, we will develop fuzzy binary relation that is an extended fuzzy preference relation(EFPR) to express the preference degree of two IFNs, and then the relation is improved to be a relative preference relation(RPR) used to rank a set of IFNs. Since EFPR on IFNs is a total ordering relation, RPR will be also a total ordering relation. Based on belonging and non-belonging components of membership functions in IFNs, using EFPR being also fuzzy preference relation(FPR) is suitable to compare FNs on pairwise, but time complexity on fuzzy operation of comparison computing is complicated. Hence, RPR is developed to avoid comparing on pairwise. Through yielding RPR values for a set of IFNs, IFNs are effectively and efficiently ranked to utilize related decision-making problems.

Keywords: Extended fuzzy preference relation(EFPR), intuitionistic fuzzy numbers(IFNs), preference degree, ranking, relative preference relation(RPR).

1 Introduction

Applying fuzzy numbers(FNs) had become a critical issue in fuzzy sets(FSs) since FSs proposed by Zadah [20]. In numerous applications of FNs, ranking [3, 4] them was an essential perspective after Jain [10], Dubois and Prade [7] indicated related concept. Hence, some methods were dedicated to the ranking of FNs. For instance, Lee et al. [13] ranked fuzzy values having satisfaction function, Fortemps and Roubens [9] ranked and defuzzified FNs based on area compensation, Cheng [5] used a distance method to rank FNs, Chen and Lu [4] ranked FNs through left and right fuzzy dominance, Chu and Tsao [6] ranked FNs having an area between the centroid point and original point, Wang [15] ranked triangular and trapezoidal FNs by relative preference relation(RPR), and then he [16] ranked triangular interval-valued FNs(IVIFNs) based on RPR. In these pasted approaches, intuitionistic FNs(IFNs) were scarcely mentioned due to the kind of FNs with membership, non-membership, and hesitation components. For instance, Koppula et al. [11] also utilized RPR for generalization and ranking of common FNs, but excluding IFNs. Although some tried to propose ranking methods for IFNs [12] or interval-valued IFNs(IVIFNs) [18, 19], ranking them was still complicated [14].

IFNs were from intuitionistic fuzzy sets(IFSs) first denoted by Atanassov [1], and IFSs were regarded as a generalization of Zadehs FSs [20]. The main difference between Atanassovs and Zadehs is in non-membership functions of FSs. As $A = \{(x, \mu_A(x), v_A(x))|x \in X\}$ is an intuitionistic fuzzy set(IFS), $\mu_A(x)$ and $v_A(x)$ respectively indicate membership function and non-membership function of x in A, where $0 \leq \mu_A(x) + v_A(x) \leq 1$. On the other hand, $A = \{(x, \mu_A(x), v_A(x))|x \in X\}$ is a Zadehs fuzzy set(FS), where $\mu_A(x)$ is the membership function of x in A, the non-membership function $v_A(x) = 1 - \mu_A(x)$, and thus $\mu_A(x) + v_A(x) = 1$. Due to $v_A(x) = 1 - \mu_A(x)$, A is usually simplified by $\{(x, \mu_A(x))|x \in X\}$. Moreover, Atanassov and Gargov [2] proposed an interval-valued IFS(IVIFS) $\widetilde{A} = \{(x, \widetilde{\mu}_A(x), \widetilde{v}_A(x))|x \in X\}$, where $\widetilde{\mu}_A(x) = [\mu_{A_L}(x), \mu_{A_U}(x)]$, $\widetilde{v}_A(x) = [v_{A_L}(x), v_{A_U}(x)]$, and $\widetilde{A} =$

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$\{(x, [\mu_{A_L}(x), \mu_{A_U}(x)], [v_{A_L}(x), v_{A_U}(x)])|x \in X\}$. In other words, the membership function and non-membership function $\widetilde{\mu}_A(x) and \widetilde{v}_A(x)$ were respectively represented by two closed intervals $[\mu_{A_L}(x), \mu_{A_U}(x)]$ and $[v_{A_L}(x), v_{A_U}(x)]$.

In the past, Lakshmana et al. [12], Xu [18], and Yue [19] simplified an interval-valued intuitionistic fuzzy number(IVIFN) as $([a_1, a_2], [a_3, a_4])$ for computation convenience they claimed. However, $([a_1, a_2], [a_3, a_4])$ representing above $\widetilde{A}$ is insufficient because x of $\widetilde{A}$ is missing. Furthermore, membership and non-membership functions expressed by two intervals $[a_1, a_2]$ and $[a_3, a_4]$ may be possible, but the two functions were not always shown by intervals. Therefore, the related computations of $([a_1, a_2], [a_3, a_4])$ including $\frac{a_1+a_2-a_3-a_4}{2}$ of score function and $\frac{a_1+a_2+a_3+a_4}{2}$ of accuracy function will be unnecessary for ranking FNs because a fuzzy number(FN) location is omitted for $([a_1, a_2], [a_3, a_4])$. Practically, membership and non-membership components of an IFN should be displayed according to corresponding function characteristics, such as $\{(x, \mu_A(x), v_A(x))|x \in X\}$. In this paper, extended fuzzy preference relation(EFPR) being also fuzzy preference relation(FPR) expanded on FNs is first proposed due to belonging and non-belonging attributes of membership functions for IFNs. Commonly, using FPR or EFPR is more suitable to rank FNs than defuzzification, but pairwise comparison operation of preference relation for FNs is complicated and difficult. Therefore, RPR is then developed to consider the characteristic of IFNs, and have the advantages of FPR (or EFPR) and defuzzification. In other words, RPR can present preference degree between two IFNs as similar as general preference relation, and rank IFNs through their corresponding crisp values as like as defuzzification. Herein, RPR is improved from EFPR. Through EFPR, two IFNs are compared on pairwise, whereas a set of IFNs are easily ranked by RPR to avoid pairwise comparison between two FNs.

For describing clearly, IFNs rationales are displayed in Section 2. Ranking computations of IFNs based on EFPR and RPR are indicated in Section 3. Several examples of ranking IFNs are shown in Section 4. Conclusions are described in final section.

2 Preliminaries

Herein, related rationales of IFNs are presented.

Definition 2.1. [20] *A Zadehs FS A of universe set U is indicated as a membership function* $\mu_A(x) \to [0, 1]$, $\mu_A(x)$, $\forall x \in U$, *denotes the degree of x in A. Obviously,* $\mu_A(x)$ *is a generalization of characteristic function for a crisp subset. A of U is characterized by a membership function with x representing the degree of membership of x in A. Therefore, A is shown as* $A = \{(x, \mu_A(x))|x \in U\}$.

Definition 2.2. *Let X be a non-empty universe of discourse. An IFS A indicated by Atanassov* [1] *is*

$$A = \{(x, \mu_A(x), v_A(x)), |x \in X\},$$

where $\mu_A(x) : X \to [0, 1]$ *and* $v_A(x) : X \to [0, 1]$ *are respectively the membership and non-membership degrees of the element* $x \in X$ *for satisfying* $0 \le \mu_A(x) + v_A(x) \le 1$. *Moreover,* $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ *denotes the hesitation degree(or called unknown degree) of x to A. Based on above, Atanassov supposed that a FS* [20] *was not equivalent to an IFS. Additionally, Atanassov and Gargov* [2] *recognized that ordinary FSs were special cases of IFSs for* $\pi_A(x) = 0$, *i.e.,* $v_A(x) = 1 - \mu_A(x)$.

Definition 2.3. *Let* $A = \{(x, \mu_A(x), v_A(x))|x \in X\}$ *and* $B = \{(x, \mu_B(x), v_B(x))|x \in X\}$ *be two IFSs. Based on Atanassovs* [1] *concepts, the related computations of IFSs are expressed as follows.*
(i) *The addition* $\bigoplus$ *of A and B is defined as*

$$A \bigoplus B = \{(x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), v_A(x)v_B(x))|x \in X\}.$$

(ii) *The multiplication* $\bigotimes$ *of A and B is denoted as*

$$A \bigotimes B = \{(x, \mu_A(x)\mu_B(x), v_A(x) + v_B(x) - v_A(x)v_B(x))|x \in X\}.$$

(iii) *The multiplication* $\bigotimes$ *of a real number* $n(> 0)$ *and A is indicated as*

$$n \bigoplus A = \{(x, 1 - (1 - \mu_A(x))^n, (v_A(x))^n)|x \in X\}.$$

Definition 2.4. *Let A be an IFN belonging to IFSs is denoted as*

$$\mu_A(x) = \begin{cases} \mu_A^L(x), a_1 \leq x < a_2 \\ \mu_A(x), a_2 \leq x \leq a_3 \\ \mu_A^U(x), a_3 < x \leq a_4 \\ 0, \ \ otherwise \end{cases},$$

is the membership function of A, and

$$v_A(x) = \begin{cases} v_A^L(x), a_1 \leq x < a_2 \\ v_A(x), a_2 \leq x \leq a_3 \\ v_A^U(x), a_3 < x \leq a_4 \\ 1, \ \ otherwise \end{cases},$$

is the non-membership function of A, where $a_1 \leq a_1, a_2 \leq a_2 \leq a_3 \leq a_3, a_4 \leq a_4 \in R$, $\mu_A^L(x)$ and $v_A^U(x)$ are non-decreasing continuous functions, whereas $\mu_A^U(x)$ and $v_A^L(x)$ are non-increasing continuous functions. For instance, an IFN A is shown as Figure 1, where $0 \leq \mu_A, v_A \leq 1$, $0 \leq \mu_A + v_A \leq 1$, and $a_1 \leq a_1 \leq a_2 \leq a_2 \leq a_3 \leq a_3 \leq a_4 \leq a_4$.

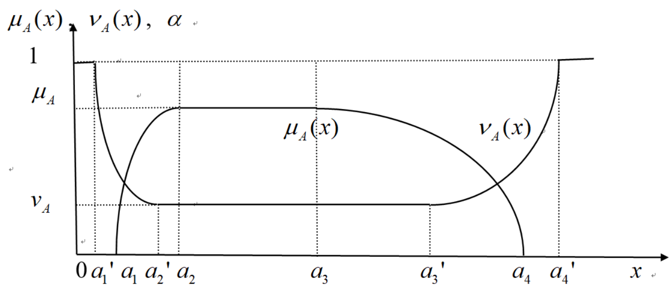

Figure 1: An IFN.

Definition 2.5. *Based on Definition 2.4, an intuitionistic trapezoidal FN(ITFN) A is assumed to be*

$$((a_1, a_2, a_3, a_4); (a_1, a_2, a_3, a_4); \mu_A, v_A),$$

is defined as

$$\mu_A(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}\mu_A, a_1 \leq x < a_2 \\ \mu_A, a_2 \leq x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3}\mu_A, a_3 < x \leq a_4 \\ 0, \ \ otherwise \end{cases},$$

is the membership function of A, and

$$v_A(x) = \begin{cases} \dfrac{a_2 - x + v_A(x - a_1)}{a_2 - a_1}, a_1 \leq x < a_2 \\ v_A, a_2 \leq x \leq a_3 \\ \dfrac{x - a_3 + v_A(a_4 - x)}{a_4 - a_3}, a_3 < x \leq a_4 \\ 1, \ \ otherwise \end{cases},$$

is the non-membership function of A, where $a_1 \leq a_1, a_2 \leq a_2 \leq a_3 \leq a_3, a_4 \leq a_4 \in R$. For instance, an ITFN A is displayed as Figure 2, where $0 \leq \mu_A, v_A \leq 1$, $0 \leq \mu_A + v_A \leq 1$, and $a_1 \leq a_1 \leq a_2 \leq a_2 \leq a_3 \leq a_3 \leq a_4 \leq a_4$.

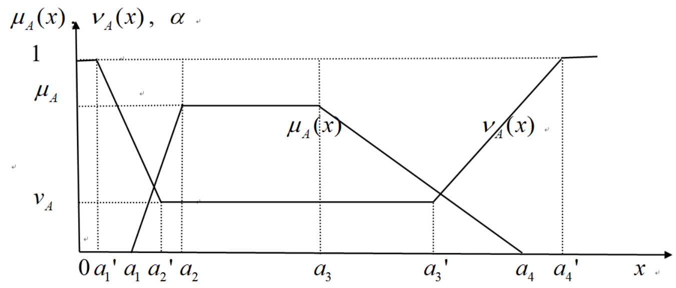

Figure 2: An ITFN.

Definition 2.6. [15] *A FPR R has a binary function $\mu_R(A,B)$ representing preference degree of FNs A over B.*
(i) *R is reciprocal iff $\mu_R(A,B) = 1 - \mu_R(B,A)$ for all FNs A and B.*
(ii) *R is transitive iff $\mu_R(A,B) \geq \frac{1}{2}$ and $\mu_R(B,C) \geq \frac{1}{2} \Rightarrow \mu_R(A,C) \geq \frac{1}{2}$ for all FNs A, B, and C.*
(iii) *R is a fuzzy total ordering iff R satisfies both (i) and (ii).*
Through R, A is preferred to B iff $\mu_R(A,B) > \frac{1}{2}$, and A is equal to B iff $\mu_R(A,B) = \frac{1}{2}$.

Definition 2.7. [17] *An EFPR R is also a FPR improved from Definition 2.6 to be a fuzzy subset of $\mathbf{R} \times \mathbf{R}$ with membership function $-\infty \leq \mu_R(A,B) \leq +\infty$ representing preference degree of FNs A over B.*
(i) R is reciprocal iff $\mu_R(A,B) = -\mu_R(B,A)$ for all FNs A and B.
(ii) R is transitive iff $\mu_R(A,B) \geq 0$ and $\mu_R(B,C) \geq 0 \Rightarrow \mu_R(A,C) \geq 0$ for all FNs A, B, and C.
(iii) R is additive iff $\mu_R(A,C) = \mu_R(A,B) + \mu_R(B,C)$.
(iv) R is a total ordering relation iff R satisfies (i), (ii), and (iii).
Based on R, A is larger than B iff $\mu_R(A,B) > 0$. Additionally, A and B are under non-difference iff $\mu_R(A,B) = 0$.

Evidently, EFPR R can be regarded as FPR R expanded. Through these definitions above, ranking computations of IFNs by EFPR and RPR will be developed.

Definition 2.8. [20] *The αcut of fuzzy set A is a crisp set $A_\alpha = \{x|\mu_A(x) \geq \alpha\}$.*

Definition 2.9. *Let A be an intuitionistic fuzzy number. Then $(A_\mu)^L_\alpha$, $(A_\mu)^U_\alpha$, $(A_v)^L_\alpha$, and $(A_v)^U_\alpha$ are respectively defined as $(A_\mu)^L_\alpha = inf_{\mu_A(x)\geq\alpha}(x)$, $(A_\mu)^U_\alpha = sup_{\mu_A(x)\geq\alpha}(x)$, $(A_v)^L_\alpha = inf_{v_A(x)\geq\alpha}(x)$, and $(A_v)^U_\alpha = sup_{v_A(x)\geq\alpha}(x)$. In addition, $A_\mu(\alpha) = (A_\mu)^L_\alpha + (A_\mu)^U_\alpha$ and $A_v(\alpha) = (A_v)^L_\alpha + (A_v)^U_\alpha$.*

3 Ranking computations of IFNs based on EFPR and RPR

Through the above definitions in Section 2, ranking computations of IFNs by EFPR and RPR are developed below.

Definition 3.1. *Let A and B be two IFNs indicated as Definition 2.4 and Definition 2.9. A binary relation F of A over B is expressed by $\mu_F(A,B)$ defined to be*

$$\mu_F(A,B) = \beta(\int_0^{\mu_A} A_\mu(\alpha)d\alpha - \int_0^{\mu_B} B_\mu(\alpha)d\alpha) + (1-\beta)(\int_{v_A}^1 A_v(\alpha)d\alpha - \int_{v_B}^1 B_v(\alpha)d\alpha),$$

where $0 \leq \beta \leq 1$, and μ_A, μ_B, v_A, and v_B are respectively the core values of $\mu_A(x)$, $\mu_B(x)$, $v_A(x)$, and $v_B(x)$. The binary relation F on IFNs will be an EFPR because it satisfies reciprocal, transitive, and additive laws. Related explanations and proofs through following lemmas are expressed.

Lemma 3.2. *F is reciprocal because $\mu_F(A,B) = -\mu_F(B,A)$ for all IFNs A and B.*

Proof.

$$\begin{aligned}\mu_F(A,B) &= \beta(\int_0^{\mu_A} A_\mu(\alpha)d\alpha - \int_0^{\mu_B} B_\mu(\alpha)d\alpha) + (1-\beta)(\int_{v_A}^1 A_v(\alpha)d\alpha - \int_{v_B}^1 B_v(\alpha)d\alpha) \\ &= -(\beta(\int_0^{\mu_B} B_\mu(\alpha)d\alpha - \int_0^{\mu_A} A_\mu(\alpha)d\alpha) + (1-\beta)(\int_{v_B}^1 B_v(\alpha)d\alpha - \int_{v_A}^1 A_v(\alpha)d\alpha)) \\ &= -\mu_F(B,A).\end{aligned}$$

□

Lemma 3.3. *F is transitive because* $\mu_F(A,B)\geq 0$ *and* $\mu_F(B,C)\geq 0 \Rightarrow \mu_F(A,C)\geq 0$ *for all IFNs A, B, and C.*

Proof. Since

$$\mu_F(A,B)=\beta(\int_0^{\mu_A}A_\mu(\alpha)d\alpha-\int_0^{\mu_B}B_\mu(\alpha)d\alpha)+(1-\beta)(\int_{v_A}^1 A_v(\alpha)d\alpha-\int_{v_B}^1 B_v(\alpha)d\alpha)\geq 0,$$

and

$$\mu_F(B,C)=\beta(\int_0^{\mu_B}B_\mu(\alpha)d\alpha-\int_0^{\mu_C}C_\mu(\alpha)d\alpha)+(1-\beta)(\int_{v_B}^1 B_v(\alpha)d\alpha-\int_{v_C}^1 C_v(\alpha)d\alpha)\geq 0,$$

we have

$$\begin{aligned}\mu_F(A,B)+\mu_F(B,C)&=\beta(\int_0^{\mu_A}A_\mu(\alpha)d\alpha-\int_0^{\mu_B}B_\mu(\alpha)d\alpha)+(1-\beta)(\int_{v_A}^1 A_v(\alpha)d\alpha-\int_{v_B}^1 B_v(\alpha)d\alpha)\\&\quad+\beta(\int_0^{\mu_B}B_\mu(\alpha)d\alpha-\int_0^{\mu_C}C_\mu(\alpha)d\alpha)+(1-\beta)(\int_{v_B}^1 B_v(\alpha)d\alpha-\int_{v_C}^1 C_v(\alpha)d\alpha)\\&=\beta(\int_0^{\mu_A}A_\mu(\alpha)d\alpha-\int_0^{\mu_B}B_\mu(\alpha)d\alpha+\int_0^{\mu_B}B_\mu(\alpha)d\alpha-\int_0^{\mu_C}C_\mu(\alpha)d\alpha)\\&\quad+(1-\beta)(\int_{v_A}^1 A_v(\alpha)d\alpha-\int_{v_B}^1 B_v(\alpha)d\alpha+\int_{v_B}^1 B_v(\alpha)d\alpha-\int_{v_C}^1 C_v(\alpha)d\alpha)\\&=\beta(\int_0^{\mu_A}A_\mu(\alpha)d\alpha-\int_0^{\mu_C}C_\mu(\alpha)d\alpha)+(1-\beta)(\int_{v_A}^1 A_v(\alpha)d\alpha-\int_{v_C}^1 C_v(\alpha)d\alpha)\\&=\mu_F(A,C).\end{aligned}$$

Due to $\mu_F(A,B)\geq 0$ and $\mu_F(B,C)\geq 0$, $\mu_F(A,C)\geq 0$ because $\mu_F(A,B)+\mu_F(B,C)=\mu_F(A,C)$. □

Lemma 3.4. *F is additive because* $\mu_F(A,C)=\mu_F(A,B)+\mu_F(B,C)$ *for all IFNs A, B, and C.*

Proof. Based on Lemma 3.3, $\mu_F(A,B)+\mu_F(B,C)=\mu_F(A,C)$. It is said that $\mu_F(A,C)=\mu_F(A,B)+\mu_F(B,C)$. □

Through Lemmas 3.2 to 3.4, the binary relation F on IFNs is an EFPR as Definition 2.7 described.

Lemma 3.5. *Let* $A=((a_1,a_2,a_3,a_4);(a_1,a_2,a_3,a_4);\mu_A,v_A)$ *and* $B=((b_1,b_2,b_3,b_4);(b_1,b_2,b_3,b_4);\mu_B,v_B)$ *be two intuitionistic trapezoidal FNs(ITFNs) shown as the same as Figure 2, where* $a_1\leq a_1,a_2\leq a_2\leq a_3\leq a_3,a_4\leq a_4$ *and* $b_1\leq b_1,b_2\leq b_2\leq b_3\leq b_3,b_4\leq b_4$*. The EFPR* $\mu_F(A,B)$ *of A over B is defined as*

$$\begin{aligned}\mu_F(A,B)&=\beta(\int_0^{\mu_A}A_\mu(\alpha)d\alpha-\int_0^{\mu_B}B_\mu(\alpha)d\alpha)+(1-\beta)(\int_{v_A}^1 A_v(\alpha)d\alpha-\int_{v_B}^1 B_v(\alpha)d\alpha)\\&=\beta(\frac{a_1+a_2+a_3+a_4}{2}\mu_A-\frac{b_1+b_2+b_3+b_4}{2}\mu_B)+(1-\beta)(\frac{a_1+a_2+a_3+a_4}{2}(1-v_A)-\\&\quad\frac{b_1+b_2+b_3+b_4}{2}(1-v_B)),\end{aligned}$$

where $0\leq\beta\leq 1$.

Definition 3.6. *Let* $A=((a_1,a_2,a_3,a_4);(a_1,a_2,a_3,a_4);\mu_A,v_A)$ *and* $B=((b_1,b_2,b_3,b_4);(b_1,b_2,b_3,b_4);\mu_B,v_B)$ *be two IFNs shown as Figure 1, where* $a_1\leq a_1,a_2\leq a_2\leq a_3\leq a_3,a_4\leq a_4$ *and* $b_1\leq b_1,b_2\leq b_2\leq b_3\leq b_3,b_4\leq b_4$*. Let S be the lower boundary of A and B expressed as* $((s_1,s_2,s_3,s_4);(s_1,s_2,s_3,s_4);\mu_S,v_S)$ *for* $s_1=min\{a_1,b_1\},s_2=min\{a_2,b_2\},s_3=min\{a_3,b_3\},s_4=min\{a_4,b_4\},s_1=min\{a_1,b_1\},s_2=min\{a_2,b_2\},s_3=min\{a_3,b_3\},s_4=min\{a_4,b_4\},$ $\mu_S=min\{\mu_A,\mu_B\},v_S=max\{v_A,v_B\}$*. Let R be the upper boundary of A and B expressed as* $((r_1,r_2,r_3,r_4);(r_1,r_2,r_3,r_4);\mu_R,v_R)$ *for* $r_1=max\{a_1,b_1\},r_2=max\{a_2,b_2\},r_3=max\{a_3,b_3\},r_4=max\{a_4,b_4\},r_1=max\{a_1,b_1\},r_2=max\{a_2,b_2\},r_3=max\{a_3,b_3\},r_4=max\{a_4,b_4\},\mu_R=max\{\mu_A,\mu_B\},v_R=min\{v_A,v_B\}$*. Practically, S and R are also used in three or more IFNs, not merely two. For a set X composed of IFNs* $X_1,X_2,,X_n$*, where* $X_i=((x_{i_1},x_{i_2},x_{i_3},x_{i_4});(x_{i_1},x_{i_2},x_{i_3},x_{i_4});\mu_{X_i},v_{X_i})$ *for* $i=1,2,,n$*. Then let* $S=((s_1,s_2,s_3,s_4);(s_1,s_2,s_3,s_4);\mu_S,v_S)$ *be the lower boundary of* $X_1,X_2,,X_n$*, for* $s_1=min_i\{x_{i1}\},s_2=min_i\{x_{i2}\},s_3=min_i\{x_{i3}\},s_4=min_i\{x_{i4}\},s_1=$

$min_i\{x_{i1}\}, s_2 = min_i\{x_{i2}\}, s_3 = min_i\{x_{i3}\}, s_4 = min_i\{x_{i4}\}, \mu_S = min_i\{\mu_{Xi}\}, v_S = max_i\{v_{Xi}\}$, and $i = 1, 2, , n$. Moreover, let $R = ((r_1, r_2, r_3, r_4); (r_1, r_2, r_3, r_4); \mu_R, v_R)$ be the upper boundary of $X_1, X_2, , X_n$, for $r_1 = max_i\{x_{i1}\}, r_2 = max_i\{x_{i2}\}, r_3 = max_i\{x_{i3}\}, r_4 = max_i\{x_{i4}\}, r_1 = max_i\{x_{i1}\}, r_2 = max_i\{x_{i2}\}, r_3 = max_i\{x_{i3}\}, r_4 = max_i\{x_{i4}\}, \mu_R = max_i\{\mu_{Xi}\}, v_R = min_i\{v_{Xi}\}$, and $i = 1, 2, , n$. Based on above, S and R will be the lower and upper boundaries of IFNs with a specifying scope. In other words, the two boundaries may be varied for different ranking set of FNs, but they are invariant within a fixed set composed of FNs.

Definition 3.7. *Let S and R be two boundaries of two IFNs A and B as shown as Figure 1, where $\mu_S(x)$ and $\mu_R(x)$ are respectively the membership functions of S and R, whereas $v_S(x)$ and $v_R(x)$ are respectively the non-membership functions of S and R. The boundary difference $D(R, S)$ between R and S will be classified into four varied situations below.*
(i) $D(R, S) = \beta(\int_0^{\mu_R} R_\mu(\alpha)d\alpha - \int_0^{\mu_S} S_\mu(\alpha)d\alpha) + (1-\beta)(\int_{v_R}^1 R_v(\alpha)d\alpha - \int_{v_S}^1 S_v(\alpha)d\alpha)$ *as* $r_1 \geq s_4$ *and* $r_1 \geq s_4$.
(ii) $D(R, S) = \beta(\int_0^{\mu_R} R_\mu(\alpha)d\alpha - \int_0^{\mu_S} S_\mu(\alpha)d\alpha + 2\mu_R(s_4 - r_1)) + (1-\beta)(\int_{v_R}^1 R_v(\alpha)d\alpha - \int_{v_S}^1 S_v(\alpha)d\alpha)$ *as* $r_1 < s_4$ *and* $r_1 \geq s_4$.
(iii) $D(R, S) = \beta(\int_0^{\mu_R} R_\mu(\alpha)d\alpha - \int_0^{\mu_S} S_\mu(\alpha)d\alpha) + (1-\beta)(\int_{v_R}^1 R_v(\alpha)d\alpha - \int_{v_S}^1 S_v(\alpha)d\alpha + 2(1-v_R)(s_4 - r_1))$ *as* $r_1 \geq s_4$ *and* $r_1 < s_4$.
(iv) $D(R, S) = \beta(\int_0^{\mu_R} R_\mu(\alpha)d\alpha - \int_0^{\mu_S} S_\mu(\alpha)d\alpha + 2\mu_R(s_4 - r_1)) + (1-\beta)(\int_{v_R}^1 R_v(\alpha)d\alpha - \int_{v_S}^1 S_v(\alpha)d\alpha + 2(1-v_R)(s_4 - r_1))$ *as $r_1 < s_4$ and $r_1 < s_4$. Evidently, $D(R, S)$ must be a positive value. Moreover, related computations of $D(R, S)$ for ITFNs in the above four varied situations are denoted in Lemma 5.*

Lemma 3.8. *Let $S = ((s_1, s_2, s_3, s_4); (s_1, s_2, s_3, s_4); \mu_S, v_S)$ and $R = ((r_1, r_2, r_3, r_4); (r_1, r_2, r_3, r_4); \mu_R, v_R)$ be two boundaries of ITFNs A and B as shown as Figure 2. The boundary difference $D(R, S)$ between R and S will be classified into four varied situations below.*
(i) $D(R, S) = \beta(\frac{r_1+r_2+r_3+r_4}{2}\mu_R - \frac{s_1+s_2+s_3+s_4}{2}\mu_S) + (1-\beta)(\frac{r_1+r_2+r_3+r_4}{2}(1-v_R) - \frac{s_1+s_2+s_3+s_4}{2}(1-v_S))$ *as* $r_1 \geq s_4$ *and* $r_1 \geq s_4$.
(ii) $D(R, S) = \beta(\frac{r_1+r_2+r_3+r_4}{2}\mu_R - \frac{s_1+s_2+s_3+s_4}{2}\mu_S + 2\mu_R(s_4 - r_1)) + (1-\beta)(\frac{r_1+r_2+r_3+r_4}{2}(1-v_R) - \frac{s_1+s_2+s_3+s_4}{2}(1-v_S))$ *as* $r_1 < s_4$ *and* $r_1 \geq s_4$.
(iii) $D(R, S) = \beta(\frac{r_1+r_2+r_3+r_4}{2}\mu_R - \frac{s_1+s_2+s_3+s_4}{2}\mu_S) + (1-\beta)(\frac{r_1+r_2+r_3+r_4}{2}(1-v_R) - \frac{s_1+s_2+s_3+s_4}{2}(1-v_S) + 2(1-v_R)(s_4 - r_1))$ *as* $r_1 \geq s_4$ *and* $r_1 < s_4$.
(iv) $D(R, S) = \beta(\frac{r_1+r_2+r_3+r_4}{2}\mu_R - \frac{s_1+s_2+s_3+s_4}{2}\mu_S + 2\mu_R(s_4 - r_1)) + (1-\beta)(\frac{r_1+r_2+r_3+r_4}{2}(1-v_R) - \frac{s_1+s_2+s_3+s_4}{2}(1-v_S) + 2(1-v_R)(s_4 - r_1))$ *as* $r_1 < s_4$ *and* $r_1 < s_4$.

Definition 3.9. *A binary relation F of A over B is expressed by $\mu_F(A, B) = \frac{1}{2}(\frac{\mu_F(A,B)}{D(R,S)} + 1)$, where S and R be two boundaries for a set of IFNs including A and B within the set. Since $\mu_F(A, B) \leq D(R, S)$, $-1 \leq \frac{\mu_F(A,B)}{D(R,S)} \leq 1$ and then $0 \leq \mu_F(A, B) \leq 1$. Through the descriptions of Definition 3.2, $\mu_F(A, B)$ may be different in varied sets of FNs, but $\mu_F(A, B)$ is fixed for a indicating set composed of FNs. Owing to the binary relation F improved from F, F on IFNs will be a FPR for it being reciprocal and transitive. Related explanations and proofs according to following lemmas are indicated.*

Lemma 3.10. *F is reciprocal because $\mu_F(A, B) = 1 - \mu_F(B, A)$ for all IFNs A and B in a specify set.*

Proof. Let S and R be two boundaries of ITFNs A and B in a specify set. Due to $\mu_F(A, B) = -\mu_F(B, A)$ in Lemma 3.2,

$$\mu_F(A, B) = \frac{1}{2}(\frac{\mu_F(A, B)}{D(R, S)} + 1) = \frac{1}{2} \times \frac{\mu_F(A, B)}{D(R, S)} + \frac{1}{2} = 1 - \frac{1}{2} \times \frac{\mu_F(B, A)}{D(R, S)} - \frac{1}{2}$$
$$= 1 - \frac{1}{2}(\frac{\mu_F(B, A)}{D(R, S)} + 1) = 1 - \mu_F(B, A).$$

□

Lemma 3.11. *F is transitive because $\mu_F(A, B) \geq \frac{1}{2}$ and $\mu_F(B, C) \geq \frac{1}{2} \Rightarrow \mu_F(A, C) \geq \frac{1}{2}$ for all IFNs A, B, and C in a specify set.*

Proof. Let S and R be two boundaries for IFNs A, B, and C in a specify set. $\mu_F(A, B) = \frac{1}{2}(\frac{\mu_F(A,B)}{D(R,S)} + 1) \geq \frac{1}{2}$ denotes $\mu_F(A, B) \geq 0$. Similarly, $\mu_F(B, C) \geq \frac{1}{2}$ indicates $\mu_F(B, C) \geq 0$. Based on Lemma 3.3, $\mu_F(A, B) \geq 0$ and $\mu_F(B, C) \geq 0 \Rightarrow \mu_F(A, C) \geq 0$. Therefore, $\mu_F(A, C) = \frac{1}{2}(\frac{\mu_F(A,C)}{D(R,S)} + 1) \geq \frac{1}{2}$ because $\mu_F(A, C) \geq 0$. □

Through Lemmas 3.10 and 3.11, the binary relation F on IFNs is a FPR as Definition 2.6 described. Evidently, F is more complicated on computation than the EFPR F. However, the contribution of F is in ranking a set of FNs, whereas F is mainly used for two FNs on pairwise comparison.

Definition 3.12. *Let a set X consist of IFNs $X_1, X_2, , X_n$, where $X_i = ((x_{i1}, x_{i2}, x_{i3}, x_{i4}); (x_{i1}, x_{i2}, x_{i3}, x_{i4}); \mu_{X_i}, v_{X_i})$ for $i = 1, 2, , n$. The mean of $X_1, X_2, , X_n$ is defined as $\bar{X} = ((\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4); (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4); \mu_{\bar{X}_i}, v_{\bar{X}_i})$, where $\bar{x}_1 = \frac{1}{n}\sum_{i=1}^{n} x_{i1}$, $\bar{x}_2 = \frac{1}{n}\sum_{i=1}^{n} x_{i2}$, $\bar{x}_3 = \frac{1}{n}\sum_{i=1}^{n} x_{i3}$, $\bar{x}_4 = \frac{1}{n}\sum_{i=1}^{n} x_{i4}$, $\bar{x}_1 = \frac{1}{n}\sum_{i=1}^{n} x_{i1}$, $\bar{x}_2 = \frac{1}{n}\sum_{i=1}^{n} x_{i2}$, $\bar{x}_3 = \frac{1}{n}\sum_{i=1}^{n} x_{i3}$, $\bar{x}_4 = \frac{1}{n}\sum_{i=1}^{n} x_{i4}$, $\mu_{\bar{X}} = 1 - (\prod_{i=1}^{n}(1 - \mu_{X_i}))^{\frac{1}{n}}$, and $v_{\bar{X}} = (\prod_{i=1}^{n} v_{X_i})^{\frac{1}{n}}$. Practically, the mean computation of $X_1, X_2, , X_n$ totally conform to the rule of Definition 2.3.*

Definition 3.13. *Let $\bar{X} = ((\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4); (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4); \mu_{\bar{X}_i}, v_{\bar{X}_i})$ be the mean of IFNs $X_1, X_2, , X_n$ in a set X, where $X_i = ((x_{i1}, x_{i2}, x_{i3}, x_{i4}); (x_{i1}, x_{i2}, x_{i3}, x_{i4}); \mu_{X_i}, v_{X_i})$ for $i = 1, 2, , n$. S and R are assumed to be the lower and upper boundaries of $X_1, X_2, , X_n$. Herein, the RPR of X_i over $\bar{X}$ in the set X is defined as $\mu_F(X_i, \bar{X}) = \frac{1}{2}(\frac{\mu_F(X_i,\bar{X})}{D(R,S)} + 1)$ for $i = 1, 2, , n$. Evidently, $0 \le \mu_F(X_i, \bar{X}) \le 1$, and the greater value of RPR denotes the larger IFNs. Therefore, $X_1, X_2, , X_n$ are easily ranked according to their corresponding RPR values $\mu_F(X_1, \bar{X}), \mu_F(X_2, \bar{X}), , \mu_F(X_n, \bar{X})$. Furthermore, $\mu_F(X_i, \bar{X}) = \mu_F(X_j, \bar{X})$ represents that $\frac{1}{2}(\frac{\mu_F(X_i,\bar{X})}{D(R,S)} + 1) = \frac{1}{2}(\frac{\mu_F(X_j,\bar{X})}{D(R,S)} + 1)$, and thus $\mu_F(X_i, \bar{X}) = \mu_F(X_j, \bar{X})$ for $1 \le i, j \le n$ and $i \ne j$. Herein, $\mu_F(X_i, \bar{X}) = \mu_F(X_j, \bar{X})$ indicates that the EFPR values of X_i and X_j over $\bar{X}$ are equivalent. It is said that X_i and X_j are undifferentiated over $\bar{X}$ in preference, and the undifferentiated situation is denoted by $X_i \sim X_j$.*

4 Numerical examples

To demonstrate the ranking computations of IFNs based on EFPR and RPR, these numerical examples are described below.

In the first example,

$$A = ((5, 7, 9, 11); (4, 6, 10, 12); 0.7, 0.15) \text{ and } C = ((-11, -9, -7, -5); (-12, -10, -6, -4); 0.5, 0.35),$$

are two IFNs. Intuitively, A is larger than C, i.e., $A > C$ according to their locations.

Based on EFPR,

$$\begin{aligned}\mu_F(A, C) = \beta(&\frac{5+7+9+11}{2}(0.7) - \frac{(-11)+(-9)+(-7)+(-5)}{2}(0.5)) + (1-\beta)(\frac{4+6+10+12}{2}(1-0.15)\\ &- \frac{(-12)+(-10)+(-6)+(-4)}{2}(1-0.35))\\ =\ & 19.2\beta + 24(1-\beta)\\ =\ & 24 - 4.8\beta.\end{aligned}$$

Due to $0 \le \beta \le 1, \mu_F(A, C) > 0$ and thus $A > C$. Ranking result is consistent with above intuition. For any value of $\beta \in [0, 1]$, A is always superior to C.

In the second example,

$$A = ((5, 7, 9, 11); (4, 6, 10, 12); 0.7, 0.15),\ B = ((-3, -1, 1, 3); (-4, -2, 2, 4); 0.6, 0.25),$$

and

$$C = ((-11, -9, -7, -5); (-12, -10, -6, -4); 0.5, 0.35),$$

are three IFNs. Intuitively, the ranking result should be $A > B > C$ due to their corresponding locations.

Through EFPR,

$$\begin{aligned}\mu_F(A, B) = \beta(&\frac{5+7+9+11}{2}(0.7) - \frac{(-3)+(-1)+1+3}{2}(0.6)) + (1-\beta)(\frac{4+6+10+12}{2}(1-0.15)\\ &- \frac{(-4)+(-2)+2+4}{2}(1-0.25))\\ =\ & 11.2\beta + 13.6(1-\beta)\\ =\ & 13.6 - 2.4\beta.\end{aligned}$$

Obviously, $\mu_F(A,B)>0$ and thus $A>B$ for $0\le\beta\le 1$. Moreover, $\mu_F(A,C)=24-4.8\beta$ according to the computations of the first example and $A>C$. Besides,

$$\begin{aligned}\mu_F(B,C)&=\beta(\frac{(-3)+(-1)+1+3}{2}(0.6)-\frac{(-11)+(-9)+(-7)+(-5)}{2}(0.5))+(1-\beta)(\frac{(-4)+(-2)+2+4}{2}(1-0.25)\\&\quad-\frac{(-12)+(-10)+(-6)+(-4)}{2}(1-0.35))\\&=8\beta+10.4(1-\beta)\\&=10.4-2.4\beta.\end{aligned}$$

Evidently, $\mu_F(B,C)>0$ and thus $B>C$ for $0\le\beta\le 1$. Owing to the previous results, the final result is $A>B>C$. Ranking result is consistent with above intuition.

Based on EFPR, A, B, and C are compared on pairwise. Further, they are viewed within a set and sorted by RPR. For A, B, and C in a specifying set,

$$R=((5,7,9,11);(4,6,10,12);0.7,0.15),\ \text{and}\ S=((-11,-9,-7,-5);(-12,-10,-6,-4);0.5,0.35).$$

Therefore, $\mu_F(R,S)=24-4.8\beta$. Moreover, the mean of three FNs is computed as

$$\bar{M}=((-3,-1,1,3);(-4,-2,2,4);0.609,0.236).$$

Then

$$\begin{aligned}\mu_F(A,\bar{M})&=\beta(\frac{5+7+9+11}{2}(0.7)-\frac{(-3)+(-1)+1+3}{2}(0.609))+(1-\beta)(\frac{4+6+10+12}{2}(1-0.15)\\&\quad-\frac{(-4)+(-2)+2+4}{2}(1-0.236))\\&=11.2\beta+13.6(1-\beta)\\&=13.6-2.4\beta>0,\end{aligned}$$

$$\begin{aligned}\mu_F(B,\bar{M})&=\beta(\frac{(-3)+(-1)+1+3}{2}(0.6)-\frac{(-3)+(-1)+1+3}{2}(0.609))+(1-\beta)(\frac{(-4)+(-2)+2+4}{2}(1-0.25)\\&\quad-\frac{(-4)+(-2)+2+4}{2}(1-0.236))=0,\end{aligned}$$

and

$$\begin{aligned}\mu_F(C,\bar{M})&=\beta(\frac{(-11)+(-9)+(-7)+(-5)}{2}(0.5)-\frac{(-3)+(-1)+1+3}{2}(0.609))\\&\quad+(1-\beta)(\frac{(-12)+(-10)+(-6)+(-4)}{2}(1-0.35)-\frac{(-4)+(-2)+2+4}{2}(1-0.236))\\&=-8\beta-10.4(1-\beta)\\&=-10.4+2.4\beta<0.\end{aligned}$$

Therefore,

$$\mu_F(A,\bar{M})=\frac{1}{2}(\frac{13.6-2.4\beta}{24-4.8\beta}+1)>\frac{1}{2},\quad \mu_F(B,\bar{M})=\frac{1}{2}(\frac{0}{24-4.8\beta}+1)=\frac{1}{2},$$

and

$$\mu_F(C,\bar{M})=\frac{1}{2}(\frac{-10.4+2.4\beta}{24-4.8\beta}+1)<\frac{1}{2}.$$

According to RPR values of $\mu_F(A,\bar{M})$, $\mu_F(B,\bar{M})$, and $\mu_F(C,\bar{M})$, $A>B>C$. Ranking result is consistent with above intuition and pairwise comparison computations by EFPR.

Practically, the computations of EFPR and RPR are similar, but time complexity of RPR is simpler on fuzzy operation than EFPR for numerous IFNs. To n FNs, time complexity of fuzzy operation by RPR is $O(n)$, whereas time complexity of fuzzy operation by EFPR is $O(n^2)$. The above two examples using special values are illustrated to emphasize the rationality of EFPR and RPR. Moreover, these special values present the same preference orders for all values of β. Sometimes, varied values of β may determine different ranking results of IFNs. The following examples will indicate these descripting situations.

In the third example,

$$X_1 = ((0.3, 0.4, 0.5, 0.6); (0.3, 0.4, 0.5, 0.6); 0.4, 0.3), \quad X_2 = ((0.2, 0.3, 0.4, 0.5); (0.2, 0.3, 0.4, 0.5); 0.6, 0.15),$$

$$X_3 = ((0.2, 0.3, 0.5, 0.6); (0.2, 0.3, 0.5, 0.6); 0.25, 0.45), \quad \text{and} \quad X_4 = ((0.6, 0.7, 0.8, 0.9); (0.6, 0.7, 0.8, 0.9); 0.35, 0.3),$$

are four IFNs within a set X. In the set X,

$$R = ((0.6, 0.7, 0.8, 0.9); (0.6, 0.7, 0.8, 0.9); 0.6, 0.15), \; S = ((0.2, 0.3, 0.4, 0.5); (0.2, 0.3, 0.4, 0.5); 0.25, 0.45),$$

and $\mu_F(R, S) = 0.89 - 0.615\beta$.

Moreover, the mean of four IFNs is derived as $\bar{X} = ((0.325, 0.425, 0.55, 0.65), (0.325, 0.43, 0.55, 0.65), 0.415, 0.279)$. Then

$$\begin{aligned}\mu_F(X_1, \bar{X}) &= \beta(\frac{0.3+0.4+0.5+0.6}{2}(0.4) - \frac{0.325+0.43+0.55+0.65}{2}(0.415)) + (1-\beta)(\frac{0.3+0.4+0.5+0.6}{2}(1-0.3) \\ &\quad - \frac{0.325+0.43+0.55+0.65}{2}(1-0.279)) \\ &= -0.073 + 0.028\beta,\end{aligned}$$

$$\begin{aligned}\mu_F(X_2, \bar{X}) &= \beta(\frac{0.2+0.3+0.4+0.5}{2}(0.6) - \frac{0.325+0.43+0.55+0.65}{2}(0.415)) + (1-\beta)(\frac{0.2+0.3+0.4+0.5}{2}(1-0.15) \\ &\quad - \frac{0.325+0.43+0.55+0.65}{2}(1-0.279)) \\ &= -0.108 + 0.123\beta,\end{aligned}$$

$$\begin{aligned}\mu_F(X_3, \bar{X}) &= \beta(\frac{0.2+0.3+0.5+0.6}{2}(0.25) - \frac{0.325+0.43+0.55+0.65}{2}(0.415)) + (1-\beta)(\frac{0.2+0.3+0.5+0.6}{2}(1-0.45) \\ &\quad - \frac{0.325+0.43+0.55+0.65}{2}(1-0.279)) \\ &= -0.263 + 0.058\beta,\end{aligned}$$

and

$$\begin{aligned}\mu_F(X_4, \bar{X}) &= \beta(\frac{0.6+0.7+0.8+0.9}{2}(0.35) - \frac{0.325+0.43+0.55+0.65}{2}(0.415)) + (1-\beta)(\frac{0.6+0.7+0.8+0.9}{2}(1-0.3) \\ &\quad - \frac{0.325+0.43+0.55+0.65}{2}(1-0.279)) \\ &= 0.347 - 0.227\beta.\end{aligned}$$

Therefore,

$$\mu_F(X_1, \bar{X}) = \frac{1}{2}(\frac{-0.073+0.028\beta}{0.89-0.615\beta} + 1) < \frac{1}{2}, \; \mu_F(X_2, \bar{X}) = \frac{1}{2}(\frac{-0.108+0.123\beta}{0.89-0.615\beta} + 1),$$

$$\mu_F(X_3, \bar{X}) = \frac{1}{2}(\frac{-0.263+0.058\beta}{0.89-0.615\beta} + 1) < \frac{1}{2}, \; \text{and} \; \mu_F(X_4, \bar{X}) = \frac{1}{2}(\frac{0.347-0.227\beta}{0.89-0.615\beta} + 1) > \frac{1}{2}.$$

For $0 \leq \beta \leq 1$, β are classified into eleven varied values that are 0,0.1,,1. According to these varied values of β, the corresponding computations of RPR and ranking results for X_1, X_2, X_3, and X_4 are presented in Table 1.

Table 1. Corresponding computations and ranking results for X_1, X_2, X_3, and X_4

β	$\mu_F(X_1, \bar{X})$	$\mu_F(X_2, \bar{X})$	$\mu_F(X_3, \bar{X})$	$\mu_F(X_4, \bar{X})$	Ranking
0	0.4591	0.4394	0.3524	0.6951	$X_4 > X_1 > X_2 > X_3$
0.1	0.4599	0.4453	0.3529	0.6857	$X_4 > X_1 > X_2 > X_3$
0.2	0.4608	0.4515	0.3534	0.6761	$X_4 > X_1 > X_2 > X_3$
0.3	0.4617	0.4578	0.3540	0.6660	$X_4 > X_1 > X_2 > X_3$
0.4	0.4626	0.4645	0.3546	0.6556	$X_4 > X_2 > X_1 > X_3$
0.5	0.4636	0.4713	0.3552	0.6447	$X_4 > X_2 > X_1 > X_3$
0.6	0.4646	0.4785	0.3559	0.6334	$X_4 > X_2 > X_1 > X_3$
0.7	0.4657	0.4860	0.3566	0.6216	$X_4 > X_2 > X_1 > X_3$
0.8	0.4668	0.4938	0.3573	0.6093	$X_4 > X_2 > X_1 > X_3$
0.9	0.4679	0.5020	0.3580	0.5964	$X_4 > X_2 > X_1 > X_3$
1	0.4691	0.5105	0.3588	0.5829	$X_4 > X_2 > X_1 > X_3$

Evidently, three varied situations are denoted below.
(i) $X_4 > X_1 > X_2 > X_3$ if $0 \leq \beta < \frac{7}{19}$,
(ii) $X_4 > X_1 \sim X_2 > X_3$ if $\beta = \frac{7}{19} \approx 0.3684$, or
(iii) $X_4 > X_2 > X_1 > X_3$ if $\frac{7}{19} < \beta \leq 1$, where $\sim$ indicates non-difference between two FNs.

In the fourth example,

$$\begin{aligned}
Y_1 &= ((0.38, 0.48, 0.57, 0.66); (0.38, 0.48, 0.57, 0.66); 0.475, 0.25),\\
Y_2 &= ((0.15, 0.24, 0.33, 0.44); (0.15, 0.24, 0.33, 0.44); 0.55, 0.175),\\
Y_3 &= ((0.52, 0.62, 0.76, 0.82); (0.52, 0.62, 0.76, 0.82); 0.575, 0.22),\\
Y_4 &= ((0.44, 0.56, 0.70, 0.84); (0.44, 0.56, 0.70, 0.84); 0.205, 0.43),
\end{aligned}$$

are four IFNs within a set Y. In the set Y,

$$R = ((0.52, 0.62, 0.76, 0.84); (0.52, 0.62, 0.76, 0.84); 0.59, 0.175),$$

$$S = ((0.15, 0.24, 0.33, 0.44); (0.15, 0.24, 0.33, 0.44); 0.205, 0.43),$$

and $\mu_F(R, S) = 0.7997 - 0.1103\beta$.
In addition, the mean of four IFNs is yielded as $\bar{Y} = ((0.3725, 0.475, 0.59, 0.69), (0.3725, 0.475, 0.59, 0.69), 0.468, 0.254)$.
Then

$$\begin{aligned}
\mu_F(Y_1, \bar{Y}) = \beta(&\frac{0.38 + 0.48 + 0.57 + 0.66}{2}(0.475) - \frac{0.3725 + 0.475 + 0.59 + 0.69}{2}(0.468))\\
&+ (1-\beta)(\frac{0.38 + 0.48 + 0.57 + 0.66}{2}(1 - 0.25) - \frac{0.3725 + 0.475 + 0.59 + 0.69}{2}(1 - 0.254))\\
= &-0.0102 + 0.0082\beta,
\end{aligned}$$

$$\begin{aligned}
\mu_F(Y_2, \bar{Y}) = \beta(&\frac{0.15 + 0.24 + 0.33 + 0.44}{2}(0.55) - \frac{0.3725 + 0.475 + 0.59 + 0.69}{2}(0.468))\\
&+ (1-\beta)(\frac{0.15 + 0.24 + 0.33 + 0.44}{2}(1 - 0.175) - \frac{0.3725 + 0.475 + 0.59 + 0.69}{2}(1 - 0.254))\\
= &-0.3154 + 0.1361\beta,
\end{aligned}$$

$$\begin{aligned}
\mu_F(Y_3, \bar{Y}) = \beta(&\frac{0.52 + 0.62 + 0.76 + 0.82}{2}(0.575) - \frac{0.3725 + 0.475 + 0.59 + 0.69}{2}(0.468))\\
&+ (1-\beta)(\frac{0.52 + 0.62 + 0.76 + 0.82}{2}(1 - 0.22) - \frac{0.3725 + 0.475 + 0.59 + 0.69}{2}(1 - 0.254))\\
= &0.2669 + 0.0168\beta,
\end{aligned}$$

and

$$\begin{aligned}
\mu_F(Y_4, \bar{Y}) = \beta(&\frac{0.44 + 0.56 + 0.7 + 0.84}{2}(0.205) - \frac{0.3725 + 0.475 + 0.59 + 0.69}{2}(0.468))\\
&+ (1-\beta)(\frac{0.44 + 0.56 + 0.7 + 0.84}{2}(1 - 0.43) - \frac{0.3725 + 0.475 + 0.59 + 0.69}{2}(1 - 0.254))\\
= &-0.07 - 0.1679\beta.
\end{aligned}$$

Therefore,

$$\mu_F(Y_1, \bar{Y}) = \frac{1}{2}(\frac{-0.0102 + 0.0082\beta}{0.7997 - 0.1103\beta} + 1), \mu_F(Y_2, \bar{Y}) = \frac{1}{2}(\frac{-0.3154 + 0.1361\beta}{0.7997 - 0.1103\beta} + 1) < \frac{1}{2},$$

$$\mu_F(Y_3, \bar{Y}) = \frac{1}{2}(\frac{0.2669 + 0.0168\beta}{0.7997 - 0.1103\beta} + 1) > \frac{1}{2}, \text{ and } \mu_F(Y_4, \bar{Y}) = \frac{1}{2}(\frac{-0.07 - 0.1679\beta}{0.7997 - 0.1103\beta} + 1) < \frac{1}{2}.$$

For $0 \leq \beta \leq 1$, β are classified into eleven varied values that are 0,0.1,,1. According to these varied values of β, the corresponding computations of RPR and ranking results for Y_1, Y_2, Y_3, and Y_4 are expressed in Table 2.

Table 2. Corresponding computations and ranking results for Y_1, Y_2, Y_3, and Y_4

β	$\mu_F(Y_1,\bar{Y})$	$\mu_F(Y_2,\bar{Y})$	$\mu_F(Y_3,\bar{Y})$	$\mu_F(Y_4,\bar{Y})$	Ranking
0	0.4936	0.3028	0.6669	0.4562	$Y_3 > Y_1 > Y_4 > Y_2$
0.1	0.4941	0.3086	0.6703	0.4449	$Y_3 > Y_1 > Y_4 > Y_2$
0.2	0.4945	0.3147	0.6738	0.4334	$Y_3 > Y_1 > Y_4 > Y_2$
0.3	0.4950	0.3209	0.6773	0.4215	$Y_3 > Y_1 > Y_4 > Y_2$
0.4	0.4954	0.3273	0.6810	0.4092	$Y_3 > Y_1 > Y_4 > Y_2$
0.5	0.4959	0.3339	0.6849	0.3966	$Y_3 > Y_1 > Y_4 > Y_2$
0.6	0.4964	0.3406	0.6888	0.3836	$Y_3 > Y_1 > Y_4 > Y_2$
0.7	0.4969	0.3476	0.6928	0.3702	$Y_3 > Y_1 > Y_4 > Y_2$
0.8	0.4975	0.3548	0.6970	0.3564	$Y_3 > Y_1 > Y_4 > Y_2$
0.9	0.4980	0.3623	0.7013	0.3421	$Y_3 > Y_1 > Y_2 > Y_4$
1	0.4986	0.3699	0.7057	0.3274	$Y_3 > Y_1 > Y_2 > Y_4$

Therefore, three varied situations are presented in the following.
(i) $Y_3 > Y_1 > Y_4 > Y_2$ if $0 \le \beta < \frac{0.1227}{0.1520}$,
(ii) $Y_3 > Y_1 > Y_4 \sim Y_2$ if $\beta = \frac{0.1227}{0.1520} \approx 0.8072$, or
(iii) $Y_3 > Y_1 > Y_2 > Y_4$ if $\frac{0.1227}{0.1520} < \beta \le 1$.

In the fifth example, a chief executive officer had to tackle an investment for his enterprise, and thus he required three senior managers to present relative ratings within 0 to 1 for five feasible investment programs Z_1, Z_2, Z_3, Z_4, and Z_5. To these investment programs, three managers considered belongingness and non-belongingness of profit to propose overall assessments in the following.
$Z_{11} = ((0.6, 0.7, 0.8, 0.9); (0.6, 0.7, 0.8, 0.9); 0.35, 0.3)$,
$Z_{12} = ((0.4, 0.5, 0.6, 0.7); (0.4, 0.5, 0.6, 0.7); 0.4, 0.4)$,
$Z_{13} = ((0.3, 0.5, 0.6, 0.7); (0.3, 0.5, 0.6, 0.7); 0.45, 0.55)$,
$Z_{21} = ((0.2, 0.3, 0.5, 0.6); (0.2, 0.3, 0.5, 0.6); 0.25, 0.45)$,
$Z_{22} = ((0.3, 0.4, 0.5, 0.6); (0.3, 0.4, 0.5, 0.6); 0.4, 0.35)$,
$Z_{23} = ((0.1, 0.3, 0.4, 0.5); (0.1, 0.3, 0.4, 0.5); 0.35, 0.5)$,
$Z_{31} = ((0.4, 0.5, 0.6, 0.7); (0.4, 0.5, 0.6, 0.7); 0.3, 0.4)$,
$Z_{32} = ((0.6, 0.7, 0.8, 0.9); (0.6, 0.7, 0.8, 0.9); 0.3, 0.3)$,
$Z_{33} = ((0.2, 0.3, 0.5, 0.6); (0.2, 0.3, 0.5, 0.6); 0.6, 0.3)$,
$Z_{41} = ((0.3, 0.4, 0.5, 0.6); (0.3, 0.4, 0.5, 0.6); 0.4, 0.3)$,
$Z_{42} = ((0.2, 0.3, 0.4, 0.5); (0.2, 0.3, 0.4, 0.5); 0.65, 0.1)$,
$Z_{43} = ((0.5, 0.7, 0.8, 0.9); (0.5, 0.7, 0.8, 0.9); 0.35, 0.35)$,
$Z_{51} = ((0.2, 0.3, 0.4, 0.5); (0.2, 0.3, 0.4, 0.5); 0.6, 0.15)$,
$Z_{52} = ((0.1, 0.4, 0.6, 0.9); (0.1, 0.4, 0.6, 0.9); 0.25, 0.5)$, and
$Z_{53} = ((0.3, 0.4, 0.5, 0.6); (0.3, 0.4, 0.5, 0.6); 0.45, 0.45)$,
where Z_{tq} denoted the rating of investment program t assessed by manager q for t=1,2,,5; q=1,2,3. Let Z_t be the mean of Z_{t1}, Z_{t2}, Z_{t3} indicated as Definition 3.5. Therefore,
$Z_1 = ((0.4333, 0.5667, 0.6667, 0.7667); (0.4333, 0.5667, 0.6667, 0.7667); 0.4014, 0.4041)$,
$Z_2 = ((0.2000, 0.3333, 0.4667, 0.5667); (0.2000, 0.3333, 0.4667, 0.5667); 0.3362, 0.4286)$,
$Z_3 = ((0.4000, 0.5000, 0.6333, 0.7333); (0.4000, 0.5000, 0.6333, 0.7333); 0.4191, 0.3302)$,
$Z_4 = ((0.3333, 0.4667, 0.5667, 0.6667); (0.3333, 0.4667, 0.5667, 0.6667); 0.4851, 0.2190)$, and
$Z_5 = ((0.2000, 0.3667, 0.5000, 0.6667); (0.2000, 0.3667, 0.5000, 0.6667); 0.4515, 0.3232)$
are five IFNs within a set Z. In the set Z,
$R = ((0.4333, 0.5667, 0.6667, 0.7667); (0.4333, 0.5667, 0.6667, 0.7667); 0.4851, 0.2190)$,
$S = ((0.2000, 0.3333, 0.4667, 0.5667); (0.2000, 0.3333, 0.4667, 0.5667); 0.3362, 0.4286)$, and $\mu_F(R, S) = 0.7109 - 0.2547\beta$.
Additionally, the mean of five programs is computed as

$$\bar{Z} = ((0.3133, 0.4467, 0.5667, 0.6800); (0.3133, 0.4467, 0.5667, 0.6800); 0.4208, 0.3322).$$

Then

$$\mu_F(Z_1, \bar{Z}) = 0.0550 + 0.0112\beta,\ \mu_F(Z_2, \bar{Z}) = -0.2224 + 0.0636\beta,\ \mu_F(Z_3, \bar{Z}) = 0.0891 - 0.0363\beta,$$
$$\mu_F(Z_4, \bar{Z}) = 0.1240 - 0.0530\beta, \quad \text{and} \quad \mu_F(Z_5, \bar{Z}) = -0.0834 - 0.0525\beta.$$

Therefore,

$$\mu_F(Z_1, \bar{Z}) = \frac{1}{2}(\frac{0.0550 + 0.0112\beta}{0.7109 - 0.2547\beta} + 1) > \frac{1}{2},\ \mu_F(Z_2, \bar{Z}) = \frac{1}{2}(\frac{-0.2224 + 0.0636\beta}{0.7109 - 0.2547\beta} + 1) < \frac{1}{2},$$

$$\mu_F(Z_3,\bar{Z})=\frac{1}{2}(\frac{0.0891-0.0363\beta}{0.7109-0.2547\beta}+1)>\frac{1}{2},\ \mu_F(Z_4,\bar{Z})=\frac{1}{2}(\frac{0.1240-0.0530\beta}{0.7109-0.2547\beta}+1)>\frac{1}{2},$$

and $\mu_F(Z_5,\bar{Z})=\frac{1}{2}(\frac{-0.0834-0.0525\beta}{0.7109-0.2547\beta}+1)<\frac{1}{2}$.
For $0\le\beta\le 1$, β are classified into eleven varied values that are 0,0.1,,1. According to varied values of β, the corresponding computations of RPR and ranking results for five programs Z_1, Z_2, Z_3, Z_4, and Z_5 are displayed in Table 3.

Table 3. Corresponding computations and ranking results for Z_1, Z_2, Z_3, Z_4, and Z_5

β	$\mu_F(Z_1,\bar{Z})$	$\mu_F(Z_2,\bar{Z})$	$\mu_F(Z_3,\bar{Z})$	$\mu_F(Z_5,\bar{Z})$	$\mu_F(Z_4,\bar{Z})$	Ranking
0	0.5387	0.3436	0.5627	0.5872	0.4413	$Z_4>Z_3>Z_1>Z_5>Z_2$
0.1	0.5409	0.3424	0.5624	0.5866	0.4430	$Z_4>Z_3>Z_1>Z_5>Z_2$
0.2	0.5433	0.3411	0.5620	0.5859	0.4448	$Z_4>Z_3>Z_1>Z_5>Z_2$
0.3	0.5460	0.3398	0.5616	0.5852	0.4467	$Z_4>Z_3>Z_1>Z_5>Z_2$
0.4	0.5488	0.3383	0.5612	0.5844	0.4488	$Z_4>Z_3>Z_1>Z_5>Z_2$
0.5	0.5519	0.3367	0.5608	0.5836	0.4510	$Z_4>Z_3>Z_1>Z_5>Z_2$
0.6	0.5553	0.3349	0.5603	0.5826	0.4535	$Z_4>Z_3>Z_1>Z_5>Z_2$
0.7	0.5590	0.3330	0.5598	0.5816	0.4562	$Z_4>Z_3>Z_1>Z_5>Z_2$
0.8	0.5630	0.3309	0.5592	0.5805	0.4592	$Z_4>Z_1>Z_3>Z_5>Z_2$
0.9	0.5675	0.3285	0.5586	0.5792	0.4625	$Z_4>Z_1>Z_3>Z_5>Z_2$
1	0.5725	0.3259	0.5579	0.5778	0.4662	$Z_4>Z_1>Z_3>Z_5>Z_2$

Hence, three varied situations of ranking programs are presented below.
(i) $Z_4>Z_3>Z_1>Z_5>Z_2$ if $0\le\beta<\frac{0.0341}{0.0475}$,
(ii) $Z_4>Z_3\sim Z_1>Z_5>Z_2$ if $\beta=\frac{0.0341}{0.0475}\approx 0.7179$, or
(iii) $Z_4>Z_1>Z_3>Z_5>Z_2$ if $\frac{0.0341}{0.0475}<\beta\le 1$.
Based on ranking results above, the chief executive officer may choose the fourth program to be the investment for his enterprise through the assessments of three senior managers.

Based on RPR, IFNs in the three examples above are rationally ranked, and avoid comparing on pairwise. Hence, the related ranking computations are reasonable and useful. Further, the ranking computations of IFNs can be proverbially utilized in fuzzy decision-making problems [17] indicated by the fifth example because they are able to obtain many messages.

5 Conclusions

In this paper, IFNs are ranked through EFPR and RPR. By EFPR, IFNs are rationally compared on pairwise to present preference degree of the two. It is evident that the operation is superior to defuzzification due to the expression of preference degree, but related computations are more complicated than defuzzification for pairwise comparison. Thus EFPR revised into RPR is used to rank a set of IFNs. Through RPR, FNs avoid comparing on pairwise, and a set of FNs are sorted according to their corresponding relation preference relation values. For ranking n FNs, time complexity of fuzzy operation by EFPR is $O(n^2)$, whereas time complexity of fuzzy operation by RPR is $O(n)$. As FNs are numerous, time complexity of fuzzy operation by EFPR on account of pairwise comparison is higher than RPR. It is the strength of RPR on fuzzy computation. Through RPR, IFNs are easily and quickly ranked. In the future, RPR will be also applied in ranking other fuzzy numbers [8] to enhance its contribution by adjusted computations.

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