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Original paper

Characterization of bivariate quadratic transformations of quasi-copulas

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Abstract

In this study, we focus on bivariate transformations of trivariate quasi-copulas. We characterize necessary and sufficient conditions of transformations that can be written in the form of compositions between two quasi-copulas and a quadratic polynomial function. The conditions only depend on the coefficients of the quadratic polynomial. The set of these coefficients is convex with linear-section boundary, and it lies on the seven-dimensional Euclidean space. All extreme points of this set have been characterized via CAS and can be used to construct quasi-copulas. Construction examples are also given.

Keywords: Aggregation function, quasi-copula, quadratic transformation.

1 Introduction

In the present, data are important for various sectors. For instance, a business sector needs data to predict the behavior of customers for improving their products or marketing plans. The public health sector needs patients' data to predict the incubation period of a disease. Thus, the efficiency of data analysis is especially important.

Sometimes, analyzing a large amount of data is overly complicated. Choosing a representation of an interesting data set is then necessary and it is usually more efficient than directly analyzing raw data. Aggregation functions play a vital role in this process. In fact, aggregation functions were used for a long time. In the third century B.C., the simple aggregation function as the arithmetic mean was first used by Babylonian astronomers. They used it to compute the position of the sun, the moon, and the planets [11]. In addition, concepts of aggregation functions were used by the Egyptian to compute the volume of a frustum of a square pyramid in 1850 B.C [2]. Now, applications of aggregation functions appear in many fields such as mathematics, statistics, information theory, and decision making. Well-known examples of aggregation functions include the arithmetic mean, the minimum function, the maximum function, the product function, and the median. More can be found, for example, in [4, 12, 14, 22].

The systematic construction of aggregation functions has become an interesting topic in the past few decades. Since data have several types, it is necessary to have many types of aggregation functions for analyzing each of them. Thus, constructions of aggregation functions become attractive. Several methods have been introduced recently. This includes, for example, transformations of aggregation functions into new ones as in [5, 6, 7, 15, 16, 17, 18, 19, 20, 21, 23] and transformations of other functions to aggregation functions as in [8, 10].

In this work, we will explore the transformation-based constructions introduced by Kolesrov et al. in 2013 [18]. In that work, Kolesrov et al. constructed bivariate copulas and bivariate quasi-copulas by transforming existing copulas with a specific quadratic polynomial. Later in 2015, Kolesrov et al. [16] characterized all quadratic polynomials that are able to transform copulas. In the same year, Kolesrov and Mesiar [17] characterized quadratic polynomials that are able to transform semi-copulas and quasi-copulas. Following on this idea, Wisadwongsa and Tasena [23] considered the idea of bivariate transformations of bivariate copulas. They were able to characterize quadratic polynomials for

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such case via linear inequalities of their coefficients. Tasena [21] also further considered multivariate transformations of bivariate copulas via higher order polynomials. Yet, all these works only focus on bivariate cases.

The work on multivariate cases was only started recently by Boonmee and Tasena [6] who completely characterized univariate quadratic transformations of a multivariate semi-copula and a multivariate quasi-copula. Later, Boonmee and Chanthorn [5] were also able to characterize quadratic transformations of multivariate semi-copulas. It turns out the structure of multivariate transformations is much more complicated than that of univariate transformations. Moreover, a characterization of multivariate quadratic transformations of multivariate quasi-copulas is not done at all.

In this work, we hope to shed light into this complex problem by focusing on the bivariate quadratic transformations of trivariate quasi-copulas. We are able to completely characterize such polynomials via linear inequalities of their coefficients. It turns out that such polynomials form a convex set with 78 extreme points which is much more complicated than the univariate case where only one nontrivial extreme point exists. We are also able to show that this class of transformations is different from the class of quadratic transformations of trivariate semi-copulas characterized by [5].

In the next section, we will introduce some related terminologies used throughout this study. In Section 3, we will present our main results and example usage. Last, the conclusion and discussion will be done in Section 4.

2 Preliminaries

Here and henceforth, denote the unit interval by $[0,1]$. The notation $\vec{x}$ stands for the vector $(x_1,\ldots,x_k)$. Most of the time, k will either be 2 or 3. The vectors $\vec{0}$ and $\vec{1}$ stand for the constant vector of zeros and the vector of ones, respectively.

A nondecreasing function $Q:[0,1]^k\to[0,1]$ is called a *quasi-copula* if it satisfies the following two conditions.

(B) For all $\vec{x}$, $Q(\vec{x})=0$ if $x_i=0$ for some i and $Q(\vec{x})=x_j$ if $x_i=1$ for all $i\neq j$.

(L) For all $\vec{x}$ and $\vec{y}$, $|Q(\vec{x})-Q(\vec{y})|\leq\sum_i|x_i-y_i|$.

The condition (B) is also called the boundary condition while the condition (L) is called the Lipschitz condition. Two well-known quasi-copulas are the Frchet-Hoeffding bounds M and W defined by

$$M(\vec{x})=\min(x_1,\ldots,x_k),$$

and

$$W(\vec{x})=\max\left(0,\sum_{i=1}^k x_i-k+1\right),$$

for all $\vec{x}\in[0,1]^k$. It is known that $W\leq Q\leq M$ for any quasi-copula Q. In fact, it can be proved that a nondecreasing Lipschitz function Q satisfies (B) if and only if $W\leq Q\leq M$.

It should be mentioned that the original definition for bivariate quasi-copulas given in [1] is different from the above conditions. Nevertheless, these two are equivalent as shown for the bivariate case in [13] and the multivariate case in [9].

Recall that a function $\Gamma:[0,1]^k\to\mathbb{R}$ with property (B) is *absolutely continuous* if and only if there is a function $\gamma:[0,1]^k\to\mathbb{R}$ such that $\Gamma(\vec{x})=\int_{[\vec{0},\vec{x}]}\gamma(\vec{t})\,d\vec{t}$ for all $\vec{x}\in[0,1]^k$. This means that $\partial_1\cdots\partial_k\Gamma$ must exist a.e. and that the fundamental theorem of calculus must hold for Γ. After all, we would have $\gamma=\partial_1\cdots\partial_k\Gamma$ a.e. It can easily be seen that not all quasi-copulas are absolutely continuous. This happens, for example, in the case of W and M. More information regarding this issue can also be found in [13]. Although, the fact that all quasi-copulas are Lipschitz implies that they are all *separately absolutely continuous* in the sense that the function $x\mapsto Q(x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,x_k)$ is absolutely continuous on $[0,1]$ for all $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_k\in[0,1]$ and for all i. In other words, they are absolutely continuous with respect to each variable [3]. Therefore, $0\leq\partial_iQ\leq1$ a.e. for any quasi-copula Q. The converse is also true by the fundamental theorem of calculus, that is, a function Q satisfying (B) is a quasi-copula if and only if Q is separately absolutely continuous and $0\leq\partial_iQ\leq1$ a.e. for all i. This fact will play an important role in the proof of our results.

Two generalizations of quasi-copulas worth mentioning in this work are aggregation functions and semi-copulas. An *aggregation function* is simply a nondecreasing function sending $\vec{0}$ to 0 and $\vec{1}$ to 1. A *semi-copula* is an aggregation function satisfying (B). Notice that a semi-copula is not necessarily continuous and even if it is, it is not necessarily a quasi-copula. On the other hands, if a quasi-copula Q satisfies

$$\sum_{\vec{x}\in\prod_{i=1}^k\{a_i,b_i\}}(-1)^{\#\{i|x_i=a_i\}}Q(\vec{x})\geq0, \qquad (V)$$

whenever $\vec{a} \leq \vec{b}$, then it will be called a *copula*. Clearly, not all quasi-copulas are copulas. Thus, the classes of aggregation functions, semi-copulas, quasi-copulas, and copulas are all distinct.

Over the years, several construction methods of these functions were presented. Here, we are interested in transformation methods via quadratic polynomials, that is, a transformation of the form

$$T_P\left(f_1, \ldots, f_n\right)(\vec{x}) = P\left(\vec{x}, f_1(\vec{x}), \ldots, f_n(\vec{x})\right),$$

where P is some quadratic polynomial. The goal is usually to characterize P in such a way that T_P preserves the class of aggregation functions (semi-copulas, quasi-copulas, copulas, etc.). Several transformations of this type have been proposed, see the introduction for more details. Here, we will only provide the most recent result, which is also the closest one to our work. In [6], it is shown that a transformation of the form

$$\tau_1(f)(\vec{x}) = \frac{1}{k-1} f(\vec{x})\left(\sum_{i=1}^{k} x_i - f(\vec{x})\right),$$

preserves the class of semi-copulas and quasi-copulas. Moreover, any univariate transformation that preserved the class of semi-copulas (quasi-copulas) must be a convex combination of τ_1 and the identity transformation. In [5], a characterization of multivariate transformations that preserved the class of semi-copulas is also given. A characterization of multivariate transformations that preserved the class of quasi-copulas, however, is never found. Thus, it is interesting to see whether a characterization can be found. It turns out that this problem is much more difficult in the case of quasi-copulas compared to the case of semi-copulas. Thus, we will only present the result in the case of bivariate transformations of trivariate quasi-copulas. We hope that our results will pave the way to the general case in the future.

3 Results

Instead of directly constructing a few bivariate quadratic transformations of (trivariate) quasi-copulas, we will characterize all such polynomials including figuring out all extreme points.

Since the set of quasi-copulas is compact and convex, the set of polynomials P for which T_P is a bivariate transformation of (trivariate) quasi-copulas must also be compact and convex. In fact, such polynomials must be given in the form

$$\begin{aligned} P(\vec{x}, z_1, z_2) = {} & (1-a)\left(\frac{1}{4}(z_1+z_2)\left(\sum_{i=1}^{3} x_i\right) - \frac{1}{8}(z_1+z_2)^2\right) + \frac{1}{2}a(z_1+z_2) \\ & + b(z_1-z_2)^2 + c(z_1-z_2) + d(z_1-z_2)\left(z_1+z_2+2-\sum_{i=1}^{3} x_i\right) \\ & + (z_1-z_2)\sum_{i=1}^{3} e_i\left(x_i - \frac{2}{3}\right). \end{aligned} \tag{1}$$

The proof will be given in the following lemma.

Lemma 3.1. *Let $P = P(x_1, x_2, x_3, z_1, z_2)$ be a quadratic polynomial such that T_P is a bivariate transformation of trivariate quasi-copulas. Then P must be given in the form of (1).*

Proof. Notice that any quadratic polynomial P must be given in the form

$$P(\vec{x}, z_1, z_2) = \sum_{p,q} a_{pq} z_p z_q + \sum_{i,q} b_{iq} x_i z_q + \sum_q c_q z_q + \sum_{i,j} d_{ij} x_i x_j + \sum_i e_i x_i + f,$$

where we assume without loss of generality that $a_{12} = a_{21}$ and $d_{ij} = d_{ji}$ for all $i, j = 1, 2, 3$. Notice that $T_P(M, M)$ is a quasi-copula by the assumption. The fact that $T_P(M, M)(x, 0, 0) = 0$ for all $x \in [0, 1]$ then forces $d_{11} = e_1 = f = 0$. Similarly, $T_P(M, M)(0, x, 0) = T_P(M, M)(0, 0, x) = 0$ for all $x \in [0, 1]$ implies $d_{22} = d_{33} = e_2 = e_3 = 0$, and the fact that $T_P(M, M)(1, 1, 0) = T_P(M, M)(0, 1, 1) = T_P(M, M)(1, 0, 1) = 0$ implies $d_{ij} = 0$ for all $i \neq j$. Therefore, only a_{pq}, b_{iq}, and c_q matter.

Next, we use the fact that $T_P(M, M)(x, 1, 1) = x$ to conclude that

$$1 = \left(\sum_{p,q} a_{pq} + \sum_q b_{1q}\right) x + \sum_q \left(b_{2q} + b_{3q} + c_q\right),$$

for all $x \in (0,1]$. This implies

$$\sum_{p,q} a_{pq} + \sum_{q} b_{1q} = 0 = 1 - \sum_{q} \left(b_{2q} + b_{3q} + c_q\right).$$

By applying similar argument to $T_P(M,M)(1,x,1) = x = T_P(M,M)(1,1,x)$, we also have

$$\sum_{p,q} a_{pq} + \sum_{q} b_{iq} = 0 = 1 - \sum_{q} \left(\sum_{j\neq i} b_{jq} + c_q\right),$$

for $i = 1,2,3$. Therefore,

$$\sum_{q} b_{iq} = -\sum_{p,q} a_{pq} = \frac{1}{2}\left(1 - \sum_{q} c_q\right), \tag{2}$$

for $i = 1,2,3$.

Last, set $a = c_1 + c_2$, $b = \frac{1}{2}(a_{11} + a_{22}) + \frac{1}{8}(1-a)$, $c = \frac{1}{3}\sum_i (b_{i1} - b_{i2}) + \frac{1}{2}(c_1 - c_2)$, $d = \frac{1}{2}(a_{11} - a_{22})$, and $e_i = d + \frac{1}{2}(b_{i1} - b_{i2})$. It can be checked that this transformation is faithful and does not violate (2). In fact, we have $a_{11} = b - \frac{1}{8}(1-a) + d$, $a_{22} = b - \frac{1}{8}(1-a) - d$, $2a_{12} = -2b - \frac{1}{4}(1-a)$, $b_{i1} = \frac{1}{4}(1-a) + e_i - d$, $b_{i2} = \frac{1}{4}(1-a) - e_i + d$, $c_1 = \frac{1}{2}a + c + 2d - \frac{2}{3}\sum_i e_i$, and $c_2 = \frac{1}{2}a - c - 2d + \frac{2}{3}\sum_i e_i$. Rewrite the polynomial P accordingly yields (1). □

Note that the form of P in (1) is not sufficient for T_P to be a bivariate transformation of (trivariate) quasi-copulas. It is only guaranteed that $T_P(Q_1,Q_2)$ satisfies the boundary condition (B) for any quasi-copulas Q_1 and Q_2. Thus, we have to characterize the conditions for which $T_P(Q_1,Q_2)$ is nondecreasing Lipschitz for all quasi-copulas Q_1 and Q_2. Since $T_P(Q_1,Q_2)$ is separately absolutely continuous, this is equivalent to

$$0 \leq \partial_i T_P(Q_1,Q_2) \leq 1,$$

a.e. for all $i = 1,2,3$. A direct computation shows that $\partial_i T_P(Q_1,Q_2)(\vec{x}) = \tau_i(\vec{x}, \partial_i Q_1(\vec{x}), \partial_i Q_2(\vec{x}), Q_1(\vec{x}), Q_2(\vec{x}))$ where

$$\begin{aligned}\tau_i\left(\vec{x},\vec{t},\vec{q}\right) &= (t_1+t_2)\left(\frac{1}{4}(1-a)\left(\sum_i x_i - q_1 - q_2\right) + \frac{1}{2}a + d(q_1-q_2)\right)\\ &\quad + (t_1-t_2)\left(c + d\left(q_1+q_2+2-\sum_i x_i\right) + \sum_i e_i\left(x_i - \frac{2}{3}\right)\right)\\ &\quad + (q_1-q_2)\left(2b(t_1-t_2) - d + e_i\right) + \frac{1}{4}(1-a)(q_1+q_2),\end{aligned}$$

whenever both $\partial_i Q_1(\vec{x})$ and $\partial_i Q_2(\vec{x})$ exist. Thus, it is sufficient to show that $0 \leq \tau_i(\vec{x}, t_1, t_2, q_1, q_2) \leq 1$ for all $t_1, t_2 \in [0,1]$, $q_1, q_2 \in [W(\vec{x}), M(\vec{x})]$ and all $\vec{x} \in [0,1]^3$. It turns out that this condition is actually equivalent to $0 \leq \partial_i T_P(Q_1,Q_2) \leq 1$. Its proof will be based on constructing appropriate quasi-copulas for each value of t_1, t_2, q_1, and q_2 as shown in the following lemma.

Lemma 3.2. *For any given $\vec{a} \in [0,1]^3$, $q \in [W(\vec{a}), M(\vec{a})]$, and a quasi-copula Q, there is a quasi-copula $C = C_{\vec{a},q,Q}$ such that*

$$C(\vec{x}) = qQ\left(\left(\frac{x_1 - a_1 + q}{q}\right)\vee 0, \left(\frac{x_2 - a_2 + q}{q}\right)\vee 0, \left(\frac{x_3 - a_3 + q}{q}\right)\vee 0\right), \tag{3}$$

whenever $\vec{x} \leq \vec{a}$.

Proof. For this to work, we need to find constants $g_1, g_2, g_3 \in [0,1]$ such that

$$\begin{aligned} g_2 + g_3 &\leq a_1 - q,\\ g_1 + g_3 &\leq a_2 - q,\\ g_1 + g_2 &\leq a_3 - q,\\ g_1 + g_2 + g_3 &\geq a_1 + a_2 + a_3 - 1 - 2q.\end{aligned} \tag{4}$$

If $\sum_i a_i - 2a_j \geq 0$ for all j, we can simply choose $g_j = \frac{1}{2}\left(\sum_i a_i - 2a_j - q\right)$ for all j. If $\sum_i a_i - 2a_j < 0$ for a specific j, we can choose $g_j = 0$ and $g_i = a_l - q$ for all distinct i, j, l. It is straightforward to verify that $g_1, g_2, g_3 \in [0,1]$ are indeed a solution to (4).

Denote $t_{r,s}(x) = 0 \vee \left(\frac{x-r}{s-r}\right) \wedge 1$ for all $x \in [0,1]$ and define $U_{i,1} = t_{0,a_i-q}$, $U_{i,2} = t_{a_i-q,a_i}$, and $U_{i,3} = t_{a_i,1}$. Now, define $C = C_{\vec{a},q,Q}$ via

$$\begin{aligned} C(\vec{x}) &= qQ\left(U_{1,2}(x_1), U_{2,2}(x_2), U_{3,2}(x_3)\right) + g_1 U_{1,3}(x_1) U_{2,1}(x_2) U_{3,1}(x_3) \\ &+ g_2 U_{1,1}(x_1) U_{2,3}(x_2) U_{3,1}(x_3) + g_3 U_{1,1}(x_1) U_{2,1}(x_2) U_{3,3}(x_3) \\ &+ (a_1 - q - g_2 - g_3) U_{1,1}(x_1) U_{2,3}(x_2) U_{3,3}(x_3) \\ &+ (a_2 - q - g_1 - g_3) U_{1,3}(x_1) U_{2,1}(x_2) U_{3,3}(x_3) \\ &+ (a_3 - q - g_1 - g_2) U_{1,3}(x_1) U_{2,3}(x_2) U_{3,1}(x_3) \\ &+ \left(1 + 2q + \sum_i g_i - \sum_i a_i\right) U_{1,3}(x_1) U_{2,3}(x_2) U_{3,3}(x_3). \end{aligned}$$

It is straightforward to check, although a bit tedious, that C satisfies (B) and (3). The fact that it is a convex combination of nondecreasing separately absolutely continuous functions also implies that it is nondecreasing and separately absolutely continuous. A direct computation also shows that $0 \le \partial_i C(\vec{x}) \le 1$ a.e. Thus, C is indeed a quasi-copula. □

Now, we are in the position to prove our claim regarding $\partial_i T_P(Q_1, Q_2)$.

Lemma 3.3. *Let $P = P(x_1, x_2, x_3, z_1, z_2)$ be a quadratic polynomial in the form of (1). For each $i = 1,2,3$, the following statements are equivalent.*

1. $0 \le \partial_i T_P(Q_1, Q_2) \le 1$ *a.e. for all quasi-copulas Q_1 and Q_2.*
2. $0 \le \tau_i(\vec{a}, t_1, t_2, q_1, q_2) \le 1$ *for all $t_1, t_2 \in [0,1]$, $q_1, q_2 \in [W(\vec{a}), M(\vec{a})]$ and all $\vec{a} \in [0,1]^3$.*
3. $0 \le \tau_i(\vec{a}, t_1, t_2, q_1, q_2) \le 1$ *for all $t_1, t_2 \in \{0,1\}$, $q_1, q_2 \in \{W(\vec{a}), M(\vec{a})\}$ and all $\vec{a} \in [0,1]^3$.*

Proof. The equivalence between the second and the last statements follows from the fact that τ_i is linear in the last four variables. Obviously, the second statement implies the first one. For the converse, let $\vec{a} \in [0,1]^3$ and $q \in \{W(\vec{a}), M(\vec{a})\}$. Set $Q_{1,q} = C_{\vec{a},q,W}$ and $Q_{0,q} = C_{\vec{a},q,M}$ so that

$$Q_{0,q}(\vec{x}) = qM\left(\left(\frac{x_1 - a_1 + q}{q}\right) \vee 0, \left(\frac{x_2 - a_2 + q}{q}\right) \vee 0, \left(\frac{x_3 - a_3 + q}{q}\right) \vee 0\right),$$

and

$$Q_{1,q}(\vec{x}) = qW\left(\left(\frac{x_1 - a_1 + q}{q}\right) \vee 0, \left(\frac{x_2 - a_2 + q}{q}\right) \vee 0, \left(\frac{x_3 - a_3 + q}{q}\right) \vee 0\right),$$

whenever $\vec{x} \le \vec{a}$. For such $\vec{x}$, $\partial_i Q_{0,q}(\vec{x}) = 0$ when $x_i - a_i > x_l - a_l$ for some $l \ne i$ and $\partial_i Q_{0,q}(\vec{x}) = 1$ otherwise. Also, $\partial_i Q_{1,q}(\vec{x}) = 1$ when $x_l > a_l - \frac{q}{3}$ for all l.

Let $q_1, q_2 \in \{W(\vec{a}), M(\vec{a})\}$. Set $Q_j = Q_{0,q_j}$. Then we can choose sequences $\vec{x}_n \uparrow \vec{a}$ and $\vec{y}_n \uparrow \vec{a}$ such that $\partial_i Q_j(\vec{x}_n) = 0$ and $\partial_i Q_j(\vec{y}_n) = 1$ for all n. It follows that $0 \le \tau_i(\vec{x}_n, 0, 0, Q_1(\vec{x}_n), Q_2(\vec{x}_n)) \le 1$ and $0 \le \tau_i(\vec{y}_n, 1, 1, Q_1(\vec{y}_n), Q_2(\vec{y}_n)) \le 1$ for all n. Since τ_i is continuous, we may let $n \to \infty$ and get both $0 \le \tau_i(\vec{a}, 0, 0, q_1, q_2) \le 1$ and $0 \le \tau_i(\vec{a}, 1, 1, q_1, q_2) \le 1$.

Instead, we can set $Q_1 = Q_{0,q_1}$ while $Q_2 = Q_{1,q_2}$. Then we can choose a sequence $\vec{x}_n \uparrow \vec{a}$ such that $\partial_i Q_1(\vec{x}_n) = 0$ and $\partial_i Q_2(\vec{x}_n) = 1$ for all n. It follows that $0 \le \tau_i(\vec{x}_n, 0, 1, Q_1(\vec{x}_n), Q_2(\vec{x}_n)) \le 1$ for all n which leads to $0 \le \tau_i(\vec{a}, 0, 1, q_1, q_2) \le 1$ by letting $n \to \infty$. The case $0 \le \tau_i(\vec{a}, 1, 0, q_1, q_2) \le 1$ can be proved similarly. □

Using Lemma 3.3, we see that characterizing T_P can be done by computing the extremum of τ_i which results in (linear) constraints (T1)-(T6) in the following result.

Theorem 3.4. *Let $P = P(x_1, x_2, x_3, z_1, z_2)$ be a quadratic polynomial in the form of (1). Then T_P is a quasi-copula transformation if and only if the following conditions hold.*

(T1) *For all $i = 1,2,3$, $\frac{1}{4}(1-a) \pm (d - e_i) \ge 0$.*

(T2) *For all $i = 1,2,3$, $\frac{1}{4}(1-a) \pm (d + e_i) \ge 0$.*

(T3) $\frac{1}{2}a \pm \left(c + 2d - \frac{2}{3}\sum_i e_i\right) \ge 0$.

(T4) $\frac{1}{4}+\frac{1}{4}a \pm \left(c+d+\frac{1}{3}\sum_i e_i\right) \geq 0$.

(T5) *For all* $i=1,2,3$,

$$\frac{2}{3}\sum_{p=1}^{2}\left[d-(-1)^p\left(2b+e_i\right)\right]\wedge 0+\frac{1}{2}+c\geq 0, \tag{5}$$

and

$$-\frac{2}{3}\sum_{p=1}^{2}\left[d-(-1)^p\left(-2b+e_i\right)\right]\vee 0+\frac{1}{2}-c\geq 0. \tag{6}$$

(T6) *For all* $i=1,2,3$,

$$\frac{2}{3}\sum_{p=1}^{2}\left[d-(-1)^p\left(2b+e_i\right)\right]\vee 0-\frac{1}{2}+c\leq 0, \tag{7}$$

and

$$-\frac{2}{3}\sum_{p=1}^{2}\left[d-(-1)^p\left(-2b+e_i\right)\right]\wedge 0-\frac{1}{2}-c\leq 0. \tag{8}$$

Proof. The proof will be based on the equivalent conditions discovered in Lemma 3.3. For convenience, denote $\alpha_{i,t_1,t_2}(\vec{x})=\min_{q_1,q_2\in\{W(\vec{x}),M(\vec{x})\}}\tau_i(\vec{x},t_1,t_2,q_1,q_2)$ and $\beta_{i,t_1,t_2}(\vec{x})=\max_{q_1,q_2\in\{W(\vec{x}),M(\vec{x})\}}\tau_i(\vec{x},t_1,t_2,q_1,q_2)$.

First, we named the condition $0\leq a\leq 1$ as (R1). We will show that (T1) - (T2) and (R1) together are equivalent to $0\leq\alpha_{i,t,t}\leq\beta_{i,t,t}\leq 1$ for all $t=0,1$ and $i=1,2,3$.

Since

$$\tau_i(\vec{x},0,0,q_1,q_2)=\sum_{p=1}^{2}\left(\frac{1}{4}(1-a)+(-1)^p(d-e_i)\right)q_p,$$

(T1) implies $\alpha_{i,0,0}\geq 0$. Also,

$$\tau_i\left(\frac{2}{3}\cdot\vec{1},0,0,W\left(\frac{2}{3}\cdot\vec{1}\right),M\left(\frac{2}{3}\cdot\vec{1}\right)\right)=\frac{2}{3}\left[\frac{1}{4}(1-a)+(d-e_i)\right],$$

and

$$\tau_i\left(\frac{2}{3}\cdot\vec{1},0,0,M\left(\frac{2}{3}\cdot\vec{1}\right),W\left(\frac{2}{3}\cdot\vec{1}\right)\right)=\frac{2}{3}\left[\frac{1}{4}(1-a)-(d-e_i)\right].$$

Thus, $\alpha_{i,0,0}\geq 0$ is actually equivalent to (T1).

Since

$$\tau_i(\vec{x},1,1,q_1,q_2)=-\sum_{p=1}^{2}\left(\frac{1}{4}(1-a)+(-1)^p(d+e_i)\right)q_p+\frac{1}{2}(1-a)\sum_i x_i+a,$$

(T2) implies $\beta_{i,1,1}\leq 1$. Also,

$$\tau_i\left(\frac{2}{3}\cdot\vec{1},1,1,W\left(\frac{2}{3}\cdot\vec{1}\right),M\left(\frac{2}{3}\cdot\vec{1}\right)\right)=-\frac{2}{3}\left[\frac{1}{4}(1-a)+(d+e_i)\right]+1,$$

and

$$\tau_i\left(\frac{2}{3}\cdot\vec{1},1,1,M\left(\frac{2}{3}\cdot\vec{1}\right),W\left(\frac{2}{3}\cdot\vec{1}\right)\right)=-\frac{2}{3}\left[\frac{1}{4}(1-a)-(d+e_i)\right]+1.$$

Thus, $\beta_{i,1,1}\leq 1$ is actually equivalent to (T2).

Under (T1),

$$\max_{\vec{x}\in[0,1]^3}\beta_{i,0,0}(\vec{x})=\max_{\vec{x}\in[0,1]^3}M(\vec{x})\sum_{p=1}^{2}\left(\frac{1}{4}(1-a)+(-1)^p(d-e_i)\right)=\frac{1}{2}(1-a).$$

directly implies $\beta_{i,0,0} \geq 0$ if and only if $a \leq 1$. Under (T2),

$$\begin{aligned}\min_{\vec{x}\in[0,1]^3} \alpha_{i,1,1}(\vec{x}) &= \min_{\vec{x}\in[0,1]^3} \left[-M(\vec{x}) \sum_{p=1}^{2} \left(\frac{1}{4}(1-a) + (-1)^p (d+e_i)\right) + \frac{1}{2}(1-a)\sum_i x_i + a\right] \\ &= \min_{\vec{x}\in[0,1]^3} \left[\frac{1}{2}(1-a)\left(\sum_i x_i - M(\vec{x})\right) + a\right] \\ &= a,\end{aligned}$$

directly implies $\alpha_{i,1,1} \geq 0$ if and only if $a \geq 0$. We note that $a \geq 0$ implies $a \geq -1$ which also implies that $\beta_{i,0,0} \leq 1$.

Now, we may assume (R1), (T1) - (T2), as well as $0 \leq \alpha_{i,t,t} \leq \beta_{i,t,t} \leq 1$ for all $t = 0, 1$ and $i = 1, 2, 3$. Under these assumptions, we will show that (T3), (T5), and (9) together are equivalent to $\alpha_{i,s,t} \geq 0$ for all $s \neq t$ and $i = 1, 2, 3$. Here,

$$\frac{3}{4} - \frac{1}{4}a \pm \left(c + d + \frac{1}{3}\sum_i e_i\right) \geq 0. \tag{9}$$

Note that the condition (R1) implies $\frac{3}{4} - \frac{1}{4}a \geq \frac{1}{4} + \frac{1}{4}a$ which means (T4) implies (9). Thus, (9) may be removed later. Now,

$$\tau_i(\vec{x}, 1, 0, q_1, q_2) = \sum_{p=1}^{2} [d - (-1)^p (2b + e_i)] q_p + \sum_i \left(\frac{1}{4}(1-a) - d + e_i\right) x_i + \frac{1}{2}a + c + 2d - \frac{2}{3}\sum_i e_i,$$

while

$$\tau_i(\vec{x}, 0, 1, q_2, q_1) = \sum_{p=1}^{2} [-d - (-1)^p (2b - e_i)] q_p + \sum_i \left(\frac{1}{4}(1-a) + d - e_i\right) x_i + \frac{1}{2}a - c - 2d + \frac{2}{3}\sum_i e_i.$$

It follows that

$$\begin{aligned}\alpha_{i,1,0}(\vec{x}) &= M(\vec{x}) \sum_{p=1}^{2} [d - (-1)^p (2b + e_i)] \wedge 0 + W(\vec{x}) \sum_{p=1}^{2} [d - (-1)^p (2b + e_i)] \vee 0 \\ &\quad + \sum_i \left(\frac{1}{4}(1-a) - d + e_i\right) x_i + \frac{1}{2}a + c + 2d - \frac{2}{3}\sum_i e_i,\end{aligned}$$

while

$$\begin{aligned}\alpha_{i,0,1}(\vec{x}) &= M(\vec{x}) \sum_{p=1}^{2} [-d - (-1)^p (2b - e_i)] \wedge 0 + W(\vec{x}) \sum_{p=1}^{2} [-d - (-1)^p (2b - e_i)] \vee 0 \\ &\quad + \sum_i \left(\frac{1}{4}(1-a) + d - e_i\right) x_i + \frac{1}{2}a - c - 2d + \frac{2}{3}\sum_i e_i.\end{aligned}$$

The difference between the latter and the former is only on the signs of c, d, e_i. Thus, the proof involving $\alpha_{i,0,1}(\vec{x})$ is analogous to those that involving $\alpha_{i,1,0}(\vec{x})$ and the results are simply the change of signs of c, d, e_i.

By the condition (T1), the function $\vec{x} \mapsto \sum_i \left(\frac{1}{4}(1-a) - d + e_i\right) x_i$ is nondecreasing. Since W is also nondecreasing, we have

$$\begin{aligned}\min_{\vec{x}\in[0,1]^3} \alpha_{i,1,0}(\vec{x}) &= \min_{x\in[0,1]} \min_{M(\vec{x})=x} \alpha_{i,1,0}(\vec{x}) \\ &= \min_{x\in[0,1]} \alpha_{i,1,0}\left(x \cdot \vec{1}\right) \\ &= \min\left(\alpha_{i,1,0}\left(\vec{0}\right), \alpha_{i,1,0}\left(\frac{2}{3} \cdot \vec{1}\right), \alpha_{i,1,0}\left(\vec{1}\right)\right).\end{aligned}$$

Similarly, $\min_{\vec{x}\in[0,1]^3} \alpha_{i,0,1}(\vec{x}) = \min\left(\alpha_{i,0,1}\left(\vec{0}\right), \alpha_{i,0,1}\left(\frac{2}{3} \cdot \vec{1}\right), \alpha_{i,0,1}\left(\vec{1}\right)\right)$. Now, (T3) is equivalent to $\alpha_{i,1,0}\left(\vec{0}\right) \geq 0$ and $\alpha_{i,0,1}\left(\vec{0}\right) \geq 0$, (9) is equivalent to $\alpha_{i,1,0}\left(\vec{1}\right) \geq 0$ and $\alpha_{i,0,1}\left(\vec{1}\right) \geq 0$, and (T5) is equivalent to $\alpha_{i,1,0}\left(\frac{2}{3} \cdot \vec{1}\right) \geq 0$ and $\alpha_{i,0,1}\left(\frac{2}{3} \cdot \vec{1}\right) \geq 0$.

Next, we will show that (T4) and (T6) together are equivalent to $\beta_{i,s,t} \leq 1$ for all $s \neq t$ and $i = 1, 2, 3$. This will be done under the assumptions that all former equivalent conditions hold. Note that

$$\begin{aligned}\beta_{i,1,0}(\vec{x}) = M(\vec{x}) \sum_{p=1}^{2} [d - (-1)^p (2b + e_i)] \vee 0 + W(\vec{x}) \sum_{p=1}^{2} [d - (-1)^p (2b + e_i)] \wedge 0 \\ + \sum_i \left(\frac{1}{4}(1-a) - d + e_i\right) x_i + \frac{1}{2}a + c + 2d - \frac{2}{3}\sum_i e_i,\end{aligned}$$

while

$$\begin{aligned}\beta_{i,0,1}(\vec{x}) = M(\vec{x}) \sum_{p=1}^{2} [-d - (-1)^p (2b - e_i)] \vee 0 + W(\vec{x}) \sum_{p=1}^{2} [-d - (-1)^p (2b - e_i)] \wedge 0 \\ + \sum_i \left(\frac{1}{4}(1-a) + d - e_i\right) x_i + \frac{1}{2}a - c - 2d + \frac{2}{3}\sum_i e_i.\end{aligned}$$

Again, $\vec{x} \mapsto M(\vec{x}) \sum_{p=1}^{2} [d - (-1)^p (2b + e_i)] \vee 0 + \sum_i \left(\frac{1}{4}(1-a) - d + e_i\right) x_i$ is nondecreasing. Thus, $\max_{\sum x_i \leq 2} \beta_{i,1,0}(\vec{x}) = \max_{\sum x_i = 2} \beta_{i,1,0}(\vec{x})$ and hence

$$\max_{\vec{x} \in [0,1]^3} \beta_{i,1,0}(\vec{x}) = \max_{\sum x_i \geq 2} \beta_{i,1,0}(\vec{x}).$$

Let $\Omega = \left\{\vec{x} \in [0,1]^3 \mid \sum_i x_i \geq 2, x_1 \leq x_2 \wedge x_3\right\}$, $\Omega_i = \{\vec{x} \in \Omega \mid x_1 = x_i\}$, and

$$\begin{aligned}\beta(\vec{x}) = x_1 \sum_{p=1}^{2} [d - (-1)^p (2b + e_i)] \vee 0 + \left(\sum_{i=1}^{3} x_i - 2\right) \sum_{p=1}^{2} [d - (-1)^p (2b + e_i)] \wedge 0 \\ + \sum_i \left(\frac{1}{4}(1-a) - d + e_i\right) x_i + \frac{1}{2}a + c + 2d - \frac{2}{3}\sum_i e_i.\end{aligned}$$

Then $\partial_1 \beta = \frac{1}{4}(1-a) + d + e_1 \geq 0$ by (T2), which implies $\beta = \beta_{i,1,0}$ is nondecreasing in the first variable on Ω. Thus, the maximum happens when $x_1 = x_2 \wedge x_3$, that is,

$$\max_{\Omega} \beta_{i,1,0} = \left(\max_{\Omega_2} \beta\right) \vee \left(\max_{\Omega_3} \beta\right).$$

Since β is linear and Ω_2 has three extreme points which are $\left(\frac{1}{2}, \frac{1}{2}, 1\right)$, $\frac{2}{3} \cdot \vec{1}$, and $\vec{1}$,

$$\max_{\Omega_2} \beta = \beta\left(\frac{1}{2}, \frac{1}{2}, 1\right) \vee \beta\left(\frac{2}{3} \cdot \vec{1}\right) \vee \beta\left(\vec{1}\right).$$

Similarly,

$$\max_{\Omega_3} \beta = \beta\left(\frac{1}{2}, 1, \frac{1}{2}\right) \vee \beta\left(\frac{2}{3} \cdot \vec{1}\right) \vee \beta\left(\vec{1}\right).$$

Thus,

$$\max_{\Omega} \beta_{i,1,0} = \beta\left(\frac{1}{2}, \frac{1}{2}, 1\right) \vee \beta\left(\frac{1}{2}, 1, \frac{1}{2}\right) \vee \beta\left(\frac{2}{3} \cdot \vec{1}\right) \vee \beta\left(\vec{1}\right).$$

By symmetry, we have

$$\max_{\sum x_i \geq 2} \beta_{i,1,0}(\vec{x}) = \beta_{i,1,0}\left(\frac{1}{2}, \frac{1}{2}, 1\right) \vee \beta_{i,1,0}\left(\frac{1}{2}, 1, \frac{1}{2}\right) \vee \beta_{i,1,0}\left(1, \frac{1}{2}, \frac{1}{2}\right) \vee \beta_{i,1,0}\left(\frac{2}{3} \cdot \vec{1}\right) \vee \beta_{i,1,0}\left(\vec{1}\right).$$

Now,

$$\begin{aligned}2\beta_{i,1,0}\left(\frac{1}{2}, \frac{1}{2}, 1\right) &= \frac{3}{2}\beta_{i,1,0}\left(\frac{2}{3} \cdot \vec{1}\right) + \left(\frac{1}{4} + \frac{1}{2}c - \frac{1}{3}(e_1 + e_2) + \frac{2}{3}e_3\right) \\ &\leq \frac{3}{2}\beta_{i,1,0}\left(\frac{2}{3} \cdot \vec{1}\right) + \left(\frac{1}{4} - (d - e_3) + \frac{1}{2}\left(c + 2d - \frac{2}{3}\sum_{i=1}^{3} e_i\right)\right) \\ &\leq \frac{3}{2}\beta_{i,1,0}\left(\frac{2}{3} \cdot \vec{1}\right) + \frac{1}{2}.\end{aligned}$$

by (T1) and (T3). By symmetry,

$$\beta_{i,1,0}\left(\frac{1}{2},\frac{1}{2},1\right)\vee\beta_{i,1,0}\left(\frac{1}{2},1,\frac{1}{2}\right)\vee\beta_{i,1,0}\left(1,\frac{1}{2},\frac{1}{2}\right)\leq\frac{3}{4}\beta_{i,1,0}\left(\frac{2}{3}\cdot\vec{1}\right)+\frac{1}{4}.$$

Therefore, $\max_{\sum x_i\geq 2}\beta_{i,1,0}(\vec{x})\leq 1$ if and only if $\beta_{i,1,0}\left(\frac{2}{3}\cdot\vec{1}\right)\vee\beta_{i,1,0}\left(\vec{1}\right)\leq 1$. Using similar arguments, we also have $\max\beta_{i,0,1}\leq 1$ if and only if $\beta_{i,0,1}\left(\frac{2}{3}\cdot\vec{1}\right)\vee\beta_{i,0,1}\left(\vec{1}\right)\leq 1$. Now, (T4) is equivalent to $\beta_{i,1,0}\left(\vec{1}\right)\leq 1$ and $\beta_{i,0,1}\left(\vec{1}\right)\leq 1$ while (T6) is equivalent to $\beta_{i,1,0}\left(\frac{2}{3}\cdot\vec{1}\right)\leq 1$ and $\beta_{i,0,1}\left(\frac{2}{3}\cdot\vec{1}\right)\leq 1$.

Last, (R1) follows from (T2) and (T3) which implies it is redundant. □

It can be seen that the conditions in Theorem 3.4 are quite complicated. These conditions can be simplified as is done in the following theorem.

Theorem 3.5. *For any quadratic polynomial P, T_P defines a bivariate transformation of trivariate quasi-copulas if and only if P is given in the form of (1) and its coefficients satisfy the following inequalities.*

(R1) $0\leq a\leq 1$.

(R2) $-\frac{1}{2}\leq c\leq\frac{1}{2}$.

(R3) *For all $i=1,2,3$,*

$$\max\begin{pmatrix}-\frac{1}{4}(1-a)+|d|\\ -\frac{3}{4}+2|b|+\left|\frac{3}{2}c+d\right|\end{pmatrix}\leq e_i\leq\min\begin{pmatrix}\frac{1}{4}(1-a)-|d|\\ \frac{3}{4}-2|b|-\left|\frac{3}{2}c+d\right|\end{pmatrix}.$$

(R4) $\max\begin{pmatrix}\frac{3}{2}c+3d-\frac{3}{4}a\\ -\frac{3}{4}-\frac{3}{4}a-3c-3d\end{pmatrix}\leq\sum_{i=1}^{3}e_i\leq\min\begin{pmatrix}\frac{3}{2}c+3d+\frac{3}{4}a\\ \frac{3}{4}+\frac{3}{4}a-3c-3d\end{pmatrix}$.

Proof. Notice that (R4) is simply a rewriting of (T3) and (T4) while (T1) and (T3) together imply (R1). A part of (R3) is also a rewriting of (T1) and (T2). Thus, it is sufficient to prove that the conditions (T5) and (T6) together in Theorem 3.4 are equivalent to (R2) and

$$-\frac{3}{4}+2|b|+\left|\frac{3}{2}c+d\right|\leq e_i\leq\frac{3}{4}-\left|\frac{3}{2}c+d\right|-2|b| \tag{10}$$

for all $i=1,2,3$.

First, we will prove the necessity part. Using the fact that the set of quasi-copula transformations is convex, it is sufficient to prove that the upper bound and the lower bound of e_i in (10) satisfy the conditions (T5) and (T6) in Theorem 3.4. Also, notice that the condition (T5) becomes (T6) when we replace e_i with $-e_i$ and vice versa. Thus, $-e_i$ satisfies the condition (T5) as long as e_i does. Therefore, it is sufficient to show that $e_i=-\frac{3}{4}+2|b|+\left|\frac{3}{2}c+d\right|$ satisfies the conditions (T5) and (T6). Again, this can be done using case-by-case analysis on the signs of quantities involved and the fact that

$$2|b|+\left|\frac{3}{2}c+d\right|\leq\frac{3}{4}, \tag{11}$$

and

$$\left|\frac{3}{2}c+2d\right|\leq\frac{3}{4}. \tag{12}$$

The first inequality is an immediate consequence of (10) while the second one follows from the condition (R4).

For condition (5), notice that $\sum\left[d-(-1)^p(2b+e_i)\right]\wedge 0$ can be either 0, $\pm(e_i+2b)+d$, or $2d$. In the first case, (5) follows from (R2) while in the last case, (5) follows from (12). In the second case, condition (5) reduces to

$$\pm\left(-\frac{3}{4}+2(|b|+b)+\left|\frac{3}{2}c+d\right|\right)-\frac{3}{2}c-d\leq\frac{3}{4}.$$

The fact that

$$2(|b|+b)+\left|\frac{3}{2}c+d\right|-\frac{3}{2}c-d\leq 4|b|+2\left|\frac{3}{2}c+d\right|,$$

immediately implies

$$\left(-\frac{3}{4}+2\left(|b|+b\right)+\left|\frac{3}{2}c+d\right|\right)-\frac{3}{2}c-d\leq\frac{3}{4},$$

while the fact that $|A|+A\geq 0$ for all real number A immediately implies

$$-\left(-\frac{3}{4}+2\left(|b|+b\right)+\left|\frac{3}{2}c+d\right|\right)-\frac{3}{2}c-d\leq\frac{3}{4}.$$

Therefore, we can conclude that (5) always holds.

Conditions (6), (7), and (8) can be proved analogously.

Now, we will prove the sufficiency part. Notice that conditions (6) and (8) trivially imply (R2). The proof of (10) can be done using case-by-case analysis. We will show the proof for $b\geq 0$. The case $b\leq 0$ can be done similarly.

Case 1. $e_i\pm d\leq -2b$. Then (T5) and (T6) reduce to

$$-\frac{3}{4}+2b+\left|\frac{3}{2}c+d\right|\leq e_i\leq -2b-|d|.$$

Case 2. $e_i-d\leq -2b\leq e_i+d\leq 2b$. Then (T5) and (T6) reduce to

$$\max\begin{pmatrix}-2b-d\\ -\frac{3}{4}+2b+\left|\frac{3}{2}c+d\right|\end{pmatrix}\leq e_i\leq -|2b-d|.$$

Case 3. $e_i+d\leq -2b\leq e_i-d\leq 2b$. Then (T5) and (T6) reduce to

$$\max\begin{pmatrix}-2b+d\\ -\frac{3}{4}+2b+\left|\frac{3}{2}c+d\right|\end{pmatrix}\leq e_i\leq -|2b+d|.$$

Case 4. $-2b\leq e_i\pm d\leq 2b$. Then (T5) and (T6) reduce to

$$\max\begin{pmatrix}-2b+|d|\\ -\frac{3}{4}+2b+\left|\frac{3}{2}c+d\right|\end{pmatrix}\leq e_i\leq \min\begin{pmatrix}2b-|d|\\ \frac{3}{4}-2b-\left|\frac{3}{2}c+d\right|\end{pmatrix}.$$

Case 5. $e_i-d\leq -2b\leq 2b\leq e_i+d$. Then (T5) and (T6) reduce to

$$2b-d\leq e_i\leq -2b+d.$$

Case 6. $e_i+d\leq -2b\leq 2b\leq e_i-d$. Then (T5) and (T6) reduce to

$$2b+d\leq e_i\leq -2b-d.$$

Case 7. $-2b\leq e_i-d\leq 2b\leq e_i+d$. Then (T5) and (T6) reduce to

$$|2b-d|\leq e_i\leq \min\begin{pmatrix}2b+d\\ \frac{3}{4}-2b-\left|\frac{3}{2}c+d\right|\end{pmatrix}.$$

Case 8. $-2b\leq e_i+d\leq 2b\leq e_i-d$. Then (T5) and (T6) reduce to

$$|2b+d|\leq e_i\leq \min\begin{pmatrix}2b-d\\ \frac{3}{4}-2b-\left|\frac{3}{2}c+d\right|\end{pmatrix}.$$

Case 9. $2b\leq e_i\pm d$. Then (T5) and (T6) reduce to

$$2b+|d|\leq e_i\leq \frac{3}{4}-2b-\left|\frac{3}{2}c+d\right|.$$

The union of ranges in these 9 cases then yields (10).

□

With the help of a CAS (specifically, the code is written in Julia languange), we are able to conclude that the set of such $(a, b, c, d, e_1, e_2, e_3)$ has exactly 78 extreme points. The list of extreme points is provided in Table 1.

a	b	c	d	e_1	e_2	e_3
0	±1/8	1/6	0	1/4	1/4	−1/4
0	±1/8	1/6	0	1/4	−1/4	1/4
0	±1/8	1/6	0	−1/4	1/4	1/4
0	±1/8	−1/6	0	−1/4	−1/4	1/4
0	±1/8	−1/6	0	−1/4	1/4	−1/4
0	±1/8	−1/6	0	1/4	−1/4	−1/4
0	±1/8	1/2	−1/4	0	0	0
0	±1/8	−1/2	1/4	0	0	0
0	±1/4	0	0	0	1/4	−1/4
0	±1/4	0	0	0	−1/4	1/4
0	±1/4	0	0	1/4	0	−1/4
0	±1/4	0	0	−1/4	0	1/4
0	±1/4	0	0	1/4	−1/4	0
0	±1/4	0	0	−1/4	1/4	0
0	±1/4	0	1/8	1/8	1/8	1/8
0	±1/4	0	−1/8	−1/8	−1/8	−1/8
0	±7/24	1/18	−1/12	1/6	−1/6	−1/6
0	±7/24	−1/18	1/12	−1/6	1/6	1/6
0	±7/24	1/18	−1/12	−1/6	1/6	−1/6
0	±7/24	−1/18	1/12	1/6	−1/6	1/6
0	±7/24	1/18	−1/12	−1/6	−1/6	1/6
0	±7/24	−1/18	1/12	1/6	1/6	−1/6
0	±13/40	1/10	−3/20	−1/10	−1/10	−1/10
0	±13/40	−1/10	3/20	1/10	1/10	1/10
0	±3/8	0	0	0	0	0
1/4	±9/32	0	0	3/16	3/16	−3/16
1/4	±9/32	0	0	3/16	−3/16	3/16
1/4	±9/32	0	0	−3/16	3/16	3/16
1/4	±9/32	0	0	−3/16	−3/16	3/16
1/4	±9/32	0	0	−3/16	3/16	−3/16
1/4	±9/32	0	0	3/16	−3/16	−3/16
1/3	±1/6	1/6	0	1/6	1/6	1/6
1/3	±1/6	−1/6	0	−1/6	−1/6	−1/6
2/5	±3/8	1/10	−3/20	0	0	0
2/5	±3/8	−1/10	3/20	0	0	0
1/2	±5/16	0	0	1/8	1/8	1/8
1/2	±5/16	0	0	−1/8	−1/8	−1/8
1	±3/8	0	0	0	0	0
1	0	±1/2	0	0	0	0

Table 1: List of the extreme points

Here, we provide a few construction examples based on those points.

Example 3.6. *Consider* $(a, b, c, d, e_1, e_2, e_3) = \left(1, \frac{3}{8}, 0, 0, 0, 0, 0\right)$ *which corresponds to the polynomial*

$$P(x_1, x_2, x_3, z_1, z_2) = \frac{3}{8}(z_1 - z_2)^2 + \frac{1}{2}(z_1 + z_2).$$

Since $(a, b, c, d, e_1, e_2, e_3)$ *is one of the extreme points, we know that* T_P *is a bivariate quadratic transformation of quasi-copulas.*

Now, T_P can be used to construct new quasi-copulas by plugging in two quasi-copulas Q_1, Q_2 to get another quasi-copula $Q = T_P(Q_1, Q_2)$. For example, we choose quasi-copulas Q_1 and Q_2 defined by

$$Q_1(x,y,z) = xyz,$$

and

$$Q_2(x,y,z) = xyz + xyz(1-x)(1-y)(1-z),$$

then

$$Q(x,y,z) = \frac{3}{8}x^2y^2z^2(1-x)^2(1-y)^2(1-z)^2 + xyz + \frac{1}{2}xyz(1-x)(1-y)(1-z).$$

If we instead choose

$$Q_1(x,y,z) = xyz - xyz(1-x)(1-y)(1-z),$$

then

$$Q(x,y,z) = \frac{3}{2}x^2y^2z^2(1-x)^2(1-y)^2(1-z)^2 + xyz.$$

Example 3.7. *Instead of directly using the extreme points, we can also use their convex combinations. Consider, for example, two extreme points $\left(0, \frac{1}{4}, 0, 0, 0, -\frac{1}{4}, \frac{1}{4}\right)$ and $\left(0, \frac{1}{4}, 0, 0, 0, \frac{1}{4}, -\frac{1}{4}\right)$. Their average is $\left(0, \frac{1}{4}, 0, 0, 0, 0, 0\right)$ which corresponds to the polynomial*

$$P(x_1,x_2,x_3,z_1,z_2) = \frac{1}{4}(z_1+z_2)\left(\sum_{i=1}^{3} x_i\right) - \frac{1}{8}(z_1+z_2)^2 + \frac{1}{4}(z_1-z_2)^2.$$

It follows that

$$T_P(Q_1,Q_2) = \frac{1}{4}(Q_1+Q_2)S - \frac{1}{8}(Q_1+Q_2)^2 + \frac{1}{4}(Q_1-Q_2)^2,$$

where S denotes the summation function, is a quasi-copula for all quasi-copulas Q_1 and Q_2. Thus, it can be used to construct quasi-copulas similar to that of the previous example. Also, notice that P is invariant with respect to x_i. Thus, $T_P(Q_1,Q_2)$ is invariant whenever Q_1 and Q_2 are.

Example 3.8. *Consider $(a,b,c,d,e_1,e_2,e_3) = \left(0, -\frac{7}{24}, -\frac{1}{18}, \frac{1}{12}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$ which corresponds to the polynomial*

$$\begin{aligned} P(x_1,x_2,x_3,z_1,z_2) &= \frac{1}{4}(z_1+z_2)\sum_{i=1}^{3} x_i - \frac{1}{8}(z_1+z_2)^2 \\ &\quad - \frac{7}{24}(z_1-z_2)^2 - \frac{1}{18}(z_1-z_2) + \frac{1}{12}(z_1-z_2)\left(z_1+z_2+2-\sum_{i=1}^{3} x_i\right) \\ &\quad - \frac{1}{6}\left(x_1-\frac{2}{3}\right)(z_1-z_2) + \frac{1}{6}\left(x_2-\frac{2}{3}\right)(z_1-z_2) + \frac{1}{6}\left(x_3-\frac{2}{3}\right)(z_1-z_2) \\ &= -\frac{1}{3}z_1^2 + \frac{1}{3}z_1z_2 - \frac{1}{2}z_2^2 + \frac{1}{3}x_2z_1 + \frac{1}{3}x_3z_1 + \frac{1}{2}x_1z_2 + \frac{1}{6}x_2z_2 + \frac{1}{6}x_3z_2. \end{aligned}$$

Again, we know that T_P is a bivariate quadratic transformation of quasi-copulas. Thus, it can be used to construct quasi-copulas similar to that of the previous example.

Since P is not symmetric under variables x_i, T_P can not be a composition of a univariate quadratic transformation and a convex combination of two quasi-copulas. (Since the set of bivariate quadratic transformations of quasi-copulas is a convex set in a 7-dimensional space, it is much more complicated than the set of univariate quadratic transformations.) It can also be checked using results in [5] that T_P is not a transformation of semi-copulas. Thus, these two classes of transformations are different which is different from the case of univariate transformations [6].

4 Conclusions and discussions

In this work, we characterize bivariate quadratic transformations of trivariate quasi-copulas in terms of their coefficients. This results in a convex subset of a seven-dimensional linear space with 78 extreme points. Each of these extreme points can then be used to construct new families of quasi-copulas. Convex combinations of these cases are also possible. We also found that our class is different from the class of bivariate quadratic transformations of semi-copulas.

Recall that the class of univariate quadratic transformations only has one nontrivial extreme point. This implies our class is much more complicated than the univariate case. In fact, the number of conditions grows exponentially in n for n-variate transformations as can be seen in the proof of the condition 3 in Lemma 3.3 and in the proof of $0 \leq \alpha_{i,t_1,\ldots,t_n} \leq \beta_{i,t_1,\ldots,t_n} \leq 1$ in Theorem 3.4. Therefore, the characterization of multivariate quadratic transformations is a real computational challenge even for computers. Thus, an alternative approach must be considered in such cases.

The fact that the bivariate case is much more complicated than the univariate case also implies that convex combinations of univariate transformations do not cover all of bivariate transformations. This is related to the fact that the polynomial $z_1 z_2$ is not a convex combination of z_1 and z_2 which implies that not all quadratic polynomials of the form $P(\vec{x}, z_1, z_2)$ are linear combinations of polynomials of the form $P_1(\vec{x}, z_1)$ and $P_2(\vec{x}, z_2)$. The situation is different when more z_i are involved. In general, all quadratic polynomials of the form $P(\vec{x}, \vec{z})$ can be written as a linear combination of polynomials of the form $P_{ij}(\vec{x}, z_i, z_j)$. Thus, it is natural to ask whether all multivariate quadratic transformations are convex combinations of bivariate quadratic transformations. The answer remains open at this point. If true, however, then there would be no need to characterize multivariate quadratic transformations. Although, characterizations of bivariate quadratic transformations of multivariate quasi-copulas remain a computational challenge. All of these are worth exploring in the future.

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