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Short note

A note on the ordinal sum of triangular norms on bounded lattices

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Abstract

The ordinal sum construction is an important way to generate new triangular norms (t-norms for short) on real unit interval from existing ones. Saminger extended the ordinal sum construction of t-norms to bounded lattices in a direct way and proved that, except for very special cases the ordinal sum of t-norms on bounded lattices may be not a t-norm. To ensure the ordinal sum of t-norms is always a t-norm, several modified ordinal sum of t-norms have been developed. This note reviews several such constructions. As a byproduct, we point out that some recent results concerning ordinal sum of t-norms by Aşici can be seen as corollaries of these construction.

Keywords: Bounded lattices, triangular norms, ordinal sum.

1 Introduction

The ordinal sum construction is an effective method to generate new semigroups from existing ones [4, 5]. Ling [12] applied this method to triangular norms (t-norms for short), a special kind of semigroup on the real unit interval. One of the most important results in the theory of t-norms is the representation theorem of continuous t-norms, which states that a t-norm is continuous if and only if it is uniquely representable as an ordinal sum of continuous Archimedean t-norms, see [11].

Due to their indispensable position in many fields including fuzzy logic [8], t-norms and related operators (uninorms, nullnorms and etc.) have been generalized from the unit interval to more general settings, bounded lattices for example [10, 9]. Since many nice properties of the real unit interval do not hold in general bounded lattices, whether a result of t-norms on the unit interval remains valid for general bounded lattices needs a careful examination.

Saminger [16, 17] extended the ordinal sum construction of t-norms to bounded lattices in a direct way and proved that, an ordinal sum of t-norms does not always yield a t-norm. To ensure an ordinal sum of t-norms on bounded lattices is again a t-norm, several modifications of Saminger's ordinal sum have been proposed, see Ouyang, Zhang and De Baets [14] and Dvořák and Holčapek [7] for example (note that these two methods have been unified and generalized in Ouyang et al. [15]).

The aim of this note is to review several ordinal sum constructions of t-norms on bounded lattices. As a byproduct, we point out that some recent results concerning ordinal sum of t-norms by Aşici can be seen as corollaries of these constructions. Thus, it gives a confirmation of the correctness of Aşici's results. The remainder of this note is organized as follows. In Section 2 we recall some basic knowledge that will be used in this paper. We recall the ordinal sum of t-norms in the sense of Saminger in Section 3 and point out that the result in [1] can be seen as a corollary of [16]. In Section 4 we recall an extension theorem of t-norms which was proved by Dvořák and Holčapek [7] and its relation to the ordinal sum construction proposed in [7, 15]. We also point out that both the results of [2, 3] can be seen as corollaries of this extension theorem.

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2 Preliminaries

A *poset* is a nonempty set L equipped with a partial order $\leq$. A *meet semilattice* (resp. *join semilattice*) is a poset L such that for any $x, y \in L$ the infimum $x \wedge y$ (resp. the supremum $x \vee y$) exists in L. A nonempty set L is called a *lattice* if it is both a meet semilattice and a join semilattice. For $a, b \in L$, if neither $a \leq b$ nor $b \leq a$, then we say that a and b are incomparable and this is denoted by $a \| b$. The set of all elements of L that are incomparable with a is denoted by I_a. A lattice (resp. semi-lattice) L is called *bounded* if it has a top element (usually denoted by 1) and a bottom element (usually denoted by 0). The above knowledge concerning lattices can be found, e.g., in [4].

Let L be a lattice and $a, b \in L$ with $a \leq b$. The subinterval $[a, b]$ of L is defined as

$$[a, b] = \{x \in L \mid a \leq x \leq b\}.$$

Other subintervals such as $[a, b[$ and $]a, b[$ can be defined similarly. Obviously, the subinterval $[a, b]$ of L is a bounded sublattice with top element b and bottom element a.

Definition 2.1. [6] *Let L be a lattice and $[a, b]$ be a subinterval of L. A binary operation $T\colon [a, b]^2 \to [a, b]$ is said to be a t-norm on $[a, b]$ if, for any $x, y, z, w \in [a, b]$, the following conditions are fulfilled:*

(*i*) $T(x, y) = T(y, x)$ *(commutativity);*

(*ii*) *If $x \leq y$ and $z \leq w$, then $T(x, z) \leq T(y, w)$ (increasingness);*

(*iii*) $T(T(x, y), z) = T(x, T(y, z))$ *(associativity);*

(*iv*) $T(b, x) = x$ *(neutrality).*

The strongest t-norm on $[a, b]$ is $\wedge$, while the weakest t-norm on $[a, b]$ is T_D defined by $T_D(x, y) = x \wedge y$ if $b \in \{x, y\}$ and a otherwise.

Definition 2.2. *Let L be a poset. A map* int: $L \to L$ *is said to be an interior operator if, for any $x, y \in L$, the following conditions are satisfied:*
(*i*) $\mathrm{int}(x) \leq x$ *(contraction);*
(*ii*) $x \leq y$ *implies* $\mathrm{int}(x) \leq \mathrm{int}(y)$ *(increasingness);*
(*iii*) $\mathrm{int}(\mathrm{int}(x)) = \mathrm{int}(x)$ *(idempotence).*

There are some alternative definitions of an interior operator in the literature. For example, instead of the increasingness, the interior operators int in the sense of Ouyang and Zhang [13] are required to preserve meet, *i.e.*, $\mathrm{int}(x \wedge y) = \mathrm{int}(x) \wedge \mathrm{int}(y)$ for all $x, y \in L$ (the interior operators int in the sense of Dvořák and Holčapek [7] are further supposed to fulfill $\mathrm{int}(1) = 1$). As a consequence, the interior operators int in the sense of Ouyang and Zhang should be defined on a meet semilattice rather than on a general poset. Obviously, preserving meet implies increasingness for interior operators on meet semilattices, but not vice versa.

3 The ordinal sum of t-norms in the sense of Saminger

In [16], Saminger extended the ordinal sum of t-norms from the real unit interval to bounded lattices in a direct way. Saminger gave the following definition:

Definition 3.1. [16] *Consider a bounded lattice L and some linearly ordered index set I. Further, let $(]a_i, b_i[)_{i\in I}$ be a family of pairwise disjoint subintervals of L and $(T_i)_{i\in I}$ a family of t-norms on the corresponding intervals $([a_i, b_i])_{i\in I}$. Then the ordinal sum $T = (\langle a_i, b_i, T_i\rangle)_{i\in I}\colon L^2 \to L$ is given by*

$$T(x, y) = \begin{cases} T_i(x, y) & \text{if } (x, y) \in [a_i, b_i]^2 \\ x \wedge y & \text{otherwise.} \end{cases} \tag{1}$$

Note that the ordinal sum defined by (1) may be not a t-norm even if only one summand is involved. See [16] for such an example on the lattice **N5**. Saminger provided the following result which characterized when the ordinal sum of an arbitrary t-norm on $[a, b]$ is again a t-norm.

Theorem 3.2. [16] *Let $[a, b]$ be a subinterval of a bounded lattice L. Then the following are equivalent:*
(i) *The ordinal sum T of an arbitrary t-norm V on $[a, b]$ is a t-norm.*
(ii) *For all $x \in L$ it holds that*
 (a) *if $x \in I_a$, then $x \| u$ for all $u \in [a, b[$;*
 (b) *if $x \in I_b$, then $x \| u$ for all $u \in]a, b]$.*

Aşici proved the following result.

Theorem 3.3. [1] *Let L be a bounded lattice and $a \in]0, 1[$. If $x \| y$ for all $x \in I_a$ and $y \in]0, a]$, and $x < y$ for all $x \in I_a$ and $y \in]a, 1]$. Then the operator defined by*

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [0,a]^2 \\ y & \text{if } (x,y) \in]a,1] \times I_a \\ x & \text{if } (x,y) \in I_a \times]a,1] \\ 0 & \text{if } (x,y) \in I_a \times [0,a] \cup [0,a] \times I_a \\ x \wedge y & \text{otherwise,} \end{cases} \tag{2}$$

is a t-norm, where V is a t-norm on $[0, a]$.

Remark 3.4. *We point out that Theorem 3.3 can be seen as a corollary of Theorem 3.2. In fact, since the lattice L in Theorem 3.3 satisfies $x \| y$ for all $x \in I_a$ and $y \in]0, a]$, for any $(x, y) \in I_a \times [0, a] \cup [0, a] \times I_a$ we have $x \wedge y = 0$. Moreover, we have $x \wedge y = y$ for $(x, y) \in]a, 1] \times I_a$ and $x \wedge y = x$ for $(x, y) \in I_a \times]a, 1]$ since $x < y$ for all $x \in I_a$ and $y \in]a, 1]$. Thus, the operator T defined by (2) is an ordinal sum of V in the sense of Saminger,* i.e.,

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [0,a]^2 \\ x \wedge y & \text{otherwise,} \end{cases}$$

Note that $x \| y$ for all $x \in I_a$ and $y \in]0, a]$ is exactly the condition ii(b) of Theorem 3.2. Since $I_0 = \emptyset$, the condition ii(a) of Theorem 3.2 is satisfied automatically. So the condition (ii) of Theorem 3.2 is satisfied and thus, by (i) of Theorem 3.2, T is a t-norm.

4 The ordinal sum of t-norms based on interior operators

Since the ordinal sum of t-norms in the sense of Saminger does not always lead to a t-norm, several modified ordinal sums of t-norms have been developed in the literature. In contrast to that of Saminger, these methods always lead to a t-norm.

For example, by appropriately dealing with those elements that are incomparable with the endpoints of the given subintervals, Ouyang, Zhang and De Baets [14] proposed an alternative definition of ordinal sum of countably many (finite or countably infinite) t-norms on subintervals of a complete lattice, where the endpoints of the subintervals constitute a chain. Note that the ordinal sum in the sense of Ouyang, Zhang and De Baets always yields a t-norm, see [14]. In [7], Dvořák and Holčapek proved an extension theorem able to extend t-norms on the range of an interior operator to the underlying bounded meet semilattice. Based on this theorem, they developed an ordinal sum construction of t-norms on a bounded meet semilattice and proved it to always result in a t-norm. Note that if there are only finite t-norms involved, the ordinal sum proposed in [14] is a special case of that of [7], but Dvořák and Holčapek's ordinal sum cannot deal with infinite t-norms. Soon afterwards, Ouyang et al. [15] unified and generalized the results in [7] and [14]. Concretely, by means of an interior operator, they proved an ordinal sum theorem for countably many t-norms on bounded meet semilattices and characterized t-norms that are representable as the ordinal sum of countably many t-norms on given bounded meet semilattices.

Dvořák and Holčapek [7] proved the following extension theorem of t-norms, which has been demonstrated to be a powerful tool of the ordinal sum construction of t-norms [7, 15].

Theorem 4.1. [7] *Let L be a bounded meet semilattice, $M \subset L$ be the range of an interior operator* int, *i.e.,* $\mathrm{int}(L) = M$ *and V be a t-norm on M. Then V can be extended to L via* int, *i.e., the binary operator $T \colon L^2 \to L$ given by*

$$T(x,y) = \begin{cases} x \wedge y & \text{if } 1 \in \{x,y\} \\ V(\mathrm{int}(x), \mathrm{int}(y)) & \text{otherwise} \end{cases} \tag{3}$$

is a t-norm on L.

Note that Dvořák and Holčapek proved this theorem for interior operators in their sense. Fortunately, this theorem remains true for interior operators in the sense of Definition 2.2, see [15].

In [2], Aşici proved the following result(see Theorem 3.1 of [2]). In the following, we give an alternative proof of this result. Our proof is based on Theorem 4.1. As a result, it can be seen as a corollary of Theorem 4.1.

Theorem 4.2. [2] *Let L be a bounded lattice with $a \in L$ and $\mathrm{int}\colon L \to L$ be an interior operator such that for all $x \in I_a$ it holds $x \wedge a = \mathrm{int}(x \wedge a)$. Given a t-norm V on $[a,1]$, then the function $T\colon L^2 \to L$ given by*

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1[^2 \\ y \wedge a & \text{if } (x,y) \in [a,1[\times I_a \\ x \wedge a & \text{if } (x,y) \in I_a \times [a,1[\\ x \wedge y \wedge a & \text{if } (x,y) \in I_a \times I_a \\ x \wedge y & \text{if } x = 1 \text{ or } y = 1 \\ \mathrm{int}(x) \wedge \mathrm{int}(y) & \text{otherwise} \end{cases} \tag{4}$$

is a t-norm.

Proof. Here we give an alternative proof which differs from that of [2]. Define $\mathrm{int}_1\colon L \to L$ by

$$\mathrm{int}_1(x) = \begin{cases} x & \text{if } x \geq a \\ x \wedge a & \text{if } x \in I_a \\ \mathrm{int}(x) & \text{if } x \in [0,a[. \end{cases}$$

It is routine to verify that int_1 is an interior operator. To finish this, it suffices to prove that int_1 satisfies the conditions (i)-(iii) of Definition 2.2.

(i) $\mathrm{int}_1(x) \leq x$ is obvious.

(ii) Let $x, y \in L$ with $x \leq y$ be given. We prove $\mathrm{int}_1(x) \leq \mathrm{int}_1(y)$ for all possible cases.

If $x \in [0,a[, y \in [0,a[$ then

$$\mathrm{int}_1(x) = \mathrm{int}(x) \leq \mathrm{int}(y) = \mathrm{int}_1(y).$$

If $x \in [0,a[, y \in I_a$ then

$$\mathrm{int}_1(x) \leq x \leq y \wedge a = \mathrm{int}_1(y).$$

If $x \in [0,a[, y \in [a,1]$ then

$$\mathrm{int}_1(x) \leq x \leq y = \mathrm{int}_1(y).$$

If $x, y \in I_a$ then

$$\mathrm{int}_1(x) = x \wedge a \leq y \wedge a = \mathrm{int}_y.$$

If $x \in I_a, y \in [a,1]$ then

$$\mathrm{int}_1(x) = x \wedge a \leq y = \mathrm{int}_1(y).$$

If $x, y \in [a,1]$ then $\mathrm{int}_1(x) = x \leq y = \mathrm{int}_1(y)$.

(iii) Let $x \in L$ be given. If $x \geq a$ then $\mathrm{int}_1(\mathrm{int}_1(x)) = \mathrm{int}_1(x) = x$. If $x < a$ then $\mathrm{int}_1(\mathrm{int}_1(x)) = \mathrm{int}(\mathrm{int}(x)) = \mathrm{int}(x) = \mathrm{int}_1(x)$. If $x \in I_a$ then by assumption it holds that $\mathrm{int}(x \wedge a) = x \wedge a < a$ and thus $\mathrm{int}_1(\mathrm{int}_1(x)) = \mathrm{int}_1(x \wedge a) = \mathrm{int}(x \wedge a) = x \wedge a = \mathrm{int}_1(x)$.

Now, let M be the range of int_1 then $M = \mathrm{int}_1([0,a[) \cup [a,1]$. Let $V'\colon M^2 \to M$ be the binary operator defined by

$$V'(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1]^2 \\ x \wedge y & \text{otherwise.} \end{cases}$$

Then V' is the ordinal sum of V in the sense of Saminger, see Definition 3.1. Since each $x \in M$ is comparable with a and 1, by Theorem 3.2 we conclude that V' is a t-norm on M. Finally, the binary operator T defined by (4) is nothing but the extension of V' to L via int_1 and thus is a t-norm. □

Remark 4.3. *From the above proof we can see that Theorem 4.2 can be seen as a corollary of Theorem 4.1. It is not difficult to prove that the interior operator int_1 preserves meet if int does. Thus, Theorem 4.2 is also a corollary of the original extension theorem of t-norms, see [7]. The t-norm T is in fact an ordinal sum in the sense of [7] (see also [15]), that is, $T = (\langle \mathrm{int}_1([0,a]), \wedge \rangle, \langle [a,1], V \rangle)$. Note that Aşici proved Theorem 4.2 for an interior operator satisfying a stronger condition $\mathrm{int}(x \wedge y) = \mathrm{int}(x) \wedge \mathrm{int}(y)$, thus Theorem 4.2 is in fact a generalized version of Theorem 3.1 in [2].*

In [3], Aşici proved the following result (see Theorem 3.1 of [3]). As we will see, it can also be seen as a corollary of Theorem 4.1.

Theorem 4.4. [3] *Let L be a bounded lattice with $a \in L$ such that for all $x \in I_a$ and $y \in]0, a]$ it holds $x \| y$ and* $\text{int}\colon L \to L$ *be an interior operator. Given a t-norm V on $[a, 1]$, then the function $T\colon L^2 \to L$ defined by*

$$T(x,y) = \begin{cases} V(x,y) & \text{if } (x,y) \in [a,1[^2 \\ \text{int}(x) \wedge \text{int}(y) & \text{if } (x,y) \in [0,a[^2 \cup [0,a[\times[a,1[\cup[a,1[\times[0,a[\\ x \wedge y & \text{if } x=1 \text{ or } y=1 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

is a t-norm.

Remark 4.5. *We point out that the t-norm defined by* (5) *is exactly the same as that defined by* (4)*. In fact, since the lattice L in Theorem 4.4 fulfills the condition "$x\|y$ holds for all $x \in I_a$ and $y \in]0, a]$", we have $x \wedge a = y \wedge a = x \wedge y \wedge a = 0$ for all $x, y \in I_a$. Thus, the last case in* (5) *is the same as the Cases 2-4 in* (4) *in this special lattice. As a consequence, Theorem 4.4 can also be seen as a corollary of Theorem 4.1.*

5 Concluding remark

In this note, we have recalled some progress of the ordinal sum of t-norms on bounded lattices. We have also provided an alternative proof of Aşici's theorems in [1, 2, 3]. As a result, these theorems can be seen as corollaries of Saminger [16] and Dvořák and Holčapek [7].

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