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Original paper

The bilateral fuzzy soft set and its application in multi-criteria multi-person decision making under conflict information

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Abstract

Fuzzy soft set is a general mathematical tool to deal with uncertain and fuzziness information. But inadequacy lies in which cannot express various uncertainty in a more natural and accurate manner, due to the reason of adopting the fuzzy membership which restricts to $[0, 1]$. As a result, it merely concerns non-negative fuzzy belongingness of elements, so as to be incapable of capturing the conflict information that supporting, neutral and opposing scenarios regarding belongingness of elements in a set to taken into account. To overcome the limitation, this paper extends fuzzy soft set to bilateral fuzzy soft set (BFSS), which considers both non-negative and negative sides for fuzzy membership and normalizes the range to $[-1, 1]$. Then defines some basic operations on BFSS, as well as a detail study of relevant properties. In order to eliminate redundant parameters and draw the essential part of BFSS, the parameters reduction on BFSS is also investigated, and the significance of parameters is introduced accordingly. On this basis, an assessment approach for multi-criteria multi-person decision making under conflict information, together with the algorithm for implementation is presented. Finally, a typical numerical example regarding voting demonstrates the validity and feasibility of the proposed method.

Keywords: Soft sets, memberships, operations, parameters reduction, significance.

1 Introduction

Since 1999 Molodtsov [27] originated soft set as a general mathematical tool to deal with uncertain, fuzzy, or not clearly defined objects, and showed some potential applications in several directions, such as game theory, operations research, soft analysis and soft probability, etc. Study on soft set theory has experienced explosive growth in the last decade. This is due to a significant change in ideology towards uncertainty and vagueness, namely establishes an approximate model of initial objects and then finds the approximate solution, which differs from the idea of precise modeling in traditional mathematical tools, so as to avoid complexity of modeling and solving processes. Molodtsov argued that classic uncertain theories such as probability theory, interval mathematics, fuzzy set theory [40], intuitionistic fuzzy set theory [5] and rough set theory [29] are not always competent for handling these uncertainties existing in economics, engineering, environment science, sociology and some other fields which always involve imprecise data, and explained the reason lies in, possibly, the inadequacy of the parametrization tool of the theory. However, the concept of soft set uses adequate parametrization if necessary that makes the theory is a powerful tool to handle uncertainty.

Based on the pioneer work of Molodtsov, Maji et al. [24] first attempted to investigate the basic operations on soft sets, and published a detailed theoretical study, including concepts of inclusion of soft sets, equality, complement of soft set, null soft sets and absolute soft sets with examples. They also defined some binary operations like *union*, *intersection*, *AND* operator, *OR* operator, together with a discussion of relevant properties. In order to better extract useful information from the synthesis of two soft sets. Ali et al. [3] proposed some restricted and relative operations

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on soft sets, and discussed their De Morgan's laws. Sezgin and Atagün [32] outlined the basic properties of operations on soft sets and illustrated their interconnections between each other. Based on the concept of soft mapping raised by Molodtsov [28], Xiao and Zou [36] proved that both fuzzy set and rough set may be considered as a special soft set with specific parameters and set-valued mapping. They also demonstrated that the basic operations of fuzzy and rough sets also established in the framework of revised operations on soft sets, as well as clarified their connections and differences. To gain the essential part of the set of parameters in a soft set. Maji et al. [25] introduced the concept of reduct soft set of a soft set, which suffices to describe all basic approximate descriptions of the soft set. Chen et al. [9] formed the notion of parameters reduction of soft sets, in which reduct soft set should keep the optimal choice objects unchangeable. Kong et al. [20] introduced a new technique of parameters reduction, called the normal parameters reduction of soft sets. Ali [2] further put forward a new idea of parameters reduction, i.e., keep the classification ability of objects in the initial universe invariant. With a growing body of theoretical studies on soft sets, there yield a great number of results in applications as well and successfully extend to many fields such as decision making [7, 8, 18, 31], combined forecasting [35], medical diagnosis [26], rule mining [14], supplied selection [34], business failure prediction [38], financial market relationships[6], classification [15], multi-attribute group decision making [22, 42], etc. It is worthy of note that soft set associated with other mathematical tools makes it more flexible and available to capture a wider range of uncertainties. A series of considerable research achievements concerning integration of soft set with other concepts of dealing with uncertainty emerged in succession, i.e. fuzzy soft set [23], intuitionistic fuzzy soft set [17], generalized interval-valued hesitant intuitionistic fuzzy soft sets [1], soft rough set [21], N-soft rough sets [41], vague soft set [37], soft multi-set [19], multi-fuzzy soft set [39], hesitant fuzzy soft set [10], belief interval-valued soft set [33], neutrosophic soft rough set [11], multi-fuzzy N-soft set [13], N-polar soft set [12], soft Q-sets [16] and soft M-open sets [4].

The problem of multi-criteria multi-person decision making under conflict information (MMDM-CI for short) is characterized with the rating of each alternative according to each criterion relying on the results of voting by a group of decision makers and the weight coefficient of each criterion. Note that the conflict information appears in various scenarios including opinion, view, voting result, etc., refers to three typical types, i.e., support, neutral and oppose. The traditional fuzzy concept for modeling this kind of uncertainty is lack of the ability due to the reason of non-negative membership setting. The aim of this paper is to introduce a new concept called bilateral fuzzy soft set as a generalization of fuzzy soft set, in which we first attempt to extend the range of fuzzy membership function from the unit interval $[0, 1]$ to bilateral unit interval $[-1, 1]$. The main advantage of the raised concept considers the conflict situation and embodies without restriction of the membership range. Some operations on bilateral fuzzy soft sets such as *complement*, *union*, *intersection*, *AND*, *OR* are defined, and relevant properties in detail are discussed, as well as a study of parameter reduction technique and definition of significance of parameter on bilateral fuzzy soft set. Based on these knowledge, a novel method together with the corresponding algorithm for implementation is specially designed for MMDM-CI. Finally, a numerical example of environmental programs selection illustrates the feasibility and effectiveness of the method. Therefore, the main contributions of this work lie in that:

(1) A novel concept called bilateral fuzzy soft set is put forward that can be regarded as an extension of fuzzy soft set, and an initial study regarding basic operations and properties on bilateral fuzzy soft sets is presented.

(2) Based on parameters reduction of BFSS, a method for handling MMDM-CI is provided. Note that the advantages of the proposed technique can be summarized as two respects: it is without limitation the range of data in the framework of bilateral fuzzy soft set; it can eliminate the redundant attributes and select the optimal attribute subset, moreover, derive the objective weight with respect to each criterion by considering significance of each criterion.

The rest of this paper is organized as follows: Section 2 briefly reviews some background knowledge about soft set theory, which is a starting point of this research. The concept of bilateral fuzzy soft set together with its theoretical framework including some basic operations and relevant properties on bilateral fuzzy soft sets are presented in Section 3. In Section 4, parameters reduction of bilateral fuzzy soft set and significance of parameter are defined, an algorithm for MMDM-CI is designed accordingly. In Section 5, we apply the proposed algorithm based on bilateral fuzzy soft set into a MMDM-CI problem about optimal selection of environmental programs. Finally, some conclusions and outlook for further research are given in the last section.

2 Preliminaries

2.1 Soft sets

In the following, we recall some background knowledge regarding soft sets in brief, which are required in the sequel of our work. Throughout this paper, let $U = \{u_1, u_2, \ldots, u_m\}$ be the initial universe of discourse, and $E = \{e_1, e_2, \ldots, e_n\}$

be a set of parameters.

Definition 2.1. [27] Let $S(U)$ denote the set of subsets of set U, a pair (F, E) is called a soft set (SS for short) over U if F is a mapping from E to $S(U)$.

Note that the soft set is a parameterized family of subsets of the set U. For $\forall e \in E$, $F(e)$ can be considered as the set of e-elements of the soft set (F, E), or as the set of e-approximate elements of the soft set. For the purpose of storing a soft set in a computer, the soft set (F, E) over U can be uniquely expressed in a binary matrix form as follows:

$$(F, E) = \begin{bmatrix} & e_1 & e_2 & \cdots & e_n \\ u_1 & \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ u_2 & \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_m & \mu_{m1} & \mu_{m2} & \cdots & \mu_{mn} \end{bmatrix},$$

where μ denotes the membership function of (F, E), is expressed as

$$\mu_{ij} = \begin{cases} 1 & \text{if } u_i \in F(e_j); \\ 0 & \text{if } u_i \notin F(e_j). \end{cases}$$

Example 2.2. Suppose that $U = \{u_1, u_2, u_3, u_4\}$ be a set of four candidates under consideration, and $E = \{e_1, e_2, e_3, e_4, e_5\}$ is a set of parameters, in which $e_i (i = 1, 2, 3, 4, 5)$ stand for *experience*, *computer knowledge*, *young age*, *higher education* and *good health*, respectively. Consider the mapping F from E to 2^U, Then soft set (F, E) can describe "capability" candidate, and

$$F(e_1) = \{u_1, u_2\}, F(e_2) = \{u_1, u_3, u_4\}, F(e_3) = U,$$
$$F(e_4) = \{u_2, u_3\}, F(e_5) = \{u_1, u_2, u_4\}.$$

No doubt the soft set (F, E) over U can be represented in a matrix form as below:

$$(F, E) = \begin{bmatrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ u_1 & 1 & 1 & 1 & 0 & 1 \\ u_2 & 1 & 0 & 1 & 1 & 1 \\ u_3 & 0 & 1 & 1 & 1 & 0 \\ u_4 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

For more details regarding soft sets, one could refer to [24, 27].

2.2 Extensions of the notion of soft sets

In this subsection, we briefly describe three basic notions of fuzzy soft sets, multi-soft sets, multi-fuzzy soft sets, which can be seen as a combination of soft sets with fuzzy sets, multi-sets and multi-fuzzy sets, respectively. Each of them models a different type of uncertainty.

Definition 2.3. [23] Let $FS(U)$ be the set of all fuzzy subsets of U, a pair $(\widetilde{F}, E)$ is called a fuzzy soft set (FSS for short) over U, where $\widetilde{F}$ is a mapping given by $\widetilde{F} : E \to FS(U)$.

Example 2.4. Consider Example 2.2. In real life much information appears in the characteristics of fuzziness. Thus one often uses a membership function to characterize the degree of belonging $u_i (u_i \in U)$ to the fuzzy set $F(e_j)(e_j \in E)$ instead of crisp number 0 and 1. A fuzzy soft set $(\widetilde{F}, E)$ can describe "capability" candidate under the fuzzy circumstances.

$$F(e_1) = \{u_1/0.6, u_2/0.2, u_3/0.1, u_4/0\},$$
$$F(e_2) = \{u_1/0.3, u_2/0, u_3/0.7, u_4/0.5\},$$
$$F(e_3) = \{u_1/0.2, u_2/0.9, u_3/0.1, u_4/0.7\},$$
$$F(e_4) = \{u_1/0.5, u_2/0.3, u_3/0.5, u_4/0.8\},$$
$$F(e_5) = \{u_1/0.8, u_2/1, u_3/0.3, u_4/0.4\}.$$

Definition 2.5. [19] Let $M^kS(U)$ be the set of all multi-sets of dimension k in U, and k be a positive integer. A pair $(\mathcal{F}, E)$ is called a multi-soft set (MSS for short) of dimension k over U, where $\mathcal{F}$ is a mapping given by $\mathcal{F} : E \to M^kS(U)$.

Example 2.6. Suppose that $U = \{u_1, u_2, u_3, u_4\}$ is a set of houses and $E = \{e_1, e_2, e_3\}$ is a set of parameters, where e_1 stands for the parameter "price" which include three linguistic variables: *expensive*, *medium* and *cheap*, e_2 stands for the parameter "location" which consists of *urban*, *suburban* and *rural*, and e_3 stands for the parameter "in the green surroundings" which scales in three categories: *good*, *common*, *bad*. We define a multi-soft set of dimension 3 as follows:

$$\mathcal{F}(e_1) = \{u_1/(1,0,0), u_2/(1,1,0), u_3/(0,1,0), u_4/(0,1,1)\},$$

$$\mathcal{F}(e_2) = \{u_1/(1,0,1), u_2/(1,1,1), u_3/(0,1,1), u_4/(1,1,1)\},$$

$$\mathcal{F}(e_3) = \{u_1/(1,1,0), u_2/(0,1,0), u_3/(0,1,1), u_4/(0,0,1)\}.$$

Definition 2.7. [39] Let $M^kFS(U)$ be the set of all multi-fuzzy sets of dimension k in U, and k be a positive integer. A pair $(\widetilde{\mathcal{F}}, E)$ is called a multi-fuzzy soft set (MFSS for short) of dimension k over U, where $\widetilde{\mathcal{F}}$ is a mapping given by $\widetilde{\mathcal{F}} : E \to M^kFS(U)$.

Example 2.8. Reconsider Example 2.6. Due to the fact that information is always aroused with multiplicity and uncertainty. We utilize multi-fuzzy membership function to capture this type of information. Then a pair $(\widetilde{\mathcal{F}}, E)$ is a multi-fuzzy soft set of dimension 3 which is defined in the following manner:

$$\widetilde{\mathcal{F}}(e_1) = \{u_1/(0.9,0.3,0.1), u_2/(0.7,0.5,0.2), u_3/(0.3,0.8,0.4), u_4/(0.1,0.4,0.7)\},$$

$$\widetilde{\mathcal{F}}(e_2) = \{u_1/(0.6,0.1,0.4), u_2/(0.5,0.8,0.4), u_3/(0.2,0.9,0.7), u_4/(0.4,0.3,0.6)\},$$

$$\widetilde{\mathcal{F}}(e_3) = \{u_1/(0.8,0.8,0.2), u_2/(0.4,0.6,0.1), u_3/(0.3,0.8,0.4), u_4/(0.1,0.2,0.9)\}.$$

2.3 Parameters reduction of soft set

A series of literatures issued soft set mainly focused on the parameters reduction [9, 20, 25]. Here we sketch a general style proposed by [2](Ali,2012), the following is going to give a brief review on Ali's main results.

Suppose the initial universe $U = \{u_1, u_2, \ldots, u_m\}$, $E = \{e_1, e_2, \ldots, e_n\}$ as the set of conditions parameters. The pair (F, E) is a soft set represented by a "$0-1$" tabular form (μ_{ij}). Define $\mu_E(u_i) = \sum_{j=1}^{n} \mu_{ij}$ as the value of decision parameter D. The process of parameters reduction is to find the minimum number of condition parameters without disturbing classification ability for D.

Definition 2.9. A condition parameter $e_\tau \subseteq E$ is dispensable if satisfying $R_D = R_{D_{e_\tau}}$, where R_* is an equivalence relation such that $R_* = \{(x, y) \in U \times U : \mu(*, x) = \mu(*, y)\}$, i.e. D and D_{e_τ} are equivalent classifiers. Here D_{e_τ} denotes $\mu_{E-e_\tau}(u_i) = \sum_{e_j \neq e_\tau} \mu_{ij}$; otherwise, e_τ is indispensable.

As further pointed out, A is dispensable iff eliminating all elements in A does not disturb classification ability of D, or else A is indispensable.

Definition 2.10. A subset $B \subseteq E$ is a reduct of E iff B is indispensable and $E - B$ is dispensable.

The algorithm of Ali's parameters reduction can be illustrated as follows:

1. Input the soft set (F, E) with the condition parameters set E;

2. Input decision parameter $D = \sum_j \mu_{ij}$ as the last column in table obtained by condition parameters;

3. Calculate all the equivalence relations $R_e(e \in E)$ and R_D;

4. Identify each parameter dispensable or indispensable as defined above. Delete all the dispensable parameters one by one;

5. Output a table with minimum number of condition parameters, i.e. one obtains a reduct of E having the same classification ability for D as the original table with D.

3 Bilateral fuzzy soft sets

3.1 Bilateral fuzzy sets

Definition 3.1. A set $\mathscr{X}$ is called a bilateral fuzzy set over U where the membership function $\mu_{\mathscr{X}}$ is the mapping $\mu_{\mathscr{X}} : U \to [-1, 1]$.

Obviously, a bilateral fuzzy set which considers both positive, neutral and negative fuzzy membership, and leads to membership function with $[-1, 1]$ as its range. The membership function of a bilateral fuzzy set is piecewise which can capture fuzziness with conflict information as illustrated in the following Table 1.

Table 1: The stage characteristics of the membership of a bilateral fuzzy set.

The range of membership	Category	Stage characteristics
$\mu_{\mathscr{X}}(u) = -1$	*Absolutely Negative*	Unanimously oppose $u \in U$
$\mu_{\mathscr{X}}(u) \in (-1, 0)$	*Partially Negative*	Opposition outnumber support on $u \in U$
$\mu_{\mathscr{X}}(u) = 0$	*Neutral*	Opposition equal to support on $u \in U$
$\mu_{\mathscr{X}}(u) \in (0, 1)$	*Partially Positive*	Support outnumber opposition on $u \in U$
$\mu_{\mathscr{X}}(u) = 1$	*Absolutely Positive*	Unanimously support $u \in U$

Note that a motivation for negative membership can refer to [30], and there is an illustrative example in detail offered in [30] which can be regarded as a special bilateral fuzzy set as shown below:

Let a finite set U be the initial universe, and Λ be a finite set of agents. The task of every agent is to decide whether elements of U obey given property e or not. A set $\mathscr{X}$ is called a bilateral fuzzy set over U which the membership function $\mu_e^{\Lambda}(u)$ is defined as follows:

$$\mu_e^{\Lambda}(u) = \frac{\sum_{a \in \Lambda} \mu_e^a(u)}{card(\Lambda)},$$

where μ_e^a is the mapping $\mu_e^a : U \to \{-1, +1\}$ such that

$$\mu_e^a(u) = \begin{cases} +1 & \text{if according to agent } a \text{ object } u \text{ obeys property } e, \\ -1 & \text{otherwise.} \end{cases}$$

Notice that $-1 \leq \mu_e^{\Lambda}(u) \leq +1$, the number $\mu_e^{\Lambda}(u)$ is called the consensus level of Λ on e in u. If $\mu_e^{\Lambda}(u) = 1$, it implies all agents agree that u obeys the property e; if $\mu_e^{\Lambda}(u) = -1$ that means all agents agree that u does not obey the property e; if $\mu_e^{\Lambda}(u) = 0$ that means one half of agents agree that u obey the property e, and the resting agents agree that does not obey the property e; and if $-1 < \mu_e^{\Lambda}(u) < +1$ means that agents opinions differ regarding whether u obeys e or not.

Let $\mathscr{X}$ and $\mathscr{Y}$ be two bilateral fuzzy sets, and $\mu_{\mathscr{X}}$, $\mu_{\mathscr{Y}}$ be their membership functions. Then we have the following operations:

(1) $\mathscr{X}$ is called to be a subset of $\mathscr{Y}$, denoted by $\mathscr{X} \subseteq \mathscr{Y}$, if $\mu_{\mathscr{X}}(u) \leq \mu_{\mathscr{Y}}(u)$ for all $u \in U$.

(2) If satisfying $\mu_{\mathscr{X}}(u) = \mu_{\mathscr{Y}}(u)$ for all $u \in U$, then we can say $\mathscr{X}$ is equal to $\mathscr{Y}$, denoted by $\mathscr{X} = \mathscr{Y}$.

(3) The union of $\mathscr{X}$ and $\mathscr{Y}$ which denoted by $\mathscr{X} \cup \mathscr{Y}$ if satisfying $\mu_{\mathscr{X} \cup \mathscr{Y}}(u) = \max(\mu_{\mathscr{X}}(u), \mu_{\mathscr{Y}}(u))$ for all $u \in U$.

(4) The intersection of $\mathscr{X}$ and $\mathscr{Y}$ which denoted by $\mathscr{X} \cap \mathscr{Y}$ if satisfying $\mu_{\mathscr{X} \cap \mathscr{Y}}(u) = \min(\mu_{\mathscr{X}}(u), \mu_{\mathscr{Y}}(u))$ for all $u \in U$.

(5) The complement of $\mathscr{X}$ which denoted by $\mathscr{X}^c$ if satisfying $\mu_{\mathscr{X}^c}(u) = 1 - \mathscr{X}(u)$ for all $u \in U$.

3.2 Concept of bilateral fuzzy soft sets

Definition 3.2. Let $BFS(U)$ be the set of all bilateral fuzzy sets of U. A pair $(\mathscr{F}, E)$ is called a bilateral fuzzy soft set (BFSS for short) over U, where $\mathscr{F}$ is a mapping given by $\mathscr{F} : E \to BFS(U)$.

It is worth noting that the BFSS is a parameterized family of bilateral fuzzy subsets of U. For $\forall e \in E$, $\mathscr{F}(e)$ may be considered as the set of e elements, or e-approximate elements of the BFSS $(\mathscr{F}, E)$.

The BFSS $(\mathscr{F}, E)$ can be uniquely characterized by a matrix form as follows:

$$(\mathscr{F}, E) = \begin{bmatrix} & e_1 & e_2 & \cdots & e_n \\ u_1 & \mu_{\mathscr{F}(e_1)}(u_1) & \mu_{\mathscr{F}(e_2)}(u_1) & \cdots & \mu_{\mathscr{F}(e_n)}(u_1) \\ u_2 & \mu_{\mathscr{F}(e_1)}(u_2) & \mu_{\mathscr{F}(e_2)}(u_2) & \cdots & \mu_{\mathscr{F}(e_n)}(u_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_m & \mu_{\mathscr{F}(e_1)}(u_m) & \mu_{\mathscr{F}(e_2)}(u_m) & \cdots & \mu_{\mathscr{F}(e_n)}(u_m) \end{bmatrix}.$$

It is worthwhile to mention in the viewpoint of the range of membership function, the membership of Cantor's set is exactly belonging or not belonging to a set, the fuzzy membership concerns the uncertainty of being a member of the set, and the multi-membership refers to multiplicities of elements in a set, while multi-fuzzy membership depicts a mixture features of multiplicity and fuzziness. Whereas bilateral fuzzy membership considers both positive and negative multi-fuzziness, which leads to membership function with $[-1, 1]$ as its range. Thus it seems natural to establish a link among the above concepts that can be summarized as follows: for $\forall u_i \in U, \forall e_j \in E$,

(1) SS (F, E) - $\mu_{F(e_j)}(u_i) \in \{0, 1\}$;

(2) FSS $(\widetilde{F}, E)$ - $\mu_{\widetilde{F}(e_j)}(u_i) \in [0, 1]$;

(3) MSS $(\mathcal{F}, E)$ - $\frac{1}{k}\sum_{m=1}^{k} \mu_{\mathcal{F}(e_j)}(u_i) \in \{0, 1/k, 2/k, \ldots, 1\}$, k is a positive integer;

(4) MFSS $(\widetilde{\mathcal{F}}, E)$ - $\frac{1}{k}\sum_{m=1}^{k} \mu_{\widetilde{\mathcal{F}}(e_j)}(u_i) \in [0, 1]$, k is a positive integer;

(5) BFSS $(\mathscr{F}, E)$ - $\mu_{\mathscr{F}(e_j)}(u_i) \in [-1, 1]$.

Example 3.3. Suppose that the initial universe consisting of six objects, i.e. $U = \{u_1, u_2, \ldots, u_6\}$, three predicates and four agents, i.e. $E = \{e_1, e_2, e_3\}$, $\Lambda = \{a, b, c, d\}$. Each agent is to give his opinion on each element of the universe whether it obeys predicate e_1, e_2 and e_3 or not. The exemplary opinion of agents are displayed in the form of $+1$(*agree*) and -1(*not agree*), as shown below (1), (2), (3), a BFSS $(\mathscr{F}, E)$ is shown as (4):

$$(1): \begin{bmatrix} & \mu_{e_1}^a & \mu_{e_1}^b & \mu_{e_1}^c & \mu_{e_1}^d & \mu_{e_1}^\Lambda \\ u_1 & +1 & +1 & -1 & +1 & 1/2 \\ u_2 & +1 & -1 & -1 & +1 & 0 \\ u_3 & -1 & +1 & -1 & -1 & -1/2 \\ u_4 & +1 & +1 & +1 & +1 & 1 \\ u_5 & -1 & +1 & -1 & +1 & 0 \\ u_6 & -1 & -1 & -1 & -1 & -1 \end{bmatrix} \quad (2): \begin{bmatrix} & \mu_{e_2}^a & \mu_{e_2}^b & \mu_{e_2}^c & \mu_{e_2}^d & \mu_{e_2}^\Lambda \\ u_1 & +1 & -1 & -1 & +1 & 0 \\ u_2 & +1 & +1 & -1 & +1 & 1/2 \\ u_3 & -1 & -1 & -1 & -1 & -1 \\ u_4 & +1 & -1 & +1 & -1 & 0 \\ u_5 & -1 & +1 & +1 & +1 & 1/2 \\ u_6 & +1 & -1 & -1 & -1 & -1/2 \end{bmatrix}$$

$$(3): \begin{bmatrix} & \mu_{e_3}^a & \mu_{e_3}^b & \mu_{e_3}^c & \mu_{e_3}^d & \mu_{e_3}^\Lambda \\ u_1 & +1 & +1 & +1 & +1 & 1 \\ u_2 & +1 & -1 & +1 & +1 & 1/2 \\ u_3 & +1 & -1 & -1 & -1 & -1/2 \\ u_4 & +1 & +1 & +1 & +1 & 1 \\ u_5 & -1 & -1 & -1 & +1 & -1/2 \\ u_6 & -1 & -1 & -1 & -1 & -1 \end{bmatrix} \quad (4): (\mathscr{F}, E) = \begin{bmatrix} & e_1 & e_2 & e_3 \\ u_1 & 1/2 & 0 & 1 \\ u_2 & 0 & 1/2 & 1/2 \\ u_3 & -1/2 & -1 & -1/2 \\ u_4 & 1 & 0 & 1 \\ u_5 & 0 & 1/2 & -1/2 \\ u_6 & -1 & -1/2 & -1 \end{bmatrix}.$$

Definition 3.4. Let $A, B \subseteq E$, and $(\mathscr{F}, A), (\mathscr{G}, B) \in BFS(U)$. A BFSS $(\mathscr{F}, A)$ is called to be a bilateral fuzzy soft subset of $(\mathscr{G}, B)$, denoted by $(\mathscr{F}, A) \widetilde{\subseteq} (\mathscr{G}, B)$, if

(1) $A \subseteq B$, and

(2) $\mu_{\mathscr{F}(e_j)}(u_i) \leqslant \mu_{\mathscr{G}(e_j)}(u_i)$ for all $e_j \in A$, $u_i \in U$.

Example 3.5. Consider Example 3.3. Let $A = \{e_1, e_2\}$, $B = E = \{e_1, e_2, e_3\}$. Obviously $A \subseteq B$. Suppose that $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are two BFSSs defined as follows:

$$(\mathscr{F}, A) = \begin{bmatrix} & e_1 & e_2 \\ u_1 & 1/2 & 0 \\ u_2 & 0 & 1/2 \\ u_3 & -1/2 & -1 \\ u_4 & 1 & 0 \\ u_5 & 0 & 1/2 \\ u_6 & -1 & -1/2 \end{bmatrix} \quad (\mathscr{G}, B) = \begin{bmatrix} & e_1 & e_2 & e_3 \\ u_1 & 1 & 1/2 & 1 \\ u_2 & 1/2 & 1/2 & 1/2 \\ u_3 & -1/2 & 0 & 1/2 \\ u_4 & 1 & 1 & 1 \\ u_5 & 1/2 & 1/2 & -1/2 \\ u_6 & 1 & 1/2 & 1/2 \end{bmatrix}.$$

Therefore we have $(\mathscr{F}, A)\widetilde{\subseteq}(\mathscr{G}, B)$.

Definition 3.6. Let $A, B \subseteq E$, and $(\mathscr{F}, A), (\mathscr{G}, B) \in BFS(U)$. Two BFSSs $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ are called to be bilateral fuzzy soft equal, denoted by $(\mathscr{F}, A)\widetilde{=}(\mathscr{G}, B)$, if

(1) $A = B$, and

(2) $\mu_{\mathscr{F}(e_j)}(u_i) = \mu_{\mathscr{G}(e_j)}(u_i)$ for all $e_j \in A$, $u_i \in U$.

Definition 3.7. Let $(\mathscr{F}, E) \in BFS(U)$.

(i) A BFSS $(\mathscr{F}, E)$ is called to be an indistinguishable BFSS, denoted by ϕ_E, if $\mu_{\mathscr{F}(e_j)}(u_i) = 0$ for all $e_j \in E$ and $u_i \in U$.

(ii) A BFSS $(\mathscr{F}, E)$ is called to be an absolute BFSS, denoted by $\widetilde{1}_E$, if $|\mu_{\mathscr{F}(e_j)}(u_i)| = 1$ for all $e_j \in E$ and $u_i \in U$.

(iii) A BFSS $(\mathscr{F}, E)$ is called to be an absolute-positive BFSS, denoted by 1_E, if $\mu_{\mathscr{F}(e_j)}(u_i) = 1$ for all $e_j \in E$ and $u_i \in U$.

(iv) A BFSS $(\mathscr{F}, E)$ is called to be an absolute-negative BFSS, denoted by -1_E, if $\mu_{\mathscr{F}(e_j)}(u_i) = -1$ for all $e_j \in E$ and $u_i \in U$.

3.3 Operations on bilateral fuzzy soft sets

Definition 3.8. let $\rceil E = \{\rceil e_1, \rceil e_2, \ldots, \rceil e_n\}$ denote the *not* set of E, and $(\mathscr{F}, E) \in BFS(U)$. The *complement* of BFSS $(\mathscr{F}, E)$ denoted by $(\mathscr{F}, E)^c$, is defined by

$$(\mathscr{F}, E)^c = (\mathscr{F}^c, \rceil E),$$

where $\mathscr{F}^c : \rceil E \to BFS(U)$ is a mapping satisfying $\mu_{\mathscr{F}^c(\rceil e_j)}(u_i) = -\mu_{\mathscr{F}(e_j)}(u_i)$ for all $\rceil e_j \in \rceil E$ and $u_i \in U$.

Clearly, $(\mathscr{F}^c)^c$ is the same as $\mathscr{F}$ and $((\mathscr{F}, E)^c)^c = (\mathscr{F}, E)$. Note that in the above definition of *complement*, the parameters set of the *complement* $(\mathscr{F}, E)$ has changed to $\rceil E$ in accordance with the notion of *complement soft sets* as defined in [24].

Example 3.9. Consider Example 3.3. We obtain $(\mathscr{F}, E)^c$ as follows:

$$(\mathscr{F}, E)^c = \begin{bmatrix} & \rceil e_1 & \rceil e_2 & \rceil e_3 \\ u_1 & -1/2 & 0 & -1 \\ u_2 & 0 & -1/2 & -1/2 \\ u_3 & 1/2 & 1 & 1/2 \\ u_4 & -1 & 0 & -1 \\ u_5 & 0 & -1/2 & 1/2 \\ u_6 & 1 & 1/2 & 1 \end{bmatrix}.$$

Definition 3.10. Let $(\mathscr{F}, E), (\mathscr{G}, E) \in BFS(U)$ with considering the same parameters set E.

(i) The union of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ denoted by $(\mathscr{H}, E) = (\mathscr{F}, E) \cup (\mathscr{G}, E)$, if for all $e \in E$ and $u \in U$, satisfying

$$\mu_{\mathscr{H}(e)}(u) = \mu_{\mathscr{F}(e) \cup \mathscr{G}(e)}(u) = \max\left(\mu_{\mathscr{F}(e)}(u), \mu_{\mathscr{G}(e)}(u)\right).$$

(ii) The intersection of $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ denoted by $(\mathscr{H}, E) = (\mathscr{F}, E) \cap (\mathscr{G}, E)$, if for all $e \in E$ and $u \in U$, satisfying

$$\mu_{\mathscr{H}(e)}(u) = \mu_{\mathscr{F}(e) \cap \mathscr{G}(e)}(u) = \min\left(\mu_{\mathscr{F}(e)}(u), \mu_{\mathscr{G}(e)}(u)\right).$$

Example 3.11. The union and intersection of the two BFSSs $(\mathscr{F}, E)$ in Example 3.3 and $(\mathscr{G}, E)$ in Example 3.5 are calculated as below:

$$Union : (\mathscr{H}, E) = \begin{bmatrix} & e_1 & e_2 & e_3 \\ u_1 & 1 & 1/2 & 1 \\ u_2 & 1/2 & 1/2 & 1/2 \\ u_3 & -1/2 & 0 & 1/2 \\ u_4 & 1 & 1 & 1 \\ u_5 & 1/2 & 1/2 & -1/2 \\ u_6 & 1 & 1/2 & 1/2 \end{bmatrix}; \quad Intersection : (\mathscr{H}, E) = \begin{bmatrix} & e_1 & e_2 & e_3 \\ u_1 & 1/2 & 0 & 1 \\ u_2 & 0 & 1/2 & 1/2 \\ u_3 & -1/2 & -1 & -1/2 \\ u_4 & 1 & 0 & 1 \\ u_5 & 0 & 1/2 & -1/2 \\ u_6 & -1 & -1/2 & -1 \end{bmatrix}.$$

Theorem 3.12. (*De Morgan's laws*). Let $(\mathscr{F}, E), (\mathscr{G}, E) \in BFS(U)$. Then

(i) $((\mathscr{F}, E) \cap (\mathscr{G}, E))^c = (\mathscr{F}, E)^c \cup (\mathscr{G}, E)^c$.

(ii) $((\mathscr{F}, E) \cup (\mathscr{G}, E))^c = (\mathscr{F}, E)^c \cap (\mathscr{G}, E)^c$.

Proof. (i) Assume that $(\mathscr{F}, E) \cap (\mathscr{G}, E) = (\mathscr{H}, E)$. Hence $((\mathscr{F}, E) \cap (\mathscr{G}, E))^c = (\mathscr{H}, E)^c = (\mathscr{H}^c, \rceil E)$. Take all $\rceil e \in \rceil E$ and $u_i \in U$. We obtain $\mathscr{H}^c(\rceil e)$ such that

$$\begin{aligned} \mu_{\mathscr{H}^c(\rceil e)}(u_i) &= -\mu_{\mathscr{H}(e)}(u_i) \\ &= -\min\left(\mu_{\mathscr{F}(e)}(u_i), \mu_{\mathscr{G}(e)}(u_i)\right) \\ &= \max\left(-\mu_{\mathscr{F}(e)}(u_i), -\mu_{\mathscr{G}(e)}(u_i)\right) \\ &= \max\left(\mu_{\mathscr{F}^c(\rceil e)}(u_i), \mu_{\mathscr{G}^c(\rceil e)}(u_i)\right). \end{aligned}$$

The last equality implies that $\mathscr{H}^c(\rceil e) = \mathscr{F}^c(\rceil e) \cup \mathscr{G}^c(\rceil e)$. Therefore this completes the proof.
(ii) The result can be proved in the similar way. □

Theorem 3.13. Let $(\mathscr{F}, E)$,$(\mathscr{G}, E)$ and $(\mathscr{H}, E) \in BFS(U)$. Then

(i) (*Associative law*).

(1) $(\mathscr{F}, E) \cap ((\mathscr{G}, E) \cap (\mathscr{H}, E)) = ((\mathscr{F}, E) \cap (\mathscr{G}, E)) \cap (\mathscr{H}, E)$.

(2) $(\mathscr{F}, E) \cup ((\mathscr{G}, E) \cup (\mathscr{H}, E)) = ((\mathscr{F}, E) \cup (\mathscr{G}, E)) \cup (\mathscr{H}, E)$.

(ii) (*Distribution law*).

(1) $(\mathscr{F}, E) \cap ((\mathscr{G}, E) \cup (\mathscr{H}, E)) = ((\mathscr{F}, E) \cap (\mathscr{G}, E)) \cup ((\mathscr{F}, E) \cap (\mathscr{H}, E))$.

(2) $(\mathscr{F}, E) \cup ((\mathscr{G}, E) \cap (\mathscr{H}, E)) = ((\mathscr{F}, E) \cup (\mathscr{G}, E)) \cap ((\mathscr{F}, E) \cup (\mathscr{H}, E))$.

Proof. Straightforward. □

Definition 3.14. Let $(\mathscr{F}, A), (\mathscr{G}, B) \in BFS(U)$.

(i) The "*AND*"operation on two BFSSs $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ denoted by $(\mathscr{F}, A) \wedge (\mathscr{G}, B)$, which is defined by $(\mathscr{F}, A) \wedge (\mathscr{G}, B) = (\mathscr{H}, A \times B)$, where $\mathscr{H}(\alpha, \beta) = \mathscr{F}(\alpha) \sqcap \mathscr{G}(\beta)$ such that

$$\mu_{\mathscr{H}(\alpha,\beta)}(u_i) = \min\left(\mu_{\mathscr{F}(\alpha)}(u_i), \mu_{\mathscr{G}(\beta)}(u_i)\right),$$

for all $(\alpha, \beta) \in A \times B$ and $u_i \in U$.

(ii) The "*OR*"operation on two BFSSs $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ denoted by $(\mathscr{F}, A) \vee (\mathscr{G}, B)$, which is defined by $(\mathscr{F}, A) \vee (\mathscr{G}, B) = (\mathscr{O}, A \times B)$, where $\mathscr{O}(\alpha, \beta) = \mathscr{F}(\alpha) \sqcup \mathscr{G}(\beta)$ such that

$$\mu_{\mathscr{O}(\alpha,\beta)}(u_i) = \max\left(\mu_{\mathscr{F}(\alpha)}(u_i), \mu_{\mathscr{G}(\beta)}(u_i)\right),$$

for all $(\alpha, \beta) \in A \times B$ and $u_i \in U$.

Example 3.15. The results of "*AND*, "*OR*" operations on the two BFSSs $(\mathscr{F}, A)$ and $(\mathscr{G}, B)$ in Example 3.5 is showed as below:

$$AND : (\mathscr{H}, A \times B) = \begin{bmatrix} & (e_1,e_1) & (e_1,e_2) & (e_1,e_3) & (e_2,e_1) & (e_2,e_2) & (e_2,e_3) \\ u_1 & 1/2 & 1/2 & 1/2 & 0 & 0 & 0 \\ u_2 & 0 & 0 & 0 & 1/2 & 1/2 & 1/2 \\ u_3 & -1/2 & -1/2 & -1/2 & -1 & -1 & -1 \\ u_4 & 1 & 1 & 1 & 0 & 0 & 0 \\ u_5 & 0 & 0 & -1/2 & 1/2 & 1/2 & -1/2 \\ u_6 & -1 & -1 & -1 & -1/2 & -1/2 & -1/2 \end{bmatrix},$$

$$OR : (\mathscr{O}, A \times B) = \begin{bmatrix} & (e_1,e_1) & (e_1,e_2) & (e_1,e_3) & (e_2,e_1) & (e_2,e_2) & (e_2,e_3) \\ u_1 & 1 & 1/2 & 1 & 1 & 1/2 & 1 \\ u_2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ u_3 & -1/2 & 0 & 1/2 & -1/2 & 0 & 1/2 \\ u_4 & 1 & 1 & 1 & 1 & 1 & 1 \\ u_5 & 1/2 & 1/2 & 0 & 1/2 & 1/2 & 1/2 \\ u_6 & 1 & 1/2 & 1/2 & 1 & 1/2 & 1/2 \end{bmatrix}.$$

Theorem 3.16. (*De Morgan's laws*). Let $(\mathscr{F}, A), (\mathscr{G}, B) \in BFS(U)$. Then

(i) $((\mathscr{F}, A) \wedge (\mathscr{G}, B))^c = (\mathscr{F}, A)^c \vee (\mathscr{G}, B)^c$.

(ii) $((\mathscr{F}, A) \vee (\mathscr{G}, B))^c = (\mathscr{F}, A)^c \wedge (\mathscr{G}, B)^c$.

Proof. (i) Suppose that $(\mathscr{F}, A) \wedge (\mathscr{G}, B) = (\mathscr{H}, A \times B)$. Therefore,

$$((\mathscr{F}, A) \wedge (\mathscr{G}, B))^c = (\mathscr{H}, A \times B)^c = (\mathscr{H}^c, \rceil(A \times B)).$$

Take all $(\rceil\alpha, \rceil\beta) \in \rceil A \times \rceil B$ and $u_i \in U$. We obtain $\mathscr{H}^c(\rceil\alpha, \rceil\beta)$ such that

$$\begin{aligned} \mu_{\mathscr{H}^c(\rceil\alpha,\rceil\beta)}(u_i) &= -\mu_{\mathscr{H}(\alpha,\beta)}(u_i) \\ &= -\min\left(\mu_{\mathscr{F}(\alpha)}(u_i), \mu_{\mathscr{G}(\beta)}(u_i)\right) \\ &= \max\left(-\mu_{\mathscr{F}(\alpha)}(u_i), -\mu_{\mathscr{G}(\beta)}(u_i)\right) \\ &= \max\left(\mu_{\mathscr{F}^c(\rceil\alpha)}(u_i), \mu_{\mathscr{G}^c(\rceil\beta)}(u_i)\right). \end{aligned}$$

The last equality implies that $\mathscr{H}^c(\rceil\alpha, \rceil\beta) = \mathscr{F}^c(\rceil\alpha) \sqcup \mathscr{G}^c(\rceil\beta)$. hence (i) is established.
(ii) The result can be proved in the similar way. □

Theorem 3.17. Let $(\mathscr{F}, A)$,$(\mathscr{G}, B)$ and $(\mathscr{H}, C) \in BFS(U)$. Then

(i) (*Associative law*).

(1) $(\mathscr{F}, A) \wedge ((\mathscr{G}, B) \wedge (\mathscr{H}, C)) = ((\mathscr{F}, A) \wedge (\mathscr{G}, B)) \wedge (\mathscr{H}, C)$.

(2) $(\mathscr{F}, A) \vee ((\mathscr{G}, B) \vee (\mathscr{H}, C)) = ((\mathscr{F}, A) \vee (\mathscr{G}, B)) \vee (\mathscr{H}, C)$.

(ii) (*Distribution law*).

(1) $(\mathscr{F}, A) \wedge ((\mathscr{G}, B) \vee (\mathscr{H}, C)) = ((\mathscr{F}, A) \wedge (\mathscr{G}, B)) \vee ((\mathscr{F}, A) \wedge (\mathscr{H}, C))$.

(2) $(\mathscr{F}, A) \vee ((\mathscr{G}, B) \wedge (\mathscr{H}, C)) = ((\mathscr{F}, A) \vee (\mathscr{G}, B)) \wedge ((\mathscr{F}, A) \vee (\mathscr{H}, C))$.

Proof. Straightforward. □

4 Parameters reduction of bilateral fuzzy soft set

Parameters reduction of the initial BFSS cannot be processed directly because of continuous value of its membership function. The universal treatment measure is to change the one into a corresponding BFSS with the discrete values of membership degree by using some techniques. In the following we provide a technology which bridges the two together.

Definition 4.1. Let $(\mathscr{F}, E) \in BFS(U)$, and let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ $(\forall \xi_j \in [0,1])$ denote a vector of the distinguishing coefficients, a crisp SS is called to be ξ-distinguishable region of $(\mathscr{F}, E)$, denoted by $(\mathscr{F}_\xi, E)$, if for $\forall\, e_j \in E$, satisfying

$$\mathscr{F}_\xi(e_j) = \left\{u_i \in U : |\mu_{\mathscr{F}(e_j)}(u_i)| \geq \xi_j\right\},$$

otherwise, it is called ξ-indistinguishable region of $(\mathscr{F}, E)$, denoted by $(\mathscr{F}_{\urcorner\xi}, E)$.

In the above definition, each ξ_j can be viewed as a given threshold of consensus level on e_j of E. For decision making in reality based on BFSS, usually these thresholds are given in advance for ascertaining that each objects of the initial universe U obeys the property e_j or not to some extent.

Furthermore, one can explore concepts of ξ-distinguishable-positive region and ξ-distinguishable-negative region of a BFSS as follows.

Definition 4.2. Let $(\mathscr{F}, E) \in BFS(U)$, and let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ $(\forall \xi_j \in [0,1])$ denote vector of the distinguishing coefficients.

(i) a crisp SS called ξ-distinguishable-positive region of $(\mathscr{F}, E)$, denoted by $(\overline{\mathscr{F}}_\xi, E)$, if for $\forall\, e_j \in E$, satisfying

$$\overline{\mathscr{F}}_\xi(e_j) = \left\{u_i \in U : \mu_{\mathscr{F}(e_j)}(u_i) \geq \xi_j\right\}.$$

(ii) a crisp SS called ξ-distinguishable-negative region of $(\mathscr{F}, E)$, denoted by $(\underline{\mathscr{F}}_\xi, E)$, if for $\forall\, e_j \in E$, satisfying

$$\underline{\mathscr{F}}_\xi(e_j) = \left\{u_i \in U : \mu_{\mathscr{F}(e_j)}(u_i) \leq -\xi_j\right\}.$$

Followed by one can synthesize three regions above, i.e. ξ-indistinguishable, ξ-distinguishable-negative and ξ-distinguishable-positive regions, into an integrated BFSS with the bilateral fuzzy membership belonging to $\{-1, 0, 1\}$. It can be viewed as an approximate description of the initial BFSS.

Definition 4.3. Let $(\mathscr{F}, E) \in BFS(U)$, and let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. A BFSS induced by $(\mathscr{F}, E)$, denoted by $(Apr\mathscr{F}_\xi, E)$, where $Apr\mathscr{F}_\xi$ is a mapping given by $Apr\mathscr{F}_\xi : E \to BFS(U)$ such that

$$\mu_{Apr\mathscr{F}(e_j)}(u_i) = \begin{cases} 1 & \text{if } \mu_{\mathscr{F}(e_j)}(u_i) \geq \xi_j; \\ -1 & \text{if } \mu_{\mathscr{F}(e_j)}(u_i) \leq -\xi_j; \\ 0 & \text{if } |\mu_{\mathscr{F}(e_j)}(u_i)| < \xi_j. \end{cases}$$

Example 4.4. Reconsider Example 3.3. Given $\xi = (0.7, 0.5, 0.8)$, then we obtain

$$(\overline{\mathscr{F}}_\xi, E) = \begin{bmatrix} & e_1 & e_2 & e_3 \\ u_1 & 0 & 0 & 1 \\ u_2 & 0 & 1 & 0 \\ u_3 & 0 & 0 & 0 \\ u_4 & 1 & 0 & 1 \\ u_5 & 0 & 1 & 0 \\ u_6 & 0 & 0 & 0 \end{bmatrix}, \quad (\underline{\mathscr{F}}_\xi, E) = \begin{bmatrix} & e_1 & e_2 & e_3 \\ u_1 & 0 & 0 & 0 \\ u_2 & 0 & 0 & 0 \\ u_3 & 0 & 1 & 0 \\ u_4 & 0 & 0 & 0 \\ u_5 & 0 & 0 & 0 \\ u_6 & 1 & 1 & 1 \end{bmatrix}.$$

$$(\mathscr{F}_{\urcorner\xi}, E) = \begin{bmatrix} & e_1 & e_2 & e_3 \\ u_1 & 1 & 1 & 0 \\ u_2 & 1 & 0 & 1 \\ u_3 & 1 & 0 & 1 \\ u_4 & 0 & 1 & 0 \\ u_5 & 1 & 0 & 1 \\ u_6 & 0 & 0 & 0 \end{bmatrix}, \quad (Apr\mathscr{F}_\xi, E) = \begin{bmatrix} & e_1 & e_2 & e_3 \\ u_1 & 0 & 0 & 1 \\ u_2 & 0 & 1 & 0 \\ u_3 & 0 & -1 & 0 \\ u_4 & 1 & 0 & 1 \\ u_5 & 0 & 1 & 0 \\ u_6 & -1 & -1 & -1 \end{bmatrix}.$$

Subsequently we introduce a new technique of parameters reduction (NPR for short) on BFSS $(Apr\mathscr{F}_\xi, E)$. The core idea on reduction of parameters is without disturbing classification ability of parameters in the original data by means of equivalence relation. Differ from Ali's method [2] in which he considered the classification ability of $\sum_j \mu_{ij}$ as the decision parameter D, we take intersection of all equivalence relations $\bigcap_{\forall e \in E} R_e$, which implies a finest partition over U generated by all these condition parameters of E, denoted by $[u]_D$, as the classification on U with decision parameter, instead of classification ability of $\sum_j \mu_{ij}$. Formal definitions are given below.

Definition 4.5. A subset $A \subseteq E$ is dispensable if satisfying $[u]_{E-A} = [u]_E = [u]_D$, where $[u]_*$ is a partition over U generated by $*$; otherwise, A is indispensable.

Definition 4.6. A subset $B \subseteq E$ is called to be a reduct of E iff B is indispensable and $E - B$ is dispensable.

Thus a reduct is a set of parameters that preserves partition. The algorithm of NPR is addressed as follows:

1. Input the BFSS $(Apr\mathscr{F}_\xi, E)$ with the condition parameters set E;
2. Compute all the patition $[u]_e (e \in E)$ and take $[u]_E$ as the classification benchmark;
3. Eliminate all the dispensable parameters of E as defined above;
4. Output a set with minimum number of condition parameters, i.e. one obtains a reduct of E keeping the same partition on the initial universe U for E.

Example 4.7. Consider Example 4.4. From Ali's algorithm of reduction, one obtains easily that the original parameters set without dispensable parameter is Ali's reduct of E on BFSS $(Apr\mathscr{F}_\xi, E)$. By using the proposed algorithm of NPR, it is easy to check that parameter e_3 is dispensable and the only reduct of E is the set $\{e_1, e_2\}$.

As it follows from considerations concerning reduction of parameters, all parameters in E cannot be equally important in many real problems, some of them may be eliminated without disturbing the partition over U generated by E. Accordingly, here we introduce a concept of significance of parameter, that quantifies how important is a parameter in E. Significance of a parameter can be evaluated by measuring effect of removing the parameter from E on classification defined by E.

Definition 4.8. Let $B \subseteq E$, and let D be the decision parameter with the condition of $[u]_D = [u]_E$. the significance of $e(e \in B)$ with regard to D is defined in the following way:

$$\sigma_{BD}(e) = \gamma(B, D) - \gamma(B - e, D).$$

Where

$$\gamma(B, D) = \frac{card\left(\{u \in U : u \in [u]_B \subseteq [u]_D\}\right)}{card(U)};$$

$$\gamma(B - e, D) = \frac{card\left(\{u \in U : u \in [u]_{B-e} \subseteq [u]_D\}\right)}{card(U)}.$$

Obviously $0 \leq \sigma_{BD}(e) \leq 1$. The more important is the parameter e, the greater is the value $\sigma_{BD}(e)$. From Definition 4.8, it is straightforward to obtain that $\sigma_{BD}(e) = 0$ iff e is a dispensable parameter.

5 Application of the proposed method based on parameters reduction of bilateral fuzzy soft set

In application of multi-criteria decision making problem, most of the previous approaches by using soft set theory did not take into account conflict situation, i.e. coexistence of positive and negative consensus level with a criterion in the objects of initial universe and set equal weight as usual to all criteria, instead of considering significance of each criterion for ranking the alternatives in the initial universe. In this section we are going to present an application of BFSS in a multi-criteria multi-person decision making under conflict information (MMDM-CI) problem. Here we also consider the importance weight of each criterion. So MMDM-CI problem can be expressed as below:

Suppose a group of decision makers (DMs) denoted Λ, consisting of l experts, take a collective evaluation to rank m alternatives, i.e. $U = \{u_1, u_2, \ldots, u_m\}$ based on a set of criteria $E = \{e_1, e_2, \ldots, e_n\}$. Let $\mu_{ij}^a \in \{-1, 0, 1\}$ be exemplary opinion of $a(a \in \Lambda)$ on criterion $e_j (j = 1, 2, \ldots, n)$ in alternative $u_i (i = 1, 2 \ldots, m)$, where $-1, 0, 1$ represents *Oppose*, *Abstention*, *Support*, respectively. And suppose $\xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$ is a vector of the distinguishing coefficients, in which $\xi_j (j = 1, 2, \ldots, n)$ represents a threshold of consensus level to judge whether it obeys the criterion e_j or not. In what follows we propose a novel method by advantage of BFSS to solve MMDM-CI above. This approach consists of three main stages, first we establish a BFSS model for MMDM-CI. The second is to delete all these parameters of E which are trivial or irrelevant under the premise of keeping the classification ability of the initial parameters set, to achieve the purpose of simplifying the criteria system. The final step is determination and integration of objective weights for comprehensive evaluation results.

The proposed algorithm for MMDM-CI based on parameters reduction of BFSS can be illustrated as follows:

5.1 Algorithm

1 Identify the alternatives and parameters of the BFSS $(\mathscr{F}, E)$;

2 Calculate the consensus level of Λ on e_j in each u_i and treat as bilateral fuzzy membership μ_{ij}^{Λ}, then obtain the BFSS $(\mathscr{F}, E)$;

3 Input the vector of distinguishing coefficients ξ, accordingly calculate ξ-indistinguishable, ξ-distinguishable-negative, ξ-distinguishable-positive regions of $(\mathscr{F}, E)$;

4 Synthesize three regions and obtain a BFSS $(Apr\mathscr{F}_{\xi}, E)$;

5 Obtain a reduct B of E by using the above algorithm of NPR;

6 Calculate significance of each parameter in B and obtain a vector of significance;

7 Normalize the significance and ascertain the weight value of each parameter e_j in B:

$$w_{e_j \in B} = \frac{\sigma_{BD}(e_j)}{\sum_{e_j \in B} \sigma_{BD}(e_j)};$$

8 Calculate the synthesis values for the evaluated alternatives as follows:

$$S_i = \sum_{e_j \in B} w_{e_j \in B} \cdot \mu_{ij}^{\Lambda}, \text{ for } i \in \{1, 2, \ldots, m\}.$$

Rank all alternatives according to the value S_i, the higher the value, the better performance will be. The optimal alternative(s) u_i whose S_i is(are) maximum.

Let us note three main novelties of the proposed method that follow directly from the relevant definitions and properties.

(1) The method by using parameters reductions of BFSS is an extension of classic model based on parameters reductions of fuzzy soft set, which could handle MMDM with conflict opinions of experts rather than single fuzziness.

(2) The method is pointed out: the optimal attribute subset selection according to parameters reductions of BFSS and attribute weight assignment according to significance of parameters on BFSS, both of which could guarantee the optimality of decision results.

(3) The processing of alternatives effectiveness evaluation can be guiding significance for other similar economics and engineering applications.

The flowchart of the algorithm above is shown as Figure 1.

5.2 Numerical example

As we all known, environmental protection is a key factor for the social-economic development of societies. Thus the correct and feasible environmental program affects economic development and human's future. Therefore, the most appropriate environmental program selection is crucially important. Suppose there are eight alternatives u_i $(i = 1, 2, 3, 4, 5, 6, 7, 8)$ to be selected, and five dimensions to be considered: e_1: *socio-political*, e_2: *economic*, e_3: *technological*, e_4: *easily conducted*, e_5: *sustainable*. There are a group of one thousand voters, denoted by Λ, are invited to vote performance of each dimension of eight alternatives, where the opinion of each voter with regard to these dimensions have three exclusive options: $+1$ denotes *Support*, -1 denotes *Oppose*, 0 denotes *Abstention*. Therefore there exist two conflict situations of "*Support*=1" and "*Oppose*=−1". It is worth emphasizing that this kind of voting problem under conflict information existing widely in reality can be abstracted as a typical MMDM-CI problem. We define the arithmetic average value $\mu_e^{\Lambda}(u) = \sum_{a \in \Lambda} \mu_e^a(u)/card(\Lambda)$ to represent result of voting regarding attribute e,

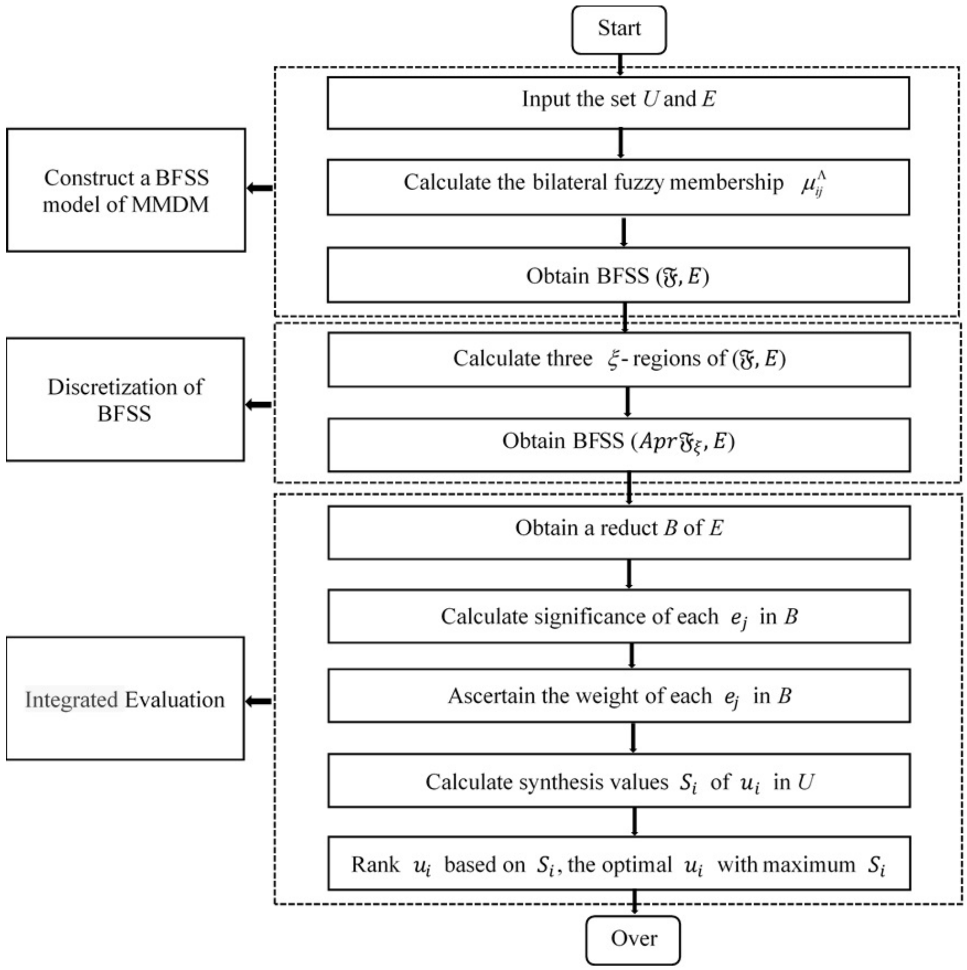

Figure 1: The flowchart of algorithm based on parameters reduction of BFSS.

means the consensus level of Λ on e in u, and naturally take it as the bilateral fuzzy membership of u with regard to e. After completing the voting procedures, then generates a BFSS $(\mathscr{F}, E)$ which is expressed as follows:

$$(\mathscr{F}, E) = \begin{bmatrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ u_1 & 0.565 & -0.522 & 0.692 & 0.482 & 0.763 \\ u_2 & 0.821 & 0.126 & -0.712 & 0.642 & 0.719 \\ u_3 & 0.267 & 0.354 & 0.722 & 0.574 & 0.063 \\ u_4 & 0.118 & -0.013 & 0.668 & 0.869 & 0.338 \\ u_5 & -0.105 & 0.472 & 0.567 & -0.211 & 0.873 \\ u_6 & -0.632 & 0.576 & 0.465 & -0.556 & 0.532 \\ u_7 & 0.634 & -0.507 & -0.707 & 0.488 & 0.477 \\ u_8 & -0.445 & 0.178 & 0.863 & 0.472 & 0.912 \end{bmatrix}.$$

With repeated discussions by a group of senior consultants, ultimately determine the threshold value ξ_j for support or opposition on each e_j $(j = 1, 2, \dots, n)$ as following Table 2.

Thus the vector of the distinguishing coefficients can be determined as $\xi = \{0.6, 0.5, 0.7, 0.5, 0.8\}$, as a consequence

Table 2: The threshold value for positive negative or uncertain on each e_j.

Voting results/The threshold values	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5
Positive	≥ 0.6	≥ 0.5	≥ 0.7	≥ 0.5	≥ 0.8
Negative	≤ -0.6	≤ -0.5	≤ -0.7	≤ -0.5	≤ -0.8
Uncertain	$(-0.6, 0.6)$	$(-0.5, 0.5)$	$(-0.7, 0.7)$	$(-0.5, 0.5)$	$(-0.8, 0.8)$

we obtain a BFSS $(Apr\mathscr{F}_\xi, E)$ as below:

$$(Apr\mathscr{F}_\xi, E) = \begin{bmatrix} & e_1 & e_2 & e_3 & e_4 & e_5 \\ u_1 & 0 & -1 & 0 & 0 & 0 \\ u_2 & 1 & 0 & -1 & 1 & 0 \\ u_3 & 0 & 0 & 1 & 1 & 0 \\ u_4 & 0 & 0 & 0 & 1 & 0 \\ u_5 & 0 & 0 & 0 & 0 & 1 \\ u_6 & -1 & 1 & 0 & -1 & 0 \\ u_7 & 1 & -1 & -1 & 0 & 0 \\ u_8 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

By implementing the algorithm above, the reduction results as follows:

$$Red1 : \{e_2, e_3, e_4\}; \quad Red2 : \{e_2, e_3, e_5\}; \quad Red3 : \{e_3, e_4, e_5\}.$$

Without loss of generality, we choose $Red1$ denoted by B as the reduct of E, and subsequently calculate significance of each parameter in B, i.e. $\sigma_{BD}(e_2) = 1/4, \sigma_{BD}(e_3) = 7/8, \sigma_{BD}(e_4) = 1/2$ and normalize it to get weight vector $w = (0.154, 0.538, 0.308)$, then obtain the comprehensive evaluation value of each u_i in U as following Table 3.

Table 3: Results obtained by parameters reduction of BFSS.

U	$e_2(w_{e_2} = 0.154)$	$e_3(w_{e_3} = 0.538)$	$e_4(w_{e_4} = 0.308)$	$Evaluation(S_i)$
u_1	−0.522	0.692	0.482	0.441
u_2	0.126	−0.712	0.642	−0.166
u_3	0.354	0.722	0.574	0.620
u_4	−0.013	0.668	0.869	0.625
u_5	0.472	0.567	−0.211	0.313
u_6	0.576	0.465	−0.556	0.168
u_7	−0.507	−0.707	0.488	−0.309
u_8	0.178	0.863	0.472	0.637

According to the evaluation results of each alternative, one can obtain

$$u_8 \succ u_4 \succ u_3 \succ u_1 \succ u_5 \succ u_6 \succ u_2 \succ u_7.$$

Therefore, the optimal environmental program is u_8 because $\max_{i=1,2,\ldots,8}(S_i) = S_8$.

Further we make a comparison of our results with those calculated as the arithmetic mean-with equal weighting. We get the evaluation results of all programs $[S_i] = [1.98, 1.596, 1.98, 1.98, 1.596, 0.385, 0.385, 1.98]$, the ranking of these alternatives are

$$u_1 \sim u_3 \sim u_4 \sim u_8 \succ u_2 \sim u_5 \succ u_6 \sim u_7.$$

There are four programs available as optimal alternatives by using aggregation approach of arithmetic mean, i.e. u_1, u_3, u_4 and u_8. The traditional method in which the parameter significance is not considered would lead to not distinguish the four programs and regard them as equal. However, by utilizing our algorithm of BFSS as mentioned above, parameters weights are taken into account and the optimal alternative is unique in this numerical example. In comparison with FSS, BFSS can quantify a wider range of uncertainty information including conflict situations, and the proposed method based on parameters reduction of BFSS can rank alternatives in MMDM-CI problem more reasonable and effectively.

6 Conclusion

In this paper, we originated a new mathematical notion called bilateral fuzzy soft set (BFSS) combining by the concepts of bilateral fuzzy set and soft set logically, which can be treated as a generalization of fuzzy soft set. From the angle of membership function, we clarified the connection and difference of soft sets, fuzzy soft sets, multi-soft sets, multi-fuzzy soft sets and bilateral fuzzy soft sets in brief. In compare with these mathematical tools mentioned above, BFSS is more capable of dealing with a wider range of fuzziness including conflict situations due to the reason of without restriction of the membership range. Based on the proposed concept, we studied some of its operations and basic properties. Under these conditions, we investigated parameters reduction of BFSS which differs from previous reduction of parameters in case of soft sets, as well as defining the notion of significance of parameters. Moreover, a method based on parameters reduction and significance of parameter on bilateral fuzzy soft set that adapting MMDM problem with conflict situation problem is put forward. It is worth noting that the method has two main advantages which can be concluded as: one can manage fuzziness with conflict situations, the other can acquire the optimal attribute subset selection and attribute weight assignment. An illustrative example regarding selection of environmental programs demonstrates the reasonability and efficiency of the proposed method. Some possible future work include sketching uncertain data with missing values using BFSS, rule extraction and applying parameters reduction of BFSS into evaluation and prediction problems under incomplete information are of interest.

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