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Original paper

On the distributivity property of uninorms locally internal on the boundary over noncontinuous t-(co)norms

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Abstract

This paper characterizes a uninorm distributive over a noncontinuous t-(co)norm, where the uninorm is locally internal on the boundary. Firstly, when the uninorm is disjunctive, the necessary conditions (resp. the sufficient conditions) for a uninorm distributive over a noncontinuous t-norm are analyzed under the certain condition. Secondly, the distributivity of a conjunctive uninorm over a noncontinuous t-conorm is characterized by duality. In particular, this paper is related to the open question recalled by Klement in the Linz2000 closing session, which provides the noncontinuous solutions of that question.

Keywords: Fuzzy connectives, aggregation operators, uninorms, distributivity.

1 Introduction

Aczél [1] firstly discussed the distributivity of two binary operations from the point of view of functional equations, which plays an important role in fuzzy sets and fuzzy logic, such as the distributivity between aggregation operations [6], t-norms and t-conorms [15] and fuzzy implications [3, 14, 27, 28, 40]. Since Yager and Rybalov [38] introduced uninorms in 1996, there has been a rapid development in both applications and theory of uninorms. Uninorms have been applied in many topics, such as fuzzy logic [11, 32, 33], fuzzy decision making [7, 35, 39], image processing [4, 12], fuzzy system modeling [34, 36, 37] and so on. Inspired by that great quantity of applications, many authors studied uninorms from a theoretical point of view, such as, uninorms in $\mathcal{U}^e_{\max}$ and $\mathcal{U}^e_{\min}$ [10], uninorms continuous in the open unit square [13, 21], idempotent uninorms [8, 20], locally internal uninorms [9], uninorms with continuous underlying operators [16, 23, 26], uninorms locally internal on the boundary [17, 22] and so on, where e is the neutral element of the uninorm.

Klement [30] recalled the question "*Characterize all (conditionally) distributive uninorms over a given (continuous) t-conorm*" in Linz2000 closing session. Especially, the question recalled by Klement is related to pseudoanalysis, which has been directly applied in nonlinear PDE, measure theory, information theory, system theory and optimization [5, 24, 25]. Moreover, the question mentioned above was studied by many authors [16, 18, 19, 29, 31]. For example, the question recalled by Klement was figured out in [29], where the uninorm was assumed to be one of the well-known uninorms, i.e., $\mathcal{U}^e_{\max}$, $\mathcal{U}^e_{\min}$, representable uninorms, uninorms continuous in the open unit square and idempotent uninorms. The characterizations of the distributivity and conditional distributivity of uninorms with continuous underlying operators over continuous t-conorms were presented by Li and Liu in [16, 18]. Li and Qin [19] investigated the conditional distributivity of a uninorm with continuous underlying operator over a given t-conorm. Especially, Su et al. [31] characterized uninorms with continuous underlying operators (conditionally) distributive over a given continuous t-norm. However, t-(co)norms in the (conditional) distributivity equations mentioned above were always assumed to be

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continuous. In fact, because t-(co)norms are not always continuous, it seems to be necessary to discuss the noncontinuous t-(co)norms in the distributivity of uninorms over t-(co)norms. Therefore, these suggest that the distributivity of uninorms over t-(co)norms should be studied more thoroughly in this paper to solve the question recalled by Klement.

In this paper, we study the distributivity of uninorms over noncontinuous t-(co)norms, where the uninorms are locally internal on the boundary. Firstly, we characterize the distributivity of a disjunctive uninorm U over a noncontinuous t-norm T with the assumption $T(e,e)=0$, where U is locally internal on the boundary and e is the neutral element of U. In particular, we discuss four different classes of disjunctive uninorms locally internal on the boundary in the distributivity equation. In contrast to the characterizations of the (conditional) distributivity of a uninorm over a continuous t-(co)norm in [16, 18, 19, 29, 31], we obtain the necessary condition (resp. the several sufficient conditions) for a disjunctive uninorm U distributive over a noncontinuous t-norm T. Secondly, we study a conjunctive uninorm U distributive over a noncontinuous t-conorm S under the condition $S(e,e)=1$ by duality. In particular, because uniorms locally internal on the boundary have the well-known classes of uninorms, that is, $\mathcal{U}^e_{\max}, \mathcal{U}^e_{\min}$, uninorms continuous in the open unit square, idempotent uninorms, locally internal uninorms and uninorms with continuous underlying operators, as special cases, this paper can be viewed as a supplement to [16, 18, 19, 29, 31], which presents the noncontinuous solutions of the question recalled by Klement. Finally, the results in this paper show when a uninorm is distributive over a noncontinuous t-(co)norm as pseudo-operators in pseudoanalysis. Moreover, uninorms and t-(co)norms are frequently applied as logical connectives in fuzzy logic, image processing, approximate reasoning, modeling some specific problems in the utility theory, where the distributivity property can be of interest.

The remaining content of the paper is organized as follows. In Section 2, we recall some basic concepts and related properties of uninorms applied in this paper. Section 3 obtains the necessary conditions (resp. the sufficient conditions) for disjunctive uninorms distributive over noncontinuous t-norms, where the uninorms are locally internal on the boundary. In Section 4, we characterize the necessary conditions and several sufficient conditions for the distributivity of conjunctive uninorms over noncontinuous t-norms by duality, respectively, where the uninorms are also assumed to be locally internal on the boundary. In the final section, we present some conclusions of our research.

2 Preliminaries

In this section, we recall some basic concepts and terminology used throughout the paper.

Definition 2.1. [38] *A binary operator $U:[0,1]^2\to[0,1]$ is called a uninorm, if it is commutative, associative, increasing with respect to each variable and it has a neutral element $e\in[0,1]$, i.e., $U(x,e)=x$ for all $x\in[0,1]$.*

Meanwhile, when $e=1$, a uninorm U degenerates into a t-norm. Similarly, when $e=0$, a uninorm U degenerates into a t-conorm. A t-norm T is said to be *positive* [2], if $T(x,y)=0$ implies $x=0$ or $y=0$. Similarly, a t-conorm S is said to be *positive* [2], if $S(x,y)=1$ implies $x=1$ or $y=1$. A uninorm U always satisfies $U(0,1)=0$ or $U(0,1)=1$. If $U(0,1)=1$, then U is called a *disjunctive* uninorm. On the contrary, if $U(0,1)=0$, then U is called a *conjunctive* uninorm. Furthermore, Fodor et al. [10] obtained the sturcture of uninorms as follows.

Proposition 2.2. [10] *Let U be a uninorm with a neutral element $e\in]0,1[$ and $A(e)=[0,e[\times]e,1]\cup]e,1]\times[0,e[$. Then U is given as*

$$U(x,y)=\begin{cases} eT\left(\frac{x}{e},\frac{y}{e}\right), & \text{if } (x,y)\in[0,e]^2;\\ e+(1-e)S\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right), & \text{if } (x,y)\in[e,1]^2;\\ B(x,y), & \text{if } (x,y)\in A(e);\end{cases}$$

where T is a t-norm, S is a t-conorm and B satisfies $\min(x,y)\leqslant B(x,y)\leqslant\max(x,y)$ for all $(x,y)\in A(e)$.

The t-norm T (resp. t-conorm S) forming a uninorm U is called the *underlying t-norm* (resp. *underlying t-conorm*) of U. The most well-known classes of uninorms are presented as follows.

- $\mathcal{U}^e_{\max}$ [10], the family of all uninorms that satisfy $B(x,y)=\max(x,y)$ for all $(x,y)\in A(e)$.
- $\mathcal{U}^e_{\min}$ [10], the family of all uninorms that satisfy $B(x,y)=\min(x,y)$ for all $(x,y)\in A(e)$.
- Representable uninorms [10], the family of all uninorms that have a multiplicative generator.
- Uninorms continuous in the open unit square [13, 21]. The representable uninorms are special cases of uninorms continuous in the open unit square. Moreover, Mas et al. [21] described uninorms continuous in the open unit square in two different types, that is, $\mathcal{U}_{\cos,\max}$ and $\mathcal{U}_{\cos,\min}$.
- Idempotent uninorms [8, 20], the family of all uninorms that satisfy $U(x,x)=x$ for all $x\in[0,1]$.

- Locally internal uninorms [9], the family of all uninorms that satisfy $B(x,y) \in \{x,y\}$ for all $(x,y) \in A(e)$.
- Uninorms with continuous underlying operators [16, 23, 26], the family of all uninorms that have the continuous underlying operators.

Mas et al. [22] proposed a new class of uninorms, that is, uninorms locally internal on the boundary.

Definition 2.3. [22] *Let U be a disjunctive uninorm. Then U is said to be locally internal on the boundary, if $U(0,x) \in \{0,x\}$ holds for all $x \in [0,1]$. Similarly, let U be a conjunctive uninorm. Then U is said to be locally internal on the boundary, if $U(x,1) \in \{x,1\}$ holds for all $x \in [0,1]$.*

Remark 2.4. *Uninorms locally internal on the boundary have the most well-known classes of uninorms mentioned above, i.e., $\mathcal{U}^e_{\max}$, $\mathcal{U}^e_{\min}$, representable uninorms, uninorms continuous in the open unit square, idempotent uninorms, locally internal uninorms and uninorms with continuous underlying operators, as special cases* [17, 22]*. Moreover, Li and Liu* [17] *pointed out that there exist uninorms, which are not locally internal on the boundary.*

Li and Liu [17] analyzed the properties of uninorms locally internal on the boundary and obtained their different cases.

Proposition 2.5. [17] *Let a disjunctive uninorm U be locally internal on the boundary with the neutral element $e \in]0,1[$ and $\omega \in]e,1[$. Then one of the following four cases is satisfied.*

(i) *$U(0,x) = 0$ for all $x \in [0,\omega]$ and $U(0,x) = x$ for all $x \in]\omega,1]$. Moreover, there exist a t-conorm S and a conjunctive uninorm U' such that U has the following form,*

$$U(x,y) = \begin{cases} \omega U'\left(\frac{x}{\omega}, \frac{y}{\omega}\right), & \text{if } (x,y) \in [0,\omega]^2; \\ \omega + (1-\omega) S\left(\frac{x-\omega}{1-\omega}, \frac{y-\omega}{1-\omega}\right), & \text{if } (x,y) \in [\omega,1]^2; \\ \max(x,y), & \text{otherwise.} \end{cases} \tag{1}$$

(ii) *$U(0,x) = 0$ for all $x \in [0,\omega[$ and $U(0,x) = x$ for all $x \in [\omega,1]$. Moreover, there exist a t-conorm S and a disjunctive uninorm U' with the positive underlying t-conorm such that U has the following form,*

$$U(x,y) = \begin{cases} \omega U'\left(\frac{x}{\omega}, \frac{y}{\omega}\right), & \text{if } (x,y) \in [0,\omega]^2; \\ \omega + (1-\omega) S\left(\frac{x-\omega}{1-\omega}, \frac{y-\omega}{1-\omega}\right), & \text{if } (x,y) \in [\omega,1]^2; \\ \max(x,y), & \text{otherwise.} \end{cases} \tag{2}$$

(iii) *$U(0,x) = 0$ for all $x \in [0,e]$ and $U(0,x) = x$ for all $x \in]e,1]$, i.e., $U \in \mathcal{U}^e_{\max}$.*

(iv) *$U(0,x) = 0$ for all $x \in [0,1[$ and $U(0,1) = 1$. Moreover, the uninorm U has a positive underlying t-conorm.*

Proposition 2.6. [17] *Let a conjunctive uninorm U be locally internal on the boundary with the neutral element $e \in]0,1[$ and $\lambda \in]0,e[$. Then one of the following four cases is satisfied.*

(i) *$U(x,1) = x$ for all $x \in [0,\lambda[$ and $U(x,1) = 1$ for all $x \in [\lambda,1]$. Moreover, there exist a t-norm T and a disjunctive uninorm U' such that U has the following form,*

$$U(x,y) = \begin{cases} \lambda T\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right), & \text{if } (x,y) \in [0,\lambda]^2; \\ \lambda + (1-\lambda) U'\left(\frac{x-\lambda}{1-\lambda}, \frac{y-\lambda}{1-\lambda}\right), & \text{if } (x,y) \in [\lambda,1]^2; \\ \min(x,y), & \text{otherwise.} \end{cases} \tag{3}$$

(ii) *$U(x,1) = x$ for all $x \in [0,\lambda]$ and $U(x,1) = 1$ for all $x \in]\lambda,1]$. Moreover, there exist a t-norm T and a conjunctive uninorm U' with the positive underlying t-norm such that U has the following form,*

$$U(x,y) = \begin{cases} \lambda T\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right), & \text{if } (x,y) \in [0,\lambda]^2; \\ \lambda + (1-\lambda) U'\left(\frac{x-\lambda}{1-\lambda}, \frac{y-\lambda}{1-\lambda}\right), & \text{if } (x,y) \in [\lambda,1]^2; \\ \min(x,y), & \text{otherwise.} \end{cases} \tag{4}$$

(iii) *$U(x,1) = x$ for all $x \in [0,e[$ and $U(x,1) = 1$ for all $x \in [e,1]$, i.e., $U \in \mathcal{U}^e_{\min}$.*

(iv) *$U(0,1) = 0$ and $U(x,1) = 1$ for all $x \in]0,1]$. Moreover, the uninorm U has a positive underlying t-norm.*

Definition 2.7. [1] *Consider $F, G : [0,1]^2 \to [0,1]$ be two binary operators. Then F is distributive over G, if the following hold for all $x, y, z, \in [0,1]$,*

$$F\big(x, G(y,z)\big) = G\big(F(x,y), F(x,z)\big) \text{ and } F\big(G(x,y), z\big) = G\big(F(x,z), F(y,z)\big).$$

3 Disjunctive uninorms locally internal on the boundary distributive over noncontinuous t-norms

Although the question recalled by Klement was studied in [16, 18, 19, 29, 31], t-(co)norms were always assumed to be continuous. In this section, we discuss the noncontinuous situations. In particular, we analyze the four different cases of disjunctive uninorms locally internal on the boundary.

3.1 Case for $U(0,x)=0$ for all $x \in [0,\omega]$ and $U(0,x)=x$ for all $x \in]\omega,1]$

Lemma 3.1. *Let a disjunctive uninorm given by Eq.* (1) *be distributive over a t-norm* T *with* $T(e,e)=0$. *Then the following hold.*

(i) $T(x,x)=0$ *for all* $x \in [0,\omega]$.

(ii) $T(x,x)=x$ *for all* $x \in]\omega,1]$.

Proof. (i) If $x \in [0,\omega]$, then it follows from Proposition 2.5(i) that the following holds,

$$0=U(0,x)=U\big(T(e,e),x\big)=T\big(U(e,x),U(e,x)\big)=T(x,x).$$

Thus $T(x,x)=0$ holds for all $x \in [0,\omega]$.
(ii) If $x \in]\omega,1]$, then we obtain

$$x=U(0,x)=U\big(T(0,0),x\big)=T\big(U(0,x),U(0,x)\big)=T(x,x).$$

Thus $T(x,x)=x$ holds for all $x \in]\omega,1]$. □

Proposition 3.2. *Let a disjunctive uninorm given by Eq.* (1) *be distributive over a t-norm* T *with* $T(e,e)=0$. *Then the t-norm* T *is noncontinuous.*

Proof. By Proposition 2.5(i), we obtain $\omega \in]e,1[$. It follows from Lemma 3.1 that the following hold,

$$\lim_{x\to\omega^-} T(x,x)=\lim_{x\to\omega^-} 0=0 \text{ and } \lim_{x\to\omega^+} T(x,x)=\lim_{x\to\omega^+} x=\omega.$$

Thus T is noncontinuous. □

The necessary condition for a disjunctive uninorm U distributive over a noncontinuous t-norm T is obtained as follows.

Proposition 3.3. *Let a disjunctive uninorm* U *given by Eq.* (1) *be distributive over a t-norm* T *with* $T(e,e)=0$. *Then there exists* $C:]\omega,1]\times[0,\omega]\to[0,\omega]$ *such that* T *has the following form*

$$T(x,y)=\begin{cases} 0, & \text{if } (x,y)\in[0,\omega]^2;\\ \min(x,y), & \text{if } (x,y)\in]\omega,1]^2;\\ C(x,y), & \text{if } (x,y)\in]\omega,1]\times[0,\omega];\\ C(y,x), & \text{if } (x,y)\in[0,\omega]\times]\omega,1]; \end{cases} \tag{5}$$

where C *is increasing with respect to each variable and satisfies* $C(1,y)=y$ *for all* $y\in[0,\omega]$, $C(x,0)=0$ *for all* $x\in]\omega,1]$ *and the following equation for all* $x\in[0,\omega]$, $y\in]\omega,1]$ *and* $z\in]\omega,1]$ *(see Figure 1),*

$$C\big(z,C(y,x)\big)=C\big(\min(y,z),x\big). \tag{6}$$

Proof. By Lemma 3.1(ii), we obtain $T(x,y)=\min(x,y)$ for all $(x,y)\in]\omega,1]^2$. Meanwhile, applying Lemma 3.1(i), we have $T(x,x)=0$ for all $x\in[0,\omega]$. Thus, $T(x,y)=0$ holds for all $(x,y)\in[0,\omega]^2$. Especially, because C is the part of t-norm T, C is obviously increasing with respect to each variable. Meanwhile, C obviously satisfies $C(1,y)=y$ for all $y\in[0,\omega]$ and $C(x,0)=0$ for all $x\in]\omega,1]$. Because T is associative, we have for all $x\in[0,\omega]$, $y\in]\omega,1]$ and $z\in]\omega,1]$,

$$T\big(T(x,y),z\big)=T\big(x,T(y,z)\big),$$

that is,

$$C\big(z,C(y,x)\big)=C\big(\min(y,z),x\big).$$

Thus C satisfies Eq. (6). □

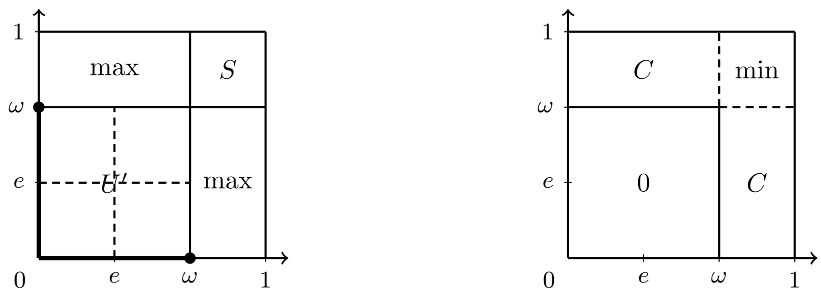

Figure 1: Structure of U (left) and T (right) in Proposition 3.3.

Considering the specific form of C in Eq. (5), we have the following sufficient condition for a disjunctive uninorm U over a noncontinuous t-norm T.

Proposition 3.4. *Let a disjunctive uninorm U be given by Eq. (1) and T be a t-norm. Then U is distributive over T, if T is given by Eq. (5) and C has the following form for all $(x,y) \in]\omega,1] \times [0,\omega]$,*

$$C(x,y) = \begin{cases} 0, & \text{if } x \in]\omega,1[; \\ y, & \text{if } x = 1. \end{cases} \tag{7}$$

Proof. We sperate the closed unit interval $[0,1]$ as $[0,\omega] \cup]\omega,1]$ to verify the conclusion. Since both the uninorm U and t-norm T are commutative, we assume $y \leqslant z$ in the following proof.

Case 1: $x \in [0,\omega]$, $y \in [0,\omega]$, $z \in [0,\omega]$
By Eq. (1), we have

$$U(x,0) = 0 \text{ for all } x \in [0,\omega] \text{ and } U(x,y) \leqslant U(x,z) \leqslant U(\omega,\omega) = \omega.$$

It follows from Eqs. (1) and (5) that the following hold,

$$U\big(x,T(y,z)\big) = U(x,0) = 0 \text{ and } T\big(U(x,y),U(x,z)\big) = 0.$$

Case 2: $x \in [0,\omega]$, $y \in [0,\omega]$, $z \in]\omega,1]$
On the account of Eq. (1), we obtain

$$U(x,0) = 0 \text{ for all } x \in [0,\omega],\ U(x,y) \leqslant U(\omega,\omega) = \omega \text{ and } U(x,z) = \max(x,z) = z.$$

Applying Eq. (7), the following holds,

$$U\big(x,T(y,z)\big) = U\big(x,C(z,y)\big) = \begin{cases} U(x,0), & \text{if } z \in]\omega,1[; \\ U(x,y), & \text{if } z = 1; \end{cases} = \begin{cases} 0, & \text{if } z \in]\omega,1[; \\ U(x,y), & \text{if } z = 1. \end{cases}$$

By Eqs. (5) and (7), we have

$$T\big(U(x,y),U(x,z)\big) = C\big(z,U(x,y)\big) = \begin{cases} 0, & \text{if } z \in]\omega,1[; \\ U(x,y), & \text{if } z = 1. \end{cases}$$

Case 3: $x \in [0,\omega]$, $y \in]\omega,1]$, $z \in]\omega,1]$
It follows from Eq. (1) that the following hold,

$$U(x,y) = \max(x,y) = y \text{ and } U(x,z) = \max(x,z) = z.$$

By Eqs. (1) and (5), we have

$$U\big(x,T(y,z)\big) = U\big(x,\min(y,z)\big) = U(x,y) = y,$$
$$T\big(U(x,y),U(x,z)\big) = T(y,z) = \min(y,z) = y.$$

Case 4: $x \in]\omega,1]$, $y \in [0,\omega]$, $z \in [0,\omega]$
Applying Eq. (1), we have

$$U(x,y) = \max(x,y) = x \text{ and } U(x,z) = \max(x,z) = x.$$

Hence it follows from Eqs. (1) and (5) that the following hold,

$$U\big(x, T(y,z)\big) = U(x,0) = \max(x,0) = x \text{ and } T\big(U(x,y), U(x,z)\big) = T(x,x) = x.$$

Case 5: $x \in]\omega, 1]$, $y \in [0, \omega]$, $z \in]\omega, 1]$
By Eq. (1), we obtain

$$U(x,y) = x \text{ and } U(x,z) \geqslant \max(x,z) > \omega.$$

Thus it follows from Eq. (1) and $T(y,z) \leqslant \min(y,z) = y \leqslant \omega$ that the following holds,

$$U\big(x, T(y,z)\big) = \max\big(x, T(y,z)\big) = x.$$

Meanwhile, by Eq. (5), we obtain

$$T\big(U(x,y), U(x,z)\big) = T\big(x, U(x,z)\big) = \min\big(x, U(x,z)\big) = x.$$

Case 6: $x \in]\omega, 1]$, $y \in]\omega, 1]$, $z \in]\omega, 1]$
It follows from Eq. (1) that the following holds

$$U(x,z) \geqslant U(x,y) \geqslant \max(x,y) > \omega.$$

Moreover, by Eqs. (1) and (5), we obtain

$$U\big(x, T(y,z)\big) = U\big(x, \min(y,z)\big) = U(x,y),$$
$$T\big(U(x,y), U(x,z)\big) = \min\big(U(x,y), U(x,z)\big) = U(x,y).$$

Therefore, U is distributive over T. □

Notice that the form of C in Eq. (5) is not unique. We have the another sufficient condition for a disjunctive uninorm U distributive over a noncontinuous t-norm T.

Proposition 3.5. *Let a disjunctive uninorm U be given by Eq. (1) and T be a t-norm. Then U is distributive over T, if T is given by Eq. (5) and C has the following form for all $(x,y) \in]\omega, 1] \times [0, \omega]$,*

$$C(x,y) = \min(x,y). \tag{8}$$

Proof. Similarly to the proof of Proposition 3.4, we always assume $y \leqslant z$. Under that premise, we only verify Case 2 in the proof of Proposition 3.4 related to C. The others can be proven in a similar way as for Proposition 3.4.

Case 2: $x \in [0, \omega]$, $y \in [0, \omega]$, $z \in]\omega, 1]$
By Eqs. (1) and (5), we obtain $U(x,y) \leqslant \omega$ and $U(x,z) = z$. Moreover, it follows from Eqs. (5) and (8) that the following hold,

$$U\big(x, T(y,z)\big) = U\big(x, \min(y,z)\big) = U(x,y),$$
$$T\big(U(x,y), U(x,z)\big) = \min\big(U(x,y), z\big) = U(x,y).$$

□

Example 3.6. *Consider a uninorm U given as*

$$U(x,y) = \begin{cases} 0.6R\big(\frac{x}{0.6}, \frac{y}{0.6}\big), & \text{if } (x,y) \in]0, 0.6[^2; \\ 0, & \text{if } (x,y) \in [0, 0.9] \times \{0\} \cup \{0\} \times [0, 0.9]; \\ \max(x,y), & \text{otherwise,} \end{cases}$$

where R is a representable uninorm with a neutral element $e' = 0.5$. Then the uninorm U is continuous in the open unit square with the neutral element $e = 0.3$. Meanwhile, consider a t-norm T_1 given as

$$T_1(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[\times [0,0.9] \cup [0,0.9] \times [0,1[; \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Then T_1 is noncontinuous. Moreover, it follows from Proposition 3.4 that the uninorm U is distributive over the noncontinuous t-norm T_1.

Similarly, consider a t-norm T_2 *given as*

$$T_2(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,0.9]^2; \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Then T_2 *is also noncontinuous. In particular, it follows from Proposition 3.5 that the uninorm* U *is distributive over the noncontinuous t-norm* T_2*.*

Inspired by Propositions 3.4 and 3.5, we present the following example of the noncontinuous t-norm T satisfying the necessary conditions listed in Proposition 3.3, but the uninorm U is not distributive over the noncontinuous t-norm T.

Example 3.7. *Consider a uninorm* U *given as*

$$U(x,y) = \begin{cases} \min(x,y), & \text{if } (x,y) \in [0,0.25[\times[0,0.5] \cup [0,0.5] \times [0,0.25[; \\ \max(x,y), & \text{otherwise.} \end{cases}$$

Then U *can be viewed as a numerical example of Eq.* (1) *with* $\omega = 0.5$ *and* $e = 0.25$*. Moreover, consider the nilpotent minimum t-norm* T *given as*

$$T(x,y) = \begin{cases} 0, & \text{if } x+y \leqslant 1; \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Then T *is noncontinuous and satisfying* $T(e,e) = 0$*. Consider* $x = 0.45$*,* $y = 0.3$ *and* $z = 0.6$*. Then we obtain*

$$U\big(x, T(y,z)\big) = U\big(0.45, T(0.3,0.6)\big) = U(0.45,0) = 0,$$
$$T\big(U(x,y), U(x,z)\big) = T\big(U(0.45,0.3), U(0.45,0.6)\big) = T(0.45,0.6) = 0.45.$$

Thus U *is not distributive over* T*.*

Remark 3.8. *There may be other forms of* C *in Eq.* (5) *such that a disjunctive uninorm* U *given by Eq.* (1) *is distributive over a noncontinuous t-norm* T*. Unfortunately, we cannot obtain all of them or the regularity of them.*

3.2 Case for $U(0,x) = 0$ for all $x \in [0,\omega[$ and $U(0,x) = x$ for all $x \in [\omega,1]$

Similarly to Lemma 3.1 and Proposition 3.2, we have the following conclusion.

Lemma 3.9. *Let a disjunctive uninorm* U *given by Eq.* (2) *be distributive over a t-norm* T *with* $T(e,e) = 0$*. Then the following hold.*

(i) $T(x,x) = 0$ *for all* $x \in [0,\omega[$.

(ii) $T(x,x) = x$ *for all* $x \in [\omega,1]$.

(iii) *The t-norm* T *is noncontinuous.*

When U is given by Eq. (2), we have the following necessary condition for the uninorm U distributive over a noncontinuous t-norm T with the assumption $T(e,e) = 0$.

Proposition 3.10. *Let a disjunctive uninorm* U *given by Eq.* (2) *be distributive over a t-norm* T *with* $T(e,e) = 0$*. Then there exists* $C : [\omega,1] \times [0,\omega[\to [0,\omega[$ *such that* T *has the following form*

$$T(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,\omega[^2; \\ \min(x,y), & \text{if } (x,y) \in [\omega,1]^2; \\ C(x,y), & \text{if } (x,y) \in [\omega,1] \times [0,\omega[; \\ C(y,x), & \text{if } (x,y) \in [0,\omega[\times[\omega,1]; \end{cases} \tag{9}$$

where C *is increasing with respect to each variable and satisfies* $C(1,y) = y$ *for all* $y \in [0,\omega[$*,* $C(x,0) = 0$ *for all* $x \in [\omega,1]$ *and the following equation for all* $x \in [0,\omega[$*,* $y \in [\omega,1]$ *and* $z \in [\omega,1]$ *(see Figure 2),*

$$C\big(z, C(y,x)\big) = C\big(\min(y,z), x\big). \tag{10}$$

Proof. It can be proven in a similar way as for Proposition 3.3. □

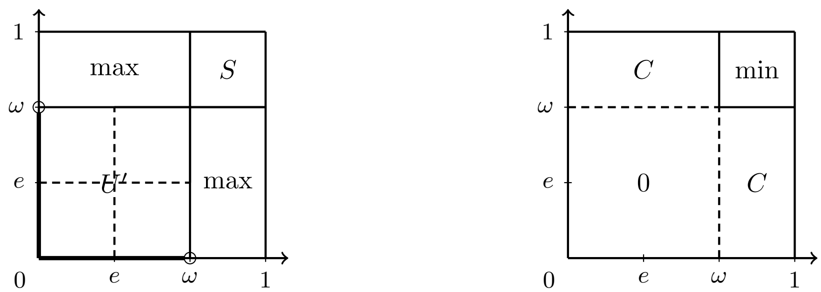

Figure 2: Structure of U (left) and T (right) in Proposition 3.10.

Consider C in Eq. (9) be specific. Then we have the following sufficient condition for U given by Eq. (2) distributive over T with the assumption $T(e,e)=0$.

Proposition 3.11. *Let a disjunctive uninorm U be given by Eq.* (2) *and a t-norm T be given by Eq.* (9)*. Then U is distributive over T, if either one of the following cases is satisfied.*

(i) *C has the following form for all $(x,y)\in[\omega,1]\times[0,\omega[$,*

$$C(x,y)=\begin{cases}0, & \text{if } x\in[\omega,1[;\\ y, & \text{if } x=1.\end{cases} \tag{11}$$

(ii) *C has the following form for all $(x,y)\in[\omega,1]\times[0,\omega[$,*

$$C(x,y)=\min(x,y). \tag{12}$$

Proof. (i) We sperate the closed unit interval $[0,1]$ as $[0,\omega[\cup[\omega,1]$. Similarly to the proof of Proposition 3.4, we always assume $y\leqslant z$. Meanwhile, because the underlying t-conorm of the disjunctive uninorm U' is positive by Proposition 2.5(ii), we only verify the following cases related to the positive underlying t-conorm of the uninorm U' as an example. The others can be proven in a similar way as for Proposition 3.4.

Case 1: $x\in[0,\omega[$, $y\in[0,\omega[$, $z\in[0,\omega[$
Because the underlying t-conorm of U' is positive and $U(0,\omega)=\omega$, we have

$$U(x,y)\leqslant U(x,z)<U(\omega,\omega)=\omega \text{ and } U(x,0)=0 \text{ for all } x\in[0,\omega[.$$

It follows from Eqs. (2) and (9) that the following hold,

$$U\big(x,T(y,z)\big)=U(x,0)=0 \text{ and } T\big(U(x,y),U(x,z)\big)=0.$$

Case 2: $x\in[0,\omega[$, $y\in[0,\omega[$, $z\in[\omega,1]$
Similarly to Case 1, we obtain $U(x,y)<\omega$ and $U(x,0)=0$ for all $x\in[0,\omega[$. By Eqs. (2) and (11), the following holds,

$$U\big(x,T(y,z)\big)=U\big(x,C(z,y)\big)=\begin{cases}U(x,0), & \text{if } z\in[\omega,1[;\\ U(x,y), & \text{if } z=1;\end{cases}=\begin{cases}0, & \text{if } z\in[\omega,1[;\\ U(x,y), & \text{if } z=1.\end{cases}$$

By Eq. (2), we have $U(x,z)=\max(x,z)=z$. Thus, we obtain

$$T\big(U(x,y),U(x,z)\big)=T\big(U(x,y),z\big)=C\big(z,U(x,y)\big)=\begin{cases}0, & \text{if } z\in[\omega,1[;\\ U(x,y), & \text{if } z=1.\end{cases}$$

Therefore, U is distributive over T.
(ii) It can be proven in a similar way as for item (i). □

Remark 3.12. *Similarly to Remark 3.8, we still cannot obtain the regularity of the form of C in Eq.* (9) *such that the disjunctive uninorm U given by Eq.* (2) *is distributive over a noncontinuous t-norm T.*

Figure 3: Structure of U (left) and T (right) in Proposition 3.14.

3.3 Case for $U(0,x)=0$ for all $x\in[0,e]$ and $U(0,x)=x$ for all $x\in]e,1]$

By Proposition 2.5(iii), if a uninorm U satisfies $U(0,x)=0$ for all $x\in[0,e]$ and $U(0,x)=x$ for all $x\in]e,1]$, then $U\in\mathcal{U}^e_{\max}$. Thus, in the sequel, we apply the symbol $\mathcal{U}^e_{\max}$ to denote the family of all uninorms fulfilling $U(0,x)=0$ for all $x\in[0,e]$ and $U(0,x)=x$ for all $x\in]e,1]$.

Lemma 3.13. *Let a uninorm $U\in\mathcal{U}^e_{\max}$ be distributive over a t-norm T with $T(e,e)=0$. Then the following hold.*

(i) *$T(x,x)=0$ for all $x\in[0,e]$ and $T(x,x)=x$ for all $x\in]e,1]$.*

(ii) *T is noncontinuous.*

Proof. (i) It follows from Proposition 2.2 and $T(e,e)=0$ that the following holds for all $x\in[0,e]$,

$$0=U(x,0)=U\big(x,T(e,e)\big)=T\big(U(x,e),U(x,e)\big)=T(x,x).$$

By Proposition 2.2, we obtain for all $x\in]e,1]$,

$$x=\max(x,0)=U(x,0)=U\big(x,T(0,0)\big)=T\big(U(x,0),U(x,0)\big)=T(x,x).$$

Thus, we obtain $T(x,x)=0$ for all $x\in[0,e]$ and $T(x,x)=x$ for all $x\in]e,1]$.
(ii) It can be proven in a similar way as for Proposition 3.2. □

Similarly to Propositions 3.3, 3.4 and 3.5, we characterize the distributivity of a uninorm $U\in\mathcal{U}^e_{\max}$ over a noncontinuous t-norm T under the condition $T(e,e)=0$ as follows.

Proposition 3.14. *Let a uninorm $U\in\mathcal{U}^e_{\max}$ be distributive over a t-norm T with $T(e,e)=0$. Then there exists $C:]e,1]\times[0,e]\to[0,e]$ such that T has the following form*

$$T(x,y)=\begin{cases}0, & \text{if } (x,y)\in[0,e]^2;\\ \min(x,y), & \text{if } (x,y)\in]e,1]^2;\\ C(x,y), & \text{if } (x,y)\in]e,1]\times[0,e];\\ C(y,x), & \text{if } (x,y)\in[0,e]\times]e,1];\end{cases}\tag{13}$$

where C is increasing with respect to each variable and satisfies $C(1,y)=y$ for all $y\in[0,e]$, $C(x,0)=0$ for all $x\in]e,1]$ and the following equation for all $x\in[0,e]$, $y\in]e,1]$ and $z\in]e,1]$ (see Figure 3),

$$C\big(z,C(y,x)\big)=C\big(\min(y,z),x\big).\tag{14}$$

Proposition 3.15. *Let a uninorm $U\in\mathcal{U}^e_{\max}$ and a t-norm T be given by Eq. (13). Then U is distributive over T, if either one of the following cases is satisfied.*

(i) *C has the following form for all $(x,y)\in]e,1]\times[0,e]$,*

$$C(x,y)=\begin{cases}0, & \text{if } x\in]e,1[;\\ y, & \text{if } x=1.\end{cases}\tag{15}$$

(ii) *C has the following form for all $(x,y)\in]e,1]\times[0,e]$,*

$$C(x,y)=\min(x,y).\tag{16}$$

Example 3.16. *Consider a uninorm $U \in \mathcal{U}_{\max}^{0.5}$ have the following form,*

$$U(x,y) = \begin{cases} 2xy, & \text{if } (x,y) \in [0,0.5]^2; \\ \max(x,y), & \text{otherwise;} \end{cases}$$

and let a t-norm T_1 given as

$$T_1(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[\times[0,0.5] \cup [0,0.5] \times [0,1[; \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Then T_1 is noncontinuous. From Proposition 3.4, we obtain the distributivity of U over T_1.

Consider a t-norm T_2 given as

$$T_2(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,0.5]^2; \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Then T_2 is a noncontinuous t-norm. It follows from Proposition 3.4 that the uninorm U is distributive over the noncontinuous t-norm T_2.

Consider the t-norm T_3 be the nilpotent minimum. Then T_3 is obviously noncontinuous and fulfills $T(0.5, 0.5) = 0$. Consider $x_0 = 0.4$, $y_0 = 0.4$ and $z_0 = 0.65$. Then the following hold,

$$U\big(x_0, T_3(y_0, z_0)\big) = U\big(0.4, T_3(0.4, 0.65)\big) = U(0.4, 0.4) = 0.32,$$
$$T_3\big(U(x_0, y_0), U(x_0, z_0)\big) = T_3\big(U(0.4, 0.4), U(0.4, 0.65)\big) = T_3(0.32, 0.65) = 0.$$

Thus U is not distributive over T_3.

3.4 Case for $U(0,x) = 0$ for all $x \in [0,1[$ and $U(0,1) = 1$

In contrast to Sections 3.1, 3.2 and 3.3, we obtain the following necessary and sufficient condition for a disjunctive uninorm distributive over a noncontinuous t-norm, where the uninorm U satisfies $U(0,x) = 0$ for all $x \in [0,1[$.

Proposition 3.17. *Let a disjunctive uninorm fulfill $U(0,x) = 0$ for all $x \in [0,1[$ with the neutral element e and a t-norm T fulfill $T(e,e) = 0$. Then U is distributive over T if and only if T has the following form,*

$$T(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[^2; \\ \min(x,y), & \text{otherwise.} \end{cases} \tag{17}$$

Proof. **Necessity.** Because $U(0,x) = 0$ holds for all $x \in [0,1[$, we obtain for all $x \in [0,1[$,

$$0 = U(0,x) = U\big(T(e,e), x\big) = T\big(U(e,x), U(e,x)\big) = T(x,x).$$

Thus we have $T(x,y) = 0$ for all $(x,y) \in [0,1[^2$. Therefore, T fulfills Eq. (17).

Sufficiency. We verify the conclusion in the following cases.

Case 1: $x = 1$, $y \in [0,1]$, $z \in [0,1]$

As the uninorm U is disjunctive, we obtain $U\big(x, T(y,z)\big) \geqslant U(1,0) = 1$, i.e., $U\big(x, T(y,z)\big) = 1$. Similarly, $U(x,y) = 1$ and $U(x,z) = 1$ hold. Thus $T\big(U(x,y), U(x,z)\big) = T(1,1) = 1$ holds.

Case 2: $x \in [0,1]$, $y = 1$, $z \in [0,1]$

Similarly to Case 1, we obtain $U(x,y) = 1$. Thus the following hold,

$$U\big(x, T(y,z)\big) = U(x,z) \text{ and } T\big(U(x,y), U(x,z)\big) = T\big(1, U(x,z)\big) = U(x,z).$$

Case 3: $x \in [0,1]$, $y \in [0,1]$, $z = 1$

It can be proven in a similar way as for Case 2.

Case 4: $x \in [0,1[$, $y \in [0,1[$, $z \in [0,1[$

It follows from $U(0,x) = 0$ for all $x \in [0,1[$ and Eq. (17) that the following holds,

$$U\big(x, T(y,z)\big) = U(x,0) = 0.$$

By Proposition 2.5(iv), the underlying t-conorm of U is positive. Thus we obtain

$$U(x,y) < U(1,1) = 1 \text{ and } U(x,z) < U(1,1) = 1.$$

By Eq. (17), we obtain $T\big(U(x,y), U(x,z)\big) = 0$. □

Remark 3.18. *In Section 3, we apply the premise $T(e,e) = 0$ to characterize the distributivity of a disjunctive uninorm U over a noncontinuous t-norm T, where U is locally internal on the boundary with the neutral element e. Unfortunately, the complete characterization is still missing.*

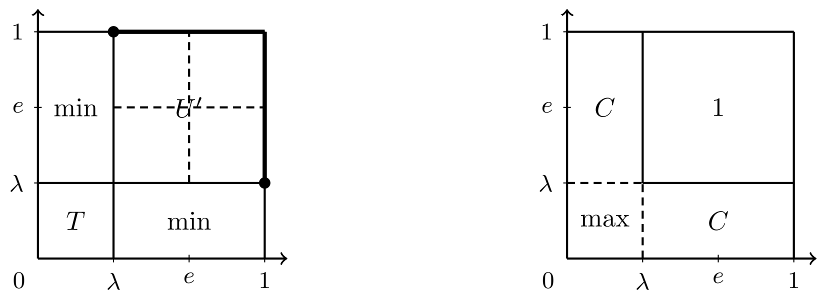

Figure 4: Structure of U (left) and S (right) in Proposition 4.2.

4 Conjunctive uninorms locally internal on the boundary distributive over noncontinuous t-conorms

In this section, we discuss the distributivity of a conjunctive uninorm U over a t-conorm S with $S(e,e)=1$ by duality.

4.1 Case for $U(x,1)=x$ for all $x\in[0,\lambda[$ and $U(x,1)=1$ for all $x\in[\lambda,1]$

Lemma 4.1. *Let a conjunctive uninorm U given by Eq. (3) be distributive over a t-conorm S with $S(e,e)=1$. Then the following hold,*

(i) *$S(x,x)=x$ for all $x\in[0,\lambda[$ and $S(x,x)=1$ for all $x\in[\lambda,1]$.*

(ii) *S is noncontinuous.*

We have the following necessary condition for a conjunctive uninorm U given by Eq. (3) distributive over a noncontinuous t-conorm S with the assumption $S(e,e)=1$.

Proposition 4.2. *Let a conjunctive uninorm U given by Eq. (3) be distributive over a t-conorm S with $S(e,e)=1$. Then there exists $C:[\lambda,1]\times[0,\lambda[\to[\lambda,1]$ such that S has the following form*

$$S(x,y)=\begin{cases}\max(x,y), & \text{if } (x,y)\in[0,\lambda[^2;\\ C(x,y), & \text{if } (x,y)\in[\lambda,1]\times[0,\lambda[;\\ C(y,x), & \text{if } (x,y)\in[0,\lambda[\times[\lambda,1];\\ 1, & \text{if } (x,y)\in[\lambda,1]^2;\end{cases}\tag{18}$$

where C is increasing with respect to each variable and satisfies $C(x,0)=x$ for all $x\in[\lambda,1]$, $C(1,y)=1$ for all $y\in[0,\lambda[$ and the following equation for all $x\in[\lambda,1]$, $y\in[0,\lambda[$ and $z\in[0,\lambda[$ (see Figure 4),

$$C\big(C(x,y),z\big)=C\big(x,\max(y,z)\big).\tag{19}$$

Similarly to Propositions 3.4 and 3.5, the sufficient conditions for a conjunctive uninorm U given by Eq. (3) over a noncontinuous t-conorm S are studied by assuming the specific forms of C in Eq. (18).

Proposition 4.3. *Let a conjunctive uninorm U be given by Eq. (3) and a t-conorm S be given by Eq. (18). Then U is distributive over S, if either one of the following cases is satisfied.*

(i) *C has the following form for all $(x,y)\in[\lambda,1]\times[0,\lambda[$,*

$$C(x,y)=\begin{cases}1, & \text{if } y\in]0,\lambda[;\\ x, & \text{if } y=0.\end{cases}\tag{20}$$

(ii) *C has the following form for all $(x,y)\in[\lambda,1]\times[0,\lambda[$,*

$$C(x,y)=\max(x,y).\tag{21}$$

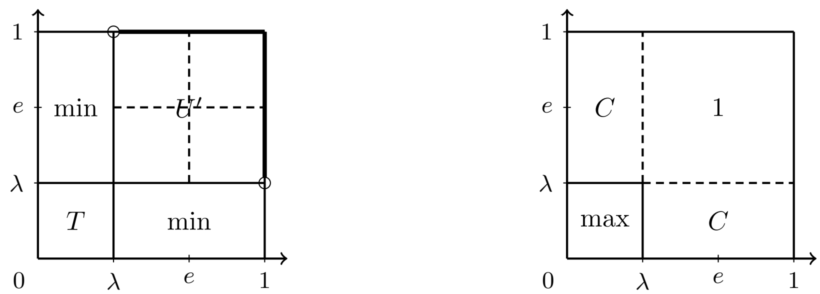

Figure 5: Structure of U (left) and S (right) in Proposition 4.5.

4.2 Case for $U(x,1)=x$ for all $x\in[0,\lambda]$ and $U(x,1)=1$ for all $x\in]\lambda,1]$

Lemma 4.4. *Let a conjunctive uninorm U given by Eq. (4) be distributive over a t-conorm S with $S(e,e)=1$. Then the following hold,*

(i) *$S(x,x)=x$ for all $x\in[0,\lambda]$ and $S(x,x)=1$ for all $x\in]\lambda,1]$.*

(ii) *S is noncontinuous.*

We obtain the following necessary condition for a conjunctive uninorm U given by Eq. (4) over a noncontinuous t-conorm S with the assumption $S(e,e)=1$.

Proposition 4.5. *Let a conjunctive uninorm U given by Eq. (4) be distributive over a t-conorm S with $S(e,e)=1$. Then there exists $C:]\lambda,1]\times[0,\lambda]\to]\lambda,1]$ such that S has the following form*

$$S(x,y)=\begin{cases}\max(x,y), & \text{if } (x,y)\in[0,\lambda]^2;\\ C(x,y), & \text{if } (x,y)\in]\lambda,1]\times[0,\lambda];\\ C(y,x), & \text{if } (x,y)\in[0,\lambda]\times]\lambda,1];\\ 1, & \text{if } (x,y)\in]\lambda,1]^2;\end{cases} \tag{22}$$

where C is increasing with respect to each variable and satisfies $C(x,0)=x$ for all $x\in]\lambda,1]$, $C(1,y)=1$ for all $y\in[0,\lambda]$ and the following equation for all $x\in]\lambda,1]$, $y\in[0,\lambda]$ and $z\in[0,\lambda]$ (see Figure 5),

$$C\big(C(x,y),z\big)=C\big(x,\max(y,z)\big). \tag{23}$$

Consider C in Eq. (22) be specific. Then we have the following sufficient conditions for a conjunctive uninorm U given by Eq. (4) over a noncontinuous t-conorm S with the assumption $S(e,e)=1$.

Proposition 4.6. *Let a conjunctive uninorm U be given by Eq. (4) and a t-conorm S be given by Eq. (22). Then U is distributive over S, if either one of the following cases is satisfied.*

(i) *C has the form for all $(x,y)\in]\lambda,1]\times[0,\lambda]$,*

$$C(x,y)=\begin{cases}1, & \text{if } y\in]0,\lambda];\\ x, & \text{if } y=0.\end{cases} \tag{24}$$

(ii) *C has the form for all $(x,y)\in]\lambda,1]\times[0,\lambda]$,*

$$C(x,y)=\max(x,y). \tag{25}$$

Remark 4.7. *Similarly to Remarks 3.8 and 3.12, we cannot obtain the regularity of the form of C in Eqs.* (18) *and* (22) *such that a conjunctive uninorm U is distributive over a t-conorm S.*

Figure 6: Structure of U (left) and S (right) in Proposition 4.9.

4.3 Case for $U(x,1)=x$ for all $x\in[0,e[$ and $U(x,1)=1$ for all $x\in[e,1]$

By duality, we study the distributivity of a uninorm $U\in\mathcal{U}^{e}_{\min}$ over a noncontinuous t-conorm S under the condition $S(e,e)=1$.

Lemma 4.8. *Let a uninorm $U\in\mathcal{U}^{e}_{\min}$ be distributive over a t-conorm S with $S(e,e)=1$. Then the following hold.*

(i) *$S(x,x)=1$ for all $x\in[e,1]$ and $S(x,x)=x$ for all $x\in[0,e[$.*

(ii) *S is noncontinuous.*

By dual property, the distributivity of uninorms in $\mathcal{U}^{e}_{\min}$ over noncontinuous t-conorms is discussed as follows.

Proposition 4.9. *Let a uninorm $U\in\mathcal{U}^{e}_{\min}$ be distributive over a t-conorm S with $S(e,e)=1$. Then there exists $C:[e,1]\times[0,e[\to[e,1]$ such that S has the following form*

$$S(x,y)=\begin{cases}\max(x,y), & \text{if } (x,y)\in[0,e[^2;\\ C(x,y), & \text{if } (x,y)\in[e,1]\times[0,e[;\\ C(y,x), & \text{if } (x,y)\in[0,e[\times[e,1];\\ 1, & \text{if } (x,y)\in[e,1]^2;\end{cases} \tag{26}$$

where C is increasing with respect to each variable and satisfies $C(x,0)=x$ for all $x\in[e,1]$, $C(1,y)=1$ for all $y\in[0,e[$ and the following equation for all $x\in[e,1]$, $y\in[0,e[$ and $z\in[0,e[$ (see Figure 6),

$$C\big(C(x,y),z\big)=C\big(x,\max(y,z)\big). \tag{27}$$

Proposition 4.10. *Let a uninorm $U\in\mathcal{U}^{e}_{\min}$ and a t-conorm S given by Eq. (26). Then U is distributive over S, if either one of the following cases is satisfied.*

(i) *C has the following form for all $(x,y)\in[e,1]\times[0,e[$,*

$$C(x,y)=\begin{cases}1, & \text{if } y\in]0,e[;\\ x, & \text{if } y=0.\end{cases} \tag{28}$$

(ii) *C has the following form for all $(x,y)\in[e,1]\times[0,e[$,*

$$C(x,y)=\max(x,y). \tag{29}$$

4.4 Case for $U(0,1)=0$ and $U(x,1)=1$ for all $x\in]0,1]$

Different from Sections 4.1, 4.2 and 4.3, the necessary and sufficient condition for a conjunctive uninorm distributive over a noncontinuous t-conorm is studied by duality, where the uninorm U satisfies $U(x,1)=1$ for all $x\in]0,1]$.

Proposition 4.11. *Let a conjunctive uninorm fulfill $U(x,1)=1$ for all $x\in]0,1]$ with the neutral element e and a t-conorm S fulfill $S(e,e)=1$. Then U is distributive over T if and only if S has the following form,*

$$S(x,y)=\begin{cases}1, & \text{if } (x,y)\in]0,1]^2;\\ \max(x,y), & \text{otherwise.}\end{cases} \tag{30}$$

Remark 4.12. *In Section 4, the auxiliary condition $S(e,e)=1$ plays an important role in studying the distributivity of a conjunctive uninorm U over a noncontinuous t-conorm S, where U is locally internal on the boundary with the neutral element e. There may be another condition playing the same role as $S(e,e)=1$. Thus, it is a future work to find the new condition(s) applied to discuss the distributivity for uninorms over noncontinuous t-conorms.*

5 Conclusions

Although the open question "*Characterize all (conditionally) distributive uninorms over a given (continuous) t-conorm*" recalled by Klement has been discussed in [16, 18, 19, 29, 31], t-(co)norms were always assumed to be continuous. We study the necessary conditions (resp. the sufficient conditions) for a uninorm distributive over a noncontinuous t-(co)norm under the certain condition to provide the noncontinuous solutions of the question recalled by Klement, where the uninorm is locally internal on the boundary. We analyze the distributivity equation on the basis of different cases of uninorms locally internal on the boundary presented by Li and Liu [17]. In particular, we obtain the necessary and sufficient condition for uninorms distributive over noncontinuous t-(co)norms for special cases of uninorms locally internal on the boundary. Meanwhile, this paper can be viewed as a supplement to [16, 18, 19, 29, 31] to some extent. We hope that the results in our paper will be helpful in applications in fuzzy logic, image processing, approximate reasoning and modeling some specific problems in the utility theory.

As we only characterize a special case of the distributivity of a uninorm U with a neutral element e over a noncontinuous t-norm T (resp. t-conorm S) under the condition $T(e,e) = 0$ (resp. $S(e,e) = 1$) in our paper, we intend to study the distributivity of uninorms over noncontinuous t-(co)norms further in the future work.

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