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Original paper

On the modularity equation for overlap (grouping) functions and semi-t-operators

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Abstract

Modularity equation plays an important role in the field of information fusion, which helps to select aggregation functions and reduce the failure rate in the process of aggregation. Based on this, we focus on the study of modularity equation between overlap (grouping) functions and semi-t-operators. We give the solutions of this equation between overlap (grouping) functions and semi-t-operators with continuous pseudo-t-norm or continuous pseudo-t-conorm in the corresponding cases.

Keywords: Modularity equation, overlap function, grouping function, semi-t-operator.

1 Introduction

In the process of information aggregation, some information from different times, places or environments usually needs to be aggregated and outputs as a representative information summary, which reflect human thinking. Aggregation has important applications in many fields: applied mathematics (decision theory, probability, statistics), computer sciences (operations research, artificial intelligence), as well as many other applied fields (see [1, 9, 10, 12]). The research on the aggregation problem shows that the selection of the commonly used aggregation function is not arbitrary, and should be based on the properties required in the applications. For example, in order to determine optimal decision-making method or improve the existing decision-making methods in decision theory, a large number of models have been proposed. Because many of these models cannot fully adapt to the complexity of the actual decision-making situations, the applicability and quality of "optimal" solution is questioned. Many scholars try to overcome these difficulties by developing modelling principles, so that they can formulate the actual model according to the actual problems to be solved. At this point, it is necessary to study the functions or operators used in human decision-making process. Generally, the error probability of information aggregation increases with the complexity of the decision. Researchers hope to reduce the failure rate, and make the modules of these programs in the aggregation process conform to the rational thinking process of human beings as much as possible. Furthermore, they hope to provide a complex decision support system for decision makers to produce reasonable and acceptable results (see [34]).

Aggregation is the process of combining several numerical values into a single representative value, and an aggregation function performs this operation (see [10]). In the above decision-making, a necessary condition for such an approach is the modelling of human evaluation categories and their aggregation. Assuming that these categories can be represented by fuzzy sets, the formal connectives of fuzzy set theory can be considered as models of aggregation (see [34]). Based on this, many aggregation functions (involving t-norms, t-conorms, uninorms, nullnorms, semi-t-operators, overlap and grouping functions, etc.) have been presented according to various application fields (see [4, 6, 13, 22, 25]). Because connectives meet different tautologies, we need to find the corresponding connectives according to applications, that is, solutions of equations. So researchers have paid attention to the modularity equation of aggregation operators in recent years: for all $x, y, z \in [0,1]$ with $z \leqslant x$, $F(x, G(y,z)) = G(F(x,y), z)$, where F, G are aggregation functions. After the

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solutions of modularity equation of related aggregation functions are given, people can provide reference for function selection and reduction of failure rate in the process of aggregation. From a mathematical point of view, the modularity equation can be regarded not only as a generalized associativity equation with some restrictions, but also as a special distributivity equation in the fuzzy set theory (see [20, 26, 27]).

In recent years, scholars have done a lot of research about modularity equation. Specifically, Mas et al. discussed the modularity equation of uninorms in $\mathcal{U}_{\min} \cup \mathcal{U}_{\max}$ and t-operators in 2002 (see [17]). Qin gave the solutions to the modularity equation for nullnorms and uninorms continuous in $(0,1)^2$ (see [20]). The modularity equation in the class of 2-uninorms were discussed by Rak in [21]. Fechner et al. considered the modularity equation for some classes of aggregation operators (see [7]). Su et al. solved the modularity of uninorms with more general structures (see [23]). Zhan et al. studied the modularity equation with semi-t-operators and semi-uninorms (see [26, 27]). Zhao et al. investigated the modularity equation for Mayor's aggregation operators and semi-t-operators (see [29, 30, 31]). In addition, overlap and grouping functions were introduced in [2, 3] as non-associative aggregation functions for the information aggregation process and applied in the classification problems (see [14, 15]), image processing (see [11]) and so forth. There are a lot of works about modularity equation for overlap and grouping functions (see [24, 28, 32, 33]). Along this research route, we continue to study the modularity equation for overlap (grouping) functions and semi-t-operators.

The rest of this paper is organized as follows. In Section 2, some concepts and results about overlap functions, grouping functions and semi-t-operators are recalled. In Section 3, we discuss the modularity equation for overlap functions and semi-t-operators. In Section 4, we study the modularity equation for grouping functions and semi-t-operators. In Section 5, we summarize this paper.

2 Preliminaries

In this section, we recall some basic concepts and results about t-norms, t-conorms, semi-t-operators, overlap and grouping functions.

Definition 2.1. [12] *A t-norm is an increasing, commutative and associative binary operation $T : [0,1]^2 \longrightarrow [0,1]$ which has neutral element 1, viz. $T(x,1) = T(1,x) = x$ for all $x \in [0,1]$. A t-conorm is an increasing, commutative and associative binary operation $S : [0,1]^2 \longrightarrow [0,1]$ which has neutral element 0, viz. $S(x,0) = S(0,x) = x$ for all $x \in [0,1]$.*

There are many t-norms, such as $T_M(x,y) = \min(x,y)$, $T_P(x,y) = xy$ for any $x,y \in [0,1]$, and many t-conorms, such as $S_M(x,y) = \max(x,y)$, $S_P(x,y) = x+y-xy$ for any $x,y \in [0,1]$.

Definition 2.2. [8] *A pseudo-t-norm $T : [0,1]^2 \longrightarrow [0,1]$ is an associative, increasing binary operation with neutral element 1. Similarly, a pseudo-t-conorm $S : [0,1]^2 \longrightarrow [0,1]$ is an associative, increasing binary operation with neutral element 0.*

Definition 2.3. [5, 16] *An associative and increasing operation $F : [0,1]^2 \longrightarrow [0,1]$ is called semi-t-operator if $F(0,0) = 0, F(1,1) = 1$ and the functions F_0, F_1, F^0, F^1 are continuous, where $F_0(x) = F(0,x), F_1(x) = F(1,x), F^0(x) = F(x,0), F^1(x) = F(x,1)$.*

Let $\mathcal{F}_{a,b}$ denote the family of all semi-t-operators such that $F(0,1) = a$ and $F(1,0) = b$.

Theorem 2.4. [5, 16] *Let $F : [0,1]^2 \longrightarrow [0,1]$, $F(0,1) = a$, $F(1,0) = b$. The operation $F \in \mathcal{F}_{a,b}$ if and only if there exist a pseudo-t-norm T_F and a pseudo-t-conorm S_F such that*

$$F(x,y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}), & (x,y) \in [0,a]^2, \\ b + (1-b)T_F(\frac{x-b}{1-b}, \frac{y-b}{1-b}), & (x,y) \in [b,1]^2, \\ a, & (x,y) \in [0,a] \times [a,1], \\ b, & (x,y) \in [b,1] \times [0,b], \\ x, & \text{otherwise}, \end{cases}$$

for $a \leqslant b$ and

$$F(x,y) = \begin{cases} bS_F(\frac{x}{b}, \frac{y}{b}), & (x,y) \in [0,b]^2, \\ a + (1-a)T_F(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & (x,y) \in [a,1]^2, \\ a, & (x,y) \in [0,a] \times [a,1], \\ b, & (x,y) \in [b,1] \times [0,b], \\ y, & \text{otherwise}, \end{cases}$$

for $b \leqslant a$.

If $a = b = 0$, then F is a pseudo-t-norm; if $a = b = 1$, then F is a pseudo-t-conorm.

Proposition 2.5. [16, 18] *Any continuous pseudo-t-norm T is commutative, and moreover, it is a continuous t-norm. Any continuous pseudo-t-conorm S is a continuous t-conorm.*

Definition 2.6. [2] *A binary function $O : [0,1]^2 \longrightarrow [0,1]$ is said to be an overlap function if it satisfies, for all $x, y \in [0,1]$, the following conditions:*

(O1) $O(x,y) = O(y,x)$;
(O2) $O(x,y) = 0$ *if and only if* $x = 0$ *or* $y = 0$;
(O3) $O(x,y) = 1$ *if and only if* $x = y = 1$;
(O4) O *is increasing in each variable;*
(O5) O *is continuous.*

A binary function $O : [0,1]^2 \longrightarrow [0,1]$ is called an 0-overlap function (see [19]) if O satisfies (O1), (O2)′, (O3), (O4), (O5), where

(O2)′ 0 is the annihilator of O.

A binary function $O : [0,1]^2 \longrightarrow [0,1]$ is called an 1-overlap function (see [19]) if O satisfies (O1), (O2), (O3)′, (O4), (O5), where

(O3)′ $x = y = 1$ implies $O(x,y) = 1$.

Definition 2.7. [3] *A binary function $G : [0,1]^2 \longrightarrow [0,1]$ is said to be a grouping function if it satisfies, for all $x, y \in [0,1]$, the following conditions:*

(G1) $G(x,y) = G(y,x)$;
(G2) $G(x,y) = 0$ *if and only if* $x = y = 0$;
(G3) $G(x,y) = 1$ *if and only if* $x = 1$ *or* $y = 1$;
(G4) G *is increasing in each variable;*
(G5) G *is continuous.*

Analogously, a binary function $G : [0,1]^2 \longrightarrow [0,1]$ is called a 0-grouping function if G satisfies (G1), (G2) ′, (G3), (G4), (G5), where

(G2)′ $x = y = 0$ implies $G(x,y) = 0$.

A binary function $G : [0,1]^2 \longrightarrow [0,1]$ is called a 1-grouping function if G satisfies (G1), (G2), (G3)′, (G4), (G5), where

(G3)′ 1 is the annihilator of G.

Definition 2.8. [17] *For two binary operations $F_1, F_2 : [0,1]^2 \longrightarrow [0,1]$, we say that F_1 is modular over F_2, if $F_1(x, F_2(y,z)) = F_2(F_1(x,y), z)$, for any $x, y, z \in [0,1]$ with $z \leqslant x$.*

3 Modularity equations between overlap functions and semi-t-operators

In this section, we discuss the modularity equation for an overlap function O and a semi-t-operator $F \in \mathcal{F}_{a,b}$ with a continuous pseudo-t-norm T_F in two parts: F over O and O over F. By Theorem 2.4, it is necessary to discuss the cases: $a \leqslant b$ and $b \leqslant a$.

3.1 The case of a semi-t-operator over an overlap function

Proposition 3.1. *Let O be an overlap function with neutral element 1 and $F \in \mathcal{F}_{a,b}$, $a \leqslant b$, be a semi-t-operator with a continuous pseudo-t-norm T_F. Then F is modular over O if and only if F is a t-norm and $O = F$.*

Proof. Let F be a t-norm and $O = F$. Then F is modular over O due to the associativity of a t-norm. Conversely, suppose that F is modular over O. Then

$$b = F(b,0) = F(b, O(0,0)) = O(F(b,0), 0) = 0,$$

and consequently, $a = 0$. So $F = T_F$ and it is a t-norm due to Proposition 2.5. Let $x, z \in [0,1]$. If $z \leqslant x$, then

$$F(x,z) = F(x, O(1,z)) = O(F(x,1), z) = O(x,z).$$

Considering the commutativity of T_F and O, we have that $O = F$. □

Remark 3.2. (i) *Note that the modularity equation for a semi-t-operator over an overlap function in Proposition 3.1 reflects the restricted associativity of a t-norm.*

(ii) *The neutral element of O plays a key role in obtaining the equivalent condition of that F is modular over O in Proposition 3.1. For example, consider the semi-t-operator $F = T_M$ and the overlap function $O(x,y) = \min(x^2, y^2)$ for all $x, y \in [0,1]$. It is obvious that O has no neutral element and F is modular over O, but $O \neq F$.*

Proposition 3.3. *Let O be an overlap function with neutral element 1 and $F \in \mathcal{F}_{a,b}$, $b \leqslant a$, be a semi-t-operator with a continuous pseudo-t-norm T_F. Then F is modular over O if and only if $b = 0$ and one of the following statements holds:*

(i) *If $a = 0$, then F is a continuous t-norm and $O = F$.*

(ii) *If $a = 1$, then $F(x,y) = y$ for all $x, y \in [0,1]$.*

(iiii) *If $0 < a < 1$, then there exists an overlap function O_1 with neutral element 1 such that*

$$O(x,y) = \begin{cases} aO_1(\frac{x}{a}, \frac{y}{a}), & x, y \in [0,a], \\ a + (1-a)T_F(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & x, y \in [a,1], \\ \min(x,y), & \text{otherwise}, \end{cases} \tag{1}$$

and

$$F(x,y) = \begin{cases} a + (1-a)T_F(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & x, y \in [a,1], \\ y, & y \in [0,a], \\ a, & x \in [0,a], y \in [a,1]. \end{cases} \tag{2}$$

Proof. Necessity. Similar to Proposition 3.1, we have that $b = 0$.

(i) If $a = 0$, then $F = T_F$ and it is a t-norm due to Proposition 2.5. Let $x, z \in [0,1]$ with $z \leqslant x$. Then

$$F(x,z) = F(x, O(1,z)) = O(F(x,1), z) = O(x,z).$$

Since O and T_F are commutative, we have that $O = F$.

(ii) If $a = 1$, then $F(x,y) = y$ for all $x, y \in [0,1]$ by Theorem 2.4.

(iii) If $0 < a < 1$, then F is given by (2). The structure of O is characterized by the following conclusions.

(a) $O(x,y) = \min(x,y)$ for any $(x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a]$. In fact, for any $y \in [0,a]$, then $F(y,1) = a$ and $O(1,y) = y$. Since F is modular over O, we have that

$$y = F(y,y) = F(y, O(1,y)) = O(F(y,1), y) = O(a,y).$$

It follows from $O(1,y) = y$ that $O(x,y) = y$ for all $x \in [a,1]$. From the commutativity of O, we can get that $O(x,y) = \min(x,y)$ for any $(x,y) \in [0,a] \times [a,1] \cup [a,1] \times [0,a]$.

(b) Define two binary functions $O_1, O_2 : [0,1]^2 \longrightarrow [0,1]$ as follows:

$$O_1(x,y) = \frac{O(ax, ay)}{a} \text{ and } O_2(x,y) = \frac{O(a + (1-a)x, a + (1-a)y) - a}{1-a},$$

for all $x, y \in [0,1]$. From (a), we have that $O(a,a) = a$, and thus, O_1 and O_2 are well defined. Then O_1 satisfies (O1), (O2), (O4), (O5) and $O_1(1,x) = x$ for all $x \in [0,1]$. If $x = y = 1$, then $O_1(1,1) = \frac{O(a,a)}{a} = 1$. If $O_1(x,y) = 1$, then $\frac{O(ax,ay)}{a} = 1$, and thus, $O(ax,ay) = a$. Assume that $x < 1$ or $y < 1$. Then $ax < a$ or $ay < a$. Thus, $O(ax,ay) \leqslant O(ax,1) = ax < a$ or $O(ax,ay) \leqslant O(1,ay) = ay < a$, which is a contradiction. So $x = y = 1$. Therefore, O_1 satisfies $(O3)$, that is, O_1 is an overlap function with neutral element 1. A simple calculation verifies that O_2 satisfies (O1), (O2)$'$, (O3), (O4), (O5) and $O_2(1,x) = x$ for all $x \in [0,1]$, that is, O_2 is an 0-overlap function with neutral element 1. Therefore, $O(x,y) = aO_1(\frac{x}{a}, \frac{y}{a})$ for all $x, y \in [0,a]$ and $O(x,y) = a + (1-a)O_2(\frac{x-a}{1-a}, \frac{y-a}{1-a})$ for all $x, y \in [a,1]$.

(c) $O_2 = T_F$. Let $x, z \in [0,1]$ with $z \leqslant x$. Denote $x^* := a + (1-a)x$, $z^* := a + (1-a)z$. Then $x^*, z^* \in [a,1]$ and $z^* \leqslant x^*$. Since F is modular over O, we have that

$$F(x^*, z^*) = F(x^*, O(1,z^*)) = O(F(x^*,1), z^*) = O(x^*, z^*),$$

and thus, $T_F(x,z) = O_2(x,z)$. It follows from the commutativity of O_2 and T_F that $O_2 = T_F$. Hence O is given by (1).

Sufficiency.

(i) According to the associativity of a t-norm, the modularity equation holds.

(ii) Let $x, y, z \in [0,1]$ with $z \leqslant x$. Then $F(x, O(y,z)) = O(y,z) = O(F(x,y), z)$.

(iii) It suffices to prove that $F(x, O(y,z)) = O(F(x,y), z)$ for any $x, y, z \in [0,1]$ with $z \leqslant x$. To do that, we need to consider the following cases.

(1) $y \in [0,a]$. Then $F(x,y) = y$, $O(y,z) \leqslant y \leqslant a$, and thus,

$$F(x, O(y,z)) = O(y,z) = O(F(x,y), z).$$

(2) $y \in (a,1]$. Then $F(x,y) \geqslant a$. If $z \in [0,a]$, then $O(y,z) = z \leqslant a$, and thus,

$$F(x, O(y,z)) = O(y,z) = z = O(F(x,y), z).$$

If $z \in (a,1]$, then $x \in (a,1]$, and thus, it follows from the associativity of T_F that

$$\begin{aligned} F(x, O(y,z)) &= a + (1-a)T_F(\tfrac{x-a}{1-a}, T_F(\tfrac{y-a}{1-a}, \tfrac{z-a}{1-a})) \\ &= a + (1-a)T_F(T_F(\tfrac{x-a}{1-a}, \tfrac{y-a}{1-a}), \tfrac{z-a}{1-a}) \\ &= O(F(x,y), z). \end{aligned}$$

Therefore, F is modular over O. □

Example 3.4. (i) *According to Proposition 3.3, let us consider* $T_F = T_M$, $O_1(x,y) = \min(x\sqrt{y}, y\sqrt{x})$ *for all* $x, y \in [0,1]$ *and* $a = \frac{1}{2}$. *Then*

$$O(x,y) = \begin{cases} \min(x\sqrt{2y}, y\sqrt{2x}), & x, y \in [0, \frac{1}{2}], \\ \min(x,y), & \text{otherwise.} \end{cases}$$

and

$$F(x,y) = \begin{cases} \min(x,y), & x, y \in [\frac{1}{2}, 1], \\ \frac{1}{2}, & x \in [0, \frac{1}{2}], y \in [\frac{1}{2}, 1], \\ y, & y \in [0, \frac{1}{2}]. \end{cases}$$

One immediately obtains that F is modular over O. In fact, this characterization has nothing to do with the specific structure of O_1.

(ii) *Consider the overlap function* $O(x,y) = \min(x^2, y^2)$ *for all* $x, y \in [0,1]$ *and the semi-t-operator F given by in* (i). *One can check that F is modular over O and O has no neutral element.*

3.2 The case of an overlap function over a semi-t-operator

Proposition 3.5. *Let* $F \in \mathcal{F}_{a,b}$, $a \leqslant b$, *be a semi-t-operator with a continuous pseudo-t-norm* T_F *and O be an overlap function. Then O is modular over F if and only if O and F are given by the following formulas:*

$$O(x,y) = \begin{cases} a + (b-a)O_1(\frac{x-a}{b-a}, \frac{y-a}{b-a}), & (x,y) \in [a,b]^2, \\ b + (1-b)T_F(\frac{x-b}{1-b}, \frac{y-b}{1-b}), & (x,y) \in [b,1]^2, \\ O(\min(x,y), 1), & (x,y) \in (a,b) \times [b,1] \cup [b,1] \times (a,b), \\ \min(x,y), & \text{otherwise,} \end{cases} \tag{3}$$

and

$$F(x,y) = \begin{cases} \max(x,y), & x, y \in [0,a], \\ b + (1-b)T_F(\frac{x-b}{1-b}, \frac{y-b}{1-b}), & x, y \in [b,1], \\ a, & x \in [0,a], y \in [a,1], \\ b, & x \in [b,1], y \in [0,b], \\ x, & \text{otherwise,} \end{cases} \tag{4}$$

where O_1 is a binary function satisfying (O1), (O2)′, (O3)′, (O4), (O5).

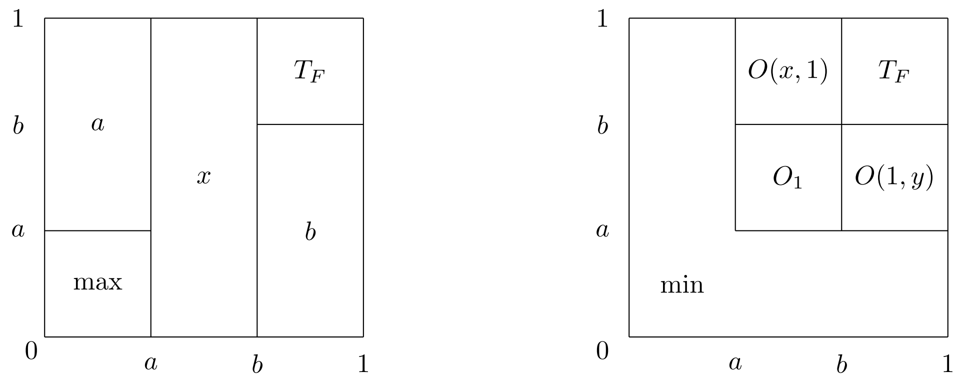

Fig. 1. Semi-t-operator (left) and overlap function (right) in Proposition 3.5

Proof. Necessity. Let O be modular over F. From Theorem 2.4, we know that $F(0,x)=x$ for all $x\in[0,a]$ and $F(1,x)=x$ for all $x\in[b,1]$. Then

$$O(1,x)=O(1,F(0,x))=F(O(1,0),x)=F(0,x)=x,$$

for all $x\in[0,a]$ and

$$O(1,x)=O(1,F(1,x))=F(O(1,1),x)=F(1,x)=x,$$

for all $x\in[b,1]$. Now we will characterize the structures of O and F through the following steps.

(a) $O(x,z)=\min(x,z)$ for all $(x,z)\in[0,a]\times[0,1]\cup[0,1]\times[0,a]$. Indeed, for any $x\in[0,1], z\in[0,a]$ with $z\leqslant x$, we have that

$$O(x,z)=O(x,F(0,z))=F(O(x,0),z)=F(0,z)=z.$$

From the commutativity of O, it follows that $O(x,z)=\min(x,z)$ for all $(x,z)\in[0,a]\times[0,1]\cup[0,1]\times[0,a]$.

(b) $O(x,y)=b+(1-b)T_F(\frac{x-b}{1-b},\frac{y-b}{1-b})$ for any $x,y\in[b,1]$. In fact, for any $x,y\in[b,1]$ with $y\leqslant x$, then $F(1,y)=y, O(x,1)=x$, and thus,

$$O(x,y)=O(x,F(1,y))=F(O(x,1),y)=F(x,y)=b+(1-b)T_F(\tfrac{x-b}{1-b},\tfrac{y-b}{1-b}).$$

By the commutativity of O and T_F, we have that $O(x,y)=b+(1-b)T_F(\frac{x-b}{1-b},\frac{y-b}{1-b})$ for any $x,y\in[b,1]$.

(c) $O(x,y)=O(\min(x,y),1)$ for all $(x,y)\in[a,b]\times[b,1]\cup[b,1]\times[a,b]$. Indeed, for any $x\in[a,b], y\in[b,1]$, $O(x,y)\geqslant O(a,b)=a$, $O(x,y)\leqslant O(b,1)=b$, and thus,

$$O(x,y)=F(O(x,y),0)=O(x,F(y,0))=O(x,b).$$

Since $y\in[b,1]$ is arbitrary, we have that $O(x,y)=O(x,b)=O(x,1)$, and thus, from the commutativity of O, we obtain that $O(x,y)=O(1,y)$ for all $(x,y)\in[b,1]\times[a,b]$. Therefore, $O(x,y)=O(\min(x,y),1)$ for all $(x,y)\in[a,b]\times[b,1]\cup[b,1]\times[a,b]$.

(d) Define a binary function $O_1:[0,1]^2\longrightarrow[0,1]$ as follows:

$$O_1(x,y)=\frac{O(a+(b-a)x,a+(b-a)y)-a}{b-a},$$

for all $x,y\in[0,1]$. By (a) and (b), we have that $O(a,a)=a$, $O(b,b)=b$, and thus, O_1 is well defined. One can check that O_1 satisfies $(O1)$, $(O2)'$, $(O3)'$, $(O4)$, $(O5)$. By (a)-(d), we have that O is given by (3).

(e) $S_F=S_M$. Let $x\in[0,a]$. Then

$$x=O(x,a)=O(x,F(a,x))=F(O(x,a),x)=F(x,x)=aS_F(\tfrac{x}{a},\tfrac{x}{a}),$$

and thus, $S_F(\frac{x}{a},\frac{x}{a})=\frac{x}{a}$, that is, $S_F(x,x)=x$ for all $x\in[0,1]$. For any $x,y\in[0,1]$, if $x\leqslant y$, then $y\leqslant S_F(x,y)\leqslant S_F(y,y)=y$, that is, $S_F(x,y)=y$; if $x>y$, then $x\leqslant S_F(x,y)\leqslant S_F(x,x)=x$, that is, $S_F(x,y)=x$. Therefore, $S_F=S_M$, that is, F is given by (4).

Sufficiency. Let $x,y,z\in[0,1]$ with $z\leqslant x$. It suffices to prove that $O(x,F(y,z))=F(O(x,y),z)$ for each case.

(1) $x\in[0,a]$. Then $z\in[0,a]$. If $y\in[0,a]$, then

$$O(x,F(y,z))=\min(x,\max(y,z))=\max(\min(x,y),z)=F(O(x,y),z).$$

If $y\in(a,b)$, then $F(y,z)=y$, and

$$O(x,F(y,z))=O(x,y)=x=F(x,z)=F(O(x,y),z).$$

If $y\in[b,1]$, then $F(y,z)=b$, and

$$O(x,F(y,z))=O(x,b)=x=F(x,z)=F(O(x,y),z).$$

(2) $x\in(a,b)$. Then $z\in[0,b)$. If $y,z\in[0,a]$, then $F(y,z)\in[0,a]$, and thus,

$$O(x,F(y,z))=F(y,z)=F(O(x,y),z).$$

If $y\in[0,a]$, $z\in(a,b)$, then $F(y,z)=a$, and thus,

$$O(x,F(y,z))=O(x,a)=a=F(y,z)=F(O(x,y),z).$$

If $y\in(a,b)$, then $F(y,z)=y, O(x,y)\in[a,b]$, and thus,

$$O(x,F(y,z))=O(x,y)=F(O(x,y),z).$$

If $y\in[b,1]$, then $F(y,z)=b$, $O(x,y)\in[a,b]$, and

$$O(x,F(y,z))=O(x,b)=O(x,y)=F(O(x,y),z).$$

(3) $x\in[b,1]$. If $y\in[0,a]$, then $F(y,z)\leqslant a$, $O(x,y)=y$, and thus,

$$O(x,F(y,z))=F(y,z)=F(O(x,y),z).$$

If $y\in(a,b)$, then $O(x,y)\in[a,b]$, and thus,

$$O(x,F(y,z))=O(x,y)=F(O(x,y),z).$$

If $y\in[b,1]$, then $O(x,y)\in[b,1]$. Whenever $z\in[0,b]$, we have that $F(y,z)=b$, and thus,

$$O(x,F(y,z))=O(x,b)=b=F(O(x,y),z);$$

Whenever $z\in(b,1]$, it follows from the associativity of T_F that $O(x,F(y,z))=F(O(x,y),z)$.

Therefore, O is modular over F. □

The structures of semi-t-operator and overlap function in Proposition 3.5 are given in Fig. 1. When a,b of Proposition 3.5 take some special values, we have the following corollary.

Corollary 3.6. *Let $F\in\mathcal{F}_{a,b}$, $a\leqslant b$, be a semi-t-operator with a continuous pseudo-t-norm T_F and O be an overlap function.*

(i) *If $a=b=0$, then O is modular over F if and only if F is a continuous t-norm and $O=F$.*

(ii) *If $a=b=1$, then O is modular over F if and only if $F=S_M$ and $O=T_M$.*

(iii) *If $a=0,b=1$, then O is modular over F if and only if $F(x,y)=x$ for all $x,y\in[0,1]$.*

(iv) *If $0=a<b<1$, then O is modular over F if and only if O and F are given by*

$$O(x,y)=\begin{cases} bO_1(\frac{x}{b},\frac{y}{b}), & x,y\in[0,b],\\ b+(1-b)T_F(\frac{x-b}{1-b},\frac{y-b}{1-b}), & x,y\in[b,1],\\ O(\min(x,y),1), & \text{otherwise,}\end{cases}$$

and

$$F(x,y)=\begin{cases} b+(1-b)T_F(\frac{x-b}{1-b},\frac{y-b}{1-b}), & x,y\in[b,1],\\ b, & x\in[b,1], y\in[0,b],\\ x, & x\in[0,b],\end{cases}$$

where O_1 is an 1-overlap function.

(v) *If $0<a<b=1$, then O is modular over F if and only if O and F are given by*

$$O(x,y)=\begin{cases} a+(1-a)O_1(\frac{x-a}{1-a},\frac{y-a}{1-a}), & x,y\in[a,1],\\ \min(x,y), & \text{otherwise},\end{cases}$$

and

$$F(x,y)=\begin{cases} \max(x,y), & x,y\in[0,a],\\ a, & x\in[0,a], y\in[a,1],\\ x, & x\in[a,1],\end{cases}$$

where O_1 is an 0-overlap function.

(vi) *If $0<a=b<1$, then*

$$O(x,y)=\begin{cases} a+(1-a)T_F(\frac{x-a}{1-a},\frac{y-a}{1-a}), & x,y\in[a,1],\\ \min(x,y), & \text{otherwise},\end{cases} \tag{5}$$

and

$$F(x,y)=\begin{cases} \max(x,y), & x,y\in[0,a],\\ a+(1-a)T_F(\frac{x-a}{1-a},\frac{y-a}{1-a}), & x,y\in[a,1],\\ a, & \text{otherwise}.\end{cases} \tag{6}$$

Remark 3.7. (i) *The overlap function solutions of modularity equation are independent of the structure of O_1 in Proposition 3.5 and Corollary 3.6 (iv) (v). For example, if $T_F=T_M, a=\frac{1}{3}, b=\frac{2}{3}$, and O_1 is an 0-overlap function with neutral element 1, then*

$$O(x,y)=\begin{cases} \frac{1}{3}+\frac{1}{3}O_1(3x-1,3y-1), & x,y\in[\frac{1}{3},\frac{2}{3}],\\ \min(x,y), & \text{otherwise},\end{cases} \tag{7}$$

is an overlap function with neutral element 1, and the semi-t-operator F (see Fig. 2) is given by

$$F(x,y)=\begin{cases} \max(x,y), & x,y\in[0,\frac{1}{3}],\\ \min(x,y), & x,y\in[\frac{2}{3},1],\\ \frac{1}{3}, & x\in[0,\frac{1}{3}], y\in[\frac{1}{3},1],\\ \frac{2}{3}, & x\in[\frac{2}{3},1], y\in[0,\frac{2}{3}],\\ x, & x\in(\frac{1}{3},\frac{2}{3}).\end{cases}$$

From Proposition 3.5, we immediately obtain that O is modular over F. More specifically, if $O_1=T_M$, then $O=T_M$ (see Fig. 3) and if $O_1=T_P$, then $O=(\langle\frac{1}{3},\frac{2}{3},T_P\rangle)$ (see Fig. 4), that is,

$$O(x,y)=\begin{cases} 3xy-x-y+\frac{2}{3}, & x,y\in[\frac{1}{3},\frac{2}{3}],\\ \min(x,y), & \text{otherwise}.\end{cases}$$

(ii) *O may not be a t-norm in Proposition 3.5 and Corollary 3.6 (iv) (v). Consider the overlap function O given by* (7), *where $O_1(x,y)=O_{mM}(x,y)=\min(x,y)\max(x^2,y^2)$. Then*

$$O(x,y)=\begin{cases} \frac{1}{3}+\min(x-\frac{1}{3},y-\frac{1}{3})\max((3x-1)^2,(3y-1)^2), & x,y\in[\frac{1}{3},\frac{2}{3}],\\ \min(x,y), & \text{otherwise}.\end{cases}$$

Since $O(O(\frac{5}{12},\frac{1}{2}),\frac{7}{12})\neq O(\frac{5}{12},O(\frac{1}{2},\frac{7}{12}))$, we have that O is not a t-norm.

(iii) *The solutions of Theorem 4.2 in [24] and Theorem 3.11 in* [28] *can be obtained by Corollary 3.6(vi). Since there are no additional assumptions in Proposition 3.5, the results are more general.*

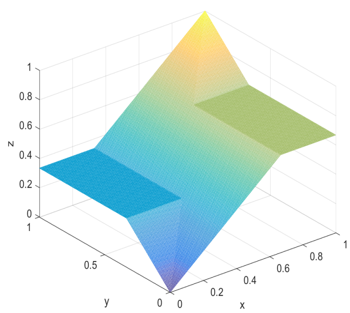

Fig. 2. The plot of F in Remark 3.7 (i)

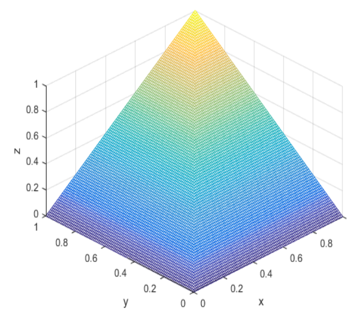

Fig. 3. The plot of O in Remark 3.7 (i) $(O_1 = T_M)$

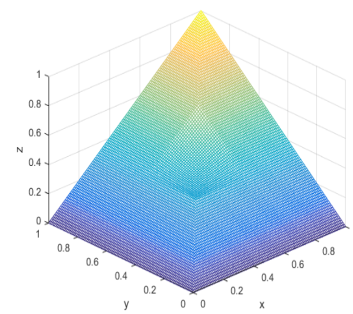

Fig. 4. The plot of O in Remark 3.7 (i) $(O_1 = T_P)$

Theorem 3.8. *Let O be an overlap function and $F \in \mathcal{F}_{a,b}$, $b \leqslant a$, be a semi-t-operator with a continuous pseudo-t-norm T_F. Then O is modular over F if and only if O is given by (5) and F is given by the following formula:*

$$F(x,y) = \begin{cases} \max(x,y), & x,y \in [0,b], \\ a + (1-a)T_F(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & x,y \in [a,1], \\ a, & x \in [0,a], y \in [a,1], \\ b, & x \in [b,1], y \in [0,b], \\ y, & y \in (a,b). \end{cases} \tag{8}$$

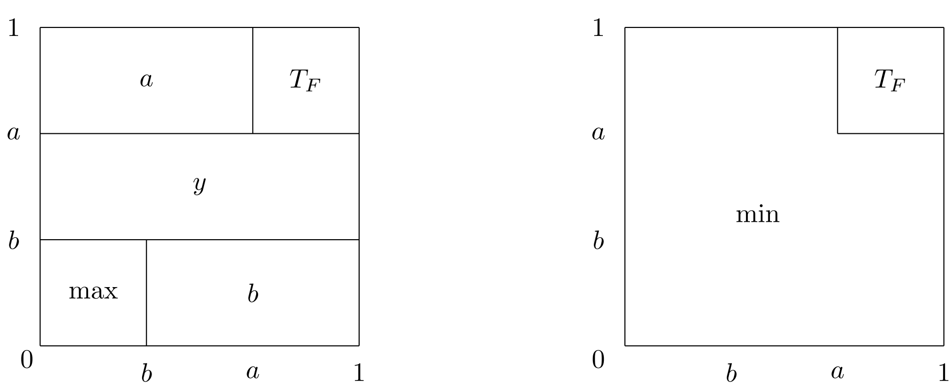

Fig. 5. Semi-t-operator (left) and overlap function (right) in Theorem 3.8

Proof. Necessity. Let F be modular over O. Due to Theorem 2.4, there is $F(0,z) = z$ for all $z \in [0,a]$ and $F(1,z) = z$ for all $z \in [a,1]$. Thus,

$$z = F(0,z) = F(O(1,0),z) = O(1,F(0,z)) = O(1,z),$$

for all $z \in [0,a]$, and

$$z = F(1,z) = F(O(1,1),z) = O(1,F(1,z)) = O(1,z),$$

for all $z \in [a,1]$. So we conclude that $O(1,x) = x$ for all $x \in [0,1]$. Due to the commutativity of O, we see that 1 is the neutral element of O. We will characterize the structures of O and F through the following steps.

(a) $O(x,y) = a + (1-a)T_F(\frac{x-a}{1-a}, \frac{y-a}{1-a})$ for any $x,y \in [a,1]$. In fact, for any $x,y \in [a,1]$ with $y \leqslant x$, there is

$$O(x,y) = O(x,F(1,y)) = F(O(x,1),y) = F(x,y) = a + (1-a)T_F(\tfrac{x-a}{1-a}, \tfrac{y-a}{1-a}).$$

By the commutativity of O and T_F, we have that $O(x,y) = a + (1-a)T_F(\frac{x-a}{1-a}, \frac{y-a}{1-a})$ for all $x,y \in [a,1]$.

(b) $O(x,y)=\min(x,y)$ for all $(x,y)\in[0,b]\times[0,1]\cup[0,1]\times[0,b]$. Indeed, for any $x\in[0,1]$, $y\in[0,b]$ with $y\leqslant x$, we have that

$$O(x,y)=O(x,F(0,y))=F(O(x,0),y)=F(0,y)=y.$$

From the commutativity of O, it follows that $O(x,y)=\min(x,y)$ for all $(x,y)\in[0,b]\times[0,1]\cup[0,1]\times[0,b]$.

(c) $O(x,y)=\min(x,y)$ for all $(x,y)\in[b,a]\times[a,1]\cup[a,1]\times[b,a]$. Taking $y,z\in[b,a]$. Then $F(z,y)=y\in[b,a]$, and due to the monotonicity of O, there is $O(a,z)\in[b,a]$, and thus,

$$O(a,y)=O(a,F(z,y))=F(O(a,z),y)=y.$$

Since $O(1,y)=y$, we have that $O(x,y)=y$ for all $x\in[a,1]$, and consequently, $O(x,y)=y$ for all $(x,y)\in[a,1]\times[b,a]$, Therefore, from the commutativity of O, we obtain that $O(x,y)=\min(x,y)$ for all $(x,y)\in[b,a]\times[a,1]\cup[a,1]\times[b,a]$.

(d) $O(x,y)=\min(x,y)$ for all $x,y\in[b,a]$. Indeed, for any $x,y\in[b,a]$ with $y\leqslant x$, we have that

$$O(x,y)=O(x,F(0,y))=F(O(x,0),y)=F(0,y)=y.$$

From the commutativity of O, it follows that $O(x,y)=\min(x,y)$ for all $x,y\in[0,1]$. Therefore, O is given by (5).

(e) $S_F=S_M$. The proof is similar to Proposition 3.5 (e). Therefore, F is given by (8).

Sufficiency. It suffices to prove $O(x,F(y,z))=F(O(x,y),z)$ for all $x,y,z\in[0,1]$ with $z\leqslant x$.

(1) $x\in[0,b]$. Then $z\in[0,b]$. If $y\in[0,b]$, then

$$\begin{aligned}O(x,F(y,z))&=\min(x,\max(y,z))=\max(\min(x,y),\min(x,z))\\&=\max(\min(x,y),z)=F(O(x,y),z).\end{aligned}$$

If $y\in(b,1]$, then $F(y,z)=b$, and thus,

$$O(x,F(y,z))=O(x,b)=x=F(x,z)=F(O(x,y),z).$$

(2) $x\in(b,a)$. Then $z\in[0,a)$. If $z\in[0,b]$, then $F(y,z)\in[0,b]$, and thus,

$$\begin{aligned}O(x,F(y,z))&=\min(x,F(y,z))=F(y,z)=\min(b,F(y,z))\\&=\min(F(x,z),F(y,z))=F(\min(x,y),z)\\&=F(O(x,y),z).\end{aligned}$$

If $z\in(b,a)$, then $O(x,F(y,z))=O(x,z)=z=F(O(x,y),z)$.

(3) $x\in[a,1]$. If $y\in[0,a]$, then $O(x,y)=y$, $F(y,z)\in[0,a]$, and thus, $O(x,F(y,z))=F(y,z)=F(O(x,y),z)$. If $y\in(a,1]$, then $O(x,y)\in[a,1]$. Whenever $z\in[0,b]$, one can check that

$$O(x,F(y,z))=O(x,b)=b=F(O(x,y),z).$$

Whenever $z\in(b,a)$, we have that

$$O(x,F(y,z))=O(x,z)=z=F(O(x,y),z).$$

Whenever $z\in[a,1]$, it follows from the associativity of T_F that $O(x,F(y,z))=F(O(x,y),z)$.

Therefore, O is modular over F. □

The structures of semi-t-operator and overlap function in Theorem 3.8 are given in Fig. 5. When a,b of Theorem 3.8 take some special values, we have the following corollary.

Corollary 3.9. *Let O be an overlap function and $F\in\mathcal{F}_{a,b}$, $b\leqslant a$, be a semi-t-operator with a continuous pseudo-t-norm T_F.*

(i) *If $b=a=0$, then O is modular over F if and only if F is a continuous t-norm and $O=F$.*

(ii) *If $b=a=1$, then O is modular over F if and only if $F=S_M$ and $O=T_M$.*

(iii) *If $b=0,a=1$, then O is modular over F if and only if $O=T_M$ and $F(x,y)=y$ for all $x,y\in[0,1]$.*

(iv) *If* $0 = b < a < 1$, *then* O *is modular over* F *if and only if* O *is given by* (5) *and*

$$F(x,y) = \begin{cases} y, & y \in [0,a], \\ a, & x \in [0,a], y \in [a,1], \\ a + (1-a)T_F(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & x,y \in [a,1]. \end{cases}$$

(v) *If* $0 < b < a = 1$, *then* O *is modular over* F *if and only if* $O = T_M$ *and*

$$F(x,y) = \begin{cases} \max(x,y), & x,y \in [0,b], \\ b, & x \in [b,1], y \in [0,b], \\ y, & y \in [b,1]. \end{cases}$$

(vi) *If* $0 < b = a < 1$, *then* O *and* F *are given by* (5) *and* (6), *respectively.*

Remark 3.10. *Note that in either case, the overlap function* O *is always a continuous t-norm in Theorem 3.8 and Corollary 3.9.*

4 Modularity equations between grouping functions and semi-t-operators

In this section, we discuss the modularity equation for a grouping function G and a semi-t-operator F with a continuous pseudo-t-conorm S_F in two parts: F over G and G over F. We still need to discuss the cases: $a \leqslant b$ and $b \leqslant a$. This section is similar to Section 3, so we only give the results.

4.1 The case of a semi-t-operator over a grouping function

Theorem 4.1. *Let* G *be a grouping function and* $F \in \mathcal{F}_{a,b}$, $a \leqslant b$, *be a semi-t-operator with a continuous pseudo-t-conorm* S_F. *Then* F *is modular over* G *if and only if* G *and* F *are given by the following formulas:*

$$G(x,y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}), & x,y \in [0,a], \\ \max(x,y), & \text{otherwise}, \end{cases} \tag{9}$$

and

$$F(x,y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}), & x,y \in [0,a], \\ \min(x,y), & x,y \in [b,1], \\ a, & x \in [0,a], y \in [a,1], \\ b, & x \in [b,1], y \in [0,b], \\ x, & x \in (a,b). \end{cases}$$

The structures of semi-t-operator and grouping function in Theorem 4.1 are given in Fig. 6.

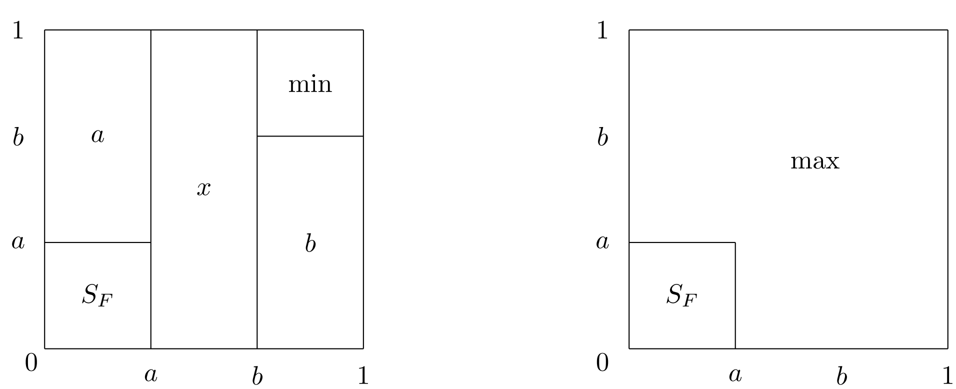

Fig. 6. Semi-t-operator (left) and grouping function (right) in Theorem 4.1

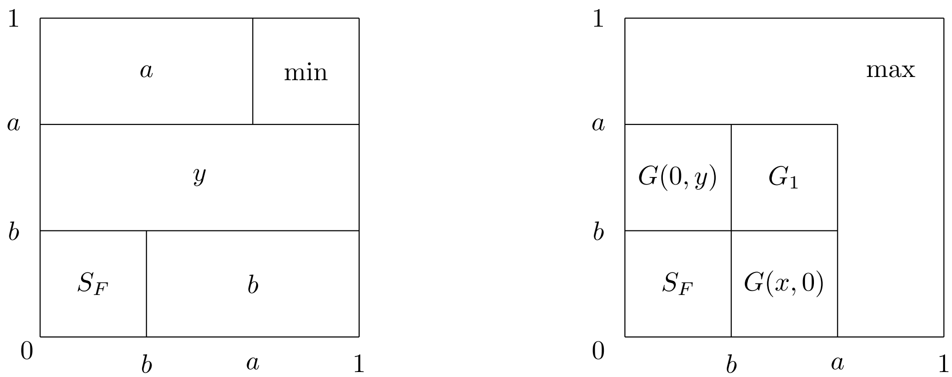

Fig. 7. Semi-t-operator (left) and grouping function (right) in Proposition 4.4

Corollary 4.2. *Let G be a grouping function and $F \in \mathcal{F}_{a,b}$, $a \leqslant b$, be a semi-t-operator with a continuous pseudo-t-conorm S_F.*

(i) *If $a = b = 0$, then F is modular over G if and only if $F = T_M$ and $G = S_M$.*

(ii) *If $a = b = 1$, then F is modular over G if and only if F is a continuous t-conorm and $G = F$.*

(iii) *If $a = 0, b = 1$, then F is modular over G if and only if $G = S_M$ and $F(x, y) = x$ for all $x, y \in [0, 1]$.*

(iv) *If $0 = a < b < 1$, then F is modular over G if and only if $G = S_M$ and*

$$F(x,y) = \begin{cases} \min(x,y), & x,y \in [b,1], \\ b, & x \in [b,1], y \in [0,b], \\ x, & x \in [0,b]. \end{cases}$$

(v) *If $0 < a < b = 1$, then F is modular over G if and only if G is given by (9) and*

$$F(x,y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}), & x,y \in [0,a], \\ a, & x \in [0,a], y \in [a,1], \\ x, & x \in [a,1]. \end{cases}$$

(vi) *If $0 < a = b < 1$, then G is given by (9) and*

$$F(x,y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}), & x,y \in [0,a], \\ \min(x,y), & x,y \in [a,1], \\ a, & \text{otherwise}. \end{cases} \tag{10}$$

Remark 4.3. (i) *Note that in either case, the grouping function G is always a continuous t-conorm in Theorem 4.1 and Corollary 4.2.*

(ii) *The solutions of Corollary 4.7 in* [21] *and Corollary 3.16 in* [29] *can be obtained by Corollary 4.2(vi). Since there are no additional assumptions in Theorem 4.1, the results are more general.*

Proposition 4.4. *Let G be a grouping function and $F \in \mathcal{F}_{a,b}$, $b \leqslant a$, be a semi-t-operator with a continuous pseudo-t-conorm S_F. Then F is modular over G if and only if G and F are given by the following formulas:*

$$G(x,y) = \begin{cases} bS_F(\frac{x}{b}, \frac{y}{b}), & x,y \in [0,b], \\ b + (a-b)G_1(\frac{x-b}{a-b}, \frac{y-b}{a-b}), & x,y \in [b,a], \\ G(\max(x,y),0), & [x \in [0,b], y \in (b,a)] \text{ or } [x \in (b,a), y \in [0,b]], \\ \max(x,y), & \text{otherwise}, \end{cases}$$

and

$$F(x,y)=\begin{cases} bS_F(\frac{x}{b},\frac{y}{b}), & x,y\in[0,b],\\ \min(x,y), & x,y\in[a,1],\\ a, & x\in[0,a], y\in[a,1],\\ b, & x\in[b,1], y\in[0,b],\\ y, & y\in(b,a),\end{cases}$$

where G_1 is a binary function satisfying (G1)(G2)$'$ (G3)$'$ (G4)(G5).

The structures of semi-t-operator and grouping function in Proposition 4.4 are given in Fig. 7.

Corollary 4.5. *Let G be a grouping function and $F\in\mathcal{F}_{a,b}$, $b\leqslant a$, be a semi-t-operator with a continuous pseudo-t-conorm S_F.*

(i) *If $b=a=0$, then F is modular over G if and only if $F=T_M$ and $G=S_M$.*

(ii) *If $b=a=1$, then F is modular over G if and only if F is a continuous t-conorm and $G=F$.*

(iii) *If $b=0, a=1$, then F is modular over G if and only if $F(x,y)=y$ for all $x,y\in[0,1]$.*

(iv) *If $0=b<a<1$, then F is modular over G if and only if*

$$G(x,y)=\begin{cases} aG_1(\frac{x}{a},\frac{y}{a}), & x,y\in[0,a],\\ \max(x,y), & \text{otherwise},\end{cases}$$

and

$$F(x,y)=\begin{cases} \min(x,y), & x,y\in[a,1],\\ a, & x\in[0,a], y\in[a,1],\\ y, & y\in[0,a],\end{cases}$$

where G_1 is a 1*-grouping function.*

(v) *If $0<b<a=1$, then F is modular over G if and only if*

$$G(x,y)=\begin{cases} bS_F(\frac{x}{b},\frac{y}{b}), & x,y\in[0,b],\\ b+(1-b)G_1(\frac{x-b}{1-b},\frac{y-b}{1-b}), & x,y\in[b,1],\\ G(\max(x,y),0), & \text{otherwise},\end{cases}$$

and

$$F(x,y)=\begin{cases} bS_F(\frac{x}{b},\frac{y}{b}), & x,y\in[0,b],\\ b, & x\in[b,1], y\in[0,b],\\ y, & y\in[b,1],\end{cases}$$

where G_1 is a 0*-grouping function.*

(vi) *If $0<a=b<1$, then F is modular over G if and only if G and F are given by* (9) *and* (10)*, respectively.*

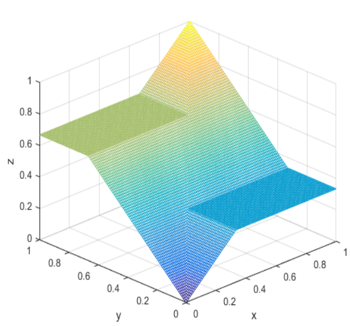

Fig. 8. The plot of F in Remark 4.6 (i)

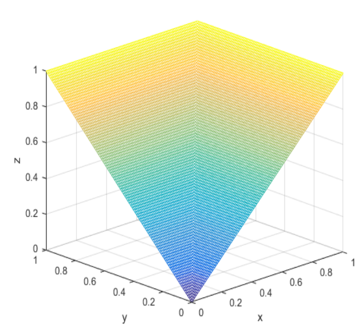

Fig. 9. The plot of G in Remark 4.6 (i) ($G_1=S_M$)

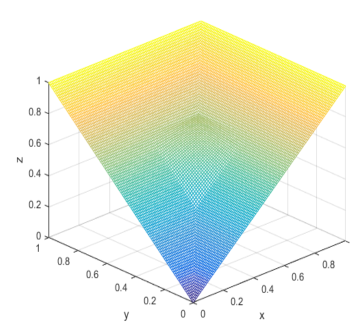

Fig. 10. The plot of G in Remark 4.6 (i) ($G_1=S_P$)

Remark 4.6. (i) *The grouping function solutions of modularity equation are independent of the structure of* G_1 *in Proposition 4.4 and Corollary 4.5 (iv)(v). For example, if* $S_F = S_M, b = \frac{1}{3}, a = \frac{2}{3}$*, and* G_1 *is a grouping function with neutral element* 0*, then*

$$G(x,y) = \begin{cases} \frac{1}{3} + \frac{1}{3}G_1(3x-1, 3y-1), & x,y \in [\frac{1}{3}, \frac{2}{3}], \\ \max(x,y), & \text{otherwise}, \end{cases} \tag{11}$$

is a grouping function with neutral element 0 *and the semi-t-operator (see Fig. 8) is given by*

$$F(x,y) = \begin{cases} \max(x,y), & x,y \in [0, \frac{1}{3}], \\ \min(x,y), & x,y \in [\frac{2}{3}, 1], \\ \frac{2}{3}, & x \in [0, \frac{2}{3}], y \in [\frac{2}{3}, 1], \\ \frac{1}{3}, & x \in [\frac{1}{3}, 1], y \in [0, \frac{1}{3}], \\ y, & y \in (\frac{1}{3}, \frac{2}{3}). \end{cases}$$

According to Proposition 4.4, F *is modular over* G*. More specifically, if* $G_1 = S_M$*, then* $G = S_M$ *(see Fig. 9) and if* $G_1 = S_P$*, then* $G = (\langle \frac{1}{3}, \frac{2}{3}, S_P \rangle)$ *(see Fig. 10), that is,*

$$G(x,y) = \begin{cases} 2x + 2y - 3xy - \frac{2}{3}, & x,y \in [\frac{1}{3}, \frac{2}{3}], \\ \max(x,y), & \text{otherwise}. \end{cases}$$

(ii) G *may not be a t-conorm in Proposition 4.4 and Corollary 4.5 (iv)(v). Consider the grouping function* G *given by* (11)*, where* $G_1(x,y) = G_{mM}(x,y) = 1 - \min(1-x, 1-y)\max((1-x)^2, (1-y)^2)$*. Then*

$$G(x,y) = \begin{cases} \frac{2}{3} - \min(\frac{2}{3} - x, \frac{2}{3} - y)\max((2-3x)^2, (2-3y)^2), & x,y \in [\frac{1}{3}, \frac{2}{3}], \\ \max(x,y), & \text{otherwise}. \end{cases}$$

Since $G(G(\frac{5}{12}, \frac{1}{2}), \frac{7}{12}) \neq G(\frac{5}{12}, G(\frac{1}{2}, \frac{7}{12}))$*, we have that* G *is not a t-conorm.*

4.2 The case of a grouping function over a semi-t-operator

Proposition 4.7. *Let* G *be a grouping function with neutral element* 0 *and* $F \in \mathcal{F}_{a,b}$*,* $a \leqslant b$*, be a semi-t-operator with a continuous pseudo-t-conorm* S_F*. Then* G *is modular over* F *if and only if* $b = 1$ *and one of the following statements holds:*

(i) *If* $a = 1$*, then* F *is a continuous t-conorm and* $G = F$*.*

(ii) *If* $a = 0$*, then* $F(x,y) = x$ *for all* $x,y \in [0,1]$*.*

(iii) *If* $0 < a < 1$*, then there exists a grouping function* G_1 *with neutral element* 0 *such that*

$$G(x,y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}), & x,y \in [0,a], \\ a + (1-a)G_1(\frac{x-a}{1-a}, \frac{y-a}{1-a}), & x,y \in [a,1], \\ \max(x,y), & \text{otherwise}, \end{cases}$$

and

$$F(x,y) = \begin{cases} aS_F(\frac{x}{a}, \frac{y}{a}), & x,y \in [0,a], \\ a, & x \in [0,a], y \in [a,1], \\ x, & x \in (a,1]. \end{cases}$$

Proposition 4.8. *Let* G *be a grouping function with neutral element* 0 *and* $F \in \mathcal{F}_{a,b}$*,* $b \leqslant a$*, be a semi-t-operator with a continuous pseudo-t-conorm* S_F*. Then* G *is modular over* F *if and only if* F *is a t-conorm and* $G = F$*.*

Remark 4.9. *Note that the modularity equation for a grouping function over a semi-t-operator in Proposition 4.8 reflects the restricted associativity of a t-conorm.*

5 Conclusion

In this paper, we studied the solutions of the modularity equation between overlap (grouping) functions and semi-t-operators. The structures of overlap (grouping) functions and semi-t-operators corresponding to each case are characterized. However, in some cases, we assume that the overlap (grouping) functions have neutral elements and semi-t-operators have the continuous pseudo-t-norm or continuous pseudo-t-conorm. In future work, we shall try to characterize the solutions of modularity equation without the above conditions, and continue to discuss the solutions of modularity equation between an overlap (grouping) function and a semicopula (quasi-copula, copula, co-copula, etc.). In addition, we will look for more application background of modularity equation to find some examples where it plays some role.

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References

[1] G. Beliakov, A. Pradera, T. Calvo, *Aggregation functions: A guide for practitioners*, Studies in Fuzziness and Soft Computing, 2007.

[2] H. Bustince, J. Fernandez, R. Mesiar, J. Montero, R. Orduna, *Overlap index, overlap functions and migrativity*, in: Proceedings of IFSA/EUSFLAT Conference, (2009), 300-305.

[3] H. Bustince, M. Pagola, R. Mesiar, E. Hüllermeier, F. Herrera, *Grouping, overlap, and generalized bientropic functions for fuzzy modeling of pairwise comparisons*, IEEE Transactions on Fuzzy Systems, **20**(3) (2012), 405-415.

[4] F. Durante, J. J. Quesada-Molina, C. Sempi, *Semicopulas: Characterizations and applicability*, Kybernetika, **42**(8) (2006), 287-302.

[5] P. Drygaś, *Distributivity between semi-t-operators and semi-nullnorms*, Fuzzy Sets and Systems, **264** (2015), 100-109.

[6] P. Drygaś, E. Rak, *Distributivity equations in the class of semi-t-operators*, Fuzzy Sets and Systems, **291** (2016), 66-81.

[7] W. Fechner, E. Rak, L. Zedam, *The modularity law in some classes of aggregation operators*, Fuzzy Sets and Systems, **332** (2018), 56-73.

[8] P. Flondor, G. Georgescu, A. Lorgulescu, *Pseudo-t-norms and pseudo-BL algebras*, Soft Computing, **5** (2001), 355-371.

[9] J. Fodor, M. Roubens, *Fuzzy preference modelling and multicriteria decision support*, Springer Netherlands, 1994.

[10] M. Grabisch, J. Marichal, R. Mesiar, E. Pap, *Aggregation functions*, Acta Polytechnica Hungarica, 2009.

[11] A. Jurio, H. Bustince, M. Pagola, A. Pradera, R. Yager, *Some properties of overlap and grouping functions and their application to image thresholding*, Fuzzy Sets and Systems, **229** (2013), 69-90.

[12] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[13] A. Kolesárová, J. Li, R. Mesiar, *k-additive aggregation functions and their characterization*, European Journal of Operational Research, **265** (2018), 985-992.

[14] G. Lucca, G. P. Dimuro, V. Mattos, B. Bedregal, H. Bustince, J. A. Sanz, *A family of Choquet-based non-associative aggregation functions for application in fuzzy rule-based classification systems*, in: 2015 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), Istanbul, (2015), 1-8.

[15] G. Lucca, J. A. Sanz, G. P. Dimuro, B. Bedregal, H. Bustince, *A proposal for tuning the α parameter in $C\alpha C$-integrals for application in fuzzy rule-based classification systems*, Natural Computing, **19** (2018), 533-546.

[16] J. Marichal, *On the associativity functional equation*, Fuzzy Sets and Systems, **114**(3) (2000), 381-389.

[17] M. Mas, G. Mayor, J. Torrens, *The modularity condition for uninorms and t-operators*, Fuzzy Sets and Systems, **126**(2) (2002), 207-218.

[18] P. S. Mostert, A. L. Shields, *On the structure of semigroups on a compact manifold with boundary*, Annals of Mathematics, **65**(1) (1957), 117-143.

[19] J. Qiao, B. Hu, *On interval additive generators of interval overlap functions and interval grouping functions*, Fuzzy Sets and Systems, **323** (2017), 19-55.

[20] F. Qin, *Uninorms solutions and (or) nullnorm solutions to the modularity condition equations*, Fuzzy Sets and Systems, **148** (2004), 231-242.

[21] E. Rak, *The modularity equation in the class of 2-uninorms*, in: Intelligent Systems, 2014, In: Advances in Intelligent Systems and Computing, **322** (2015), 45-54.

[22] A. F. Roldán López de Hierro, H. Bustince, J. Fernández, R. Mesiar, C. Roldán, *Two novel methodologies for considering aggregation functions by implicit equations and minimization problems*, European Journal of Operational Research, **270** (2018), 670-681.

[23] Y. Su, H. Liu, J. V. Riera, D. R. Aguilera, J. Torrens, *The modularity condition for uninorms revisited*, Fuzzy Sets and Systems, **357** (2019), 27-46.

[24] Y. Wang, H. Liu, *The modularity condition for overlap and grouping function*, Fuzzy Sets and Systems, **372** (2019), 97-110.

[25] C. Wang, L. Wan, B. Zhang, *Distributivity and conditional distributivity of semi-t-operators over S-uninorms*, Fuzzy Sets and Systems, **441** (2022), 224-240.

[26] H. Zhan, Y. Wang, H. Liu, *The modularity condition for semi-t-operators and semi-uninorms*, Fuzzy Sets and Systems, **334** (2018), 36-59.

[27] H. Zhan, Y. Wang, H. Liu, *The modularity condition for semi-t-operators*, Fuzzy Sets and Systems, **346** (2018), 108-126.

[28] T. Zhang, F. Qin, H. Liu, Y. Wang, *Modularity conditions between overlap (grouping) function and uni-nullnorm or null-uninorm*, Fuzzy Sets and Systems, **414** (2021), 94-114.

[29] Y. Zhao, H. Liu, *The modularity equation for semi-t-operators and T-uninorms*, International Journal of Approximate Reasoning, **146** (2022), 106-118.

[30] Y. Zhao, H. Liu, *The modularity equation for Mayor's aggregation operators and uninorms*, Iranian Journal of Fuzzy Systems, **19**(4) (2022), 137-145.

[31] Y. Zhao, H. Zhan, H. Liu, *The modularity equation with Mayor's aggregation operators and semi-t-operators*, Fuzzy Sets and Systems, **403** (2020), 108-118.

[32] K. Zhu, J. Wang, Y. Yang, *New results on the modularity condition for overlap and grouping functions*, Fuzzy Sets and Systems, **403** (2021), 139-147.

[33] K. Zhu, J. Wang, Y. Yang, *A note on the modularity condition for overlap and grouping functions*, Fuzzy Sets and Systems, **408** (2021), 108-117.

[34] H. J. Zimmermann, P. Zysno, *Decisions and evaluations by hierarchical aggregation of information*, Fuzzy Sets and Systems, **10**(1-3) (1983), 243-260.