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Interval-valued fuzzy logical connectives with respect to admissible orders

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Abstract

Interval-valued fuzzy logical connectives are extensions of fuzzy logical connectives to the interval-valued framework. The extensions require linear orders to define the monotonicity between intervals. As a significant linear order, the admissible order is introduced to compare any interval in interval-valued fuzzy logic. In this work, we examine several widely-used interval-valued fuzzy logical connectives with respect to admissible orders. We are concerned with interval-valued fuzzy negations, automorphisms, fuzzy implications and aggregation functions with respect to $K_{\alpha,\beta}$ orders and arbitrary intervals on $L([0,1])$. We also make a discussion of width-preserving interval-valued fuzzy equivalence functions and dissimilarity functions with respect to arbitrary admissible orders and the intervals with the same width on $L([0,1])$. Then we bring some approaches to constructing the proposed interval-valued fuzzy logical connectives with respect to admissible orders. The introduced interval-valued fuzzy logical connectives with respect to admissible orders may have a deep impact on some fields exploiting fuzzy methods dealing with intervals.

Keywords: Admissible order, fuzzy logical connective, interval-valued fuzzy negation, interval-valued automorphism, interval-valued fuzzy implication, interval-valued aggregation function, interval-valued fuzzy equivalence function, interval-valued fuzzy dissimilarity function.

1 Introduction

Fuzzy logical connectives are important combination tools in fuzzy logic. Fuzzy negation, automorphism, fuzzy implication, fuzzy equivalence function and fuzzy dissimilarity function are widely-used fuzzy logical connectives which have proved to be useful not only in fuzzy control and fuzzy approximate reasoning, but also in many applications in computing with words, data mining and information retrieval, decision making and so on [17]. The study of them is still intense in these days since the field of fuzzy logic is under continuous development and further questions about fuzzy logical connectives may be raised [1, 3, 4, 5, 16, 20, 22, 24, 26]. All this study has been extended to other domains like discrete scales, bounded lattices and interval-valued framework.

Interval-valued fuzzy sets were introduced independently by Grattan-Guiness [18], Jahn [19], Sambuc [32] and Zadeh [38], in order to treat intuitively vagueness joint with uncertainty. Interval-valued fuzzy reasoning plays an essential role in interval-valued fuzzy logic, which has been applied in network, control systems and so on [9, 15, 25, 29, 31, 34]. The results of reasoning completely depend on the choice of interval-valued fuzzy sets of interval-valued fuzzy antecedent and fuzzy consequences as well as interval-valued fuzzy logical connectives linking interval-valued fuzzy antecedents and fuzzy consequences. Thus characterizing and representing interval-valued fuzzy logical connectives is one of the most important and interesting mathematical problems in interval-valued fuzzy reasoning.

Interval-valued fuzzy logical connectives are extensions of fuzzy logical connectives to the interval-valued framework. The extensions are not straightforward since the monotonicity between intervals is hard to define. The main difficulty is

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to find a natural linear order for intervals. Although interval-valued fuzzy logical connectives have been systematically studied in [6, 7, 8], where interval extensions of some fuzzy logical connectives were constructed as their interval representations, this study is based on natural partial orders. According to the concept of partial order, only some intervals are comparable. If we need to compare any possible two intervals arising in application, a linear order is required. In order to solve this problem, Bustince et al. [12] introduced the notion of admissible order as a linear order that extends the usual partial order between intervals. After that, admissible orders were applied in [33, 35, 36]. In recent years, a number of scholars have attempted to use admissible orders to introduce some interval-valued fuzzy logical connectives. However, the construction of interval-valued fuzzy logical connectives with respect to arbitrary admissible orders and arbitrary intervals on $L([0,1])$ is not a trivial task. Therefore, the scholars considered to construct interval-valued fuzzy logical connectives with respect to special admissible orders and arbitrary intervals on $L([0,1])$ or those with respect to arbitrary admissible orders and special intervals on $L([0,1])$ [2, 14]. As a wide class of admissible orders, $K_{\alpha,\beta}$ orders characterized by two weighted averages can encompass the most widely known and used linear orders in the literature such as the lexicographical orders $\preceq_{Lex1}$ and $\preceq_{Lex2}$ and Xu and Yager's order $\preceq_{XY}$ [12]. In [2], Asiain et al. considered the construction of interval-valued fuzzy negations with respect to $K_{\alpha,\beta}$ orders and arbitrary intervals on $L([0,1])$. In [14], Bustince et al. proposed interval-valued restricted equivalence functions with respect to arbitrary admissible orders and the intervals with the same width on $L([0,1])$. With the exception of the above-mentioned two interval-valued fuzzy logical connectives, a deeper study of other interval-valued fuzzy logical connectives with respect to admissible orders is still missing in the literature. Note that such interval-valued fuzzy logical connectives could be seen as a significant step towards a deeper study of interval-valued fuzzy logic related to admissible orders. For this reason, it seems useful to find such missing interval-valued fuzzy logical connectives.

In this work, we aim to construct several widely-used interval-valued fuzzy logical connectives. Different construction methods are adopted according to the characteristics of each kind of interval-valued fuzzy logical connective. For some interval-valued fuzzy logical connectives including interval-valued fuzzy negation, automorphism, fuzzy implication and aggregation function, we construct them with respect to $K_{\alpha,\beta}$ orders and arbitrary intervals on $L([0,1])$. This is our first objective. When examining the above-mentioned interval-valued fuzzy logical connectives on $K_{\alpha,\beta}$ orders, no more than two intervals' $K_{\alpha,\beta}$ orders may be considered. However, for some other interval-valued fuzzy logical connectives including interval-valued fuzzy equivalence functions and dissimilarity functions, if we want to examine them on $K_{\alpha,\beta}$ orders, three intervals' $K_{\alpha,\beta}$ orders should be considered. The construction of interval-valued fuzzy logical connectives with respect to three intervals' $K_{\alpha,\beta}$ orders and arbitrary intervals on $L([0,1])$ is not a trivial task. Therefore, for interval-valued fuzzy equivalence functions and dissimilarity functions, we construct them with respect to arbitrary admissible orders and the intervals with the same width on $L([0,1])$. This is our second objective.

The present paper is organized into five sections. We begin with some preliminary results and definitions in Section 2. We are concerned with interval-valued fuzzy negations, automorphisms, fuzzy implications and aggregation functions with respect to $K_{\alpha,\beta}$ orders and arbitrary intervals on $L([0,1])$ and then bring some approaches to constructing them in Section 3. The fourth section makes a discussion of width-preserving interval-valued fuzzy equivalence functions and dissimilarity functions with respect to arbitrary admissible orders and the intervals with the same width on $L([0,1])$. We conclude the paper with a discussion about the contributions and limitations of the study in the last section.

2 Preliminaries

Consider the real unit interval $[0,1] \subseteq \mathbb{R}$ and the set $L([0,1]) = \{[a,b] \mid 0 \leq a \leq b \leq 1\}$ of subintervals of $[0,1]$. The interval $X \in L([0,1])$ can be denoted by $X = [\underline{X}, \overline{X}]$, where $\underline{X}$ and $\overline{X}$ are respectively the left and right projections of X. The width of the interval $X \in L([0,1])$ is denoted by $w(X)$, where $w(X) = \overline{X} - \underline{X}$. An interval function $f : L([0,1])^n \longrightarrow L([0,1])$ is called width-preserving if for $X_1, \ldots, X_n \in L([0,1])$ such that $w(X_1) = \ldots = w(X_n)$, it holds that $w(f(X_1, \ldots, X_n)) = w(X_1)$. Given a non-empty set P, a partial order $\preceq$ on P is a binary relation on P such that for $x, y, z \in P$: (1)$x \preceq x$; (2)$x \preceq y$ and $y \preceq x$ imply $x = y$; (3)$x \preceq y$ and $y \preceq z$ imply $x \preceq z$. An order relation on P is called linear or total if for every $x, y \in P$, $x \preceq y$ or $y \preceq x$. The usual partial orders between intervals are the product order and the inclusion order. For $X, Y \in L([0,1])$, the product order $\preceq_P$ is defined as: $X \preceq_P Y$ if and only if $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$, and the inclusion order $\subseteq$ is defined as: $X \subseteq Y$ if and only if $\underline{X} \geq \underline{Y}$ and $\overline{X} \leq \overline{Y}$. They are not linear orders. However, we require linear orders of intervals to compare any possible two intervals [37]. Thus Bustince et al. [12] introduced the concept of admissible order as a linear order that extends the product order. Then Bustince et al. [13] constructed interval-valued ordered weighted aggregation functions by means of admissible orders.

Definition 2.1. [12] *Let $(L([0,1]), \preceq)$ be a poset. The order $\preceq$ is called an admissible order, if:*
(1) *$\preceq$ is a linear order on $L([0,1])$;*
(2) *For all $[\underline{X}, \overline{X}], [\underline{Y}, \overline{Y}] \in L([0,1])$, $[\underline{X}, \overline{X}] \preceq [\underline{Y}, \overline{Y}]$ whenever $[\underline{X}, \overline{X}] \preceq_P [\underline{Y}, \overline{Y}]$.*

Simply said, an order $\preceq$ on $L([0,1])$ is admissible, if it is linear and refines the product order $\preceq_P$. Let $\preceq$ be an admissible order on $L([0,1])$. Then $1_L = [1,1]$ and $0_L = [0,0]$ are the greatest and the smallest elements of $(L([0,1]), \preceq)$.

Example 2.2. [12] *The following are three particular cases of admissible orders on $L([0,1])$.*
(1) $[\underline{X}, \overline{X}] \preceq_{Lex1} [\underline{Y}, \overline{Y}]$ *if and only if* $\underline{X} < \underline{Y}$ *or* ($\underline{X} = \underline{Y}$ *and* $\overline{X} \leq \overline{Y}$).
(2) $[\underline{X}, \overline{X}] \preceq_{Lex2} [\underline{Y}, \overline{Y}]$ *if and only if* $\overline{X} < \overline{Y}$ *or* ($\overline{X} = \overline{Y}$ *and* $\underline{X} \leq \underline{Y}$).
(3) $[\underline{X}, \overline{X}] \preceq_{XY} [\underline{Y}, \overline{Y}]$ *if and only if* $\underline{X} + \overline{X} < \underline{Y} + \overline{Y}$ *or* ($\underline{X} + \overline{X} = \underline{Y} + \overline{Y}$ *and* $\overline{X} - \underline{X} \leq \overline{Y} - \underline{Y}$).
Note that $\preceq_{Lex1}$ and $\preceq_{Lex2}$ are lexicographical orders and $\preceq_{XY}$ was defined for Atanassov intuitionistic fuzzy pairs [37].

Let $\preceq_L$ stand for any order on $L([0,1])$ with 0_L as its minimal element and 1_L as its maximal element. Let $\preceq_{TL}$ be a linear order on $L([0,1])$ with the same minimal and maximal elements. In [12], Bustince et al. proposed a wide class of admissible orders characterized by two weighted averages which encompasses the most widely known and used linear orders in the literature such as the lexicographical orders $\preceq_{Lex1}$ and $\preceq_{Lex2}$ and Xu and Yager's order $\preceq_{XY}$.

Definition 2.3. [30] *An n-ary function $M : [0,1]^n \longrightarrow [0,1]$ ($n \in \mathbb{N}, n \geq 2$) is an aggregation function if*
(1) $M(0, \ldots, 0) = 0$.
(2) $M(1, \ldots, 1) = 1$.
(3) *M is non-decreasing in all of its arguments.*
An aggregation function M is idempotent if $M(x, \ldots, x) = x$ for all $x \in [0,1]$, and it is called symmetric if $M(x_1, \ldots, x_n) = M(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for all $x_1, \ldots, x_n \in [0,1]$ and all permutation $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$.

Definition 2.4. [14] *A function $M_{IV} : L([0,1])^n \longrightarrow L([0,1])$ is an interval-valued aggregation function w.r.t. $\preceq_L$ if*
(1) $M_{IV}(0_L, \ldots, 0_L) = 0_L$.
(2) $M_{IV}(1_L, \ldots, 1_L) = 1_L$.
(3) *M_{IV} is a non-decreasing function w.r.t. $\preceq_L$.*

Proposition 2.5. [12, 30, 39] *Let $M_1, M_2 : [0,1]^2 \longrightarrow [0,1]$ be two aggregation functions, $M_1(\underline{X}, \overline{X}) = M_1(\underline{Y}, \overline{Y})$ and $M_2(\underline{X}, \overline{X}) = M_2(\underline{Y}, \overline{Y})$ can only hold simultaneously if $X = Y$ for $X, Y \in L([0,1])$. The order $\preceq_{M_1,M_2}$ on $L([0,1])$ given by*

$$X \preceq_{M_1,M_2} Y \text{ if } \begin{cases} M_1(\underline{X}, \overline{X}) < M_1(\underline{Y}, \overline{Y}) \text{ or} \\ M_1(\underline{X}, \overline{X}) = M_1(\underline{Y}, \overline{Y}) \text{ and } M_2(\underline{X}, \overline{X}) \leq M_2(\underline{Y}, \overline{Y}) \end{cases}$$

is an admissible order on $L([0,1])$.

If we define the aggregation function $K_\alpha(x,y) = \alpha x + (1-\alpha)y$ for $\alpha, \beta \in [0,1]$, then we get the $K_{\alpha,\beta}$ order.

Definition 2.6. [12] *For each interval $X = [\underline{X}, \overline{X}] \in L([0,1])$ and for any $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$, let K_α and K_β be the functions given by $K_\alpha(X) = (1-\alpha)\underline{X} + \alpha\overline{X}$ and $K_\beta(X) = (1-\beta)\underline{X} + \beta\overline{X}$. We use $\preceq_{\alpha,\beta}$ to denote the linear order relation on $L([0,1])$: $X \preceq_{\alpha,\beta} Y$ if and only if $K_\alpha(X) < K_\alpha(Y)$ or $K_\alpha(X) = K_\alpha(Y)$ and $K_\beta(X) \leq K_\beta(Y)$, which is called a $K_{\alpha,\beta}$ order.*

The relation $\preceq_{\alpha,\beta}$ is an admissible order on $L([0,1])$, hence a linear order. The lexicographical orders $\preceq_{Lex1}$ and $\preceq_{Lex2}$ are particular cases of $K_{\alpha,\beta}$ orders, for $\alpha = 0, \beta \in (0,1]$ and $\alpha = 1, \beta \in [0,1)$, respectively. The Xu and Yager's order $\preceq_{XY}$ coincides with any $K_{\alpha,\beta}$ order for $\alpha = \frac{1}{2}$ and $\beta \in (\frac{1}{2}, 1]$.

Definition 2.7. [22] *If a decreasing function $N : [0,1] \longrightarrow [0,1]$ satisfies the boundary conditions $N(0) = 1$ and $N(1) = 0$, then N is called a fuzzy negation. A fuzzy negation is said to be involutive if $N(N(x)) = x$ for all $x \in [0,1]$. Fuzzy negations that are involutive are called strong fuzzy negations.*

Example 2.8. [6] *Zadeh's fuzzy negation defined by $N_Z(x) = 1 - x$ for all $x \in [0,1]$ is a strong fuzzy negation.*

Definition 2.9. [8] *A function $N_{IV} : L([0,1]) \longrightarrow L([0,1])$ is an interval-valued fuzzy negation w.r.t. $\preceq_L$ if it is a decreasing function w.r.t. $\preceq_L$ satisfying $N_{IV}(0_L) = 1_L$ and $N_{IV}(1_L) = 0_L$. N_{IV} is strong if it is involutive.*

Definition 2.10. [11] *A function $\varphi : [0,1] \longrightarrow [0,1]$ is an automorphism if it is continuous and strictly increasing such that $\varphi(0) = 0$ and $\varphi(1) = 1$. The automorphism is bijective and the inverse of an automorphism is also an automorphism.*

Definition 2.11. [23] *A function $\varphi_{IV} : L([0,1]) \longrightarrow L([0,1])$ is an interval-valued automorphism w.r.t. $\preceq_L$ if it is a continuous and strictly increasing function w.r.t. $\preceq_L$ and such that $\varphi_{IV}(0_L) = 0_L$ and $\varphi_{IV}(1_L) = 1_L$.*

Table 1: Some t-norms and t-conorms

Name	Symbol	$T(x,y)$	Name	Symbol	$S(x,y)$
Minimum	T_M	$\min(x,y)$	Maximum	S_M	$\max(x,y)$
Product	T_P	xy	Probabilistic sum	S_P	$x+y-xy$
Łukasiewicz	T_L	$\max(x+y-1,0)$	Łukasiewicz	S_L	$\min(x+y,1)$

Definition 2.12. [22] *An associative, commutative and increasing function $T:[0,1]^2 \longrightarrow [0,1]$ is called a t-norm if $T(x,1)=x$ for all $x \in [0,1]$. An associative, commutative and increasing function $S:[0,1]^2 \longrightarrow [0,1]$ is called a t-conorm if $S(x,0)=x$ for all $x \in [0,1]$.*

Example 2.13. [22] *Table 1 lists some widely used t-norms and t-conorms.*

Definition 2.14. [22] *A function $I:[0,1]^2 \longrightarrow [0,1]$ is called a fuzzy implication if*
(1) *$x \leq z$ implies $I(x,y) \geq I(z,y)$ for all $x,y,z \in [0,1]$.*
(2) *$y \leq z$ implies $I(x,y) \leq I(x,z)$ for all $x,y,z \in [0,1]$.*
(3) *$I(0,0)=I(1,1)=1$ and $I(1,0)=0$.*

Definition 2.15. [7] *A function $I_{IV}:L([0,1])^2 \longrightarrow L([0,1])$ is an interval-valued fuzzy implication w.r.t. $\preceq_L$ if*
(1) *$X \preceq_L Z$ implies $I_{IV}(Z,Y) \preceq_L I_{IV}(X,Y)$ for all $X,Y,Z \in L([0,1])$.*
(2) *$Y \preceq_L Z$ implies $I_{IV}(X,Y) \preceq_L I_{IV}(X,Z)$ for all $X,Y,Z \in L([0,1])$.*
(3) *$I_{IV}(0_L,0_L)=I_{IV}(1_L,1_L)=1_L$ and $I_{IV}(1_L,0_L)=0_L$.*

Definition 2.16. [16] *A function $E:[0,1]^2 \longrightarrow [0,1]$ is called a fuzzy equivalence function if*
(1) *$E(x,y)=E(y,x)$ for all $x,y \in [0,1]$.*
(2) *$E(x,x)=1$ for all $x \in [0,1]$.*
(3) *$E(0,1)=E(1,0)=0$.*
(4) *For all $x,y,z \in [0,1]$, if $x \leq y \leq z$, then $E(x,z) \leq E(x,y)$ and $E(x,z) \leq E(y,z)$.*

Definition 2.17. [28] *A function $E_{IV}:L([0,1])^2 \longrightarrow L([0,1])$ is an interval-valued fuzzy equivalence function w.r.t. $\preceq_L$ if*
(1) *$E_{IV}(X,Y)=E_{IV}(Y,X)$ for all $X,Y \in L([0,1])$.*
(2) *$E_{IV}(X,X)=1_L$ for all $X \in L([0,1])$.*
(3) *$E_{IV}(0_L,1_L)=E_{IV}(1_L,0_L)=0_L$.*
(4) *For all $X,Y,Z \in L([0,1])$, $X \preceq_L Y \preceq_L Z$ implies $E_{IV}(X,Z) \preceq_L E_{IV}(X,Y), E_{IV}(X,Z) \preceq_L E_{IV}(Y,Z)$.*

Definition 2.18. [10] *A function $RE:[0,1]^2 \longrightarrow [0,1]$ is called a restricted fuzzy equivalence function if*
(1) *$RE(x,y)=RE(y,x)$ for all $x,y \in [0,1]$.*
(2) *$RE(x,y)=1$ if and only if $x=y$ for all $x,y \in [0,1]$.*
(3) *$RE(x,y)=0$ if and only if $|x-y|=1$.*
(4) *For all $x,y,z \in [0,1]$, if $x \leq y \leq z$, then $RE(x,z) \leq RE(x,y)$ and $RE(x,z) \leq RE(y,z)$.*

Definition 2.19. [21] *A function $RE_{IV}:L([0,1])^2 \longrightarrow L([0,1])$ is called an interval-valued restricted fuzzy equivalence function w.r.t. $\preceq_L$ if*
(1) *$RE_{IV}(X,Y)=RE_{IV}(Y,X)$ for all $X,Y \in L([0,1])$.*
(2) *$RE_{IV}(X,Y)=1_L$ if and only if $X=Y$ for all $X,Y \in L([0,1])$.*
(3) *$RE_{IV}(X,Y)=0_L$ if and only if $\{X,Y\}=\{0_L,1_L\}$.*
(4) *$RE_{IV}(X,Y)=RE_{IV}(N_{IV}(X),N_{IV}(Y))$ for $X,Y \in L([0,1])$, where N_{IV} is an involutive interval-valued fuzzy negation.*
(5) *For all $X,Y,Z \in L([0,1])$, if $X \preceq_L Y \preceq_L Z$, then $RE_{IV}(X,Z) \preceq_L RE_{IV}(X,Y)$ and $RE_{IV}(X,Z) \preceq_L RE_{IV}(Y,Z)$.*

Definition 2.20. [27] *A function $d:[0,1]^2 \longrightarrow [0,1]$ is called a fuzzy dissimilarity function if*
(1) *$d(x,y)=d(y,x)$ for all $x,y \in [0,1]$.*
(2) *$d(x,x)=0$ for all $x \in [0,1]$.*
(3) *$d(0,1)=d(1,0)=1$.*
(4) *For all $x,y,z \in [0,1]$, if $x \leq y \leq z$, then $d(x,z) \geq d(x,y)$ and $d(x,z) \geq d(y,z)$.*

Definition 2.21. *A function $d_{IV} : L([0,1])^2 \longrightarrow L([0,1])$ is an interval-valued fuzzy dissimilarity function w.r.t. $\preceq_L$ if*
(1) $d_{IV}(X,Y) = d_{IV}(Y,X)$ *for all* $X, Y \in L([0,1])$.
(2) $d_{IV}(X,X) = 0_L$ *for all* $X \in L([0,1])$.
(3) $d_{IV}(0_L, 1_L) = d_{IV}(1_L, 0_L) = 1_L$.
(4) *For all* $X, Y, Z \in L([0,1])$, *if* $X \preceq_L Y \preceq_L Z$, *then* $d_{IV}(X,Y) \preceq_L d_{IV}(X,Z)$ *and* $d_{IV}(Y,Z) \preceq_L d_{IV}(X,Z)$.

3 Interval-valued fuzzy logical connectives with respect to $K_{\alpha,\beta}$ orders

The construction of interval-valued fuzzy logical connectives with respect to arbitrary admissible orders is not a trivial task, see [2]. In this section, we are concerned with interval-valued fuzzy logical connectives w.r.t. $K_{\alpha,\beta}$ orders and arbitrary intervals on $L([0,1])$.

Proposition 3.1. *Let $\alpha \in (0,1)$ and N be a fuzzy negation. For any $X = [\underline{X}, \overline{X}] \in L([0,1])$, let $w(X) = \overline{X} - \underline{X}$ and $f : [0,1]^2 \longrightarrow [0,1]$ be a function satisfying the following properties:*
(1) *For any $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then $f(K_\alpha(X_1), w(X_1)) \leq f(K_\alpha(X_2), w(X_2))$.*
(2) $N(K_\alpha(X)) \geq \alpha f(K_\alpha(X), w(X))$.
(3) $N(K_\alpha(X)) + (1-\alpha) f(K_\alpha(X), w(X)) \leq 1$.
Then the function $N_{IV} : L([0,1]) \longrightarrow L([0,1])$ given by

$$\begin{cases} N_{IV}(0_L) = 1_L \\ N_{IV}(1_L) = 0_L \\ N_{IV}(X) = Y,\ X \in L([0,1]) \backslash \{0_L, 1_L\}, Y \in L([0,1]), \end{cases} \quad \text{where} \begin{cases} K_\alpha(Y) = N(K_\alpha(X)) \\ w(Y) = f(K_\alpha(X), w(X)) \end{cases}$$

is an interval-valued fuzzy negation on $L([0,1])$ w.r.t. the $K_{\alpha,\beta}$ order for any $\beta \neq \alpha$.

Proof. Since $K_\alpha(Y) = N(K_\alpha(X)) = (1-\alpha)\underline{Y} + \alpha\overline{Y}$ and $w(Y) = f(K_\alpha(X), w(X)) = \overline{Y} - \underline{Y}$, we have $\underline{Y} = N(K_\alpha(X)) - \alpha f(K_\alpha(X), w(X))$ and $\overline{Y} = N(K_\alpha(X)) + (1-\alpha) f(K_\alpha(X), w(X))$. Since $0 \leq N(K_\alpha(X)) \leq 1$, we obtain that $N(K_\alpha(X)) \leq 1 + \alpha f(K_\alpha(X), w(X))$ and thus $0 \leq N(K_\alpha(X)) - \alpha f(K_\alpha(X), w(X)) \leq 1$. Meanwhile, $0 \leq N(K_\alpha(X)) + (1-\alpha) f(K_\alpha(X), w(X)) \leq 1$. Therefore, we have $\underline{Y}, \overline{Y} \in [0,1]$ and $Y \in L([0,1])$. Thus N_{IV} is well defined. Since $N_{IV}(0_L) = 1_L$ and $N_{IV}(1_L) = 0_L$, the boundary conditions of an interval-valued fuzzy negation are satisfied. Now we prove that N_{IV} is a decreasing function w.r.t. $\preceq_{\alpha,\beta}$. For any $X, Z \in L([0,1])$, if $X \preceq_{\alpha,\beta} Z$, then $K_\alpha(X) < K_\alpha(Z)$ or $K_\alpha(X) = K_\alpha(Z)$ and $K_\beta(X) \leq K_\beta(Z)$. Let $N_{IV}(X) = Y$ and $N_{IV}(Z) = W$, then $K_\alpha(Y) = N(K_\alpha(X))$ and $K_\alpha(W) = N(K_\alpha(Z))$.

(1) If $K_\alpha(X) < K_\alpha(Z)$, then $N(K_\alpha(X)) > N(K_\alpha(Z))$ and thus $K_\alpha(Y) > K_\alpha(W)$, i.e., $W \preceq_{\alpha,\beta} Y$. Thus $N_{IV}(Z) \preceq_{\alpha,\beta} N_{IV}(X)$.

(2) If $K_\alpha(X) = K_\alpha(Z)$ and $K_\beta(X) \leq K_\beta(Z)$, then $N(K_\alpha(X)) = N(K_\alpha(Z))$ and thus $K_\alpha(Y) = K_\alpha(W)$. Now we prove $K_\beta(W) \leq K_\beta(Y)$. Since $K_\beta(W) = (1-\beta)\underline{W} + \beta\overline{W}$, $K_\beta(Y) = (1-\beta)\underline{Y} + \beta\overline{Y}$, $K_\alpha(W) = (1-\alpha)\underline{W} + \alpha\overline{W}$ and $K_\alpha(Y) = (1-\alpha)\underline{Y} + \alpha\overline{Y}$, we have $K_\beta(W) - K_\alpha(W) = (\beta-\alpha)(\overline{W} - \underline{W}) = (\beta-\alpha) f(K_\alpha(Z), w(Z))$ and $K_\beta(Y) - K_\alpha(Y) = (\beta-\alpha)(\overline{Y} - \underline{Y}) = (\beta-\alpha) f(K_\alpha(X), w(X))$. According to $K_\alpha(X) = K_\alpha(Z)$ and $K_\beta(X) \leq K_\beta(Z)$, we obtain $K_\beta(X) - K_\alpha(X) \leq K_\beta(Z) - K_\alpha(Z)$, i.e., $(\beta-\alpha)w(X) \leq (\beta-\alpha)w(Z)$. If $\beta - \alpha > 0$, then $w(X) \leq w(Z)$. By the property of f, we have $f(K_\alpha(Z), w(Z)) \leq f(K_\alpha(X), w(X))$ and $(\beta-\alpha) f(K_\alpha(Z), w(Z)) \leq (\beta-\alpha) f(K_\alpha(X), w(X))$. Thus $K_\beta(W) - K_\alpha(W) \leq K_\beta(Y) - K_\alpha(Y)$, i.e., $K_\beta(W) \leq K_\beta(Y)$. If $\beta - \alpha < 0$, then $w(X) \geq w(Z)$. By the property of f, we have $f(K_\alpha(Z), w(Z)) \geq f(K_\alpha(X), w(X))$ and $(\beta-\alpha) f(K_\alpha(Z), w(Z)) \leq (\beta-\alpha) f(K_\alpha(X), w(X))$. Thus $K_\beta(W) - K_\alpha(W) \leq K_\beta(Y) - K_\alpha(Y)$, i.e., $K_\beta(W) \leq K_\beta(Y)$. Through $K_\alpha(Y) = K_\alpha(W)$ and $K_\beta(W) \leq K_\beta(Y)$, we get $W \preceq_{\alpha,\beta} Y$, i.e., $N_{IV}(Z) \preceq_{\alpha,\beta} N_{IV}(X)$. □

Example 3.2. *Let $\alpha \in (0,1)$ and N be a fuzzy negation, according to Proposition 3.1, f should satisfy properties:*
(1) $N(K_\alpha(X)) \geq \alpha f(K_\alpha(X), w(X))$.
(2) $N(K_\alpha(X)) + (1-\alpha) f(K_\alpha(X), w(X)) \leq 1$.
(3) *For $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then $f(K_\alpha(X_1), w(X_1)) \leq f(K_\alpha(X_2), w(X_2))$.*
For properties (1) and (2), f should satisfy $f(K_\alpha(X), w(X)) \leq \min(\frac{N(K_\alpha(X))}{\alpha}, \frac{1-N(K_\alpha(X))}{1-\alpha})$. Take $f_1(K_\alpha(X), w(X)) = T(N(K_\alpha(X) + (1-\alpha)w(X)), 1 - N(K_\alpha(X) - \alpha w(X)))$, $f_2(K_\alpha(X), w(X)) = \min(\frac{N(K_\alpha(X) + (1-\alpha)w(X))}{\alpha}, \frac{1-N(K_\alpha(X) - \alpha w(X))}{1-\alpha})$, for any $X \in L([0,1])$ in Proposition 3.1, where T is a t-norm. Now we prove that f_1 and f_2 satisfy properties (1)-(3). Since $\overline{X} = K_\alpha(X) + (1-\alpha)w(X)$ and $K_\alpha(X) = \underline{X} + \alpha w(X)$, we have $N(K_\alpha(X) + (1-\alpha)w(X)) \leq \frac{N(K_\alpha(X) + (1-\alpha)w(X))}{\alpha} \leq \frac{N(K_\alpha(X))}{\alpha}$ and $1 - N(K_\alpha(X) - \alpha w(X)) \leq \frac{1 - N(K_\alpha(X) - \alpha w(X))}{1-\alpha} \leq \frac{1-N(K_\alpha(X))}{1-\alpha}$. For arbitrary t-norm T, we obtain that $T(N(K_\alpha(X) + (1-\alpha)w(X)), 1 - N(K_\alpha(X) - \alpha w(X))) \leq \min(N(K_\alpha(X) + (1-$

$\alpha)w(X)), 1-N(K_\alpha(X)-\alpha w(X))) \leq \min(\frac{N(K_\alpha(X))}{\alpha}, \frac{1-N(K_\alpha(X))}{1-\alpha})$. For $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then $N(K_\alpha(X_1)+(1-\alpha)w(X_1)) \leq N(K_\alpha(X_2)+(1-\alpha)w(X_2))$ and $1-N(K_\alpha(X_1)-\alpha w(X_1)) \leq 1-N(K_\alpha(X_2)-\alpha w(X_2))$ and thus $T(N(K_\alpha(X_1)+(1-\alpha)w(X_1)), 1-N(K_\alpha(X_1)-\alpha w(X_1))) \leq T(N(K_\alpha(X_2)+(1-\alpha)w(X_2)), 1-N(K_\alpha(X_2)-\alpha w(X_2)))$. Thus f_1 is a function satisfying properties (1)-(3). Meanwhile, we get $\min(\frac{N(K_\alpha(X)+(1-\alpha)w(X))}{\alpha}, \frac{1-N(K_\alpha(X)-\alpha w(X))}{1-\alpha}) \leq \min(\frac{N(K_\alpha(X))}{\alpha}, \frac{1-N(K_\alpha(X))}{1-\alpha})$. If $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then $\frac{N(K_\alpha(X_1)+(1-\alpha)w(X_1))}{\alpha} \leq \frac{N(K_\alpha(X_2)+(1-\alpha)w(X_2))}{\alpha}$ and $\frac{1-N(K_\alpha(X_1)-\alpha w(X_1))}{1-\alpha} \leq \frac{1-N(K_\alpha(X_2)-\alpha w(X_2))}{1-\alpha}$ and therefore we get $\min(\frac{N(K_\alpha(X_1)+(1-\alpha)w(X_1))}{\alpha}, \frac{1-N(K_\alpha(X_1)-\alpha w(X_1))}{1-\alpha}) \leq \min(\frac{N(K_\alpha(X_2)+(1-\alpha)w(X_2))}{\alpha}, \frac{1-N(K_\alpha(X_2)-\alpha w(X_2))}{1-\alpha})$. Thus f_2 is a function satisfying properties (1)-(3). Specially, if we take $T = T_P, N = N_A$, then f_1 and f_2 are $f_1(K_\alpha(X), w(X)) = (1-K_\alpha(X)-(1-\alpha)w(X))(K_\alpha(X)-\alpha w(X))$, $f_2(K_\alpha(X), w(X)) = \min(\frac{1-K_\alpha(X)-(1-\alpha)w(X)}{\alpha}, \frac{K_\alpha(X)-\alpha w(X)}{1-\alpha})$.

Proposition 3.3. *Let $\alpha \in (0,1)$ and N be a fuzzy negation. Let T be a t-norm and $g : [0,1]^2 \longrightarrow [0,1]$ be a function satisfying the property: For any $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then $g(K_\alpha(X_1), w(X_1)) \leq g(K_\alpha(X_2), w(X_2))$. Then the functions $f', f'' : [0,1]^2 \longrightarrow [0,1]$ defined by $f'(K_\alpha(X), w(X)) = T(T(N(K_\alpha(X)), 1-N(K_\alpha(X))), g(K_\alpha(X), w(X)))$, $f''(K_\alpha(X), w(X)) = T(\min(\frac{N(K_\alpha(X))}{\alpha}, \frac{1-N(K_\alpha(X))}{1-\alpha}), g(K_\alpha(X), w(X)))$ are functions satisfying properties (1)-(3) demanded in Proposition 3.1.*

Proof. Let N be an arbitrary fuzzy negation, according to Proposition 3.1, f should satisfy the following properties:

(1) $N(K_\alpha(X)) \geq \alpha f(K_\alpha(X), w(X))$.

(2) $N(K_\alpha(X)) + (1-\alpha)f(K_\alpha(X), w(X)) \leq 1$.

(3) For $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then $f(K_\alpha(X_1), w(X_1)) \leq f(K_\alpha(X_2), w(X_2))$.

For properties (1) and (2), f should satisfy $f(K_\alpha(X), w(X)) \leq \min(\frac{N(K_\alpha(X))}{\alpha}, \frac{1-N(K_\alpha(X))}{1-\alpha})$. For t-norm T, we have $T(N(K_\alpha(X)), 1-N(K_\alpha(X))) \leq \min(N(K_\alpha(X)), 1-N(K_\alpha(X))) \leq \min(\frac{N(K_\alpha(X))}{\alpha}, \frac{1-N(K_\alpha(X))}{1-\alpha})$. Thus $T(T(N(K_\alpha(X)), 1-N(K_\alpha(X))), g(K_\alpha(X), w(X))) \leq T(\min(\frac{N(K_\alpha(X))}{\alpha}, \frac{1-N(K_\alpha(X))}{1-\alpha}), g(K_\alpha(X), w(X))) \leq \min(\frac{N(K_\alpha(X))}{\alpha}, \frac{1-N(K_\alpha(X))}{1-\alpha})$ since $g(K_\alpha(X), w(X)) \in [0,1]$. For any $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then $g(K_\alpha(X_1), w(X_1)) \leq g(K_\alpha(X_2), w(X_2))$ and $T(T(N(K_\alpha(X_1)), 1-N(K_\alpha(X_1))), g(K_\alpha(X_1), w(X_1))) \leq T(T(N(K_\alpha(X_2)), 1-N(K_\alpha(X_2))), g(K_\alpha(X_2), w(X_2)))$ and $T(\min(\frac{N(K_\alpha(X_1))}{\alpha}, \frac{1-N(K_\alpha(X_1))}{1-\alpha}), g(K_\alpha(X_1), w(X_1))) \leq T(\min(\frac{N(K_\alpha(X_2))}{\alpha}, \frac{1-N(K_\alpha(X_2))}{1-\alpha}), g(K_\alpha(X_2), w(X_2)))$. □

Example 3.4. *Let $\alpha \in (0,1)$ and N' and T be a fuzzy negation and a t-norm, respectively. Since $K_\alpha(X) = \underline{X} + \alpha w(X)$ and $1-K_\alpha(X) = 1-\overline{X}+(1-\alpha)w(X)$, we get $w(X) \leq \min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})$ and thus $\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})} \in [0,1]$. For $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then we get $N'(\frac{w(X_1)}{\min(\frac{K_\alpha(X_1)}{\alpha}, \frac{1-K_\alpha(X_1)}{1-\alpha})}) \leq N'(\frac{w(X_2)}{\min(\frac{K_\alpha(X_2)}{\alpha}, \frac{1-K_\alpha(X_2)}{1-\alpha})})$. Since $1-K_\alpha(X)-(1-\alpha)w(X) = 1-\overline{X}$ and $K_\alpha(X)-\alpha w(X) = \underline{X}$, we have $1-K_\alpha(X)-(1-\alpha)w(X), K_\alpha(X)-\alpha w(X) \in [0,1]$. For any $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then $1-K_\alpha(X_1)-(1-\alpha)w(X_1) \leq 1-K_\alpha(X_2)-(1-\alpha)w(X_2)$ and $K_\alpha(X_1)-\alpha w(X_1) \leq K_\alpha(X_2)-\alpha w(X_2)$. Thus we can take $g_1(K_\alpha(X), w(X)) = N'(\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})})$, or $g_2(K_\alpha(X), w(X)) = 1-K_\alpha(X)-(1-\alpha)w(X)$, or $g_3(K_\alpha(X), w(X)) = K_\alpha(X)-\alpha w(X)$ in Proposition 3.3. As a result, the functions*

$$f_3(K_\alpha(X), w(X)) = T(T(N(K_\alpha(X)), 1-N(K_\alpha(X))), N'(\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})})),$$
$$f_4(K_\alpha(X), w(X)) = T(\min(\frac{N(K_\alpha(X))}{\alpha}, \frac{1-N(K_\alpha(X))}{1-\alpha}), N'(\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})})),$$
$$f_5(K_\alpha(X), w(X)) = T(T(N(K_\alpha(X)), 1-N(K_\alpha(X))), 1-K_\alpha(X)-(1-\alpha)w(X)),$$
$$f_6(K_\alpha(X), w(X)) = T(T(N(K_\alpha(X)), 1-N(K_\alpha(X))), K_\alpha(X)-\alpha w(X)),$$
$$f_7(K_\alpha(X), w(X)) = T(\min(\frac{N(K_\alpha(X))}{\alpha}, \frac{1-N(K_\alpha(X))}{1-\alpha}), K_\alpha(X)-\alpha w(X)),$$
$$f_8(K_\alpha(X), w(X)) = T(\min(\frac{N(K_\alpha(X))}{\alpha}, \frac{1-N(K_\alpha(X))}{1-\alpha}), 1-K_\alpha(X)-(1-\alpha)w(X))$$

are functions satisfying properties (1)-(3) demanded in Proposition 3.1, where we can choose $T = T_M, T_P$ and T_L.

Example 3.5. *According to Examples 3.2 and 3.4, we obtain several functions f satisfying properties (1)-(3) demanded in Proposition 3.1. Consider these f, we get some interval-valued fuzzy negations on $L([0,1])$ w.r.t. the $K_{\alpha,\beta}$ order.*

$N_{IV_1}(X)=[\underline{Y}_1,\overline{Y}_1]$, *where* $\underline{Y}_1=N(K_\alpha(X))-\alpha T(N(K_\alpha(X)+(1-\alpha)w(X)),1-N(K_\alpha(X)-\alpha w(X))))$,
$\overline{Y}_1=N(K_\alpha(X))+(1-\alpha)T(N(K_\alpha(X)+(1-\alpha)w(X)),1-N(K_\alpha(X)-\alpha w(X)))$.
$N_{IV_2}(X)=[\underline{Y}_2,\overline{Y}_2]$, *where* $\underline{Y}_2=N(K_\alpha(X))-\alpha\min(\frac{N(K_\alpha(X)+(1-\alpha)w(X))}{\alpha},\frac{1-N(K_\alpha(X)-\alpha w(X))}{1-\alpha})$,
$\overline{Y}_2=N(K_\alpha(X))+(1-\alpha)\min(\frac{N(K_\alpha(X)+(1-\alpha)w(X))}{\alpha},\frac{1-N(K_\alpha(X)-\alpha w(X))}{1-\alpha})$.

$N_{IV_3}(X)=[\underline{Y}_3,\overline{Y}_3]$, *where* $\underline{Y}_3=N(K_\alpha(X))-\alpha T(T(N(K_\alpha(X)),1-N(K_\alpha(X))),N'(\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha},\frac{1-K_\alpha(X)}{1-\alpha})}))$,

$\overline{Y}_3=N(K_\alpha(X))+(1-\alpha)T(T(N(K_\alpha(X)),1-N(K_\alpha(X))),N'(\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha},\frac{1-K_\alpha(X)}{1-\alpha})}))$.

$N_{IV_4}(X)=[\underline{Y}_4,\overline{Y}_4]$, *where* $\underline{Y}_4=N(K_\alpha(X))-\alpha T(\min(\frac{N(K_\alpha(X))}{\alpha},\frac{1-N(K_\alpha(X))}{1-\alpha}),N'(\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha},\frac{1-K_\alpha(X)}{1-\alpha})}))$,

$\overline{Y}_4=N(K_\alpha(X))+(1-\alpha)T(\min(\frac{N(K_\alpha(X))}{\alpha},\frac{1-N(K_\alpha(X))}{1-\alpha}),N'(\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha},\frac{1-K_\alpha(X)}{1-\alpha})}))$.

$N_{IV_5}(X)=[\underline{Y}_5,\overline{Y}_5]$, *where* $\underline{Y}_5=N(K_\alpha(X))-\alpha T(T(N(K_\alpha(X)),1-N(K_\alpha(X))),1-K_\alpha(X)-(1-\alpha)w(X))$,
$\overline{Y}_5=N(K_\alpha(X))+(1-\alpha)T(T(N(K_\alpha(X)),1-N(K_\alpha(X))),1-K_\alpha(X)-(1-\alpha)w(X))$.
$N_{IV_6}(X)=[\underline{Y}_6,\overline{Y}_6]$, *where* $\underline{Y}_6=N(K_\alpha(X))-\alpha T(T(N(K_\alpha(X)),1-N(K_\alpha(X))),K_\alpha(X)-\alpha w(X))$,
$\overline{Y}_6=N(K_\alpha(X))+(1-\alpha)T(T(N(K_\alpha(X)),1-N(K_\alpha(X))),K_\alpha(X)-\alpha w(X))$.
$N_{IV_7}(X)=[\underline{Y}_7,\overline{Y}_7]$, *where* $\underline{Y}_7=N(K_\alpha(X))-\alpha T(\min(\frac{N(K_\alpha(X))}{\alpha},\frac{1-N(K_\alpha(X))}{1-\alpha}),K_\alpha(X)-\alpha w(X))$,
$\overline{Y}_7=N(K_\alpha(X))+(1-\alpha)T(\min(\frac{N(K_\alpha(X))}{\alpha},\frac{1-N(K_\alpha(X))}{1-\alpha}),K_\alpha(X)-\alpha w(X))$.
$N_{IV_8}(X)=[\underline{Y}_8,\overline{Y}_8]$, *where* $\underline{Y}_8=N(K_\alpha(X))-\alpha T(\min(\frac{N(K_\alpha(X))}{\alpha},\frac{1-N(K_\alpha(X))}{1-\alpha}),1-K_\alpha(X)-(1-\alpha)w(X))$,
$\overline{Y}_8=N(K_\alpha(X))+(1-\alpha)T(\min(\frac{N(K_\alpha(X))}{\alpha},\frac{1-N(K_\alpha(X))}{1-\alpha}),1-K_\alpha(X)-(1-\alpha)w(X))$.

Remark 3.6. *Take $N=N_A$ in N_{IV_2}, we have $f_2(K_\alpha(X),w(X))=\min(\frac{1-K_\alpha(X)-(1-\alpha)w(X)}{\alpha},\frac{K_\alpha(X)-\alpha w(X)}{1-\alpha})$ and $N_{IV_2}(X)=[\underline{Y}_2,\overline{Y}_2]$, where $\underline{Y}_2=1-K_\alpha(X)-\alpha\min(\frac{1-K_\alpha(X)-(1-\alpha)w(X)}{\alpha},\frac{K_\alpha(X)-\alpha w(X)}{1-\alpha})$, $\overline{Y}_2=1-K_\alpha(X)+(1-\alpha)\min(\frac{1-K_\alpha(X)-(1-\alpha)w(X)}{\alpha},\frac{K_\alpha(X)-\alpha w(X)}{1-\alpha})$. Take $N'=N_A$, $T=T_P$ and a strong fuzzy negation N in N_{IV_4}, we get $f_4(K_\alpha(X),w(X))=\min(\frac{N(K_\alpha(X))}{\alpha},\frac{1-N(K_\alpha(X))}{1-\alpha})\times(1-\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha},\frac{1-K_\alpha(X)}{1-\alpha})})$ and $N_{IV_4}(X)=[\underline{Y}_4,\overline{Y}_4]$, where $\underline{Y}_4=N(K_\alpha(X))-\alpha(\min(\frac{N(K_\alpha(X))}{\alpha},\frac{1-N(K_\alpha(X))}{1-\alpha})\times(1-\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha},\frac{1-K_\alpha(X)}{1-\alpha})}))$, $\overline{Y}_4=N(K_\alpha(X))+(1-\alpha)(\min(\frac{N(K_\alpha(X))}{\alpha},\frac{1-N(K_\alpha(X))}{1-\alpha})\times(1-\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha},\frac{1-K_\alpha(X)}{1-\alpha})}))$. It is shown that the obtained interval-valued fuzzy negations N_{IV_2} and N_{IV_4} are strong interval-valued fuzzy negations on $L([0,1])$ w.r.t. $K_{\alpha,\beta}$ orders provided by Asiain et al. [2]. In this sense, some existing interval-valued fuzzy negations on $L([0,1])$ w.r.t. $K_{\alpha,\beta}$ orders can be constructed through our approach given in Proposition 3.1. If $\alpha=\frac{1}{2}$ and $N=N_A$, then we get $f_2(K_\alpha(X),w(X))=\min(\frac{1-K_\alpha(X)-\frac{1}{2}w(X)}{\frac{1}{2}},\frac{K_\alpha(X)-\frac{1}{2}w(X)}{\frac{1}{2}})=\min(2\underline{X},2(1-\overline{X}))$ and $f_4(K_\alpha(X),w(X))=\min(\frac{1-K_\alpha(X)}{\frac{1}{2}},\frac{K_\alpha(X)}{\frac{1}{2}})\times(1-\frac{w(X)}{\min(\frac{K_\alpha(X)}{\frac{1}{2}},\frac{1-K_\alpha(X)}{\frac{1}{2}})})=f_2(K_\alpha(X),w(X))$. In this case, f_2 and f_4 are equivalent.*

Proposition 3.7. *Let $\alpha\in(0,1)$ and φ be an automorphism. For any $X=[\underline{X},\overline{X}]\in L([0,1])$, let $w(X)=\overline{X}-\underline{X}$ and $h:[0,1]^2\longrightarrow[0,1]$ be a continuous and strictly monotonic function satisfying the following properties:*
(1) *For any $X_1,X_2\in L([0,1])$, if $w(X_1)\geq w(X_2)$ and $K_\alpha(X_1)=K_\alpha(X_2)$, then $h(K_\alpha(X_1),w(X_1))\geq h(K_\alpha(X_2),w(X_2))$.*
(2) *$\varphi(K_\alpha(X))\geq\alpha h(K_\alpha(X),w(X))$.*
(3) *$\varphi(K_\alpha(X))+(1-\alpha)h(K_\alpha(X),w(X))\leq 1$.*
Then the function $\varphi_{IV}:L([0,1])\longrightarrow L([0,1])$ given by

$$\begin{cases}\varphi_{IV}(0_L)=0_L\\ \varphi_{IV}(1_L)=1_L\\ \varphi_{IV}(X)=Y,\ X\in L([0,1])\backslash\{0_L,1_L\},Y\in L([0,1]),\end{cases}\quad\text{where}\begin{cases}K_\alpha(Y)=\varphi(K_\alpha(X))\\ w(Y)=h(K_\alpha(X),w(X))\end{cases}$$

is an interval-valued automorphism on $L([0,1])$ w.r.t. the $K_{\alpha,\beta}$ order for any $\beta \neq \alpha$.

Proof. Since $K_\alpha(Y) = \varphi(K_\alpha(X)) = (1-\alpha)\underline{Y} + \alpha\overline{Y}$ and $w(Y) = h(K_\alpha(X), w(X)) = \overline{Y} - \underline{Y}$, we have $\underline{Y} = \varphi(K_\alpha(X)) - \alpha h(K_\alpha(X), w(X))$ and $\overline{Y} = \varphi(K_\alpha(X)) + (1-\alpha)h(K_\alpha(X), w(X))$. Since $0 \leq \varphi(K_\alpha(X)) \leq 1$, we obtain that $\varphi(K_\alpha(X)) \leq 1 + \alpha h(K_\alpha(X), w(X))$ and thus $0 \leq \varphi(K_\alpha(X)) - \alpha h(K_\alpha(X), w(X)) \leq 1$. Meanwhile, $0 \leq \varphi(K_\alpha(X)) + (1-\alpha)h(K_\alpha(X), w(X)) \leq 1$. Thus $\underline{Y}, \overline{Y} \in [0,1]$ and $Y \in L([0,1])$. Thus φ_{IV} is well defined. By definition, $\varphi_{IV}(0_L) = 0_L$ and $\varphi_{IV}(1_L) = 1_L$, hence the boundary conditions of an interval-valued automorphism are satisfied. Now we prove that φ_{IV} is an increasing function w.r.t. $\preceq_{\alpha,\beta}$. For any $X, Z \in L([0,1])$, if $X \preceq_{\alpha,\beta} Z$, then $K_\alpha(X) < K_\alpha(Z)$ or $K_\alpha(X) = K_\alpha(Z)$ and $K_\beta(X) \leq K_\beta(Z)$. Let $\varphi_{IV}(X) = Y$ and $\varphi_{IV}(Z) = W$, then $K_\alpha(Y) = \varphi(K_\alpha(X))$ and $K_\alpha(W) = \varphi(K_\alpha(Z))$.

(1) If $K_\alpha(X) < K_\alpha(Z)$, then $\varphi(K_\alpha(X)) < \varphi(K_\alpha(Z))$, $K_\alpha(Y) < K_\alpha(W)$, i.e., $Y \preceq_{\alpha,\beta} W$. Thus $\varphi_{IV}(X) \preceq_{\alpha,\beta} \varphi_{IV}(Z)$.

(2) If $K_\alpha(X) = K_\alpha(Z)$ and $K_\beta(X) \leq K_\beta(Z)$, then $\varphi(K_\alpha(X)) = \varphi(K_\alpha(Z))$ and thus $K_\alpha(Y) = K_\alpha(W)$. Now we prove that $K_\beta(Y) \leq K_\beta(W)$. Since $K_\beta(W) = (1-\beta)\underline{W} + \beta\overline{W}, K_\beta(Y) = (1-\beta)\underline{Y} + \beta\overline{Y}$, $K_\alpha(W) = (1-\alpha)\underline{W} + \alpha\overline{W}$ and $K_\alpha(Y) = (1-\alpha)\underline{Y} + \alpha\overline{Y}$, we have $K_\beta(W) - K_\alpha(W) = (\beta-\alpha)(\overline{W} - \underline{W}) = (\beta-\alpha)h(K_\alpha(Z), w(Z))$ and $K_\beta(Y) - K_\alpha(Y) = (\beta-\alpha)(\overline{Y} - \underline{Y}) = (\beta-\alpha)h(K_\alpha(X), w(X))$. According to $K_\alpha(X) = K_\alpha(Z)$ and $K_\beta(X) \leq K_\beta(Z)$, we get $K_\beta(X) - K_\alpha(X) \leq K_\beta(Z) - K_\alpha(Z)$, i.e., $(\beta-\alpha)w(X) \leq (\beta-\alpha)w(Z)$. If $\beta - \alpha > 0$, then $w(X) \leq w(Z)$. By the property of h, we have $h(K_\alpha(Z), w(Z)) \geq h(K_\alpha(X), w(X))$ and $(\beta-\alpha)h(K_\alpha(Z), w(Z)) \geq (\beta-\alpha)h(K_\alpha(X), w(X))$. Thus $K_\beta(W) - K_\alpha(W) \geq K_\beta(Y) - K_\alpha(Y)$, i.e., $K_\beta(Y) \leq K_\beta(W)$. If $\beta - \alpha < 0$, then $w(X) \geq w(Z)$. By the property of h, we have $h(K_\alpha(Z), w(Z)) \leq h(K_\alpha(X), w(X))$ and $(\beta-\alpha)h(K_\alpha(Z), w(Z)) \geq (\beta-\alpha)h(K_\alpha(X), w(X))$. Thus $K_\beta(W) - K_\alpha(W) \geq K_\beta(Y) - K_\alpha(Y)$, i.e., $K_\beta(Y) \leq K_\beta(W)$. Through $K_\alpha(Y) = K_\alpha(W)$ and $K_\beta(Y) \leq K_\beta(W)$, we get $Y \preceq_{\alpha,\beta} W$, i.e., $\varphi_{IV}(X) \preceq_{\alpha,\beta} \varphi_{IV}(Z)$.

Since h is a continuous and strictly monotonic function, we have φ_{IV} is continuous and strictly increasing and thus it is an interval-valued automorphism on $L([0,1])$ w.r.t. the $K_{\alpha,\beta}$ order for any $\beta \neq \alpha$. □

Proposition 3.8. *Let $\alpha \in (0,1)$ and φ be an automorphism. Let T be a continuous and strictly monotonic t-norm and $k : [0,1]^2 \longrightarrow [0,1]$ be a continuous and strictly monotonic function. Then $h', h'' : [0,1]^2 \longrightarrow [0,1]$ defined by $h'(K_\alpha(X), w(X)) = T(T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))), k(K_\alpha(X), w(X)))$, $h''(K_\alpha(X), w(X)) = T(\min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha}), k(K_\alpha(X), w(X)))$ are continuous and strictly monotonic functions satisfying properties (1)-(3) of Proposition 3.7.*

Proof. Let φ be an arbitrary automorphism, according to Proposition 3.7, h should satisfy the following properties:

(1) $\varphi(K_\alpha(X)) \geq \alpha h(K_\alpha(X), w(X))$.

(2) $\varphi(K_\alpha(X)) + (1-\alpha)h(K_\alpha(X), w(X)) \leq 1$.

(3) For $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then $h(K_\alpha(X_1), w(X_1)) \geq h(K_\alpha(X_2), w(X_2))$.

For properties (1) and (2), h should satisfy $h(K_\alpha(X), w(X)) \leq \min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha})$. For continuous and strictly monotonic t-norm T, $T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))) \leq \min(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))) \leq \min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha})$. Thus $T(T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))), k(K_\alpha(X), w(X))) \leq T(\min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha}), k(K_\alpha(X), w(X))) \leq \min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha})$ since $k(K_\alpha(X), w(X)) \in [0,1]$. For $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then $k(K_\alpha(X_1), w(X_1)) \geq k(K_\alpha(X_2), w(X_2))$, $T(T(\varphi(K_\alpha(X_1)), 1-\varphi(K_\alpha(X_1))), k(K_\alpha(X_1), w(X_1))) \geq T(T(\varphi(K_\alpha(X_2)), 1-\varphi(K_\alpha(X_2))), k(K_\alpha(X_2), w(X_2)))$, $T(\min(\frac{\varphi(K_\alpha(X_1))}{\alpha}, \frac{1-\varphi(K_\alpha(X_1))}{1-\alpha}), k(K_\alpha(X_1), w(X_1))) \geq T(\min(\frac{\varphi(K_\alpha(X_2))}{\alpha}, \frac{1-\varphi(K_\alpha(X_2))}{1-\alpha}), k(K_\alpha(X_2), w(X_2)))$. □

Example 3.9. *Let $\alpha \in (0,1)$ and T be a continuous and strictly monotonic t-norm (for example, T_P). Since $K_\alpha(X) = \underline{X} + \alpha w(X)$ and $1 - K_\alpha(X) = 1 - \overline{X} + (1-\alpha)w(X)$, we have $w(X) \leq \min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})$ and thus $\frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})} \in [0,1]$. For any $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then we obtain that $\frac{w(X_1)}{\min(\frac{K_\alpha(X_1)}{\alpha}, \frac{1-K_\alpha(X_1)}{1-\alpha})} \geq \frac{w(X_2)}{\min(\frac{K_\alpha(X_2)}{\alpha}, \frac{1-K_\alpha(X_2)}{1-\alpha})}$. Meanwhile, since $1 - K_\alpha(X) + \alpha w(X) = 1 - \underline{X}$ and $K_\alpha(X) + (1-\alpha)w(X) = \overline{X}$, we have $1 - K_\alpha(X) + \alpha w(X), K_\alpha(X) + (1-\alpha)w(X) \in [0,1]$. For any $X_1, X_2 \in L([0,1])$, if $w(X_1) \geq w(X_2)$ and $K_\alpha(X_1) = K_\alpha(X_2)$, then $1 - K_\alpha(X_1) + \alpha w(X_1) \geq 1 - K_\alpha(X_2) + \alpha w(X_2)$ and $K_\alpha(X_1) + (1-\alpha)w(X_1) \geq K_\alpha(X_2) + (1-\alpha)w(X_2)$. Therefore, we can take $k_1(K_\alpha(X), w(X)) = \frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})}$,* *or* $k_2(K_\alpha(X), w(X)) = 1 - K_\alpha(X) + \alpha w(X)$, *or* $k_3(K_\alpha(X), w(X)) = K_\alpha(X) + (1-\alpha)w(X)$ *in Proposition 3.8. Thus*

$$h_1(K_\alpha(X), w(X)) = T(T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))), \frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})}),$$
$$h_2(K_\alpha(X), w(X)) = T(\min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha}), \frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})}),$$
$$h_3(K_\alpha(X), w(X)) = T(T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))), 1-K_\alpha(X)+\alpha w(X)),$$
$$h_4(K_\alpha(X), w(X)) = T(\min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha}), 1-K_\alpha(X)+\alpha w(X)),$$
$$h_5(K_\alpha(X), w(X)) = T(T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))), K_\alpha(X)+(1-\alpha)w(X)),$$
$$h_6(K_\alpha(X), w(X)) = T(\min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha}), K_\alpha(X)+(1-\alpha)w(X)),$$

are functions satisfying properties (1)-(3) demanded in Proposition 3.7.

Example 3.10. *According to Examples 3.2 and 3.4, we obtain several functions* h *demanded in Proposition 3.7. Consider these* h *and* φ *in Proposition 3.7, we get interval-valued automorphisms on* $L([0,1])$ *w.r.t.* $K_{\alpha,\beta}$ *orders:*

$$\varphi_{IV_1}(X) = [\underline{Y}_1, \overline{Y}_1], \text{ where } \underline{Y}_1 = \varphi(K_\alpha(X)) - \alpha T(T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))), \frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})}),$$
$$\overline{Y}_1 = \varphi(K_\alpha(X)) + (1-\alpha)T(T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))), \frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})}).$$
$$\varphi_{IV_2}(X) = [\underline{Y}_2, \overline{Y}_2], \text{ where } \underline{Y}_2 = \varphi(K_\alpha(X)) - \alpha T(\min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha}), \frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})}),$$
$$\overline{Y}_2 = \varphi(K_\alpha(X)) + (1-\alpha)T(\min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha}), \frac{w(X)}{\min(\frac{K_\alpha(X)}{\alpha}, \frac{1-K_\alpha(X)}{1-\alpha})}).$$
$$\varphi_{IV_3}(X) = [\underline{Y}_3, \overline{Y}_3], \text{ where } \underline{Y}_3 = \varphi(K_\alpha(X)) - \alpha T(T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))), 1-K_\alpha(X)+\alpha w(X)),$$
$$\overline{Y}_3 = \varphi(K_\alpha(X)) + (1-\alpha)T(T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))), 1-K_\alpha(X)+\alpha w(X)).$$
$$\varphi_{IV_4}(X) = [\underline{Y}_4, \overline{Y}_4], \text{ where } \underline{Y}_4 = \varphi(K_\alpha(X)) - \alpha T(\min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha}), 1-K_\alpha(X)+\alpha w(X)),$$
$$\overline{Y}_4 = \varphi(K_\alpha(X)) + (1-\alpha)T(\min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha}), 1-K_\alpha(X)+\alpha w(X)).$$
$$\varphi_{IV_5}(X) = [\underline{Y}_5, \overline{Y}_5], \text{ where } \underline{Y}_5 = \varphi(K_\alpha(X)) - \alpha T(T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))), K_\alpha(X)+(1-\alpha)w(X)),$$
$$\overline{Y}_5 = \varphi(K_\alpha(X)) + (1-\alpha)T(T(\varphi(K_\alpha(X)), 1-\varphi(K_\alpha(X))), K_\alpha(X)+(1-\alpha)w(X)).$$
$$\varphi_{IV_6}(X) = [\underline{Y}_6, \overline{Y}_6], \text{ where } \underline{Y}_6 = \varphi(K_\alpha(X)) - \alpha T(\min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha}), K_\alpha(X)+(1-\alpha)w(X)),$$
$$\overline{Y}_6 = \varphi(K_\alpha(X)) + (1-\alpha)T(\min(\frac{\varphi(K_\alpha(X))}{\alpha}, \frac{1-\varphi(K_\alpha(X))}{1-\alpha}), K_\alpha(X)+(1-\alpha)w(X)).$$

Proposition 3.11. *Let* $\alpha \in (0,1)$ *and* N_{IV} *be an interval-valued fuzzy negation on* $L([0,1])$ *w.r.t. the* $K_{\alpha,\beta}$ *order for any* $\beta \neq \alpha$*. Suppose* φ_{IV} *is an interval-valued automorphism w.r.t. the* $K_{\alpha,\beta}$ *order for any* $\beta \neq \alpha$*, then the function* $N_{IV_\varphi} : L([0,1]) \longrightarrow L([0,1])$ *defined by* $N_{IV_\varphi}(X) = \varphi_{IV}^{-1}(N_{IV}(\varphi_{IV}(X)))$ *is a strong interval-valued fuzzy negation w.r.t. the* $K_{\alpha,\beta}$ *order for any* $\beta \neq \alpha$*.*

Proof. φ_{IV} is bijective and φ_{IV}^{-1} is also an automorphism. Since $N_{IV_\varphi}(X) \in L([0,1])$, $N_{IV_\varphi}(0_L) = 1_L$ and $N_{IV_\varphi}(1_L) = 0_L$, the boundary conditions of an interval-valued fuzzy negation are satisfied. Now we prove that N_{IV_φ} is a decreasing function w.r.t. $\preceq_{\alpha,\beta}$. For any $X, Z \in L([0,1])$, if $X \preceq_{\alpha,\beta} Z$, then $\varphi_{IV}^{-1}(N_{IV}(\varphi_{IV}(Z))) \preceq_{\alpha,\beta} \varphi_{IV}^{-1}(N_{IV}(\varphi_{IV}(X)))$ and thus $N_{IV_\varphi}(Z) \preceq_{\alpha,\beta} N_{IV_\varphi}(X)$. For any $X \in L([0,1])$, we have $N_{IV_\varphi}(N_{IV_\varphi}(X)) = \varphi_{IV}^{-1}(N_{IV}(\varphi_{IV}(\varphi_{IV}^{-1}(N_{IV}(\varphi_{IV}(X)))))) = X$ and thus N_{IV_φ} is involutive. □

Proposition 3.12. *Let* $\alpha \in (0,1)$ *and* I *be a fuzzy implication. For any* $X = [\underline{X}, \overline{X}] \in L([0,1])$*, let* $w(X) = \overline{X} - \underline{X}$ *and* $l : [0,1]^4 \longrightarrow [0,1]$ *be a function satisfying the following properties:*

(1) *For any* $X_1, X_2, Y_1, Y_2 \in L([0,1])$*, if* $w(X_1) \geq w(Y_1)$*,* $w(X_2) \leq w(Y_2)$ *and* $K_\alpha(X_i) = K_\alpha(Y_i)$ *for* $i = 1, 2$*, then* $l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2)) \leq l(K_\alpha(Y_1), K_\alpha(Y_2), w(Y_1), w(Y_2))$*.*

(2) $I(K_\alpha(X_1), K_\alpha(X_2)) \geq \alpha l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2))$*.*

(3) $I(K_\alpha(X_1), K_\alpha(X_2)) + (1-\alpha) l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2)) \leq 1$*.*

(4) $l(0,0,0,0) = 0, l(1,1,0,0) = 0, l(1,0,0,0) = 0$*.*

Then the function $I_{IV} : L([0,1])^2 \longrightarrow L([0,1])$ *given by*

$$I_{IV}(X_1, X_2) = Y \text{ for } X_1, X_2, Y \in L([0,1]), \text{ where } \begin{cases} K_\alpha(Y) = I(K_\alpha(X_1), K_\alpha(X_2)) \\ w(Y) = l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2)) \end{cases}$$

is an interval-valued fuzzy implication on $L([0,1])$ w.r.t. the $K_{\alpha,\beta}$ order for any $\beta \neq \alpha$.

Proof. Since $K_\alpha(Y) = (1-\alpha)\underline{Y} + \alpha\overline{Y} = I(K_\alpha(X_1), K_\alpha(X_2))$ and $w(Y) = \overline{Y} - \underline{Y} = l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2))$, we get $\underline{Y} = I(K_\alpha(X_1), K_\alpha(X_2)) - \alpha l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2))$ and $\overline{Y} = I(K_\alpha(X_1), K_\alpha(X_2)) + (1-\alpha) l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2))$. Since $0 \leq I(K_\alpha(X_1), K_\alpha(X_2)) \leq 1$, we get $I(K_\alpha(X_1), K_\alpha(X_2)) \leq 1 + \alpha l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2))$ and thus $0 \leq I(K_\alpha(X_1), K_\alpha(X_2)) - \alpha l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2)) \leq 1$. Since $0 \leq I(K_\alpha(X_1), K_\alpha(X_2)) + (1-\alpha) l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2)) \leq 1$, $\underline{Y}, \overline{Y} \in [0,1]$ and $Y \in L([0,1])$. Thus I_{IV} is well defined. If $X_1 = X_2 = 0_L$, then $K_\alpha(X_1) = K_\alpha(X_2) = 0$, $w(X_1) = w(X_2) = 0$. Since $l(0,0,0,0) = 0$, we have $\underline{Y} = \overline{Y} = 1$. If $X_1 = X_2 = 1_L$, then $K_\alpha(X_1) = K_\alpha(X_2) = 1$, $w(X_1) = w(X_2) = 0$. Since $l(1,1,0,0) = 0$, we have $\underline{Y} = \overline{Y} = 1$. If $X_1 = 1_L, X_2 = 0_L$, then $K_\alpha(X_1) = 1, K_\alpha(X_2) = 0$, $w(X_1) = w(X_2) = 0$. Since $l(1,0,0,0) = 0$, we have $\underline{Y} = \overline{Y} = 0$. Thus $I_{IV}(0_L, 0_L) = I_{IV}(1_L, 1_L) = 1_L$ and $I_{IV}(1_L, 0_L) = 0_L$. Now we prove that for $X_1, Y_1, Z \in L([0,1])$, if $X_1 \preceq_{\alpha,\beta} Y_1$, then $I_{IV}(Y_1, Z) \preceq_{\alpha,\beta} I_{IV}(X_1, Z)$. If $X_1 \preceq_{\alpha,\beta} Y_1$, then $K_\alpha(X_1) < K_\alpha(Y_1)$ or $K_\alpha(X_1) = K_\alpha(Y_1)$ and $K_\beta(X_1) \leq K_\beta(Y_1)$. Let $I_{IV}(X_1, Z) = V_1$ and $I_{IV}(Y_1, Z) = W_1$, then $K_\alpha(V_1) = I(K_\alpha(X_1), K_\alpha(Z))$ and $K_\alpha(W_1) = I(K_\alpha(Y_1), K_\alpha(Z))$.

(1) If $K_\alpha(X_1) < K_\alpha(Y_1)$, then $I(K_\alpha(Y_1), K_\alpha(Z)) < I(K_\alpha(X_1), K_\alpha(Z))$ and thus $K_\alpha(W_1) < K_\alpha(V_1)$, i.e., $W_1 \preceq_{\alpha,\beta} V_1$. Therefore, we have $I_{IV}(Y_1, Z) \preceq_{\alpha,\beta} I_{IV}(X_1, Z)$.

(2) If $K_\alpha(X_1) = K_\alpha(Y_1)$ and $K_\beta(X_1) \leq K_\beta(Y_1)$, then $I(K_\alpha(X_1), K_\alpha(Z)) = I(K_\alpha(Y_1), K_\alpha(Z))$ and thus $K_\alpha(V_1) = K_\alpha(W_1)$. In the following, we prove that $K_\beta(W_1) \leq K_\beta(V_1)$. Since $K_\beta(V_1) = (1-\beta)\underline{V_1} + \beta\overline{V_1}$, $K_\beta(W_1) = (1-\beta)\underline{W_1} + \beta\overline{W_1}$, $K_\alpha(V_1) = (1-\alpha)\underline{V_1} + \alpha\overline{V_1}$ and $K_\alpha(W_1) = (1-\alpha)\underline{W_1} + \alpha\overline{W_1}$, we have $K_\beta(V_1) - K_\alpha(V_1) = (\beta-\alpha)(\overline{V_1} - \underline{V_1}) = (\beta-\alpha) l(K_\alpha(X_1), K_\alpha(Z), w(X_1), w(Z))$ and $K_\beta(W_1) - K_\alpha(W_1) = (\beta-\alpha)(\overline{W_1} - \underline{W_1}) = (\beta-\alpha) l(K_\alpha(Y_1), K_\alpha(Z), w(Y_1), w(Z))$. According to $K_\alpha(X_1) = K_\alpha(Y_1)$ and $K_\beta(X_1) \leq K_\beta(Y_1)$, we obtain $K_\beta(X_1) - K_\alpha(X_1) \leq K_\beta(Y_1) - K_\alpha(Y_1)$, i.e., $(\beta-\alpha) w(X_1) \leq (\beta-\alpha) w(Y_1)$. If $\beta-\alpha > 0$, then $w(X_1) \leq w(Y_1)$. By the property of l, we have $l(K_\alpha(Y_1), K_\alpha(Z), w(Y_1), w(Z)) \leq l(K_\alpha(X_1), K_\alpha(Z), w(X_1), w(Z))$ and $(\beta-\alpha) l(K_\alpha(Y_1), K_\alpha(Z), w(Y_1), w(Z)) \leq (\beta-\alpha) l(K_\alpha(X_1), K_\alpha(Z), w(X_1), w(Z))$. Thus we have $K_\beta(W_1) - K_\alpha(W_1) \leq K_\beta(V_1) - K_\alpha(V_1)$, i.e., $K_\beta(W_1) \leq K_\beta(V_1)$. If $\beta - \alpha < 0$, then $w(X_1) \geq w(Y_1)$. By the property of l, we get $l(K_\alpha(Y_1), K_\alpha(Z), w(Y_1), w(Z)) \geq l(K_\alpha(X_1), K_\alpha(Z), w(X_1), w(Z))$ and $(\beta-\alpha) l(K_\alpha(Y_1), K_\alpha(Z), w(Y_1), w(Z)) \leq (\beta-\alpha) l(K_\alpha(X_1), K_\alpha(Z), w(X_1), w(Z))$. Thus $K_\beta(W_1) - K_\alpha(W_1) \leq K_\beta(V_1) - K_\alpha(V_1)$, i.e., $K_\beta(W_1) \leq K_\beta(V_1)$. Through $K_\alpha(V_1) = K_\alpha(W_1)$ and $K_\beta(W_1) \leq K_\beta(V_1)$, we get $W_1 \preceq_{\alpha,\beta} V_1$, i.e., $I_{IV}(Y_1, Z) \preceq_{\alpha,\beta} I_{IV}(X_1, Z)$.

Then we prove that for $X_1, Y_1, Z \in L([0,1])$, if $X_1 \preceq_{\alpha,\beta} Y_1$, then $I_{IV}(Z, X_1) \preceq_{\alpha,\beta} I_{IV}(Z, Y_1)$. If $X_1 \preceq_{\alpha,\beta} Y_1$, then $K_\alpha(X_1) < K_\alpha(Y_1)$ or $K_\alpha(X_1) = K_\alpha(Y_1)$ and $K_\beta(X_1) \leq K_\beta(Y_1)$. Let $I_{IV}(Z, X_1) = V_2$ and $I_{IV}(Z, Y_1) = W_2$, then $K_\alpha(V_2) = I(K_\alpha(Z), K_\alpha(X_1))$ and $K_\alpha(W_2) = I(K_\alpha(Z), K_\alpha(Y_1))$.

(1) If $K_\alpha(X_1) < K_\alpha(Y_1)$, then $I(K_\alpha(Z), K_\alpha(X_1)) < I(K_\alpha(Z), K_\alpha(Y_1))$ and thus $K_\alpha(V_2) < K_\alpha(W_2)$, i.e., $V_2 \preceq_{\alpha,\beta} W_2$. Therefore, we have $I_{IV}(Z, X_1) \preceq_{\alpha,\beta} I_{IV}(Z, Y_1)$.

(2) If $K_\alpha(X_1) = K_\alpha(Y_1)$ and $K_\beta(X_1) \leq K_\beta(Y_1)$, then $I(K_\alpha(Z), K_\alpha(X_1)) = I(K_\alpha(Z), K_\alpha(Y_1))$ and thus $K_\alpha(V_2) = K_\alpha(W_2)$. In the following, we prove that $K_\beta(V_2) \leq K_\beta(W_2)$. Since $K_\beta(V_2) = (1-\beta)\underline{V_2} + \beta\overline{V_2}$, $K_\beta(W_2) = (1-\beta)\underline{W_2} + \beta\overline{W_2}$, $K_\alpha(V_2) = (1-\alpha)\underline{V_2} + \alpha\overline{V_2}$ and $K_\alpha(W_2) = (1-\alpha)\underline{W_2} + \alpha\overline{W_2}$, we have $K_\beta(V_2) - K_\alpha(V_2) = (\beta-\alpha)(\overline{V_2} - \underline{V_2}) = (\beta-\alpha) l(K_\alpha(Z), K_\alpha(X_1), w(Z), w(X_1))$ and $K_\beta(W_2) - K_\alpha(W_2) = (\beta-\alpha)(\overline{W_2} - \underline{W_2}) = (\beta-\alpha) l(K_\alpha(Z), K_\alpha(Y_1), w(Z), w(Y_1))$. According to $K_\alpha(X_1) = K_\alpha(Y_1)$ and $K_\beta(X_1) \leq K_\beta(Y_1)$, we obtain $K_\beta(X_1) - K_\alpha(X_1) \leq K_\beta(Y_1) - K_\alpha(Y_1)$, i.e., $(\beta-\alpha) w(X_1) \leq (\beta-\alpha) w(Y_1)$. If $\beta-\alpha > 0$, then $w(X_1) \leq w(Y_1)$. By the property of l, we have $l(K_\alpha(Z), K_\alpha(X_1), w(Z), w(X_1)) \leq l(K_\alpha(Z), K_\alpha(Y_1), w(Z), w(Y_1))$ and $(\beta-\alpha) l(K_\alpha(Z), K_\alpha(X_1), w(Z), w(X_1)) \leq (\beta-\alpha) l(K_\alpha(Z), K_\alpha(Y_1), w(Z), w(Y_1))$. Thus we have $K_\beta(V_2) - K_\alpha(V_2) \leq K_\beta(W_2) - K_\alpha(W_2)$, i.e., $K_\beta(V_2) \leq K_\beta(W_2)$. If $\beta - \alpha < 0$, then $w(X_1) \geq w(Y_1)$. By the property of l, we get $l(K_\alpha(Z), K_\alpha(X_1), w(Z), w(X_1)) \geq l(K_\alpha(Z), K_\alpha(Y_1), w(Z), w(Y_1))$ and $(\beta-\alpha) l(K_\alpha(Z), K_\alpha(X_1), w(Z), w(X_1)) \leq (\beta-\alpha) l(K_\alpha(Z), K_\alpha(Y_1), w(Z), w(Y_1))$. Thus $K_\beta(V_2) - K_\alpha(V_2) \leq K_\beta(W_2) - K_\alpha(W_2)$, i.e., $K_\beta(V_2) \leq K_\beta(W_2)$. Through $K_\alpha(V_2) = K_\alpha(W_2)$ and $K_\beta(V_2) \leq K_\beta(W_2)$, we get $V_2 \preceq_{\alpha,\beta} W_2$, i.e., $I_{IV}(Z, X_1) \preceq_{\alpha,\beta} I_{IV}(Z, Y_1)$. □

Proposition 3.13. *Let $\alpha \in (0,1)$, I and T be a fuzzy implication and a t-norm, respectively. Then $l', l'' : [0,1]^4 \longrightarrow [0,1]$ defined by $l'(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2)) = T(T(I(K_\alpha(X_1), K_\alpha(X_2)), 1 - I(K_\alpha(X_1), K_\alpha(X_2))), I(w(X_1), w(X_2)))$, $l''(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2)) = T(\min(\frac{I(K_\alpha(X_1), K_\alpha(X_2))}{\alpha}, \frac{1 - I(K_\alpha(X_1), K_\alpha(X_2))}{1-\alpha}), I(w(X_1), w(X_2)))$ are functions satisfying properties (1)-(4) demanded in Proposition 3.12.*

Proof. Let I be an arbitrary fuzzy implication, according to Proposition 3.12, l should satisfy the following properties:

(1) $I(K_\alpha(X_1), K_\alpha(X_2)) \geq \alpha l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2))$.

(2) $I(K_\alpha(X_1), K_\alpha(X_2)) + (1-\alpha) l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2)) \leq 1$.

(3) For any $X_1, X_2, Y_1, Y_2 \in L([0,1])$, if $w(X_1) \geq w(Y_1)$, $w(X_2) \leq w(Y_2)$ and $K_\alpha(X_i) = K_\alpha(Y_i)$ for $i = 1, 2$, then $l(K_\alpha(X_1), K_\alpha(X_2), w(X_1), w(X_2)) \leq l(K_\alpha(Y_1), K_\alpha(Y_2), w(Y_1), w(Y_2))$.

(4) $l(0,0,0,0)=0, l(1,1,0,0)=0, l(1,0,0,0)=0$.
For properties (1),(2), l should satisfy $l(K_\alpha(X_1),K_\alpha(X_2),w(X_1),w(X_2))\leq \min(\frac{I(K_\alpha(X_1),K_\alpha(X_2))}{\alpha},\frac{1-I(K_\alpha(X_1),K_\alpha(X_2))}{1-\alpha})$. For a t-norm T, we get $T(I(K_\alpha(X_1),K_\alpha(X_2)),1-I(K_\alpha(X_1),K_\alpha(X_2)))\leq \min(\frac{I(K_\alpha(X_1),K_\alpha(X_2))}{\alpha},\frac{1-I(K_\alpha(X_1),K_\alpha(X_2))}{1-\alpha})$. Thus $T(T(I(K_\alpha(X_1),K_\alpha(X_2)),1-I(K_\alpha(X_1),K_\alpha(X_2))),I(w(X_1),w(X_2)))\leq \min(\frac{1-I(K_\alpha(X_1),K_\alpha(X_2))}{1-\alpha},\frac{I(K_\alpha(X_1),K_\alpha(X_2))}{\alpha})$ since $I(l(X_1),l(X_2))\in[0,1]$. For $X_1,X_2,Y_1,Y_2\in L([0,1])$, if $w(X_1)\geq w(Y_1)$, $w(X_2)\leq w(Y_2)$ and $K_\alpha(X_i)=K_\alpha(Y_i)$ for $i=1,2$, then $I(w(X_1),w(X_2))\leq I(w(Y_1),w(Y_2))$ and thus $T(T(I(K_\alpha(X_1),K_\alpha(X_2)),1-I(K_\alpha(X_1),K_\alpha(X_2))),I(w(X_1),w(X_2)))\leq T(T(I(K_\alpha(X_1),K_\alpha(X_2)),1-I(K_\alpha(X_1),K_\alpha(X_2))),I(w(Y_1),w(Y_2)))$ and therefore $T(\min(\frac{I(K_\alpha(X_1),K_\alpha(X_2))}{\alpha},\frac{1-I(K_\alpha(X_1),K_\alpha(X_2))}{1-\alpha}),I(w(X_1),w(X_2)))\leq T(\min(\frac{I(K_\alpha(X_1),K_\alpha(X_2))}{\alpha},\frac{1-I(K_\alpha(X_1),K_\alpha(X_2))}{1-\alpha}),I(w(Y_1),w(Y_2)))$. If $K_\alpha(X_1)=K_\alpha(X_2)=0$, then we have $I(K_\alpha(X_1),K_\alpha(X_2))=1, 1-I(K_\alpha(X_1),K_\alpha(X_2))=0$. If $K_\alpha(X_1)=K_\alpha(X_2)=1$, then $I(K_\alpha(X_1),K_\alpha(X_2))=1, 1-I(K_\alpha(X_1),K_\alpha(X_2))=0$. If $K_\alpha(X_1)=1, K_\alpha(X_2)=0$, then $I(K_\alpha(X_1),K_\alpha(X_2))=0, 1-I(K_\alpha(X_1),K_\alpha(X_2))=1$. Thus we have $T(I(K_\alpha(X_1),K_\alpha(X_2)),1-I(K_\alpha(X_1),K_\alpha(X_2)))=0$ and $\min(\frac{I(K_\alpha(X_1),K_\alpha(X_2))}{\alpha},\frac{1-I(K_\alpha(X_1),K_\alpha(X_2))}{1-\alpha})=0$. Thus l' and l'' are functions demanded in Proposition 3.12. □

Example 3.14. *According to Proposition 3.13, we can obtain functions l satisfying properties (1)-(4) demanded in Proposition 3.12. Consider these functions and arbitrary fuzzy implication I in Proposition 3.12, we get interval-valued fuzzy implications on $L([0,1])$ w.r.t. $K_{\alpha,\beta}$ orders:* $I_{IV_1}(X_1,X_2)=[\underline{Y}_1,\overline{Y}_1], \underline{Y}_1=I(K_\alpha(X_1),K_\alpha(X_2))-\alpha T(T(I(K_\alpha(X_1),K_\alpha(X_2)),1-I(K_\alpha(X_1),K_\alpha(X_2))),I(w(X_1),w(X_2))), \overline{Y}_1=I(K_\alpha(X_1),K_\alpha(X_2))+(1-\alpha)T(T(I(K_\alpha(X_1),K_\alpha(X_2)),1-I(K_\alpha(X_1),K_\alpha(X_2))),I(w(X_1),w(X_2)))$. $I_{IV_2}(X_1,X_2)=[\underline{Y}_2,\overline{Y}_2], \underline{Y}_2=I(K_\alpha(X_1),K_\alpha(X_2))-\alpha T(\min(\frac{I(K_\alpha(X_1),K_\alpha(X_2))}{\alpha},\frac{1-I(K_\alpha(X_1),K_\alpha(X_2))}{1-\alpha}),I(w(X_1),w(X_2))), \overline{Y}_2=I(K_\alpha(X_1),K_\alpha(X_2))+(1-\alpha)T(\min(\frac{I(K_\alpha(X_1),K_\alpha(X_2))}{\alpha},\frac{1-I(K_\alpha(X_1),K_\alpha(X_2))}{1-\alpha}),I(w(X_1),w(X_2)))$.

Proposition 3.15. *Let $\alpha\in(0,1)$. For any $X=[\underline{X},\overline{X}]\in L([0,1])$, let $w(X)=\overline{X}-\underline{X}$ and $M_1:[0,1]^n\longrightarrow[0,1]$ be an arbitrary aggregation function. Suppose $M_2:[0,1]^{2n}\longrightarrow[0,1]$ is an aggregation function satisfying:*
(1) For any $X_1,\dots,X_n,Y_1,\dots,Y_n\in L([0,1])$, if $w(X_i)\geq w(Y_i)$ and $K_\alpha(X_i)=K_\alpha(Y_i)$ for all $i=1,\dots,n$, then $M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n))\geq M_2(K_\alpha(Y_1),\dots,K_\alpha(Y_n),w(Y_1),\dots,w(Y_n))$.
(2) $M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))\geq \alpha M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n))$.
(3) $M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))+(1-\alpha)M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n))\leq 1$.
(4) $M_2(0,\dots,0)=0$, $M_2(1,\dots,1,0,\dots,0)=0$. Then the function $M_{IV}:L([0,1])^n\longrightarrow L([0,1])$ given by

$$M_{IV}(X_1,\dots,X_n)=Y,\ X_1,\dots,X_n,Y\in L([0,1]),\ \text{where}\ \begin{cases}K_\alpha(Y)=M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))\\ w(Y)=M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n))\end{cases}$$

is an interval-valued aggregation function on $L([0,1])$ w.r.t. the $K_{\alpha,\beta}$ order for any $\beta\neq\alpha$.

Proof. Since $K_\alpha(Y)=M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))=(1-\alpha)\underline{Y}+\alpha\overline{Y}$ and $w(Y)=M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n))=\overline{Y}-\underline{Y}$, $\underline{Y}=M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))-\alpha M_2(K_\alpha(X_1),\dots,w(X_n))$ and $\overline{Y}=M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))+(1-\alpha)M_2(K_\alpha(X_1),\dots,w(X_n))$. Since $0\leq M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))\leq 1$, we have $M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))\leq 1+\alpha M_2(K_\alpha(X_1),\dots,w(X_n))$ and thus $0\leq M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))-\alpha M_2(K_\alpha(X_1),\dots,W(X_n))\leq 1$. Meanwhile, we get $0\leq M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))+(1-\alpha)M_2(K_\alpha(X_1),\dots,w(X_n))\leq 1$. Thus we get $\underline{Y},\overline{Y}\in[0,1]$ and $Y\in L([0,1])$. Thus M_{IV} is well defined. If $X_1=\dots=X_n=0_L$, then $K_\alpha(X_1)=\dots=K_\alpha(X_n)=0$, $w(X_1)=\dots=w(X_n)=0$ and thus $K_\alpha(Y)=M_1(0,\dots,0)=0$ and $w(Y)=M_2(0,\dots,0)=0$. Thus $\underline{Y}=\overline{Y}=0$. If $X_1=\dots=X_n=1_L$, then $K_\alpha(X_1)=\dots=K_\alpha(X_n)=1$, $w(X_1)=\dots=w(X_n)=0$ and thus $K_\alpha(Y)=M_1(1,\dots,1)=1$ and $w(Y)=M_2(1,\dots,1,0,\dots,0)=0$. Thus $\underline{Y}=\overline{Y}=1$. Thus we get $M_{IV}(0_L,\dots,0_L)=0_L$ and $M_{IV}(1_L,\dots,1_L)=1_L$. Now we prove that M_{IV} is a non-decreasing function w.r.t. $\preceq_L$. For any $X_i,Y_i\in L([0,1])$, if $X_i\preceq_{\alpha,\beta}Y_i$ for all $i=1,\dots,n$, then there exists $j\in\{1,\dots,n\}$ such that $K_\alpha(X_j)<K_\alpha(Y_j)$ or $K_\alpha(X_i)=K_\alpha(Y_i)$ and $K_\beta(X_i)\leq K_\beta(Y_i)$. Let $M_{IV}(X_1,\dots,X_n)=Y$ and $M_{IV}(Y_1,\dots,Y_n)=W$, then $K_\alpha(Y)=M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))$ and $K_\alpha(W)=M_1(K_\alpha(Y_1),\dots,K_\alpha(Y_n))$.

(1) If there exists $j\in\{1,2,\dots,n\}$ such that $K_\alpha(X_j)<K_\alpha(Y_j)$, we have $M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))<M_1(K_\alpha(Y_1),\dots,K_\alpha(Y_n))$ and thus $K_\alpha(Y)<K_\alpha(W)$, i.e., $Y\preceq_{\alpha,\beta}W$. Thus $M_{IV}(X_1,\dots,X_n)\preceq_{\alpha,\beta}M_{IV}(Y_1,\dots,Y_n)$.

(2) If $K_\alpha(X_i)=K_\alpha(Y_i)$ and $K_\beta(X_i)\leq K_\beta(Y_i)$ for all $i=1,\dots,n$, then $M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))=M_1(K_\alpha(Y_1),\dots,K_\alpha(Y_n))$ and thus $K_\alpha(Y)=K_\alpha(W)$. Now we prove that $K_\beta(Y)\leq K_\beta(W)$. Since $K_\beta(W)=(1-\beta)\underline{W}+\beta\overline{W}, K_\beta(Y)=(1-\beta)\underline{Y}+\beta\overline{Y}$, $K_\alpha(W)=(1-\alpha)\underline{W}+\alpha\overline{W}$ and $K_\alpha(Y)=(1-\alpha)\underline{Y}+\alpha\overline{Y}$, we have $K_\beta(Y)-K_\alpha(Y)=(\beta-\alpha)(\overline{Y}-\underline{Y})=(\beta-\alpha)M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n))$ and $K_\beta(W)-K_\alpha(W)=(\beta-\alpha)(\overline{W}-\underline{W})=(\beta-\alpha)M_2(K_\alpha(Y_1),\dots,K_\alpha(Y_n),w(Y_1),\dots,w(Y_n))$. According to $K_\alpha(X_i)=K_\alpha(Y_i)$ and $K_\beta(X_i)\leq K_\beta(Y_i)$, we have $K_\beta(X_i)-K_\alpha(X_i)\leq K_\beta(Y_i)-K_\alpha(Y_i)$, i.e., $(\beta-\alpha)w(X_i)\leq(\beta-\alpha)w(Y_i)$. If $\beta-\alpha>0$, then $w(X_i)\leq w(Y_i)$.

By the property of M_2, $M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \le M_2(K_\alpha(Y_1),\dots,K_\alpha(Y_n),w(Y_1),\dots,w(Y_n))$ and $(\beta-\alpha)M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \le (\beta-\alpha)M_2(K_\alpha(Y_1),\dots,K_\alpha(Y_n),w(Y_1),\dots,w(Y_n))$. Thus $K_\beta(Y)-K_\alpha(Y) \le K_\beta(W)-K_\alpha(W)$, i.e., $K_\beta(Y) \le K_\beta(W)$. If $\beta-\alpha<0$, then $w(X_i)\ge w(Y_i)$. By the property of M_2, we get $M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \ge M_2(K_\alpha(Y_1),\dots,K_\alpha(Y_n),w(Y_1),\dots,w(Y_n))$ and $(\beta-\alpha)M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \le (\beta-\alpha)M_2(K_\alpha(Y_1),\dots,K_\alpha(Y_n),w(Y_1),\dots,w(Y_n))$. Thus $K_\beta(Y)-K_\alpha(Y) \le K_\beta(W)-K_\alpha(W)$, i.e., $K_\beta(Y)\le K_\beta(W)$. Through $K_\alpha(Y)=K_\alpha(W)$ and $K_\beta(Y)\le K_\beta(W)$, we get $Y \preceq_{\alpha,\beta} W$, i.e., $M_{IV}(X_1,\dots,X_n) \preceq_{\alpha,\beta} M_{IV}(Y_1,\dots,Y_n)$. □

Proposition 3.16. *Let $\alpha\in(0,1)$, T and $M_1, M_3:[0,1]^n \longrightarrow [0,1]$ be a t-norm and two aggregation functions, respectively. Let $k:[0,1]^2\longrightarrow[0,1]$ be a function satisfying: For any $X_1,X_2\in L([0,1])$, if $w(X_1)\ge w(X_2)$ and $K_\alpha(X_1)=K_\alpha(X_2)$, then $k(K_\alpha(X_1),w(X_1))\ge k(K_\alpha(X_2),w(X_2))$. Let Δ denote $M_3(k(K_\alpha(X_1),w(X_1)),\dots,k(K_\alpha(X_n),w(X_n)))$, then $M_2', M_2'':[0,1]^{2n}\longrightarrow[0,1]$ with $M_2'(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) = T(T(M_1(K_\alpha(X_1),\dots,K_\alpha(X_n)),1-M_1(K_\alpha(X_1),$ $\dots,K_\alpha(X_n))),\Delta)$, $M_2''(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) = T(\min(\frac{M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{\alpha},\frac{1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{1-\alpha}),$ $\Delta)$ are functions satisfying properties (1)-(4) demanded in Proposition 3.15.*

Proof. Let M_1 be an arbitrary aggregation function, according to Proposition 3.15, M_2 should satisfy properties:

(1) $M_1(K_\alpha(X_1),\dots,K_\alpha(X_n)) \ge \alpha M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n))$.

(2) $M_1(K_\alpha(X_1),\dots,K_\alpha(X_n)) + (1-\alpha)M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \le 1$.

(3) For any $X_1,\dots,X_n,Y_1,\dots,Y_n\in L([0,1])$, if $w(X_i)\ge w(Y_i)$ and $K_\alpha(X_i)=K_\alpha(Y_i)$ for all $i=1,\dots,n$, then $M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \ge M_2(K_\alpha(Y_1),\dots,K_\alpha(Y_n),w(Y_1),\dots,w(Y_n))$.

(4) $M_2(0,\dots,0)=0$, $M_2(1,\dots,1,0,\dots,0)=0$.

For property (1), M_2 should satisfy $M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \le \frac{M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{\alpha}$. For property (2), M_2 should satisfy $M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \le \frac{1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{1-\alpha}$. Therefore, M_2 should satisfy $M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \le \min(\frac{M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{\alpha},\frac{1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{1-\alpha})$.

For t-norm T, $T(M_1(K_\alpha(X_1),\dots,K_\alpha(X_n)),1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))) \le \min(M_1(K_\alpha(X_1),\dots,K_\alpha(X_n)),1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))) \le \min(\frac{M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{\alpha},\frac{1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{1-\alpha})$. Since $\Delta = M_3(k(K_\alpha(X_1),w(X_1)),\dots,k(K_\alpha(X_n),w(X_n))) \in [0,1]$, we obtain that $T(T(M_1(K_\alpha(X_1),\dots,K_\alpha(X_n)),1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))),\Delta) \le T(\min(\frac{M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{\alpha},\frac{1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{1-\alpha}),\Delta) \le \min(\frac{1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{1-\alpha},\frac{M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{\alpha})$ and $T(T(M_1(K_\alpha(X_1),\dots,K_\alpha(X_n)),1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))),\Delta) \le T(\min(\frac{M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{\alpha},\frac{1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{1-\alpha}),$ $\Delta) \le \min(\frac{1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{1-\alpha},\frac{M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{\alpha})$.

For $X_1,\dots,X_n,Y_1,\dots,Y_n\in L([0,1])$, if $w(X_i)\ge w(Y_i)$ and $K_\alpha(X_i)=K_\alpha(Y_i)$ for all $i=1,2,\dots,n$, then $k(K_\alpha(X_i),w(X_i)) \ge k(K_\alpha(Y_i),w(Y_i))$ and $M_3(k(K_\alpha(X_1),w(X_1)),\dots,k(K_\alpha(X_n),w(X_n))) \ge M_3(k(K_\alpha(Y_1),w(Y_1)),\dots,k(K_\alpha(Y_n),w(Y_n)))$. Thus $M_2'(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \ge M_2'(K_\alpha(Y_1),\dots,K_\alpha(Y_n),w(Y_1),\dots,w(Y_n))$ and $M_2''(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \ge M_2''(K_\alpha(Y_1),\dots,K_\alpha(Y_n),w(Y_1),\dots,w(Y_n))$. If $K_\alpha(X_1)=\dots=K_\alpha(X_n)=0$, then $M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))=0$ and $M_2'(0,\dots,0)=0$. If $K_\alpha(X_1)=\dots=K_\alpha(X_n)=0$, then $M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))=0$ and $M_2'(0,\dots,0)=0, M_2''(0,\dots,0)=0$. If $K_\alpha(X_1)=\dots=K_\alpha(X_n)=1$, then $1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))=0, M_2'(1,\dots,1,0,\dots,0)=0$ and $M_2''(1,\dots,1,0,\dots,0)=0$. Thus M_2' and M_2'' are functions demanded in Proposition 3.15. □

Remark 3.17. *Proposition 3.16 provides a way how to construct functions M_2 satisfying properties (1)-(4) demanded in Proposition 3.15. The key step of constructing M_2 is to make it satisfy $M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) \le \min(\frac{M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{\alpha},\frac{1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{1-\alpha})$. Suppose that M_1 satisfies $M(\alpha x_1,\dots,\alpha x_n)=\alpha M(x_1,\dots,x_n)$ and $M(1-x_1,\dots,1-x_n) \le 1-M(x_1,\dots,x_n)$ for all $\alpha,x_1,\dots,x_n\in[0,1]$ and M_4 is an aggregation function satisfying $M_4(x_1,\dots,x_n)\le M_1(x_1,\dots,x_n)$. Since $w(X_i)\le\min(\frac{K_\alpha(X_i)}{\alpha},\frac{1-K_\alpha(X_i)}{1-\alpha})$, we get $M_4(w(X_1),\dots,w(X_n)) \le M_1(w(X_1),\dots,w(X_n)) \le M_1(\frac{1-K_\alpha(X_1)}{1-\alpha},\dots,\frac{1-K_\alpha(X_n)}{1-\alpha}) = \frac{M_1(1-K_\alpha(X_1),\dots,1-K_\alpha(X_n))}{1-\alpha} \le \frac{1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{1-\alpha}$, $M_4(w(X_1),$ $\dots,w(X_n)) \le M_4(\frac{K_\alpha(X_1)}{\alpha},\dots,\frac{K_\alpha(X_n)}{\alpha}) \le M_1(\frac{K_\alpha(X_1)}{\alpha},\dots,\frac{K_\alpha(X_n)}{\alpha}) = \frac{M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{\alpha}$. Thus $M_4(w(X_1),\dots,w(X_n)) \le \min(\frac{M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{\alpha},\frac{1-M_1(K_\alpha(X_1),\dots,K_\alpha(X_n))}{1-\alpha})$. For any $X_1,\dots,X_n,Y_1,\dots,Y_n\in L([0,1])$, if $w(X_i)\ge w(Y_i)$ and $K_\alpha(X_i)=K_\alpha(Y_i)$ for all $i=1,\dots,n$, then we get $M_4(w(X_1),\dots,w(X_n)) \ge M_4(w(Y_1),\dots,w(Y_n))$. Note that $M_4(0,\dots,0)=0$. Thus M_4 is a function satisfying properties (1)-(4) demanded in Proposition 3.15 and we can take $M_2(K_\alpha(X_1),\dots,K_\alpha(X_n),w(X_1),\dots,w(X_n)) = M_4(w(X_1),\dots,w(X_n))$. Consider M_4 and M_1, we get an interval-valued aggregation function on $L([0,1])$ w.r.t. the $K_{\alpha,\beta}$ order for any $\beta\ne\alpha$: $M_{IV_1}(X_1,\dots,X_n)=[\underline{Y}_1,\overline{Y}_1]$, where $\underline{Y}_1 = M_1(K_\alpha(X_1),K_\alpha(X_2),\dots,K_\alpha(X_n)) - \alpha M_4(w(X_1),w(X_2),\dots,w(X_n))$, $\overline{Y}_1 = M_1(K_\alpha(X_1),K_\alpha(X_2),\dots,K_\alpha(X_n)) + (1-\alpha)M_4(w(X_1),w(X_2),\dots,w(X_n))$. This interval-valued aggregation function is provided by Bustince et al. [14]. In this* *sense, the existing interval-valued aggregation function on $L([0,1])$ w.r.t. the $K_{\alpha,\beta}$ order can be constructed through our approach given in Proposition 3.15.*

Corollary 3.18. *Let $\alpha \in (0,1)$, T and $M_1, M_3 : [0,1]^n \longrightarrow [0,1]$ be a t-norm and two aggregation functions. Let Δ denote $M_3(w(X_1),\ldots,w(X_n))$ or $M_3(\frac{w(X_1)}{\min(\frac{K_\alpha(X_1)}{\alpha},\frac{1-K_\alpha(X_1)}{1-\alpha})},\ldots,\frac{w(X_n)}{\min(\frac{K_\alpha(X_n)}{\alpha},\frac{1-K_\alpha(X_n)}{1-\alpha})}))$. Then $M_2', M_2'' : [0,1]^{2n} \longrightarrow [0,1]$ defined by $M_2'(K_\alpha(X_1),\ldots,K_\alpha(X_n),w(X_1),\ldots,w(X_n)) = T(T(M_1(K_\alpha(X_1),\ldots,K_\alpha(X_n)),1-M_1(K_\alpha(X_1),\ldots,K_\alpha(X_n)))),\Delta)$, $M_2''(K_\alpha(X_1),\ldots,K_\alpha(X_n),w(X_1),\ldots,w(X_n)) = T(\min(\frac{M_1(K_\alpha(X_1),\ldots,K_\alpha(X_n))}{\alpha},\frac{1-M_1(K_\alpha(X_1),\ldots,K_\alpha(X_n))}{1-\alpha}),\Delta)$ are functions satisfying properties (1)-(4) demanded in Proposition 3.15.*

Example 3.19. *In Corollary 3.18, if we take M_1, M_3 and T_P, then $M_2''(K_\alpha(X_1),\ldots,K_\alpha(X_n),w(X_1),\ldots,w(X_n)) = \min(\frac{M_1(K_\alpha(X_1),\ldots,K_\alpha(X_n))}{\alpha},\frac{1-M_1(K_\alpha(X_1),\ldots,K_\alpha(X_n))}{1-\alpha}) \times M_3(\frac{w(X_1)}{\min(\frac{K_\alpha(X_1)}{\alpha},\frac{1-K_\alpha(X_1)}{1-\alpha})},\ldots,\frac{w(X_n)}{\min(\frac{K_\alpha(X_n)}{\alpha},\frac{1-K_\alpha(X_n)}{1-\alpha})})$. Consider M_2'' and M_1 in Proposition 3.15, we get an interval-valued aggregation function on $L([0,1])$ w.r.t. the $K_{\alpha,\beta}$ order for any $\beta \neq \alpha$: $M_{IV_2}(X_1,\ldots,X_n) = [\underline{Y}_2,\overline{Y}_2]$, where $\underline{Y}_2 = M_1(K_\alpha(X_1),\ldots,K_\alpha(X_n)) - \alpha M_2''(K_\alpha(X_1),\ldots,w(X_n))$, $\overline{Y}_2 = M_1(K_\alpha(X_1),\ldots,K_\alpha(X_n)) + (1-\alpha)M_2''(K_\alpha(X_1),\ldots,\ldots,w(X_n))$. This interval-valued aggregation function is provided by Bustince et al.* [14]. *In this sense, the existing interval-valued aggregation function on $L([0,1])$ w.r.t. the $K_{\alpha,\beta}$ order can be constructed through our approach given in Proposition 3.15.*

4 Width-preserving interval-valued fuzzy logical connectives with respect to arbitrary admissible orders $\preceq_{TL}$

It is not easy to construct interval-valued fuzzy equivalence functions and dissimilarity functions w.r.t. a special admissible order $K_{\alpha,\beta}$ and arbitrary intervals on $L([0,1])$. This is because when discussing about interval-valued fuzzy negations, automorphisms, aggregation functions and fuzzy implications w.r.t. admissible orders, only the admissible order between two intervals is involved. For example, given two intervals $X, Y \in L([0,1])$, the $K_{\alpha,\beta}$ order between X and Y is defined as: $X \preceq_{\alpha,\beta} Y$ if and only if $K_\alpha(X) < K_\alpha(Y)$ or $K_\alpha(X) = K_\alpha(Y)$ and $K_\beta(X) \leq K_\beta(Y)$. The $K_{\alpha,\beta}$ order between two intervals can be converted to relationships among $K_\alpha(X), K_\alpha(Y), K_\beta(X)$ and $K_\beta(Y)$. By Definitions 2.17 and 2.21, if we want to discuss about interval-valued fuzzy equivalence functions and dissimilarity functions w.r.t. admissible orders, the admissible order among three intervals should be involved. Given $X, Y, Z \in L([0,1])$, the $K_{\alpha,\beta}$ order among X, Y and Z is defined as: $X \preceq_{\alpha,\beta} Y \preceq_{\alpha,\beta} Z$, i.e., $X \preceq_{\alpha,\beta} Y$ and $Y \preceq_{\alpha,\beta} Z$ and $X \preceq_{\alpha,\beta} Z$. Thus we should consider

$$\begin{cases} K_\alpha(X) < K_\alpha(Y) \text{ or} \\ K_\alpha(X) = K_\alpha(Y) \text{ and } K_\beta(X) \leq K_\beta(Y) \end{cases}, \begin{cases} K_\alpha(Y) < K_\alpha(Z) \text{ or} \\ K_\alpha(Y) = K_\alpha(Z) \text{ and } K_\beta(Y) \leq K_\beta(Z) \end{cases}, \begin{cases} K_\alpha(X) < K_\alpha(Z) \text{ or} \\ K_\alpha(X) = K_\alpha(Z) \text{ and } K_\beta(X) \leq K_\beta(Z). \end{cases}$$

That is to say, the $K_{\alpha,\beta}$ order among three intervals can be converted to the relationships among $K_\alpha(X), K_\alpha(Y), K_\alpha(Z), K_\beta(X), K_\beta(Y)$ and $K_\beta(Z)$, which can be expressed as the following four cases:

Case 1: $K_\alpha(X) < K_\alpha(Y) < K_\alpha(Z)$ or
Case 2: $K_\alpha(X) < K_\alpha(Y) = K_\alpha(Z)$ and $K_\beta(Y) \leq K_\beta(Z)$ or
Case 3: $K_\alpha(X) = K_\alpha(Y) < K_\alpha(Z)$ and $K_\beta(X) \leq K_\beta(Y)$ or
Case 4: $K_\alpha(X) = K_\alpha(Y) = K_\alpha(Z)$ and $K_\beta(X) \leq K_\beta(Y) \leq K_\beta(Z)$.

Suppose there exists a d_{IV} w.r.t. the $K_{\alpha,\beta}$ order, by Definition 2.21, d_{IV} should satisfy: For $X, Y, Z \in L([0,1])$, if $X \preceq_{\alpha,\beta} Y \preceq_{\alpha,\beta} Z$, then $d_{IV}(X,Y) \preceq_{\alpha,\beta} d_{IV}(X,Z)$ and $d_{IV}(Y,Z) \preceq_{\alpha,\beta} d_{IV}(X,Z)$. That is to say, d_{IV} should satisfy:

1. $K_\alpha(d_{IV}(X,Y)) < K_\alpha(d_{IV}(X,Z))$ and $K_\alpha(d_{IV}(Y,Z)) < K_\alpha(d_{IV}(X,Z))$ or
2. $K_\alpha(d_{IV}(X,Y)) < K_\alpha(d_{IV}(Y,Z)) = K_\alpha(d_{IV}(X,Z))$ and $K_\beta(d_{IV}(Y,Z)) \leq K_\beta(d_{IV}(X,Z))$ or
3. $K_\alpha(d_{IV}(Y,Z)) < K_\alpha(d_{IV}(X,Z)) = K_\alpha(d_{IV}(X,Y))$ and $K_\beta(d_{IV}(X,Y)) \leq K_\beta(d_{IV}(X,Z))$ or
4. $K_\alpha(d_{IV}(X,Y)) = K_\alpha(d_{IV}(Y,Z)) = K_\alpha(d_{IV}(X,Z))$ and $K_\beta(d_{IV}(X,Y)) \leq K_\beta(d_{IV}(X,Z))$ and $K_\beta(d_{IV}(Y,Z)) \leq K_\beta(d_{IV}(X,Z))$.

It is not easy to construct a function d_{IV} defined on $L([0,1])^2 \longrightarrow L([0,1])$, which satisfies properties 1-4 under Cases 1-4. The reason is that relationships among the left and right projections of the intervals X, Y and Z cannot be directly obtained through relationships among $K_\alpha(X), K_\alpha(Y), K_\alpha(Z), K_\beta(X), K_\beta(Y)$ and $K_\beta(Z)$. Thus relationships among left and right projections of the intervals $d_{IV}(X,Y), d_{IV}(Y,Z)$ and $d_{IV}(X,Z)$ cannot be obtained, which makes relationships among $K_\alpha(d_{IV}(X,Y)), K_\alpha(d_{IV}(Y,Z)), K_\alpha(d_{IV}(X,Z)), K_\beta(d_{IV}(X,Y)), K_\beta(d_{IV}(Y,Z))$ and $K_\beta(d_{IV}(X,Z))$ unavailable. Thus constructing interval-valued fuzzy dissimilarity functions w.r.t. $K_{\alpha,\beta}$ orders and arbitrary intervals on $L([0,1])$ is not easy. The same problems are faced when we construct interval-valued fuzzy equivalence functions w.r.t. $K_{\alpha,\beta}$ orders and arbitrary intervals on $L([0,1])$. In [14], Bustince et al. proposed a definition of interval-valued restricted equivalence function which takes into account the width of the inputs.

Definition 4.1. [14] *A function $RE_{IV} : L([0,1])^2 \longrightarrow L([0,1])$ is an interval-valued fuzzy restricted equivalence function w.r.t. $\preceq_L$ if*

(1) $RE_{IV}(X,Y) = RE_{IV}(Y,X)$ *for all* $X,Y \in L([0,1])$.

(2) $RE_{IV}(X,X) = [1-w(X),1]$ *for all* $X \in L([0,1])$.

(3) $RE_{IV}(X,Y) = 0_L$ *if and only if* $\{X,Y\} = \{0_L,1_L\}$.

(4) *If* $X,Y,Z \in L([0,1])$ *are such that* $X \preceq_L Y \preceq_L Z$ *and* $w(X) = w(Y) = w(Z)$, *then* $RE_{IV}(X,Z) \preceq_L RE_{IV}(X,Y)$ *and* $RE_{IV}(X,Z) \preceq_L RE_{IV}(Y,Z)$.

Considering that there is no research on interval-valued fuzzy dissimilarity functions w.r.t. arbitrary admissible orders, we will weaken the condition of "arbitrary intervals on $L([0,1])$" and discuss interval-valued fuzzy dissimilarity functions w.r.t. the intervals with the same width on $L([0,1])$. Under this certain condition, we cannot obtain interval-valued fuzzy dissimilarity functions w.r.t. arbitrary intervals on $L([0,1])$, but we can extend the condition of "a special admissible order $K_{\alpha,\beta}$" to "an arbitrary admissible order". That is to say, we will discuss interval-valued fuzzy dissimilarity functions w.r.t. arbitrary admissible orders and the intervals with the same width on $L([0,1])$.

Definition 4.2. *A function $d_{IV} : L([0,1])^2 \longrightarrow L([0,1])$ is an interval-valued fuzzy dissimilarity function w.r.t. $\preceq_L$ if*

(1) $d_{IV}(X,Y) = d_{IV}(Y,X)$ *for all* $X,Y \in L([0,1])$.

(2) $d_{IV}(X,X) = [0,w(X)]$ *for all* $X \in L([0,1])$.

(3) $d_{IV}(0_L,1_L) = d_{IV}(1_L,0_L) = 1_L$.

(4) *If* $X,Y,Z \in L([0,1])$ *are such that* $X \preceq_L Y \preceq_L Z$ *and* $w(X) = w(Y) = w(Z)$, *then* $d_{IV}(X,Y) \preceq_L d_{IV}(X,Z)$ *and* $d_{IV}(Y,Z) \preceq_L d_{IV}(X,Z)$.

Remark 4.3. *It is shown that Axioms 1 and 3 retain the properties required in the definition of fuzzy dissimilarity function in the real-valued setting. However, Axioms 2 and 4 arise some differences. Axiom 2 comes from the idea of Bustince et al. in* [14], *where it was pointed out that if two elements have the same interval memberships, this does not mean that their underlying real-valued degree of membership are the same. Thus we cannot get less uncertainty when comparing them. Regarding Axiom 4, the order relationships of $d_{IV}(X,Y), d_{IV}(Y,Z)$ and $d_{IV}(X,Z)$ are easy to obtain w.r.t. the partial order. However, their order relationships are flexible w.r.t. the total order. Note that if two intervals have the same width, then their admissible order can equivalently convert to their partial order. Thus if we want to consider interval-valued fuzzy dissimilarity functions w.r.t. a total order (i.e., the admissible order), we should impose the restriction that $w(X) = w(Y) = w(Z)$ in Axiom 4.*

Lemma 4.4. [14] *Let $X,Y \in L([0,1])$ satisfying $w(X) = w(Y)$. Then $X \preceq_P Y$ if and only $X \preceq_{TL} Y$ for $\preceq_{TL}$.*

Proposition 4.5. *Let $\alpha \in (0,1)$ and M be a 2-ary idempotent symmetric aggregation function and let d be a fuzzy dissimilarity function. Suppose T and I are respectively a t-norm and a fuzzy implication. If T and I satisfy $I(1,y) = y, I(1-x,0) = x$ and $I(x,y) - T(x,y) = 1-x$, then the function $d_{IV} : L([0,1])^2 \longrightarrow L([0,1])$ given by $d_{IV}(X,Y) = [T(1-M(w(X),w(Y)), d(K_\alpha(X),K_\alpha(Y))), I(1-M(w(X),w(Y)), d(K_\alpha(X),K_\alpha(Y)))], X,Y \in L([0,1])$ is a width-preserving interval-valued fuzzy dissimilarity function w.r.t. arbitrary admissible order $\preceq_{TL}$.*

Proof. Note that $0 \le T(1-M(w(X),w(Y)), d(K_\alpha(X),K_\alpha(Y))) \le d(K_\alpha(X),K_\alpha(Y)) \le 1$ and $0 \le d(K_\alpha(X),K_\alpha(Y)) = I(1, d(K_\alpha(X),K_\alpha(Y))) \le I(1-M(w(X),w(Y)), d(K_\alpha(X),K_\alpha(Y))) \le 1$. Thus d_{IV} is well-defined. Since T and I satisfy $I(x,y) - T(x,y) = 1-x$, we have $w(d_{IV}(X,Y)) = I(1-M(w(X),w(Y)), d(K_\alpha(X),K_\alpha(Y))) - T(1-M(w(X),w(Y)), d(K_\alpha(X),K_\alpha(Y))) = M(w(X),w(Y))$. If $w(X) = w(Y)$, then $w(d_{IV}(X,Y)) = M(w(X),w(Y)) = w(X)$ and thus d_{IV} is width-preserving. Since $d_{IV}(Y,X) = [T(1-M(w(Y),w(X)), d(K_\alpha(Y),K_\alpha(X))), I(1-M(w(Y), w(X)), d(K_\alpha(Y),K_\alpha(X)))] = [T(1-M(w(X),w(Y)), d(K_\alpha(X),K_\alpha(Y))), I(1-M(w(X),w(Y)), d(K_\alpha(X),K_\alpha(Y)))] = d_{IV}(X,Y)$, $d_{IV}(X,X) = [T(1-M(w(X),w(X)), d(K_\alpha(X),K_\alpha(X))), I(1-M(w(X),w(X)), d(K_\alpha(X),K_\alpha(X)))] = [T(1-w(X),0), I(1-w(X),0)] = [0,w(X)]$ and $d_{IV}(1_L,0_L) = [T(1-M(0,0),d(1,0)), I(1-M(0,0),d(1,0))] = 1_L$, we obtain the first three conditions in Definition 4.2. Regarding the fourth condition, if $X \preceq_{TL} Y \preceq_{TL} Z$ and $w(X) = w(Y) = w(Z)$, then from Lemma 4.4 we have $\underline{X} \le \underline{Y} \le \underline{Z}, \overline{X} \le \overline{Y} \le \overline{Z}$ and $K_\alpha(X) \le K_\alpha(Y) \le K_\alpha(Z)$. According to the definition of d, we have $d(K_\alpha(X),K_\alpha(Z)) \ge d(K_\alpha(X),K_\alpha(Y))$ and $d(K_\alpha(X),K_\alpha(Z)) \ge d(K_\alpha(Y),K_\alpha(Z))$. Observe that $d_{IV}(X,Z) = [T(1-w(X), d(K_\alpha(X),K_\alpha(Z))), I(1-w(X), d(K_\alpha(X),K_\alpha(Z)))]$, $d_{IV}(X,Y) = [T(1-w(X), d(K_\alpha(X),K_\alpha(Y))), I(1-w(X), d(K_\alpha(X),K_\alpha(Y)))]$ and $d_{IV}(Y,Z) = [T(1-w(X), d(K_\alpha(Y),K_\alpha(Z))), I(1-w(X), d(K_\alpha(Y),K_\alpha(Z)))]$, we obtain that $d_{IV}(X,Y) \preceq_P d_{IV}(X,Z)$ and $d_{IV}(Y,Z) \preceq_P d_{IV}(X,Z)$. Since T and I satisfy $I(x,y) - T(x,y) = 1-x$ for all $x,y \in [0,1]$, we have $w(d_{IV}(X,Z)) = w(d_{IV}(X,Y)) = w(d_{IV}(Y,Z)) = w(X)$ in case that $X \preceq_{TL} Y \preceq_{TL} Z$. According to Lemma 4.4 we get $d_{IV}(X,Y) \preceq_{TL} d_{IV}(X,Z)$ and $d_{IV}(Y,Z) \preceq_{TL} d_{IV}(X,Z)$. □

Lemma 4.6. *Let T be a t-norm. If $x \le z$ implies $x - T(x,y) \le z - T(z,y)$ for all $x,y,z \in [0,1]$, then the function $I : [0,1]^2 \longrightarrow [0,1]$ given by $I(x,y) = T(x,y) + 1 - x$ for all $x,y \in [0,1]$ is a fuzzy implication satisfying $I(1,y) = y, I(1-x,0) = x$ and $I(x,y) - T(x,y) = 1-x$.*

Proof. For all $x, y, z \in [0,1]$, if $x \leq z$, then $I(x,y) = T(x,y) + 1 - x \geq T(z,y) + 1 - z = I(z,y)$. If $y \leq z$, then $I(x,y) = T(x,y) + 1 - x \leq T(x,z) + 1 - x = I(x,z)$. Note that $I(0,0) = T(0,0) + 1 = 1$, $I(1,1) = T(1,1) + 0 = 1$ and $I(1,0) = T(1,0) + 0 = 0$. Thus I is a fuzzy implication. Furthermore, I satisfies $I(1,y) = T(1,y) + 0 = y$, $I(1-x,0) = T(1-x,0) + x = x$ and $I(x,y) - T(x,y) = 1 - x$. □

Example 4.7. *Let us consider T satisfying $x - T(x,y) \leq z - T(z,y)$ for $0 \leq x \leq z \leq 1$. If $T = T_M$, then $x - \min(x,y) = \max(0, x-y)$ and $z - \min(z,y) = \max(0, z-y)$. If $x \leq z$, then $\max(0, x-y) \leq \max(0, z-y)$ and thus $x - T_M(x,y) \leq z - T_M(z,y)$. If $T = T_L$, then $x - \max(x+y-1, 0) = \min(1-y, x)$ and $z - \max(z+y-1, 0) = \min(1-y, z)$. If $x \leq z$, then $\min(1-y, x) \leq \min(1-y, z)$ and thus $x - T_L(x,y) \leq z - T_L(z,y)$. If $T = T_P$, then $x - xy = x(1-y)$ and $z - zy = z(1-y)$. If $x \leq z$, then $x(1-y) \leq z(1-y)$ and thus $x - T_P(x,y) \leq z - T_P(z,y)$. Thus T_M, T_L and T_P are three examples of t-norms satisfying the conditions of T demanded in Lemma 4.6. Then we get three corresponding fuzzy implications $I_1(x,y) = \min(1-x+y, 1)$, $I_2(x,y) = \max(1-x, y)$ and $I_3(x,y) = 1 - x + xy$ satisfying the conditions of I demanded in Lemma 4.6. Thus $(T_M, I_1), (T_L, I_2)$ and (T_P, I_3) are three pairs of t-norms and fuzzy implications can be used for constructing width-preserving interval-valued fuzzy dissimilarity functions w.r.t. arbitrary admissible orders in Proposition 4.5. If we take (T_M, I_1), (T_L, I_2) and (T_P, I_3), then we get width-preserving d_{IV}: $d_{IV}(X,Y) = [\min(1 - M(w(X), w(Y)), d(K_\alpha(X), K_\alpha(Y))), \min(M(w(X), w(Y)) + d(K_\alpha(X), K_\alpha(Y)), 1)]$, $d_{IV}(X,Y) = [\max(d(K_\alpha(X), K_\alpha(Y)) - M(w(X), w(Y)), 0), \max(M(w(X), w(Y)), d(K_\alpha(X), K_\alpha(Y)))]$, $d_{IV}(X,Y) = [(1 - M(w(X), w(Y)))d(K_\alpha(X), K_\alpha(Y)), M(w(X), w(Y)) + (1 - M(w(X), w(Y)))d(K_\alpha(X), K_\alpha(Y))]$.*

Proposition 4.8. *Let M be a 2-ary idempotent symmetric aggregation function and M' be a 2-ary aggregation function satisfying $M'(0,x) = 0$ for all $x \in [0,1]$. Suppose T and I are a t-norm and a fuzzy implication. If T and I satisfy $I(1,y) = y, I(1-x, 0) = x$ and $I(x,y) - T(x,y) = 1 - x$ for all $x, y \in [0,1]$, then the function $d_{IV} : L([0,1])^2 \longrightarrow L([0,1])$*

$$\begin{aligned} d_{IV}(X,Y) = [&T(1 - M(w(X), w(Y)), M'(\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})), \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X}))), \\ &I(1 - M(w(X), w(Y)), M'(\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})), \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X})))],\ X, Y \in L([0,1]) \end{aligned}$$

is a width-preserving interval-valued fuzzy dissimilarity function w.r.t. arbitrary admissible order $\preceq_{TL}$.

Proof. Note that $0 \leq T(1 - M(w(X), w(Y)), M'(\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})), \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X}))) \leq M'(\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})), \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X})) \leq 1$ and $0 \leq M'(\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})), \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X})) = I(1, M'(\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})), \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X}))) \leq I(1 - M(w(X), w(Y)), M'(\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})), \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X}))) \leq 1$. Thus d_{IV} is well-defined. Since T and I satisfy $I(x,y) - T(x,y) = 1 - x$, we have $w(d_{IV}(X,Y)) = I(1 - M(w(X), w(Y)), M'(\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})), \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X}))) - T(1 - M(w(X), w(Y)), M'(\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})), \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X}))) = M(w(X), w(Y))$. If $w(X) = w(Y)$, then $w(d_{IV}(X,Y)) = M(w(X), w(Y)) = w(X)$ and thus d_{IV} is width-preserving. It is easy to see $d_{IV}(Y,X) = d_{IV}(X,Y)$, $d_{IV}(X,X) = [T(1 - w(X), 0), I(1 - w(X), 0)] = [0, w(X)]$ and $d_{IV}(1_L, 0_L) = [T(1, M'(1,1)), I(1, M'(1,1))] = 1_L$, we obtain the first three conditions in Definition 4.2.

Regarding the fourth condition, if $X \preceq_{TL} Y \preceq_{TL} Z$ and $w(X) = w(Y) = w(Z)$, then from Lemma 4.4 we have $\underline{X} \leq \underline{Y} \leq \underline{Z}$ and $\overline{X} \leq \overline{Y} \leq \overline{Z}$. Therefore, $\max(0, \max(\underline{X}, \underline{Z}) - \min(\overline{X}, \overline{Z})) = \max(0, \underline{Z} - \overline{X})$, $\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})) = \max(0, \underline{Y} - \overline{X})$ and $\max(0, \max(\underline{Y}, \underline{Z}) - \min(\overline{Y}, \overline{Z})) = \max(0, \underline{Z} - \overline{Y})$. Since $\max(0, \underline{Z} - \overline{X}) \geq \max(0, \underline{Y} - \overline{X})$ and $\max(0, \underline{Z} - \overline{X}) \geq \max(0, \underline{Z} - \overline{Y})$, we have $\max(0, \max(\underline{X}, \underline{Z}) - \min(\overline{X}, \overline{Z})) \geq \max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y}))$ and $\max(0, \max(\underline{X}, \underline{Z}) - \min(\overline{X}, \overline{Z})) \geq \max(0, \max(\underline{Y}, \underline{Z}) - \min(\overline{Y}, \overline{Z}))$. Meanwhile, since $\max(\overline{X} - \underline{Z}, \overline{Z} - \underline{X}) = \overline{Z} - \underline{X}$, $\max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X}) = \overline{Y} - \underline{X}$ and $\max(\overline{Y} - \underline{Z}, \overline{Z} - \underline{Y}) = \overline{Z} - \underline{Y}$, we have $\max(\overline{X} - \underline{Z}, \overline{Z} - \underline{X}) \geq \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X})$ and $\max(\overline{X} - \underline{Z}, \overline{Z} - \underline{X}) \geq \max(\overline{Y} - \underline{Z}, \overline{Z} - \underline{Y})$. Therefore, $d_{IV}(X,Y) \preceq_P d_{IV}(X,Z)$ and $d_{IV}(Y,Z) \preceq_P d_{IV}(X,Z)$. Since T and I satisfy $I(x,y) - T(x,y) = 1 - x$, we have $w(d_{IV}(X,Z)) = w(d_{IV}(X,Y)) = w(d_{IV}(Y,Z)) = w(X)$ in case that $X \preceq_{TL} Y \preceq_{TL} Z$. By Lemma 4.6, we get $d_{IV}(X,Y) \preceq_{TL} d_{IV}(X,Z)$ and $d_{IV}(Y,Z) \preceq_{TL} d_{IV}(X,Z)$. □

Example 4.9. *Take (T_M, I_1) and $M(x,y) = M'(x,y) = \sqrt{xy}$, we get the width-preserving interval-valued fuzzy dissimilarity function w.r.t. $\preceq_{TL}$: $d_{IV}(X,Y) = [\min(1 - \sqrt{w(X)w(Y)}, \sqrt{\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})) \cdot \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X})}$, $\min(\sqrt{w(X)w(Y)} + \sqrt{\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})) \cdot \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X})}, 1)]$.*

In the framework of standard fuzzy theory, given a fuzzy dissimilarity function d, then E defined as $E(x,y) = 1 - d(x,y)$ for all $x, y \in [0,1]$ will be a fuzzy equivalence function. Based on the relationship of fuzzy dissimilarity function and fuzzy equivalence function, we propose approaches to constructing interval-valued fuzzy equivalence functions w.r.t. arbitrary admissible orders in the same manner as Propositions 4.5 and 4.8.

Definition 4.10. *A function $E_{IV}: L([0,1])^2 \longrightarrow L([0,1])$ is an interval-valued fuzzy equivalence function w.r.t. $\preceq_L$ if*

(1) $E_{IV}(X,Y) = E_{IV}(Y,X)$ *for all* $X, Y \in L([0,1])$.

(2) $E_{IV}(X,X) = [1-w(X), 1]$ *for all* $X \in L([0,1])$.

(3) $E_{IV}(0_L, 1_L) = E_{IV}(1_L, 0_L) = 0_L$.

(4) *If* $X, Y, Z \in L([0,1])$ *are such that* $X \preceq_L Y \preceq_L Z$ *and* $w(X) = w(Y) = w(Z)$*, then* $E_{IV}(X,Z) \preceq_L E_{IV}(X,Y)$ *and* $E_{IV}(X,Z) \preceq_L E_{IV}(Y,Z)$.

Remark 4.11. *It is found that the definition of interval-valued fuzzy equivalence function can be seen as a simpler form of definition of interval-valued fuzzy restricted equivalence function proposed by Bustince et al. in Definition 4.1. That is to say, the definition of interval-valued fuzzy restricted equivalence function is stronger than that of interval-valued fuzzy equivalence function when formulating the equivalence of two intervals.*

Proposition 4.12. *Let* $\alpha \in (0,1)$ *and* M *be a 2-ary idempotent symmetric aggregation function and let* E *be fuzzy equivalence function. Suppose* S *and* I *are respectively a t-conorm and a fuzzy implication. If* I *and* S *satisfy* $I(1,y) = y$ *and* $I(x,y) + S(x,y) - 1 = y$ *for all* $x, y \in [0,1]$*, then* $E_{IV}: L([0,1])^2 \longrightarrow L([0,1])$ *given by* $E_{IV}(X,Y) = [1 - I(E(K_\alpha(X), K_\alpha(Y)), M(w(X), w(Y))), S(E(K_\alpha(X), K_\alpha(Y)), M(w(X), w(Y)))], X, Y \in L([0,1])$ *is a width-preserving interval-valued fuzzy equivalence function w.r.t. arbitrary admissible order* $\preceq_{TL}$.

Proof. It can be proved in the same manner as Proposition 4.5. □

Lemma 4.13. *Let* S *be a t-conorm. If* $y \le z$ *implies* $y - S(x,y) \le z - S(x,z)$ *for all* $x, y, z \in [0,1]$*, then the function* $I: [0,1]^2 \longrightarrow [0,1]$ *given by* $I(x,y) = 1 + y - S(x,y)$ *for all* $x, y \in [0,1]$ *is a fuzzy implication satisfying* $I(1,y) = y$ *and* $I(x,y) + S(x,y) - 1 = y$.

Proof. For all $x, y, z \in [0,1]$, if $x \le z$, then $I(x,y) = 1 + y - S(x,y) \ge 1 + y - S(z,y) = I(z,y)$. If $y \le z$, then $I(x,y) = 1 + y - S(x,y) \le 1 + z - S(x,z) = I(x,z)$. Note that $I(0,0) = 1 + 0 - S(0,0) = 1$, $I(1,1) = 1 + 1 - S(1,1) = 1$ and $I(1,0) = 1 + 0 - S(1,0) = 0$. Thus I is a fuzzy implication. Furthermore, I satisfies $I(1,y) = 1 + y - S(1,y) = y$ and $I(x,y) + S(x,y) - 1 = y$. □

Example 4.14. *It is easy to proved that* S_M, S_L *and* S_P *are three examples of t-conorms satisfying the conditions of* S *demanded in Lemma 4.13. Then we can obtain three corresponding fuzzy implications* $I_1(x,y) = \min(1 - x + y, 1)$, $I_2(x,y) = \max(1-x, y)$ *and* $I_3(x,y) = 1 - x + xy$ *satisfying the conditions of* I *demanded in Lemma 4.13. Thus* $(S_M, I_1), (S_L, I_2)$ *and* (S_P, I_3) *are three pairs of t-conorms and fuzzy implications can be used for constructing width-preserving interval-valued fuzzy equivalence functions w.r.t. arbitrary admissible orders in Proposition 4.12. If we take* (S_M, I_1), (S_L, I_2) *and* (S_P, I_3)*, then we get the width-preserving* E_{IV}*:* $E_{IV}(X,Y) = [\max(E(K_\alpha(X), K_\alpha(Y)) - M(w(X), w(Y)), 0), \max(E(K_\alpha(X), K_\alpha(Y)), M(w(X), w(Y)))]$, $E_{IV}(X,Y) = [\min(1 - M(w(X), w(Y)), E(K_\alpha(X), K_\alpha(Y))), \min(E(K_\alpha(X), K_\alpha(Y)) + M(w(X), w(Y)), 1)]$, $E_{IV}(X,Y) = [(1 - M(w(X), w(Y)))E(K_\alpha(X), K_\alpha(Y)), E(K_\alpha(X), K_\alpha(Y)) + M(w(X), w(Y)) - E(K_\alpha(X), K_\alpha(Y))M(w(X), w(Y))]$.

Proposition 4.15. *Let* M *be a 2-ary idempotent symmetric aggregation function and let* M' *be a 2-ary aggregation function satisfying* $M'(0,x) = 0$ *for all* $x \in [0,1]$*. Suppose* S *and* I *are a t-conorm and a fuzzy implication. If* S *and* I *satisfy* $I(1,y) = y$ *and* $I(x,y) + S(x,y) - 1 = y$ *for all* $x, y \in [0,1]$*, then the function* $E_{IV}: L([0,1])^2 \longrightarrow L([0,1])$

$$E_{IV}(X,Y) = [1 - I(1 - M'(\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})), \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X})), M(w(X), w(Y))),$$
$$S(1 - M'(\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})), \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X})), M(w(X), w(Y)))], X, Y \in L([0,1])$$

is a width-preserving interval-valued fuzzy equivalence function w.r.t. arbitrary admissible order $\preceq_{TL}$.

Proof. It can be proved in the same manner as Proposition 4.8. □

Example 4.16. *Take* (S_M, I_1) *and* $M(x,y) = M'(x,y) = \sqrt{xy}$*, we get the width-preserving interval-valued fuzzy equivalence function w.r.t.* $\preceq_{TL}$*:* $E_{IV}(X,Y) = [\max(1 - \sqrt{w(X)w(Y)} - \sqrt{\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})) \cdot \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X})}, 0), \max(\sqrt{w(X)w(Y)}, 1 - \sqrt{\max(0, \max(\underline{X}, \underline{Y}) - \min(\overline{X}, \overline{Y})) \cdot \max(\overline{X} - \underline{Y}, \overline{Y} - \underline{X})})]$.

5 Conclusions

In this work, we investigated interval-valued fuzzy negations, automorphisms, fuzzy implications and aggregation functions with respect to $K_{\alpha,\beta}$ orders and arbitrary intervals on $L([0,1])$ and proposed some approaches to constructing these interval-valued fuzzy logical connectives. We introduced new concepts of interval-valued fuzzy equivalence functions and dissimilarity functions with respect to total orders considering the width of the intervals and described a construction method for width-preserving interval-valued fuzzy equivalence functions and dissimilarity functions with respect to the intervals with the same width and arbitrary admissible orders on $L([0,1])$. The proposed interval-valued fuzzy logical connectives with respect to admissible orders could be seen as a significant step towards a deeper study of interval-valued fuzzy logic related to admissible orders. Note that the study of interval-valued t-norms and t-conorms (denoted as T_{IV} and S_{IV}) with respect to admissible orders is still missing. In Proposition 3.15, we provided an approach to constructing interval-valued aggregation functions on $L([0,1])$ w.r.t. the $K_{\alpha,\beta}$ order for any $\beta \neq \alpha$. Although T_{IV} and S_{IV} are special cases of interval-valued aggregation functions, we cannot directly obtain them through Proposition 3.15. If we want to construct T_{IV} and S_{IV} through functions M_1 and M_2, then M_1 and M_2 are required to satisfy strict conditions such that T_{IV} and S_{IV} are associative and satisfy $T_{IV}(X, 1_L) = X$ and $S_{IV}(X, 0_L) = X$. However, finding those M_1 and M_2 is not a trivial task. Thus we intend to study the constructions of T_{IV} and S_{IV} with respect to admissible orders through analyzing the structures of M_1 and M_2 in our future work. Note that Xu and Yager's order $\preceq_{XY}$ is an order relation between two intuitionistic fuzzy values [37], which was adopted for intervals due to one-to-one correspondence between closed subintervals of $[0,1]$ and Atanassov intuitionistic fuzzy pairs, thus we could consider to reformulate all the obtained results for intuitionistic fuzzy pairs.

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