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Original paper

An addendum to "Construction of 2-uninorms on bounded lattices"

Y. M. Wang[1]

[1] *School of Mathematics, Shandong University, 250100 Jinan, PR China*

623073044@qq.com

Abstract

In this work, we supplement two methods to construct a 2-uninorm on a bounded lattice using two uninorms, and illustrate that the method proposed by Ertuğrul is a special case of the new ones. Furthermore, the relationships between 2-uninorms on bounded lattices obtained by all existing methods are showed at the end of this work.

Keywords: Uninorms, 2-uninorms, bounded lattices.

1 Introduction

In 2017, Ertuğrul [2] introduced the concept of 2-uninorms on bounded lattices, and provided one method for constructing 2-uninorms on bounded lattices. In that method, the uninorm U_1 on $[0_L, k]$ is required to be disjunctive and U_2 on $[k, 1_L]$ is required to be conjunctive. In 2022, Xie and Yi [5] proposed two ways to obtain 2-uninorms on bounded lattices, and proved that the 2-uninorm constructed by the first (second) way is the weakest (strongest) one among all 2-uninorms on bounded lattices. In addition, they showed that the structures of 2-uninorms constructed by their methods differed from that proposed by Ertuğrul even though U_1 on $[0_L, k]$ is required to be disjunctive and U_2 on $[k, 1_L]$ is required to be conjunctive. In this work, we will supplement two new methods for constructing 2-uninorms on bounded lattices, and illustrate that the method in [2] is a special case of the new ones. Furthermore, we will show the relationships between 2-uninorms on bounded lattices obtained by all existing methods.

In Section 2, we recall some notions, concepts and theorems that will be used in the following sections. In Section 3, we present two new methods to construct a 2-uninorm on bounded lattices using two uninorms, and show the relationship between all known 2-uninorms on L. Section 4 is the conclusion and future work.

2 Preliminaries

In this section, we recall some notions, concepts and theorems that will be used in Section 3.

Definition 2.1. [1] *If a lattice $(L, \leq)$ has the top element (written as 1_L) and bottom element (written as 0_L), then we call it a bounded lattice (written as $(L, \leq, 0_L, 1_L)$).*

For convenience, we use L to denote a bounded lattice instead of $(L, \leq, 0_L, 1_L)$. The notation $m \parallel n$ denotes that m is *incomparable* with n for $m, n \in L$, that is, $m \nleq n$ and $m \ngeq n$. The notation $m \nparallel n$ denotes that m is *comparable* with n, that is, $m \leq n$ or $n \leq m$. The notation I_m is defined as $I_m = \{x \in L \mid x \parallel m\}$. In addition, We define $[m, n] = \{x \in L \mid m \leq x \leq n\}$, $(m, n] = \{x \in L \mid m < x \leq n\}$, $[m, n) = \{x \in L \mid m \leq x < n\}$ and $(m, n) = \{x \in L \mid m < x < n\}$ as *subintervals* of L.

Definition 2.2. [4] *If an operation $U : L^2 \to L$ satisfies commutativity, associativity, monotonicity and has a neutral element $e \in L$, i.e., $U(e, x) = x$ for all $x \in L$, then we call U a uninorm on L. If $U(0_L, 1_L) = 0_L$ we call U a conjunctive uninorm. If $U(0_L, 1_L) = 1_L$ we call U a disjunctive uninorm.*

Corresponding Author: Y. M. Wang

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Definition 2.3. [2] *If an operation $F : L^2 \to L$ satisfies commutativity, associativity, monotonicity and there exist $e, f \in L$ and $k \in L \backslash \{0_L, 1_L\}$ such that $0_L \leq e \leq k \leq f \leq 1_L$, $F(x, e) = x$ for all $x \in [0_L, k]$ and $F(x, f) = x$ for all $x \in [k, 1_L]$, then we call F a 2-uninorm on L.*

For any 2-uninorm F on L, it is obvious that $F|_{[0_L,k]} : [0_L, k]^2 \to [0_L, k]$ and $F|_{[k,1_L]} : [k, 1_L]^2 \to [k, 1_L]$ are uninorms.

Definition 2.4. [3] *Let $H : L^2 \to L$ and $K : L^2 \to L$ be two operations. A partial order $H \leq K$ is defined as $H(x, y) \leq K(x, y)$ for all $x, y \in L$.*

Theorem 2.5. [2] *Let $k \in L \backslash \{0_L, 1_L\}$, $U_1 : [0_L, k]^2 \to [0_L, k]$ be a disjunctive uninorm with neutral element e and $U_2 : [k, 1_L]^2 \to [k, 1_L]$ be a conjunctive uninorm with neutral element f. Then the function $F : L^2 \to L$ given by Eq. (1) is a 2-uninorm on L.*

$$F(x, y) = \begin{cases} U_1(x, y) & \text{if } (x, y) \in [0_L, k]^2, \\ U_2(x, y) & \text{if } (x, y) \in [k, 1_L]^2, \\ k & \text{otherwise.} \end{cases} \tag{1}$$

Theorem 2.6. [5] *Let $k \in L \backslash \{0_L, 1_L\}$, $U_1 : [0_L, k]^2 \to [0_L, k]$ be a uninorm with neutral element e and $U_2 : [k, 1_L]^2 \to [k, 1_L]$ be a uninorm with neutral element f. Then the function $F_W : L^2 \to L$ given by Eq. (2) is a 2-uninorm on L if and only if U_2 is conjunctive.*

$$F_W(x, y) = \begin{cases} U_1(x, y) & \text{if } (x, y) \in [0_L, k]^2, \\ U_2(x, y) & \text{if } (x, y) \in [k, 1_L]^2, \\ U_1(x \wedge k, y \wedge k) & \text{otherwise.} \end{cases} \tag{2}$$

Theorem 2.7. [5] *Let $k \in L \backslash \{0_L, 1_L\}$, $U_1 : [0_L, k]^2 \to [0_L, k]$ be a uninorm with neutral element e and $U_2 : [k, 1_L]^2 \to [k, 1_L]$ be a uninorm with neutral element f. Then the function $F_S : L^2 \to L$ given by Eq. (3) is a 2-uninorm on L if and only if U_1 is disjunctive.*

$$F_S(x, y) = \begin{cases} U_1(x, y) & \text{if } (x, y) \in [0_L, k]^2, \\ U_2(x, y) & \text{if } (x, y) \in [k, 1_L]^2, \\ U_2(x \vee k, y \vee k) & \text{otherwise.} \end{cases} \tag{3}$$

Xie and Yi have proved that F_W in Theorem 2.6 and F_S in Theorem 2.7 are the weakest and strongest 2-uninorms on L respectively.

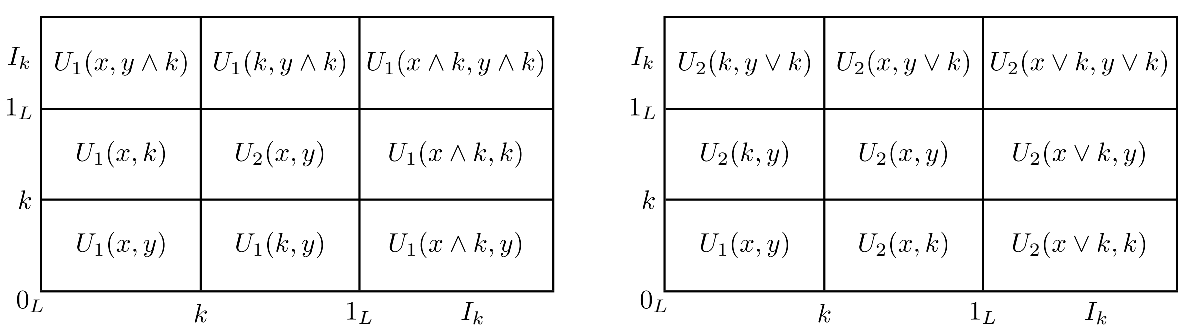

Figure 1: F_W (left) and F_S (right) on L

3 Two methods for constructing 2-uninorms on bounded lattices

In this section, we will present two approaches to obtain a 2-uninorm on a bounded lattice by extending uninorms U_1 and U_2. The first extension, denoted F_{U_1}, is a 2-uninorm on L if and only if U_2 is a conjunctive uninorm. The second extension, denoted by F_{U_2}, is a 2-uninorm on L if and only if U_1 is a disjunctive uninorm.

Theorem 3.1. *Let $k \in L \backslash \{0_L, 1_L\}$, U_1 be a uninorm on $[0_L, k]$ with neutral element e and U_2 be a uninorm on $[k, 1_L]$ with neutral element f. If the operation $F_{U_1} : L^2 \to L$ is given by Eq. (4), then F_{U_1} is a 2-uninorm on L if and only if U_2 is conjunctive.*

$$F_{U_1}(x,y) = \begin{cases} U_1(x,y) & \text{if } (x,y) \in [0_L,k]^2, \\ U_2(x,y) & \text{if } (x,y) \in [k,1_L]^2, \\ U_1(x,k) & \text{if } (x,y) \in [0_L,k) \times \{L\backslash[0_L,k]\}, \\ U_1(k,y) & \text{if } (x,y) \in \{L\backslash[0_L,k]\} \times [0_L,k), \\ k & \text{otherwise.} \end{cases} \tag{4}$$

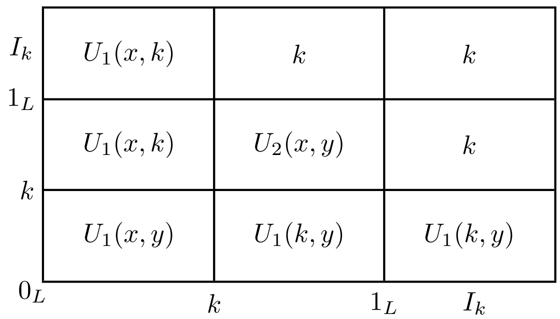

Figure 2: F_{U_1} on L

Proof. Necessity. By the associativity of 2-uninorm F_{U_1}, we have $U_2(k, 1_L) = F_{U_1}(k, 1_L) = F_{U_1}(F_{U_1}(e, 1_L), 1_L) = F_{U_1}(e, F_{U_1}(1_L, 1_L)) = F_{U_1}(e, 1_L) = U_1(e, k) = k$. That is, U_2 is a conjunctive uninorm on $[k, 1_L]$.

Sufficiency. It is obvious that F_{U_1} satisfies the commutativity, $F_{U_1}(x, e) = U_1(x, e) = x$ for any $x \in [0_L, k]$ and $F_{U_1}(x, f) = U_2(x, f) = x$ for any $x \in [k, 1_L]$. We can easily obtain that F_{U_1} satisfies the monotonicity from the fact that $U_2(k, y) = k = U_1(k, k)$ for any $y \in [k, 1_L]$. Now let us verify that F_{U_1} satisfies the associativity, that is, $F_{U_1}(x, F_{U_1}(y, z)) = F_{U_1}(F_{U_1}(x, y), z)$ for all $x, y, z \in L$.

1. For the first case $x \in [0_L, k]$.

1.1. When $y \in [0_L, k]$.

1.1.1. If $z \in [0_L, k]$, then $F_{U_1}(x, F_{U_1}(y, z)) = F_{U_1}(x, U_1(y, z)) = U_1(x, U_1(y, z)) = U_1(U_1(x, y), z) = F_{U_1}(U_1(x, y), z) = F_{U_1}(F_{U_1}(x, y), z)$.

1.1.2. If $z \in [k, 1_L] \cup I_k$, then $F_{U_1}(x, F_{U_1}(y, z)) = F_{U_1}(x, U_1(y, k)) = U_1(x, U_1(y, k)) = U_1(U_1(x, y), k) = F_{U_1}(U_1(x, y), z) = F_{U_1}(F_{U_1}(x, y), z)$.

1.2. When $y \in [k, 1_L]$.

1.2.1. If $z \in [0_L, k]$, then $F_{U_1}(x, F_{U_1}(y, z)) = F_{U_1}(x, U_1(k, z)) = U_1(x, U_1(k, z)) = U_1(U_1(x, k), z) = F_{U_1}(U_1(x, k), z) = F_{U_1}(F_{U_1}(x, y), z)$.

1.2.2. If $z \in [k, 1_L]$, then $F_{U_1}(x, F_{U_1}(y, z)) = F_{U_1}(x, U_2(y, z)) = U_1(x, k) = U_1(x, U_1(k, k)) = U_1(U_1(x, k), k) = F_{U_1}(U_1(x, k), z) = F_{U_1}(F_{U_1}(x, y), z)$.

1.2.3. If $z \in I_k$, then $F_{U_1}(x, F_{U_1}(y, z)) = F_{U_1}(x, k) = U_1(x, k) = U_1(x, U_1(k, k)) = U_1(U_1(x, k), k) = F_{U_1}(U_1(x, k), z) = F_{U_1}(F_{U_1}(x, y), z)$.

1.3. When $y \in I_k$.

1.3.1. If $z \in [0_L, k]$, then $F_{U_1}(x, F_{U_1}(y, z)) = F_{U_1}(x, U_1(k, z)) = U_1(x, U_1(k, z)) = U_1(U_1(x, k), z) = F_{U_1}(U_1(x, k), z) = F_{U_1}(F_{U_1}(x, y), z)$.

1.3.2. If $z \in [k, 1_L] \cup I_k$, then $F_{U_1}(x, F_{U_1}(y, z)) = F_{U_1}(x, k) = U_1(x, k) = U_1(x, U_1(k, k)) = U_1(U_1(x, k), k) = F_{U_1}(U_1(x, k), z) = F_{U_1}(F_{U_1}(x, y), z)$.

2. For the second case $x \in [k, 1_L]$.

2.1. When $y \in [0_L, k]$.

2.1.1. If $z \in [0_L, k]$, then $F_{U_1}(x, F_{U_1}(y, z)) = F_{U_1}(x, U_1(y, z)) = U_1(k, U_1(y, z)) = U_1(U_1(k, y), z) = F_{U_1}(U_1(k, y), z) = F_{U_1}(F_{U_1}(x, y), z)$.

2.1.2. If $z \in [k, 1_L] \cup I_k$, then $F_{U_1}(x, F_{U_1}(y, z)) = F_{U_1}(x, U_1(y, k)) = U_1(k, U_1(y, k)) = U_1(U_1(k, y), k) = F_{U_1}(U_1(k, y), z) = F_{U_1}(F_{U_1}(x, y), z)$.

2.2. When $y \in [k, 1_L]$.

2.2.1. If $z \in [0_L, k]$, then $F_{U_1}(x, F_{U_1}(y, z)) = F_{U_1}(x, U_1(k, z)) = U_1(k, U_1(k, z)) = U_1(U_1(k, k), z) = U_1(k, z) = F_{U_1}(U_2(x, y), z) = F_{U_1}(F_{U_1}(x, y), z)$.

2.2.2. If $z \in [k, 1_L]$, then $F_{U_1}(x, F_{U_1}(y,z)) = F_{U_1}(x, U_2(y,z)) = U_2(x, U_2(y,z)) = U_2(U_2(x,y),z) = F_{U_1}(U_2(x,y), z) = F_{U_1}(F_{U_1}(x,y),z)$.

2.2.3. If $z \in I_k$, then $F_{U_1}(x, F_{U_1}(y,z)) = F_{U_1}(x,k) = U_2(x,k) = k = F_1(U_2(x,y),z) = F_{U_1}(F_{U_1}(x,y),z)$.

2.3. When $y \in I_k$.

2.3.1. If $z \in [0_L, k]$, then $F_{U_1}(x, F_{U_1}(y,z)) = F_{U_1}(x, U_1(k,z)) = U_1(k, U_1(k,z)) = U_1(k,z) = F_{U_1}(k,z) = F_{U_1}(F_{U_1}(x,y),z)$.

2.3.2. If $z \in [k, 1_L] \cup I_k$, then $F_{U_1}(x, F_{U_1}(y,z)) = F_{U_1}(x,k) = U_2(x,k) = k = F_{U_1}(k,z) = F_{U_1}(F_{U_1}(x,y),z)$.

3. For the third case $x \in I_k$.

3.1. When $y \in [0_L, k]$.

3.1.1. If $z \in [0_L, k]$, then $F_{U_1}(x, F_{U_1}(y,z)) = F_{U_1}(x, U_1(y,z)) = U_1(k, U_1(y,z)) = U_1(U_1(k,y),z) = F_{U_1}(U_1(k,y), z) = F_{U_1}(F_{U_1}(x,y),z)$.

3.1.2. If $z \in [k, 1_L] \cup I_k$, then $F_{U_1}(x, F_{U_1}(y,z)) = F_{U_1}(x, U_1(y,k)) = U_1(k, U_1(y,k)) = U_1(U_1(k,y),k) = F_{U_1}(U_1(k,y),z) = F_{U_1}(F_{U_1}(x,y),z)$.

3.2. When $y \in [k, 1_L]$.

3.2.1. If $z \in [0_L, k]$, then $F_{U_1}(x, F_{U_1}(y,z)) = F_{U_1}(x, U_1(k,z)) = U_1(k, U_1(k,z)) = U_1(k,z) = F_{U_1}(k,z) = F_{U_1}(F_{U_1}(x,y),z)$.

3.2.2. If $z \in [k, 1_L]$, then $F_{U_1}(x, F_{U_1}(y,z)) = F_{U_1}(x, U_2(y,z)) = k = U_2(k,z) = F_{U_1}(k,z) = F_{U_1}(F_{U_1}(x,y),z)$.

3.2.3. If $z \in I_k$, then $F_{U_1}(x, F_{U_1}(y,z)) = F_{U_1}(x,k) = k = F_{U_1}(k,z) = F_{U_1}(F_{U_1}(x,y),z)$.

3.3. When $y \in I_k$.

3.3.1. If $z \in [0_L, k]$, then $F_{U_1}(x, F_{U_1}(y,z)) = F_{U_1}(x, U_1(k,z)) = U_1(k, U_1(k,z)) = U_1(k,z) = F_{U_1}(k,z) = F_{U_1}(F_{U_1}(x,y),z)$.

3.3.2. If $z \in [k, 1_L] \cup I_k$, then $F_{U_1}(x, F_{U_1}(y,z)) = F_{U_1}(x,k) = k = F_{U_1}(k,z) = F_{U_1}(F_{U_1}(x,y),z)$. □

In order for the readers to clearly distinguish the differences between the 2-uninorms F_W in Theorem 2.6 and F_{U_1} in Theorem 3.1, we present the following example.

Example 3.2. *Let $(L' = \{0_{L'}, a, e, k, f, m, s, 1_{L'}\}, \leq, 0_{L'}, 1_{L'})$ be a bounded lattice, and its lattice diagram is shown in Figure 3.*

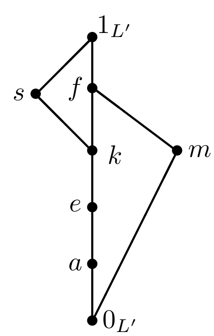

Figure 3: The lattice L'

Firstly, we define the uninorms U_1 on $[0_{L'}, k]$ and U_2 on $[k, 1_{L'}]$ as follows.

Table 1
U_1 on $[0_{L'}, k]$

U_1	$0_{L'}$	a	e	k
$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$
a	$0_{L'}$	a	a	a
e	$0_{L'}$	a	e	k
k	$0_{L'}$	a	k	k

Table 2
U_2 on $[k, 1_{L'}]$

U_2	k	f	s	$1_{L'}$
k	k	k	k	k
f	k	f	s	$1_{L'}$
s	k	s	s	$1_{L'}$
$1_{L'}$	k	$1_{L'}$	$1_{L'}$	$1_{L'}$

Then we can obtain the 2-uninorms F_W and F_{U_1} on L' by the construction methods from Theorems 2.6 and 3.1. The red fonts in Tables 3 and 4 are the differences between the 2-uninorms F_W and F_{U_1} on L'.

Table 3

F_W *on* L'

F_W	$0_{L'}$	a	e	k	f	s	$1_{L'}$	m
$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$
a	$0_{L'}$	a	a	a	a	a	a	$0_{L'}$
e	$0_{L'}$	a	e	k	k	k	k	$0_{L'}$
k	$0_{L'}$	a	k	k	k	k	k	$0_{L'}$
f	$0_{L'}$	a	k	k	f	s	$1_{L'}$	$0_{L'}$
s	$0_{L'}$	a	k	k	s	s	$1_{L'}$	$0_{L'}$
$1_{L'}$	$0_{L'}$	a	k	k	$1_{L'}$	$1_{L'}$	$1_{L'}$	$0_{L'}$
m	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$

Table 4

F_{U_1} *on* L'

F_{U_1}	$0_{L'}$	a	e	k	f	s	$1_{L'}$	m
$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$	$0_{L'}$
a	$0_{L'}$	a	a	a	a	a	a	a
e	$0_{L'}$	a	e	k	k	k	k	k
k	$0_{L'}$	a	k	k	k	k	k	k
f	$0_{L'}$	a	k	k	f	s	$1_{L'}$	k
s	$0_{L'}$	a	k	k	s	s	$1_{L'}$	k
$1_{L'}$	$0_{L'}$	a	k	k	$1_{L'}$	$1_{L'}$	$1_{L'}$	k
m	$0_{L'}$	a	k	k	k	k	k	k

Theorem 3.3. *Let $k \in L\backslash\{0_L, 1_L\}$, U_1 be a uninorm on $[0_L, k]$ with neutral element e and U_2 be a uninorm on $[k, 1_L]$ with neutral element f. If the operation $F_{U_2} : L^2 \to L$ is given by Eq. (5), then F_{U_2} is a 2-uninorm on L if and only if U_1 is disjunctive.*

$$F_{U_2}(x,y) = \begin{cases} U_1(x,y) & \text{if } (x,y) \in [0_L,k]^2, \\ U_2(x,y) & \text{if } (x,y) \in [k,1_L]^2, \\ U_2(x,k) & \text{if } (x,y) \in (k,1_L] \times \{L\backslash[k,1_L]\}, \\ U_2(k,y) & \text{if } (x,y) \in \{L\backslash[k,1_L]\} \times (k,1_L], \\ k & \text{otherwise.} \end{cases} \tag{5}$$

Proof. Necessity. By the associativity of 2-uninorm F_{U_2}, we have $U_1(0_L, k) = F_{U_2}(0_L, k) = F_{U_2}(0_L, U_2(k, f)) = F_{U_2}(0_L, F_{U_2}(0_L, f)) = F_{U_2}(F_{U_2}(0_L, 0_L), f) = F_{U_2}(0_L, f) = U_2(k, f) = k$. That is, U_1 is a disjunctive uninorm on $[0_L, k]$.

The proof of sufficiency is similar to that of Theorem 3.1. □

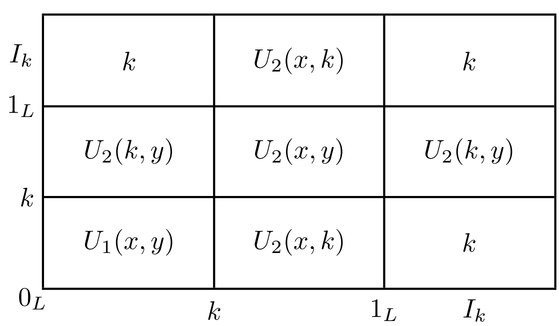

Figure 4: F_{U_2} on L

On one hand, if the uninorm U_1 on $[0_L, k]$ is disjunctive in Theorem 3.1, then we have $U_1(x, k) = k$ for any $x \in [0_L, k]$. Thus, $F_{U_1}(x, y) = k$ for all $(x, y) \in L^2\backslash\{[0_L, k]^2 \cup [k, 1_L]^2\}$. Conversely, if $F_{U_1}(x, y) = k$ for any $(x, y) \in L^2\backslash\{[0_L, k]^2 \cup [k, 1_L]^2\}$, then we can obtain that $U_1(0_L, k) = F_{U_1}(0_L, k) = F_{U_1}(0_L, F_{U_1}(0_L, f)) = F_{U_1}(F_{U_1}(0_L, 0_L), f) = F_{U_1}(0_L, f) = k$, that is, U_1 is disjunctive. On the other hand, if the uninorm U_2 on $[k, 1_L]$ is conjunctive in Theorem 3.3, then we have $U_2(x, k) = k$ for any $x \in [k, 1_L]$. Thus, $F_{U_2}(x, y) = k$ for all $(x, y) \in L^2\backslash\{[0_L, k]^2 \cup [k, 1_L]^2\}$. Conversely, if $F_{U_2}(x, y) = k$ for any $(x, y) \in L^2\backslash\{[0_L, k]^2 \cup [k, 1_L]^2\}$, then we can obtain that $U_2(k, 1_L) = F_{U_2}(k, 1_L) = F_{U_2}(F_{U_2}(e, 1_L), 1_L) = F_{U_2}(e, F_{U_2}(1_L, 1_L)) = F_{U_2}(e, 1_L) = k$, that is, U_2 is conjunctive. Therefore, we have the following corollary.

Corollary 3.4. *Let $k \in L\backslash\{0_L, 1_L\}$, U_1 be a uninorm on $[0_L, k]$ with neutral element e and U_2 be a uninorm on $[k, 1_L]$ with neutral element f. If the operation $F_k : L^2 \to L$ is given by Eq. (6), then F_k is a 2-uninorm on L if and only if U_1 is disjunctive and U_2 is conjunctive.*

$$F_k(x,y) = \begin{cases} U_1(x,y) & \text{if } (x,y) \in [0_L,k]^2, \\ U_2(x,y) & \text{if } (x,y) \in [k,1_L]^2, \\ k & \text{otherwise.} \end{cases} \tag{6}$$

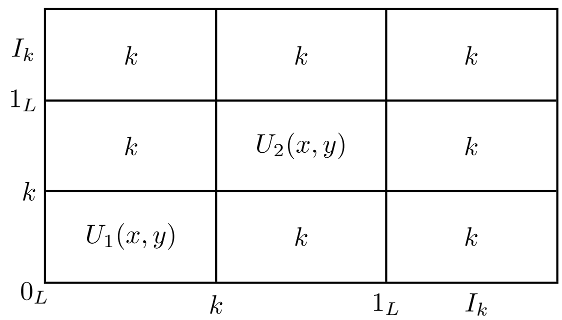

Figure 5: F_k on L

Remark 3.5. *Clearly, the structure of F_k in Corollary 3.4 is exactly the same as F in Theorem 2.5.*

We end the work with the relationships between 2-uninorms on L constructed by these methods.

Proposition 3.6. *Let $k \in L \backslash \{0_L, 1_L\}$.*

(i) *If U_1 is an arbitrary uninorm on $[0_L, k]$ and U_2 is a conjunctive uninorm on $[k, 1_L]$, then we have $F_{U_1} \geq F_W$.*

(ii) *If U_1 is a disjunctive uninorm on $[0_L, k]$ and U_2 is an arbitrary uninorm on $[k, 1_L]$, then we have $F_{U_2} \leq F_S$.*

(iii) *If U_1 is a disjunctive uninorm on $[0_L, k]$ and U_2 is a conjunctive uninorm on $[k, 1_L]$, then we have*

$$F_W \leq F_{U_1} = F_k = F = F_{U_2} \leq F_S.$$

Proof. The proofs can be easily obtained from Theorems 2.6, 2.7, 3.1, 3.3 and Corollary 3.4. □

4 Conclusion and future work

In this work, we provide two ways to construct a 2-uninorm on L using two uninorms. The structure of 2-uninorm constructed by the method in [2] is a special case of those constructed by the new methods. We also discuss the relationship between 2-uninorms on L constructed by all existing methods in Section 3.

In a future paper, we will focus on the methods for obtaining a 2-uninorm on bounded lattices by using a uni-nullnorm and a t-conorm (a t-norm and a null-uninorm).

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