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Binary option pricing formulas for fuzzy financial market based on the exponential Ornstein-Uhlenbeck model

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Abstract

Binary option is an exotic option which is popular in Over the Counter market for hedging and speculation. According to their different payoffs, there are two types of binary options, that is, cash-or-nothing and asset-or-nothing option. This paper investigates the fuzzy financial market based on the exponential Ornstein-Uhlenbeck model and derives binary option pricing formulas. In order to better understand the mathematical properties of these formulas, we give a few numerical examples and some figures to illustrate the changes of binary option price with different parameters when others are fixed.

Keywords: Credibility theory, fuzzy differential equation, Liu process, option pricing, exponential Ornstein-Uhlenbeck model.

1 Introduction

Black and Scholes [1] constructed the replication portfolio of options on the basis of geometric Brownian motion (GBM), and put forward the option pricing method in 1973. In the same year, Merton [14] discussed a similar idea to price option. Their creative contributions have laid a theoretical foundation for various options pricing based on market price changes in emerging derivative markets, including stocks, bonds, currencies and commodities.

Exotic options are more complex derivative securities than conventional options including European options and American options. The introduction of such options can broaden the scope of banking business or meet the hedging and speculation needs of investors. Binary option is an exotic option, which is popular in the Over the Counter (OTC) market. The difference between binary options and standard options is that the returns of binary option are discontinuous, while the returns of standard options are continuous. As a basic option product, it is of great significance to build more compound option products. Due to the simple return structure of binary options, it is easy to price them in a random environment. Wu and He [21] derived the analytical solution of the binary option pricing model in which the risk-free interest rate and the volatility of stock price are constants. Assuming that the risk-free interest rate and volatility change with time, Wang [20] used the heat conduction equation to derive the pricing formula of binary options under the risk neutral condition. Zhang et al. [29] described the concept, classification and advantages and disadvantages of the binary digital options, and deduced the pricing method of binary option combination. In addition, Kang and Chen [8] studied the pricing problem of binary options, in which stock price obeys the mixed fractional Brownian motion model with jump, by using differential equation method, actuarial method and numerical method.

As we all know, probability theory can be used only when the estimated probability distribution is close enough to the fundamental premise of long-term cumulative frequency. Otherwise, the law of large numbers will no longer be valid and probability theory will no longer be applicable. However, in the financial market, it is sometimes difficult to obtain sufficient and accurate sample data, which means that it is unreliable to insist on using probability theory

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to consider option pricing. An alternative solution is to give the belief degrees of financial events by comprehensively considering the suggestions of the domain experts and our knowledge. In that case, we can recourse to fuzzy set theory to solve the problem with imprecise data. Zadeh [27] gave birth to fuzzy set in 1965. Then, in order to measure a fuzzy event, Zadeh [28] came up with the concept of possibility measure in 1978. Subsequently, fuzzy set theory was widely used in the pricing of binary options. For instance, Thavaneswaran et al. [19] studied binary options by fuzzifying the maturity value of the stock price using trapezodial, parabolic and adaptive fuzzy nunbers. Qin et al. [17] developed a new framework for pricing the binary option by using fuzzy set theory based on fractional Brownian motion, and obtained fuzzy price of the binary option by using a risk-neutral pricing principle and quasi-conditional expectation. It is not difficult to find that in the above papers, the method of evaluating options utilizing fuzzy set theory is to fuzzify the coefficients in the stochastic stock model.

However, there are some defects in fuzzy set theory, that is, the possibility measure dissatisfy the self-duality and it only fuzzifies the coefficient without considering the fuzzy disturbance of the price process. In fact, whether in theory or in practice, there is an absolute need for a self-dual measure. Therefore, for the sake of defining a self-dual measure, credibility measure was put forward by Liu and Liu [13] in 2002. Thereafter, credibility theory was established and refined by Liu [10, 11]. In order to depict fuzzy systems, Liu [12] introduced Liu process, fuzzy calculus and fuzzy differential equation driven by Liu process. Subsequently, the existence and uniqueness theorem of solution of fuzzy differential equation was verified by You et al. [26], Chen and Qin [3]. Some fuzzy differential equations cannot be solved analytically, therefore, You and Hao [24, 25], Cheng and You [4], Ji and You [7] developed numerical solving methods to solve this kind of fuzzy differential equation. In the aspect of fuzzy finance, based on an assumption that the underlying asset price follows a geometric Liu process, Liu [12] first proposed a fuzzy stock model by drawing support from credibility theory. Then, Gao [6] and Peng [15] presented some complex underlying fuzzy models. Based on generalized Gao′s stock model, You and Bo [22] derived standard options pricing formulas. Subsequently, You and Bo [23] extended Gao′s stock model to fractional version and obtained its European, American, Asian, power options pricing formulas. Besides, Zhang and You [30] studied a portfolio model in which stock prices are driven by geometric Liu process and option pricing for convertible stock model. Though Gao′s stock model [6] incorporated a general economic behavior: mean reversion, it is a linear mean reversion. While in our paper, a nonlinear mean reversion behavior is considered.

The following is some literature on the exponential Ornstein-Uhlenbeck model [18]. Li et al. [9] proposed a stock model that pays dividends and obeys the exponential Ornstein-Uhlenbeck process, and the pricing formulas of the convertible bond were obtained by means of martingale approach. Zhao [31] used the exponential Ornstein-Uhlencack process to describe the stock price, and obtained the pricing formulas of power European options. Carlos and Mejia [2] presented a methodological procedure to estimate the parameters of the exponential Ornstein-Uhlenbeck process. The above papers are pricing different financial products and financial derivatives based on exponential Ornstein-Uhlencack process under the framework of probability theory. However, they fail to consider the case of few or even no samples available, which is dealt with by our paper. The main contribution of this paper is to derive the pricing formulas for binary options including cash-or-nothing option and asset-or-nothing option based on fuzzy stock model in stead of stochastic stock model.

The rest of this paper is organized as follows. Section 2 recalls some preliminary concepts of credibility theory. Some fuzzy stock models including exponential Ornstein-Uhlenbeck model are introduced in Section 3. The pricing formulas of two types of binary options are derived and numerical experiments are also performed in Section 4. At last, some conclusions are made in Section 5.

2 Preliminaries

Credibility theory is a branch of mathematics for studying the behavior of fuzzy phenomena. It has been applied to fuzzy programming, fuzzy finance, fuzzy control, and so on. In this section, some useful definitions and theorems are introduced as follows.

Definition 2.1. [13] *Let $\mathcal{P}$ be a σ-algebra on a non-empty set Θ. A set function $(\Theta, \mathcal{P}, \mathrm{Cr}) \to [0,1]$ is called a credibility measure if it satisfies the following axioms:*
Axiom 1. (Normality) $\mathrm{Cr}\{\Theta\}=1$.
Axiom 2. (Monotonicity) $\mathrm{Cr}\{A\} \le \mathrm{Cr}\{B\}$ *whenever* $A \subset B$.
Axiom 3. (Self-Duality) $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$ *for any event* A.
Axiom 4. (Maximality) $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ *for any events* A_i *with* $\sup_i \mathrm{Cr}\{A_i\} < 0.5$.

A fuzzy variable ξ is a measurable function from a credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers. The membership function $\mu(x)$ of ξ is defined by $\mu(x) = (2\mathrm{Cr}\{\xi = x\}) \wedge 1, x \in \mathcal{R}$. For example, a normal fuzzy variable

$\xi \sim \mathcal{N}(e,\sigma)$ has a membership function

$$\mu(x) = 2\Big(1 + \exp\Big(\frac{\pi|x-e|}{\sqrt{6}\sigma}\Big)\Big)^{-1}, \quad x \in \mathcal{R},\ \sigma > 0.$$

In order to use the membership function to calculate the credibility of fuzzy events, the following theorem is given.

Theorem 2.2. (Credibility Inversion Theorem, [11]) *Let ξ be a fuzzy variable with membership function μ. Then for any set B of real numbers, we have*

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\Big(\sup_{x\in B}\mu(x) + 1 - \sup_{x\in B^c}\mu(x)\Big).$$

The numerical characteristics of fuzzy variables can be described by expected value. The following definition of expected value applies not only to continuous fuzzy variables, but also to discrete ones.

Definition 2.3. [13] *Let ξ be a fuzzy variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty}\mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r,$$

provided that at least one of the two integrals is finite.

Furthermore, let ξ be a fuzzy variable, and $f : \mathcal{R} \to \mathcal{R}$ a function. Then the expected value $f(\xi)$ is

$$E[f(\xi)] = \int_0^{+\infty}\mathrm{Cr}\{f(\xi) \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{f(\xi) \leq r\}\mathrm{d}r.$$

In order to model the evolution of fuzzy phenomena, the concept of fuzzy process is proposed as a sequence of fuzzy variables indexed by time.

Definition 2.4. [12] *C_t is said to be a Liu process if*
(i) $C_0 = 0$,
(ii) *for any sets $B_1,\ B_2,\ \cdots,\ B_k$ of real numbers,*

$$\mathrm{Cr}\{\bigcap_{j=1}^{k}\{(C_{t_j} - C_{t_{j-1}}) \in B_i\}\} = \min_{1\leq j\leq k}\mathrm{Cr}\{(C_{t_j} - C_{t_{j-1}}) \in B_i\},$$

(iii) *for any given $t > 0$, the $C_{s+t} - C_s$ are identically distributed fuzzy variables for all $s > 0$,*
(iv) *every increment $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$.*

Note that Liu process is said to be a standard Liu process if the drift coefficient $e = 0$ and diffusion coefficient $\sigma = 0$.

Definition 2.5. (Liu Integral [12]) *Let X_t be a fuzzy process and let C_t be a standard Liu process. For any partition of closed interval $[a,b]$ with $a = t_1 < t_2 < ... < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1\leq i\leq K}|t_{i+1} - t_i|.$$

Then the Liu integral of X_t with respect to C_t is

$$\int_a^b X_t\mathrm{d}C_t = \lim_{\Delta\to 0}\sum_{i=1}^{k}X_{t_i}(C_{t_{i+1}} - C_{t_i}),$$

provided that the limit exists almost surely and is a fuzzy variable.

It is worth noting that the subscript of X_{t_i} is the left end point of interval $[t_i, t_{i+1}]$. This is also the difference from the traditional Riemann integral.

Theorem 2.6. [16] *Let C_t be a standard Liu process. If $f(s)$ is a Riemann integral function on $[0,t]$, then Liu integral $\int_0^t f(s)\mathrm{d}C_s$ exists. Moreover, for any fixed time t, $\int_0^t f(s)\mathrm{d}C_s$ is a normally distributed fuzzy variable with expected value 0 and variance $(\int_0^t |f(s)|\mathrm{d}s)^2$.*

Definition 2.7. (Fuzzy Differential Equation [12]) *Suppose C_t is a standard Liu process, and f and g are some given functions. Then*

$$\mathrm{d}X_t = f(t,X_t)\mathrm{d}t + g(t,X_t)\mathrm{d}C_t,$$

is called a fuzzy differential equation.

3 Exponential Ornstein-Uhlenbeck model

In this section, some models widely used in the field of fuzzy financial field are reviewed, and the exponential Ornstein-Uhlenbeck model for fuzzy financial market is introduced.

Suppose the bond price is B_t and the stock price is S_t, [12] formulated a fuzzy stock model as

$$\begin{cases} \mathrm{d}B_t = rB_t\mathrm{d}t \\ \mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}C_t, \end{cases} \tag{3.1}$$

where $r > 0$ represents the risk-free interest rate, $\mu > 0$ and $\sigma > 0$ are the instantaneous return rate and volatility of stocks, respectively, and C_t is a standard Liu process.

The model (3.1) shows that the stock price has constant expected rate of return, and it just can describe stock price in short-run properly. As a consequence, a stock model with mean-reverting property was proposed by Gao [6] as follows,

$$\begin{cases} \mathrm{d}B_t = rB_t\mathrm{d}t \\ \mathrm{d}S_t = a(b - S_t)\mathrm{d}t + \sigma S_t\mathrm{d}C_t, \end{cases} \tag{3.2}$$

where a represents the speed of reversion, b is long-run average level. When stock price is too high, the drift term is negative, which makes the stock price have a downward trend; when stock price is too low, the drift term is positive, which causes the price have a upward trend.

The above models are both linear models, and in model (3.2), the stock price may be negative values. To overcome these shortcomings, similar to Dai et al. [5], an exponential Ornstein-Uhlenbeck model driven by Liu process was investigated as follows,

$$\begin{cases} \mathrm{d}B_t = rB_t\mathrm{d}t \\ \mathrm{d}S_t = \mu(1 - c\ln S_t)S_t\mathrm{d}t + \sigma S_t\mathrm{d}C_t, \end{cases} \tag{3.3}$$

where μ and c are two positive constants that affect the instantaneous drift of the process, $\sigma > 0$ is the diffusion coefficient, and C_t is a standard Liu process. This nonlinear stock model also describes the reversion behavior of stock price. It is clear that when $c = 0$, the model (3.3) degenerates into a geometric Liu process. It follows from You et al. [26] that the analytical solution of the second differential equation in (3.3) is

$$S_t = \exp\Big(\exp(-\mu ct)\ln S_0 + \frac{1}{c}(1 - \exp(-\mu ct)) + \sigma\exp(-\mu ct)\int_0^t \exp(\mu cs)\mathrm{d}C_s\Big). \tag{3.4}$$

4 Binary option pricing formula

Binary option is a new type of option produced by the change of contract terms, which has the characteristics of discontinuous return. For binary call options, according to their set returns, they can be divided into two types : cash-or-nothing call (CONC) option and asset-or-nothing call (AONC) option. The former means that when the stock price is lower than the strike price on the maturity date, the option return is 0, but when the stock price is higher than the strike price, the option return is a fixed amount Q. The difference between the latter and the former is that when the stock price is higher than the strike price, the option return of the latter is the price of the stock on the maturity date.

4.1 Cash-or-nothing option pricing formula

In this subsection, we will discuss the price of a cash-or-nothing option of model (3.3). A CONC allows the holder to obtain fixed income Q, if the strike price K is greater than the stock price S_T at the expiration time T. That is, the profit and loss of the option at maturity date is

$$C(S_T, T) = \begin{cases} Q, & \text{if} \;\; S_T > K, \\ 0, & \text{if} \;\; S_T \le K. \end{cases}$$

Let f_{conc} represent the price of this contract. Then the investor pays f_{conc} to buy the contract at time 0, and has a payoff $C(S_T, T)$ at time T. Considering the time value of money resulted from the bond, the present value of the payoff is $\exp(-rT)C(S_T, T)$. So the net return of the investor at time 0 is

$$-f_{conc} + \exp(-rT)C(S_T, T).$$

On the other hand, the bank receives f_{conc} for selling the contract at time 0, and pays $C(S_T, T)$ at the expiration time T. Thus the net return of the bank at time 0 is

$$f_{conc} - \exp(-rT)C(S_T, T).$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$-f_{conc} + \exp(-rT)E[C(S_T, T)] = f_{conc} - \exp(-rT)E[C(S_T, T)].$$

Thus $f_{conc} = \exp(-rT)E[C(S_T, T)]$.

Next, we define two indicator functions

$$I_{\{S_T>K\}} = \begin{cases} 1, & \text{if } \ S_T > K, \\ 0, & \text{if } \ S_T \leq K, \end{cases}$$

and

$$I_{\{S_T<K\}} = \begin{cases} 1, & \text{if } \ S_T < K, \\ 0, & \text{if } \ S_T \geq K. \end{cases}$$

Then the definition option price of the CONC and the corresponding option pricing formula are given.

Definition 4.1. The CONC option price f_{conc} for model (3.3) is defined as

$$f_{conc} = \exp(-rT)E[QI_{\{S_T>K\}}],$$

where K is the strike price at expiration time T.

According to Definition 4.1, we obtain the following CONC option pricing formula.

Theorem 4.2. *The CONC option pricing formula for model (3.3) is given by*

$$f_{conc} = \frac{Q\exp(B - rT)}{\exp(B) + K^A},$$

where $A = \frac{\pi\mu c\exp(\mu cT)}{\sqrt{6}\sigma(\exp(\mu cT)-1)}$ *and* $B = \frac{\pi\mu(c\ln S_0+\exp(\mu cT)-1)}{\sqrt{6}\sigma(\exp(\mu cT)-1)}$.

Proof. In the light of Definition 4.1, the functional form in the expected value is the product of a constant and an indicator function, so we just need to get the expected value of the indicator function $I_{S_T>K}$. It follows from Definition 2.3, that

$$E[I_{S_T>K}] = \int_0^{+\infty} \text{Cr}\{I_{S_T>K} \geq x\}\mathrm{d}x = \int_0^1 \text{Cr}\{I_{S_T>K} \geq x\}\mathrm{d}x.$$

Therefore, we obtain

$$\begin{aligned} f_{conc} &= \exp(-rT)E[QI_{\{S_T>K\}}] \\ &= \exp(-rT)\int_0^{+\infty} \text{Cr}\{QI_{\{S_T>K\}} \geq x\}\mathrm{d}x \\ &= Q\exp(-rT)\int_0^1 \text{Cr}\{I_{\{S_T>K\}} \geq x\}\mathrm{d}x \\ &= Q\exp(-rT)\text{Cr}\{S_T > K\} \\ &= Q\exp(-rT)\text{Cr}\Big\{\exp\Big(\exp(-\mu cT)\ln S_0 \\ &\quad + \frac{1}{c}(1-\exp(-\mu cT)) + \sigma\exp(-\mu cT)\int_0^T \exp(\mu cs)\mathrm{d}C_s\Big) > K\Big\} \\ &= Q\exp(-rT)\text{Cr}\Big\{\int_0^T \exp(\mu cs)\mathrm{d}C_s > \frac{c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1}{c\sigma}\Big\}. \end{aligned}$$

According to Theorem 2.6, we can know $\int_0^T \exp(\mu cs)\mathrm{d}C_s$ is a normally distributed fuzzy variable with expected value 0 and variance $(\exp(\mu cT) - 1)^2/(\mu c)^2$. Thus, the discussion can be divided into two situations.

If $c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1 \geq 0$, then

$$\begin{aligned} f_{conc} &= Q\exp(-rT)\mathrm{Cr}\Big\{\int_0^T \exp(\mu cs)\mathrm{d}C_s > \frac{c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1}{c\sigma}\Big\} \\ &= Q\exp(-rT)\Big(1 + \exp(\frac{\pi\mu(c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1)}{\sqrt{6}\sigma(\exp(\mu cT) - 1)})\Big)^{-1} \\ &= \frac{Q\exp(B - rT)}{\exp(B) + K^A}, \end{aligned}$$

where $A = \frac{\pi\mu c\exp(\mu cT)}{\sqrt{6}\sigma(\exp(\mu cT)-1)}$ and $B = \frac{\pi\mu(c\ln S_0 + \exp(\mu cT) - 1)}{\sqrt{6}\sigma(\exp(\mu cT)-1)}$.

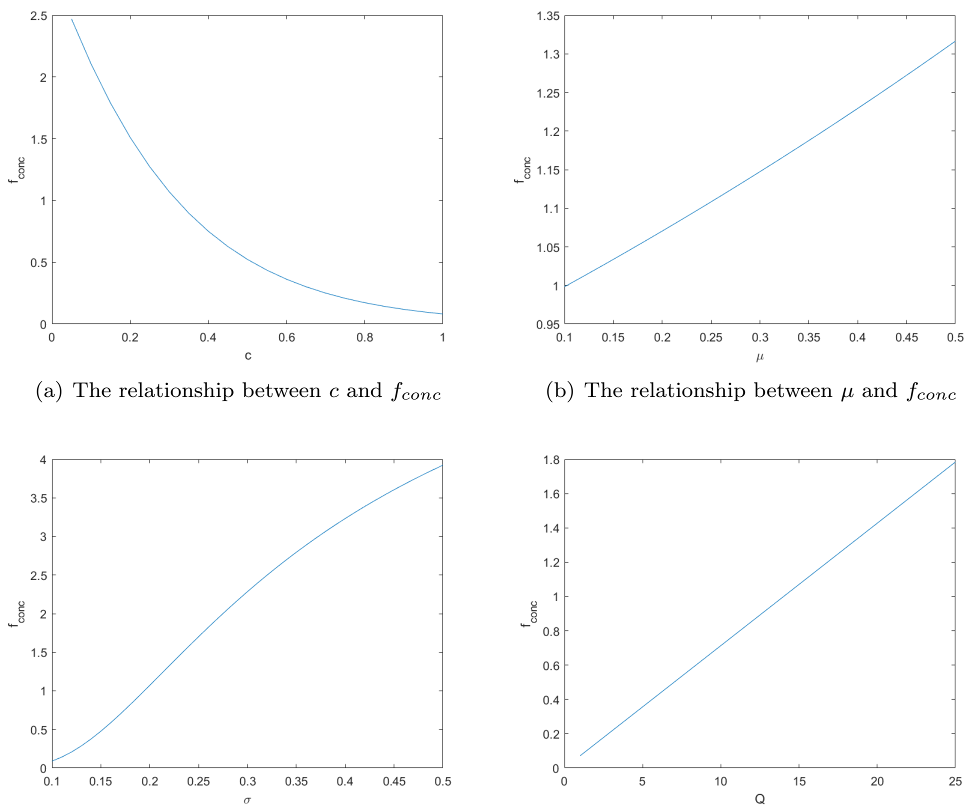
If $c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1 < 0$, then

$$\begin{aligned} f_{conc} &= Q\exp(-rT)\mathrm{Cr}\Big\{\int_0^T \exp(\mu cs)\mathrm{d}C_s > \frac{c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1}{c\sigma}\Big\} \\ &= Q\exp(-rT)\Big(1 - (1 + \exp(\frac{-\pi\mu(c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1)}{\sqrt{6}\sigma(\exp(\mu cT) - 1)}))^{-1}\Big) \\ &= \frac{Q\exp(B - rT)}{\exp(B) + K^A}, \end{aligned}$$

By summarizing, we have

$$f_{conc} = \frac{Q\exp(B - rT)}{\exp(B) + K^A}.$$

The proof is completed. □



(a) The relationship between c and f_{conc} (b) The relationship between μ and f_{conc}

(c) The relationship between σ and f_{conc} (d) The relationship between Q and f_{conc}

Figure 1: The relations of f_{conc} and c, μ, σ, Q

Example 4.3. *Suppose that the strike price of a CONC option is $K = 20$, and the duration of the contract is three months ($T = 0.25$). The return of the option is the determined amount $Q = 15$. The underlying stock is currently being sold at a price of $S_0 = 18$, the risk-less interest rate r is 5% per annum. Other parameters of stock price are $c = 0.3$, $\mu = 0.2$, $\sigma = 0.2$. According to Theorem 4.2, we can get the CONC option price is $f_{conc} = 1.0704$.*

Then, we give the figures of CONC option pricing formulas with different parameters as follows. Figure 1 and Figure 2 show that f_{conc} is a decreasing function of K, c, and f_{conc} is an increasing function of T, μ, σ.

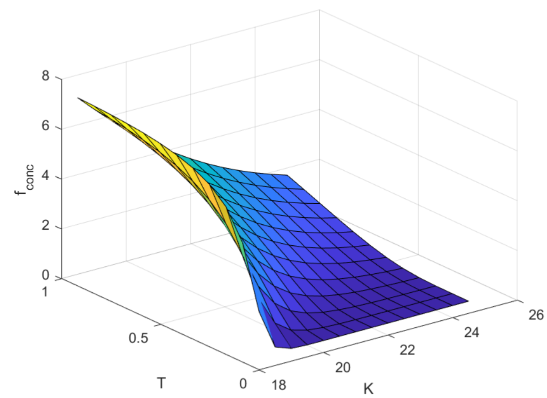


Figure 2: The relationship between T, K and f_{conc}

A cash-or-nothing put (CONP) option allows the holder to obtain fixed income Q, if the strike price K is less than the stock price S_T at the expiration time T. That is, the profit and loss of the option at maturity date is

$$P(S_T, T) = \begin{cases} 0, & \text{if } \ S_T \geq K, \\ Q, & \text{if } \ S_T < K. \end{cases}$$

Applying the fair price principle, we obtain the definition of the CONP option price.

Definition 4.4. The CONP option price f_{conp} for model (3.3) is defined as

$$f_{conp} = \exp(-rT)E[QI_{\{S_T<K\}}],$$

where K is the strike price at expiration time T.

According to Definition 4.4, the following CONP option pricing formula is derived.

Theorem 4.5. *The CONP option pricing formula for model (3.3) is given by*

$$f_{conp} = \frac{QK^A \exp(-rT)}{\exp(B) + K^A},$$

where $A = \frac{\pi\mu c \exp(\mu cT)}{\sqrt{6}\sigma(\exp(\mu cT)-1)}$ *and* $B = \frac{\pi\mu(c \ln S_0 + \exp(\mu cT) - 1)}{\sqrt{6}\sigma(\exp(\mu cT)-1)}$.

Proof. By the definition of expected value of fuzzy variable and (3.4), in a similar deduction process in the proof of Theorem 4.2, we obtain

$$\begin{aligned} f_{conp} &= \exp(-rT)E[QI_{\{S_T<K\}}] \\ &= \exp(-rT)\int_0^{+\infty} \mathrm{Cr}\{QI_{\{S_T<K\}} \geq x\}\mathrm{d}x \\ &= Q\exp(-rT)\int_0^{1} \mathrm{Cr}\{I_{\{S_T<K\}} \geq x\}\mathrm{d}x \\ &= Q\exp(-rT)\mathrm{Cr}\{S_T < K\} \\ &= Q\exp(-rT)\mathrm{Cr}\Big\{\int_0^{T} \exp(\mu cs)\mathrm{d}C_s < \frac{c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1}{c\sigma}\Big\}. \end{aligned}$$

If $c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1 > 0$, then

$$\begin{aligned}
f_{conp} &= Q\exp(-rT)\mathrm{Cr}\Big\{\int_0^T \exp(\mu cs)\mathrm{d}C_s < \frac{c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1}{c\sigma}\Big\} \\
&= Q\exp(-rT)\Big(1 - \Big(1 + \exp(\frac{\pi\mu(c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1)}{\sqrt{6}\sigma(\exp(\mu cT) - 1)})\Big)^{-1}\Big) \\
&= \frac{QK^A\exp(-rT)}{\exp(B) + K^A}.
\end{aligned}$$

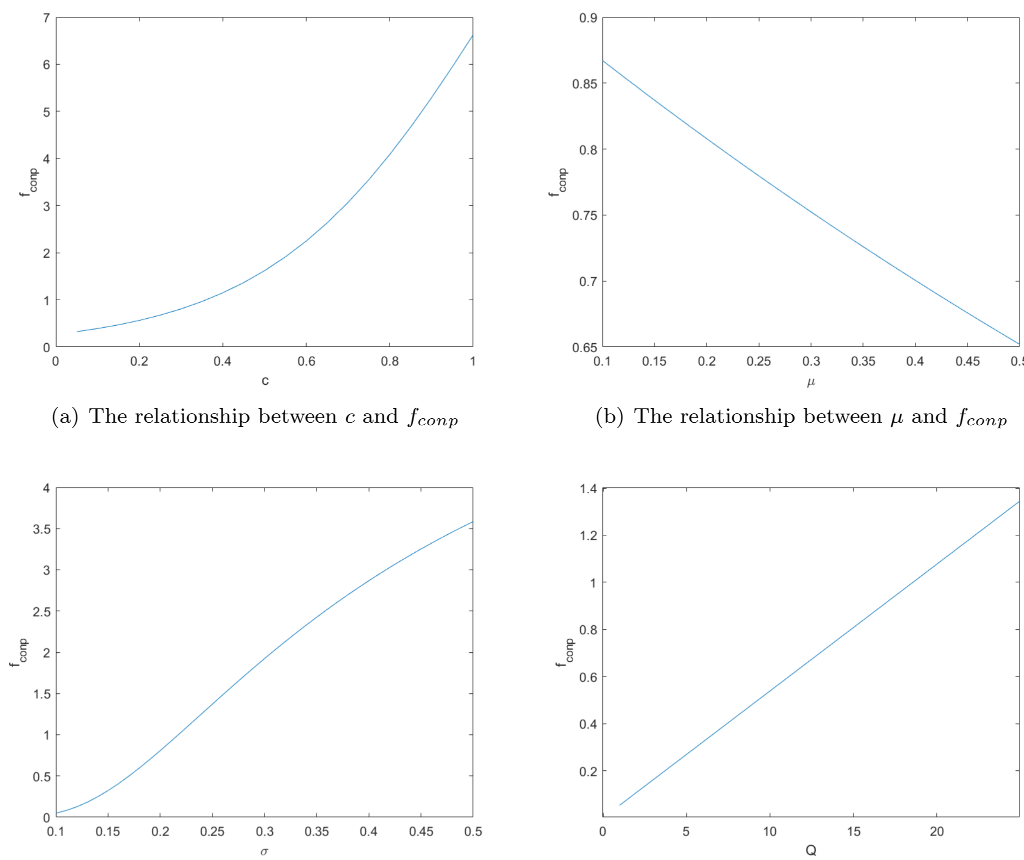
If $c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1 < 0$, then

$$\begin{aligned}
f_{conp} &= Q\exp(-rT)\mathrm{Cr}\Big\{\int_0^T \exp(\mu cs)\mathrm{d}C_s < \frac{c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1}{c\sigma}\Big\} \\
&= Q\exp(-rT)\Big(1 + \exp(\frac{-\pi\mu(c\exp(\mu cT)\ln K - c\ln S_0 - \exp(\mu cT) + 1)}{\sqrt{6}\sigma(\exp(\mu cT) - 1)})\Big)^{-1} \\
&= \frac{QK^A\exp(-rT)}{\exp(B) + K^A}.
\end{aligned}$$

Thus, we get

$$f_{conc} = \frac{QK^A\exp(-rT)}{\exp(B) + K^A}.$$

The proof is completed. □



(a) The relationship between c and f_{conp}

(b) The relationship between μ and f_{conp}

(c) The relationship between σ and f_{conc}

(d) The relationship between Q and f_{conp}

Figure 3: The relations of f_{conp} and c, μ, σ, Q

Example 4.6. *Suppose that the strike price of a CONP option is $K = 18$, and the duration of the contract is three months ($T = 0.25$). The return of the option is the determined amount $Q = 15$. The underlying stock is currently being sold at a price of $S_0 = 20$, the risk-less interest rate r is 5% per annum. Other parameters of stock price are $c = 0.3$, $\mu = 0.2$, $\sigma = 0.2$. According to Theorem 4.5, we can get the CONP option price is $f_{conp} = 0.8079$.*

Figure 3 and 4 present the relations of price f_{conp} and some significant parameters in Example 4.6. We find that f_{conp} is positively correlated with c, σ, Q, T and K, but inversely correlated with μ.

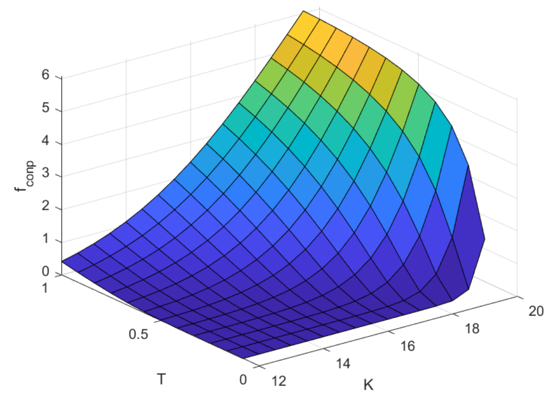


Figure 4: The relationship between T, K and f_{conp}

4.2 Asset-or-nothing option pricing formula

In this subsection, we will discuss the price of an asset-or-nothing option for model (3.3). An AONC option allows the holder to obtain fixed income S_T, if the strike price K is greater than the stock price S_T at the expiration time T. In other words, the payoff the option at maturity date is

$$C_A(S_T, T) = \begin{cases} S_T, & \text{if } S_T > K, \\ 0, & \text{if } S_T \leq K. \end{cases}$$

In a similar analysis in Subsection 4.1, we propose the definition below.

Definition 4.7. The AONC option price f_{aonc} for model (3.3) is defined as

$$f_{aonc} = \exp(-rT)E[S_T I_{\{S_T > K\}}],$$

where K is the strike price at expiration time T.

According to Definition 4.7, the following AONC option pricing formula is derived.

Theorem 4.8. *The AONC option pricing formula for model (3.3) is given by*

$$f_{aonc} = \frac{K\exp(B - rT)}{\exp(B) + K^A} + \int_K^{+\infty} \frac{\exp(B - rT)}{\exp(B) + x^A} \mathrm{d}x,$$

where $A = \frac{\pi\mu c\exp(\mu cT)}{\sqrt{6}\sigma(\exp(\mu cT) - 1)}$ *and* $B = \frac{\pi\mu(c\ln S_0 + \exp(\mu cT) - 1)}{\sqrt{6}\sigma(\exp(\mu cT) - 1)}$.

Proof. In Definition 4.7, the functional form in the expected value is $S_T I_{S_T > K}$. By using Definition 2.3, we obtain

$$\begin{aligned} E[S_T I_{S_T > K}] &= \int_0^{+\infty} \mathrm{Cr}\{S_T I_{S_T > K} \geq x\}\mathrm{d}x \\ &= \int_0^{+\infty} \mathrm{Cr}\{(S_T \geq x) \bigcap (S_T > K)\}\mathrm{d}x \\ &= \int_0^{K} \mathrm{Cr}\{S_T > K\}\mathrm{d}x + \int_K^{+\infty} \mathrm{Cr}\{S_T \geq x\}\mathrm{d}x. \end{aligned}$$

Thus,

$$\begin{aligned} f_{aonc} &= \exp(-rT)E[S_T I_{\{S_T>K\}}] \\ &= \exp(-rT)\Big(\int_0^K \mathrm{Cr}\{S_T > K\}\mathrm{d}x + \int_K^{+\infty} \mathrm{Cr}\{S_T \geq x\}\mathrm{d}x\Big) \\ &= f_{c1} + f_{c2}. \end{aligned}$$

According to the proof of Theorem 4.2, we get

$$f_{c1} = K\exp(-rT)\int_0^K \mathrm{Cr}\{S_T > K\}\mathrm{d}x = \frac{K\exp(B-rT)}{exp(B)+K^A}.$$

And

$$f_{c2} = \exp(-rT)\int_K^{+\infty} \mathrm{Cr}\Big\{\int_0^T \exp(\mu cs)\mathrm{d}C_s \geq \frac{c\exp(\mu cT)\ln x - c\ln S_0 - \exp(\mu cT) + 1}{c\sigma}\Big\}\mathrm{d}x.$$

Denote $J(x) = c\exp(\mu cT)\ln x - c\ln S_0 - \exp(\mu cT) + 1$. Obviously, $J(x)$ is a monotonically increasing function. Therefore, if $J(K) > 0$, then $J(x)$ is always greater than 0 on $[K, +\infty)$. Thus,

$$\begin{aligned} f_{c2} &= \exp(-rT)\int_K^{+\infty} \Big(1 + \exp(\frac{\pi\mu J(x)}{\sqrt{6}\sigma(\exp(\mu cT)-1)})\Big)^{-1}\mathrm{d}x \\ &= \exp(-rT)\int_K^{+\infty} \frac{1}{1+\exp(A\ln x - B)}\mathrm{d}x \\ &= \int_K^{+\infty} \frac{\exp(B-rT)}{\exp(B)+x^A}\mathrm{d}x. \end{aligned}$$

If $J(K) \leq 0$, then there exists a constant K_0 such that $J(K_0) = 0$. Consequently,

$$\begin{aligned} f_{c2} &= \exp(-rT)\Big[\int_K^{K_0} \mathrm{Cr}\{\int_0^T \exp(\mu cs)\mathrm{d}C_s \geq \frac{J(x)}{c\sigma}\}\mathrm{d}x + \int_{K_0}^{+\infty} \mathrm{Cr}\{\int_0^T \exp(\mu cs)\mathrm{d}C_s \geq \frac{J(x)}{c\sigma}\}\mathrm{d}x\Big] \\ &= \exp(-rT)\Big[\int_K^{K_0} \Big(1 - (1 + \exp(\frac{-\pi\mu J(x)}{\sqrt{6}\sigma(\exp(\mu cT)-1)}))^{-1}\Big)\mathrm{d}x \\ &\quad + \int_{K_0}^{+\infty} \Big(1 + \exp(\frac{\pi\mu J(x)}{\sqrt{6}\sigma(\exp(\mu cT)-1)})\Big)^{-1}\mathrm{d}x\Big] \\ &= \exp(-rT)\Big[\int_K^{K_0} (1 - \frac{1}{\exp(-A\ln x + B)})\mathrm{d}x + \int_{K_0}^{+\infty} \frac{1}{1+\exp(A\ln x - B)}\mathrm{d}x\Big] \\ &= \exp(-rT)\int_K^{+\infty} \frac{\exp(B)}{\exp(B)+x^A}\mathrm{d}x. \end{aligned}$$

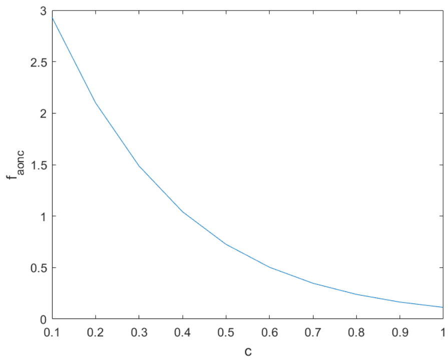
Hence,

$$f_{aonc} = f_{c1} + f_{c2} = \frac{K\exp(B-rT)}{\exp(B)+K^A} + \int_K^{+\infty} \frac{\exp(B-rT)}{\exp(B)+x^A}\mathrm{d}x.$$

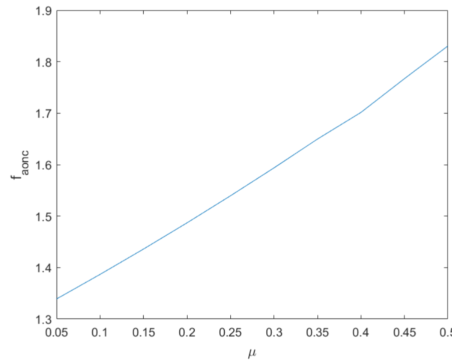
The theorem is verified. □

Example 4.9. *Suppose that the strike price of a AONC option is $K = 20$, and the duration of the contract is three months ($T = 0.25$). The underlying stock is currently being sold at a price of $S_0 = 18$, the risk-less interest rate r is 5% per annum. Other parameters of stock price are $c = 0.3$, $\mu = 0.2$, $\sigma = 0.2$. According to Theorem 4.8, we can get the AONC option price is $f_{aonc} = 1.4869$.*

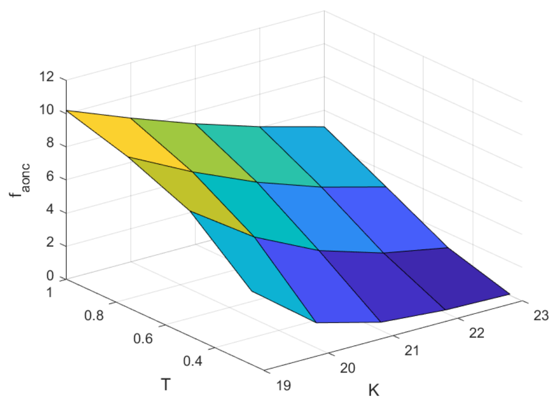
We find from Figure 5 that the option price increases with the increase of μ and T, and decreases with the increases c and K. This is reasonable owing to the impact of μ on the instantaneous return of the stock price. The larger μ is, the greater the corresponding instantaneous return of the stock, hence, the option price increases accordingly. And c also affects the level price of the stock price. When $S_t < \exp(\frac{1}{c})$, the drift is positive, and when $S_t > \exp(\frac{1}{c})$, the drift is negative. In consequence, the greater the c is, the greater the possibility that the drift term is negative, which indicates that the instantaneous stock price decreases, the option price also decreases. In addition, the closer is the time to expiration, the greater is the value of the option, because the stock has more potential for movement and thus



(a) The relationship between c and f_{aonc}



(b) The relationship between μ and f_{aonc}



(c) The relationship between K, T and f_{aonc}

Figure 5: The relations of f_{aonc} and c, μ, σ, Q

the option price increases. Additionally, strike price and expiration time play important roles in option pricing. The results show that even for the same underlying asset, the option price will be different if the strike price and expiration time are different. An asset-or-nothing put (AONP) option allows the holder to obtain income S_T, if the strike price K is greater than the stock price S_T at the expiration time T. In other words, the payoff the option at maturity date is

$$P_A(S_T, T) = \begin{cases} 0, & \text{if } \ S_T \geq K, \\ S_T, & \text{if } \ S_T < K. \end{cases}$$

Do the similar analysis as Subsection 4.1, we put forward the concept of the AONP option price for model (3.3).

Definition 4.10. The AONP option price f_{aonp} for model (3.3) is defined as

$$f_{aonp} = \exp(-rT)E[S_T I_{\{S_T<K\}}],$$

where K is the strike price at expiration time T.

According to Definition 4.10, we get the following AONP option pricing formula.

Theorem 4.11. *The AONP option pricing formula for model (3.3) is given by*

$$f_{aonp} = \begin{cases} \dfrac{\exp(-rT)K^{A+1}}{\exp B + K^A}, & \textit{if } \ J(K) \leq 0, \\ \dfrac{K^{A-1}\exp(\frac{2B}{A} - rT)}{K^A + \exp(B)} + \displaystyle\int_{\frac{\exp(\frac{2B}{A})}{K}}^{K} \dfrac{\exp(B - rT)}{x^A + \exp(B)} \mathrm{d}x, & \textit{if } \ J(K) > 0, \end{cases}$$

where $A = \frac{\pi\mu c\exp(\mu cT)}{\sqrt{6}\sigma(\exp(\mu cT)-1)}$, $B = \frac{\pi\mu(c\ln S_0 + \exp(\mu cT)-1)}{\sqrt{6}\sigma(\exp(\mu cT)-1)}$ *and* $J(x) = c\exp(\mu cT)\ln x - c\ln S_0 - \exp(\mu cT) + 1$.

Proof. By using the definition of expected value of fuzzy variable and (3.4), in a similar deduction process in the proof of Theorem 4.8, we obtain

$$\begin{aligned} f_{aonp} &= \exp(-rT)E[S_T I_{\{S_T<K\}}] \\ &= \exp(-rT)\int_0^{+\infty} \mathrm{Cr}\{S_T I_{\{S_T<K\}} \geq x\}\mathrm{d}x \\ &= \exp(-rT)\int_0^{+\infty} \mathrm{Cr}\{(S_T \geq x)\bigcap(S_T<K)\}\mathrm{d}x \\ &= \exp(-rT)\int_0^{K} \mathrm{Cr}\{x \leq S_T < K\}\mathrm{d}x \\ &= \exp(-rT)\int_0^{K} \mathrm{Cr}\Big\{\frac{J(x)}{c\sigma} \leq \int_0^T \exp(\mu c s)\mathrm{d}C_s < \frac{J(K)}{c\sigma}\Big\}\mathrm{d}x, \end{aligned}$$

where $J(x) = c\exp(\mu cT)\ln x - c\ln S_0 - \exp(\mu cT) + 1$.

If $J(K) \leq 0$, then for $\forall x \in (0, K)$, we have $J(x) < 0$. Thus,

$$f_{aonp} = \exp(-rT)\int_0^K \Big(1+\exp(\frac{-\pi\mu J(K)}{\sqrt{6}\sigma(\exp(\mu cT)-1)})\Big)^{-1}\mathrm{d}x = \frac{\exp(-rT)K^{A+1}}{\exp(B)+K^A}.$$

Otherwise, there must exist a number x_0 so that $J(x_0) = 0$. Then $J(x)$ is always negative in $(0, x_0)$ and positive in (x_0, K). In consequence,

$$\begin{aligned} f_{aonp} &= \exp(-rT)\int_0^{x_0} \mathrm{Cr}\Big\{\frac{J(x)}{c\sigma} \leq \int_0^T \exp(\mu cs)\mathrm{d}C_s < \frac{J(K)}{c\sigma}\Big\}\mathrm{d}x \\ &\quad + \exp(-rT)\int_{x_0}^{K} \mathrm{Cr}\Big\{\frac{J(x)}{c\sigma} \leq \int_{x_0}^T \exp(\mu cs)\mathrm{d}C_s < \frac{J(K)}{c\sigma}\Big\}\mathrm{d}x \\ &= \exp(-rT)\int_0^{x_0} \Big(1-(1+\exp(\frac{\pi\mu\min\{-J(x),J(K)\}}{\sqrt{6}(\exp(\mu cT)-1)}))^{-1}\Big)\mathrm{d}x \\ &\quad + \exp(-rT)\int_{x_0}^{K} \Big(1+\exp(\frac{\pi\mu J(x)}{\sqrt{6}\sigma(\exp(\mu cT)-1)})\Big)^{-1}\mathrm{d}x \\ &= f_{p1} + f_{p2}. \end{aligned}$$

It follows from $-J(x) = J(K)$ that $x = \frac{\exp(\frac{2B}{A})}{K}$. Therefore, when $x > \frac{\exp(\frac{2B}{A})}{K}$, $\min\{-J(x), J(K)\} = -J(x)$; when $x \leq \frac{\exp(\frac{2B}{A})}{K}$, $\min\{-J(x), J(K)\} = J(K)$. By dividing f_{p1} into two parts, we get

$$\begin{aligned} f_{p1} &= \exp(-rT)\int_0^{\frac{\exp(\frac{2B}{A})}{K}} \Big(1-(1+\exp(\frac{\pi\mu J(K)}{\sqrt{6}(\exp(\mu cT)-1)}))^{-1}\Big)\mathrm{d}x \\ &\quad + \exp(-rT)\int_{\frac{\exp(\frac{2B}{A})}{K}}^{x_0} \Big(1-(1+\exp(\frac{-\pi\mu J(x)}{\sqrt{6}(\exp(\mu cT)-1)}))^{-1}\Big)\mathrm{d}x \\ &= \frac{K^{A-1}\exp(\frac{2B}{A}-rT)}{K^A+\exp(B)} + \int_{\frac{\exp(\frac{2B}{A})}{K}}^{x_0} \frac{\exp(B)}{x^A+\exp(B)}\mathrm{d}x, \end{aligned}$$

and

$$f_{p2} = \exp(-rT)\int_{x_0}^{K} \Big(1+\exp(\frac{\pi\mu J(x)}{\sqrt{6}\sigma(\exp(\mu cT)-1)})\Big)^{-1}\mathrm{d}x = \exp(-rT)\int_{x_0}^{K} \frac{\exp(B)}{x^A+\exp(B)}\mathrm{d}x.$$

Thus,

$$f_{aonp} = f_{p1} + f_{p2} = \frac{K^{A-1}\exp(\frac{2B}{A}-rT)}{K^A+\exp(B)} + \int_{\frac{\exp(\frac{2B}{A})}{K}}^{K} \frac{\exp(B-rT)}{x^A+\exp(B)}\mathrm{d}x.$$

The theorem is verified. □

Example 4.12. *Suppose that the strike price of an AONP option is $K = 18$, and the duration of the contract is three months ($T = 0.25$). The underlying stock is currently being sold at a price of $S_0 = 20$, the risk-less interest rate r is 5% per annum. Other parameters of stock price are $c = 0.3$, $\mu = 0.2$, $\sigma = 0.2$.*

In this example, $J(K) = -0.0336 < 0$. According to Theorem 4.11, we can get the AONP price is $f_c = 0.9694$.

Example 4.13. *Suppose that the strike price of an AONP option is $K = 19.5$, and the duration of the contract is three months ($T = 0.25$). The underlying stock is currently being sold at a price of $S_0 = 20$, the risk-less interest rate r is 5% per annum. Other parameters of stock price are $c = 0.6$, $\mu = 0.2$, $\sigma = 0.2$.*

In this case, $J(K) = 0.0086 > 0$. According to Theorem 4.11, we can get the AONP price is $f_c = 11.3122$.

Figure 6 shows the relationship between c and f_{aonp} with different K. It can be seen intuitively that the option price increases with the increase of K and c. In figure 6, a point is marked on each curve except the curve with $K = 18$. The marked point is the point such that $J(K) = 0$ when other parameters are determined. Hence, $J(K) < 0$ on the left and $J(K) > 0$ on the right of the marked point. Obviously, with the increase of K, the marked point continues to move to the right. On the whole, f_{aonp} is monotonically increasing with respect to c. Locally, when $J(K) < 0$ (the part of the curve on the left of the marked point), the growth rate of option price f_{aonp} increases with the increase of c. When $J(K) > 0$ (the part of the curve on the right of the marked point), the growth rate of option price decreases gradually. Figure 7 shows the relationship between μ and f_{aonp} with different T. We can get from the figure that f_{aonp} is negatively correlated with μ and positively correlated with T.

It is easy to find that no matter what kind of binary option, its option price is positively correlated with the expiration time, which shows that the longer the time, the greater the stock movement potential, but the greater the risk faced by investors. Therefore, the option buyer should reasonably choose the option contract according to his own risk bearing ability when purchasing the option.

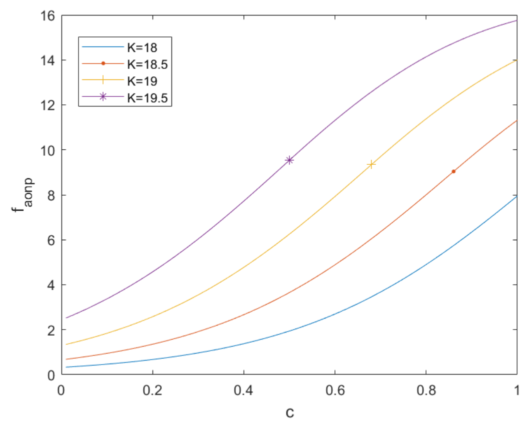


Figure 6: The relationship between c and f_{aonp} with different K

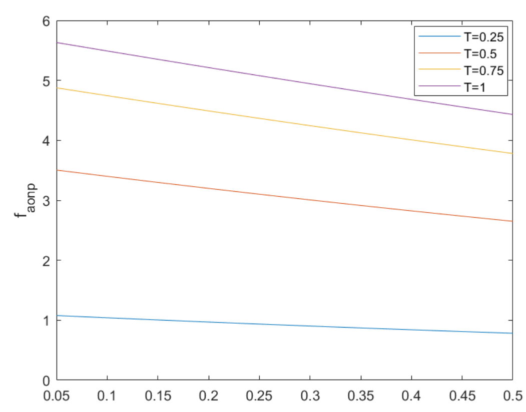


Figure 7: The relationship between μ and f_{aonp} with different T

5 Conclusions

In this paper, a stock model with nonlinear mean reversion economic behavior was introduced, that is, exponential Ornstein-Uhlenbeck model. This model is a kind of process to ensure that the stock price is nonnegative. It ensures that the stock price will not fluctuate greatly in the short term, and avoids the restriction of one-way change of stock price with time in the traditional lognormal distribution. Then, we obtained the pricing formulas of cash-or-nothing call option, cash-or-nothing put option, asset-or-nothing call option and asset-or-nothing put option. At last, we gave some numerical examples to analyze the impact of the changes of various parameters on the option price. Therefore, the longer the expiration date, the greater the potential of stock price changes, which also indicates that investors will take greater risks.

In the following research, we will conduct pricing research on more complex option contract types, such as barrier look-back options. In addition, the volatility in the stock model is constant, and we will further improve this model to get a more realistic stock model.

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References

[1] F. Black, M. Scholes, *The pricing of option and corporate liabilities*, Journal of Political Economy, **81** (1973), 637-654.

[2] A. Carlos, V. Mejia, *Calibration of the expomemtial ornstein-uhlenbeck process when spot prices are visible through the maximum log-likelihood method. Example with gold prices*, Advances in Difference Equations, **2018** (2018), 269.

[3] X. Chen, Z. Qin, *A new existence and unqueness theorem for fuzzy differential equation*, International Journal of Fuzzy Systems, **13**(2) (2011), 148-151.

[4] Y. Cheng, C. You, *Convergence of numerical methods for fuzzy differential equations*, Journal of Intelligent and Fuzzy Systems, **38**(4) (2020), 5257-5266.

[5] L. Dai, Z. Fu, Z. Huang, *Option pricing formulas for uncertain financial market based on the exponential Ornstein-Uhlenbeck model*, Journal of Intelligent Manufacturing, **28**(3) (2017), 597-604.

[6] J. Gao, *Credibilistic option pricing: A new model*, Journal of Uncertain Systems, **2**(4) (2008), 243-247.

[7] L. Ji, C. You, *Milstein method for solving fuzzy differential equation*, Iranian Journal of Fuzzy Systems, **18**(3) (2021), 129-141.

[8] L. Kang, S. Chen, *Binary option pricing with jump based on mixed fractional Brownian motion*, Advances in Applied Mathematics, **8**(5) (2019), 883-891.

[9] W. Li, J. Li, K. Qiao, *The martingale pricing for convertible bond under stochastic interest and exponential Ornstein-Uhlenbeck process model*, Journal of Quantitative Economics, **28**(4) (2011), 71-74.

[10] B. Liu, *Uncertainty theory: An introduction to its axiomatic foundations*, Berlin, Springer-Verlag, 2004.

[11] B. Liu, *Uncertainty theory*, 2nd ed., Berlin, Springer-Verlag, 2007.

[12] B. Liu, *Fuzzy process, hybrid process and uncertain process*, Journal of Uncertain Systems, **2**(1) (2008), 3-16.

[13] B. Liu, Y. K. Liu, *Expected value of fuzzy variable and fuzzy expected value models*, IEEE Transactions on Fuzzy Systems, **10**(4) (2002), 445-450.

[14] R. Merton, *The theory of rational option pricing*, Bell Journal of Economics and Management Science, **4**(1) (1973), 141-183.

[15] J. Peng, *A general stock model for fuzzy markets*, Journal of Uncertain Systems, **2**(4) (2008), 248-254.

[16] Z. Qin, X. Gao, *Fracrional Liu process with application to finance*, Mathematical and Computer Modelling, **50**(9-10) (2009), 1538-1543.

[17] X. Qin, X. Lin, Q. Shang, *Fuzzy pricing of binary option based on the long memory property of financial markets*, Journal of Intelligent and Fuzzy Systems, **38** (2020), 4889-4900.

[18] E. Schwartz, *The stochastic behavior of commodity prices: Implications for valuation and hedging*, Journal of Finance, **52**(3) (1997), 923-973.

[19] A. Thavaneswaran, S. Appadoo, J. Frank, *Binary option pricing using fuzzy numbers*, Applied Mathematics Letters, **26** (2013), 65-72.

[20] H. Wang, *Promotion of the pricing formula for binary option*, Journal of Sanming University, **33**(4) (2016), 1-5.

[21] Y. Wu, J. He, *The pricing model of binary option and its solution*, Journal of Industrial Engineering, **16**(4) (2002), 108-110.

[22] C. You, L. Bo, *Option pricing formulas for generalized fuzzy stock model*, Journal of Industrial and Management Optimization, **16**(1) (2020), 387-396.

[23] C. You, L. Bo, *Option pricing based on a type of fuzzy process*, Journal of Ambient Intelligence and Humanized Computing, **13**(8) (2022), 3771-3785.

[24] C. You, Y. Hao, *Fuzzy Euler approximation and its local convergence*, Journal of Computational and Applied Mathematics, **343** (2018), 55-61.

[25] C. You, Y. Hao, *Numerical solution of fuzzy differential equation based on Taylor expansion*, Journal of Hebei University (Nature Science), **38**(2) (2018), 113-118.

[26] C. You, W. Wang, H. Huo, *Existence and uniqueness theorems for fuzzy differential equations*, Journal of Uncertain Systems, **7**(4) (2013), 303-315.

[27] L. Zadeh, *Fuzzy sets*, Information and Control, **8**(3) (1965), 338-353.

[28] L. Zadeh, *Fuzzy sets as a basis for a theory of possibility*, Fuzzy Sets and Systems, **1** (1978), 3-28.

[29] Y. Zhang, R. Wang, B. Song, *A study on pricing of Kanzhangbao income documeny product based on binary options*, Advances in Applied Mathematics, **7**(7) (2018), 934-946.

[30] Y. Zhang, C. You, *Option pricing formula for a new stock model*, Journal of Applied Mathematics Progress, **7**(10) (2018), 1225-1232.

[31] P. Zhao, *Pricing of power European options based on the exponential Ornstein-Uhlenback process*, Journal of Guizhou Normal University (Natural Sciences), **32**(1) (2014), 44-47.