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The g-sum of two posets and its application to construct idempotent uninorms

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Abstract

In this study, a new method, called g-sum, is presented to join two posets together that generalizes the linear sum of two posets. Idempotent uninorms on a bounded chain are in one-to-one correspondence with special linear orders on it and g-sum can be used to construct such special linear orders.

Keywords: Aggregation function, uninorm, poset, meet operation, linear sum.

1 Introduction

Let $(L, \leq, 0, 1)$ be a bounded lattice. A *uninorm* [7] is a binary operation $U : L^2 \to L$ such that (L, U) is an abelian, ordered semigroup with a neutral element $e \in L$. The operation U degenerates into a t-norm for $e = 1$ and into a t-conorm for $e = 0$. For any other value $e \in L\backslash\{0, 1\}$, $U \mid_{[0,e]^2}$ is a t-norm on $[0, e]$ and $U \mid_{[e,1]^2}$ is a t-conorm on $[e, 1]$.

Uninorms were first introduced in the framework of the unit interval $[0, 1]$ by Yager and Rybalov [14] as a class of aggregation functions. Nowadays, uninorms have been extended from the unit interval to bounded lattices because aggregation on lattices has become an important new branch of aggregation theory [9]. For an aggregation function, the idempotency is a natural property. Thus, a great deal of attention is being paid to the structure of idempotent uninorms [3, 4, 6, 8, 10, 11].

Ordinal sums of two abstract semigroups were introduced by Clifford in [2] to generate a new semigroup structure on the union of two pairwise disjoint semigroups. Mesiarová-Zemnková [10, Proposition 3] showed that each idempotent uninorm can be decomposed to an ordinal sum of singleton semigroups, which shows that idempotent uninorms on the unit interval $[0, 1]$ are in one-to-one correspondence with linear orders $\preceq$ on $[0, 1]$ such that the linear order $\preceq$ coincides with the order $\leq$ on $[0, e]_{\leq}$ and is the dual of the order $\leq$ on $[e, 1]_{\leq}$. This result is valid also for idempotent uninorms defined on a bounded chain as is shown in [13, Theorem 3.5]:

Theorem 1.1. *Let $(L, \leq)$ be a chain, $e \in L\backslash\{0, 1\}$ and let $\preceq$ be a linear order on L. Then the meet operation $\curlywedge$ of $\preceq$ is a locally internal uninorm with the neutral element e on $(L, \leq)$, i.e., $U(x, y) \in \{x, y\}$ for all $x, y \in L$ if and only if, for all $x, y \in L$,*

(i) $x \prec y$ *if* $x < y \leq e$.

(ii) $x \prec y$ *if* $e \leq y < x$.

We denote by $\mathcal{L}_e$ the set of all linear orders on a bounded chain $(L, \leq)$ that fulfill conditions (i) and (ii) in Theorem 1.1. Thus, characterizing idempotent uninorms is transformed into joining two chains together. Jointing two posets together is an interesting topic in theory of ordered sets. There are several different ways to do this. For example, the disjoint union of two posets, the linear sum of two posets and the Cartesian product of posets [5, Section 1.24], the horizontal sum of the bounded posets [12]. In this paper, we will present a new method, called g-sum, to join two posets together that generalizes the linear sum of two posets. Then we use g-sum to construct idempotent uninorms.

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2 Preliminaries

In this section, we recall some basic notions and results concerning orders (for more information, see[1, 5]).

A *poset* $(P, \leq)$ is a nonempty set P with a single binary, reflexive, antisymmetric and transitive relation $\leq$. The relation $a < b$ means $a \leq b$ and $a \neq b$. For $a, b \in P$ with $a \leq b$, the subinterval $[a, b]_{\leq}$ is defined by $[a, b]_{\leq} = \{x \in P \mid a \leq x \leq b\}$. The subintervals $]a, b]_{\leq}$, $[a, b[_{\leq}$ and $]a, b[_{\leq}$ of P are defined similarly. The relation $a \leq b$ is also written $b \geq a$, and read b contains a. The *dual* $(P, \preceq)$ of the poset $(P, \leq)$ is defining by $x \leq y$ if and only if $y \preceq x$; we shall denote it by P^{∂}. A *chain* is a poset $(P, \leq)$ in which any two elements are comparable, i.e., either $a \leq b$ or $b \leq a$ for all $a, b \in P$. A *least* element of any subset H of P is an element $a \in H$ such that $a \leq h$ for all $h \in H$. A *greatest* element of H is an element $a \in H$ such that $h \leq a$ for all $h \in H$. An *upper bound* of a subset H of a poset P is an element $a \in P$ containing every $h \in H$. The *least upper bound* is an upper bound contained in every other upper bound; it is denoted l.u.b.H or $\sup H$. The notions of *lower bound* of H and *greatest lower bound* (g.l.b.H or $\inf H$) of H are defined dually. A *lattice* is a poset $(L, \leq)$ any two of whose elements have a g.l.b. or "meet" denoted by $a \wedge_{\leq} b$, and a l.u.b. or "join" denoted by $a \vee_{\leq} b$. A *bounded lattice* $(L, \leq)$ is a lattice which has the greatest and least elements, written as 1 and 0, respectively.

3 Main results

Let $(P, \leq_1)$ and $(Q, \leq_2)$ be two disjoint posets. Then the *linear sum* $P \oplus Q$ [5] (or ordinal sum of posets P and Q in the sense of Birkhoff [1]) is defined by taking the following relation on $P \cup Q$: $x \preceq y$ if and only if

$$\begin{array}{ll} & x, y \in P \text{ and } x \leq_1 y \text{ in } P, \\ \text{or} & x, y \in Q \text{ and } x \leq_2 y \text{ in } Q, \\ \text{or} & x \in P \text{ and } y \in Q. \end{array} \tag{1}$$

For any posets $(P, \leq_1)$, $(Q, \leq_2)$ and $(R, \leq_3)$, we have the following results

- the linear sum $(P \oplus Q, \preceq)$ is a chain if and only if $(P, \leq_1)$ and $(Q, \leq_2)$ are chain.
- $\oplus$ is associative [5], i.e., $(P \oplus Q) \oplus R = P \oplus (Q \oplus R)$.

Next, we introduce the g-sum of two posets.

Let $(P, \leq_1)$ and $(Q, \leq_2)$ be two posets with $P \cap Q = \emptyset$ and $g : P \to Q$ be a decreasing operation. The *g-sum* $P \oplus_g Q$ is defined by taking the following relation on $P \cup Q$: $x \preceq y$ if and only if

$$\begin{array}{ll} & x, y \in P \text{ and } x \leq_1 y \text{ in } P, \\ \text{or} & x, y \in Q \text{ and } y \leq_2 x \text{ in } Q, \\ \text{or} & x \in P, y \in Q \text{ and } y <_2 g(x) \text{ in } Q, \\ \text{or} & x \in Q, y \in P \text{ and } g(y) \leq_2 x \text{ in } Q. \end{array} \tag{2}$$

Theorem 3.1. *Let $(P, \leq_1)$ and $(Q, \leq_2)$ be two posets with $P \cap Q = \emptyset$. Assume that $g : P \to Q$ is a decreasing operation. If there is no $x, y \in P$ such that $x \| y$ and $g(x) <_2 g(y)$, then the g-sum $(P \oplus_g Q, \preceq)$ is a poset.*

Proof. The reflexivity follows from the reflexivity of $\leq_1$ on P and reflexivity of $\leq_2$ on Q.
Antisymmetry: Suppose that $x \preceq y$ and $y \preceq x$ in $P \cup Q$. If $x \in P$ and $y \in Q$, then, by the definition, we have $y <_2 g(x) \leq_2 y$, a contradiction. A similar argument shows that the case $x \in Q$ and $y \in P$ is impossible. Thus, either $x, y \in P$ or $x, y \in Q$. Hence $x = y$ because P and Q are posets.
Transitivity: Suppose that $x \preceq y$ and $y \preceq z$ in $P \cup Q$.

1. If $x, y, z \in P$, then $x \leq_1 y$ and $y \leq_1 z$ in P and further $x \leq_1 z$ in P. Thus, $x \preceq z$ in $P \cup Q$.
2. If $x, y, z \in Q$, then $y \leq_2 x$ and $z \leq_2 y$ in Q and further $z \leq_2 x$ in Q. Thus, $x \preceq z$ in $P \cup Q$.
3. If $x, y \in P$ and $z \in Q$, then $x \leq_1 y$ in P and $z <_2 g(y)$ in Q by the definition of g-sum. From the monotonicity of g, it follows $z <_2 g(y) \leq_2 g(x)$ in Q. Thus, $x \preceq z$ in $P \cup Q$.
4. If $x \in P$ and $y, z \in Q$, then $z \leq_2 y$ in Q and $y <_2 g(x)$ in Q, implying $x \preceq z$ in $P \cup Q$.
5. If $x, y \in Q$ and $z \in P$, then $y \leq_2 x$ in Q and $g(z) \leq_2 y$ in Q, implying $x \preceq z$ in $P \cup Q$.

6. If $x \in Q$ and $y, z \in P$, then $g(y) \leq_2 x$ in Q and $y \leq_1 z$ in P. The monotonicity of g yields $g(z) \leq_2 g(y) \leq_2 x$ in Q and hence $x \preceq z$ in $P \cup Q$.

7. If $x, z \in Q$ and $y \in P$, then $g(y) \leq_2 x$ and $z <_2 g(y)$ in Q. Thus, $z <_2 x$ by the transitivity of $\leq_2$ and further $x \preceq z$.

8. If $x, z \in P$ and $y \in Q$, then $y <_2 g(x)$ and $g(z) \leq_2 y$ in Q. Thus, $g(z) <_2 g(x)$. So, $x \nparallel z$, i.e., either $z \leq_1 x$ or $x \leq_1 z$. Since g is decreasing, we then have $x \leq_1 z$ and further $x \preceq z$.

Combining the above findings, we know that $(P \oplus_g Q, \preceq)$ is a poset. □ □

Note that there is no $x, y \in P$ such that $x \| y$ and $g(x) <_2 g(y)$ whenever P is a chain. Thus, the following corollary holds.

Corollary 3.2. *Let $(P, \leq_1)$ be a chain and let $(Q, \leq_2)$ be a poset with $P \cap Q = \emptyset$. Assume that $g : P \to Q$ is a decreasing operation. Then the g-sum $(P \oplus_g Q, \preceq)$ is a poset.*

Let $(P, \leq_1)$ and $(Q, \leq_2)$ be two posets such that $P \cap Q = \emptyset$. Suppose that the g-sum $P \oplus_g Q^\partial$ coincides with the linear sum of P and Q. Consider arbitrary $x \in P$ and $y \in Q$. The linear sum $P \oplus Q$ yields $x \preceq y$, which, together with the g-sum $P \oplus_g Q^\partial$, implies $g(x) <_2 y$, impossible. However, a technical adjustment is enough to make the linear sum as a special case of the g-sum.

Given an ordered set $(P, \leq)$ (with or without the least element $\perp$), we form $P_\perp$ (called P '*lifted*') as follows: taken an element $\alpha \notin P$ and define $\leq$ on $P_\perp = P \cup \{\alpha\}$ by $x \leq y$ if and only if $x = \alpha$ or $x \leq y$ in P. Let $(P, \leq_1)$ and $(Q, \leq_2)$ be two posets with $P \cap Q_\perp = \emptyset$. Consider the decreasing operation $g^\flat : P \to (Q_\perp)^\partial$ given by $g^\flat(x) = \alpha$, where α is the least element of $Q_\perp$. Then the $g^\flat$-sum $P \oplus_{g^\flat} Q^\partial$ is just the linear sum $P \oplus Q$.

Next, we present the necessary and sufficient conditions such that the g-sum is a chain.

Proposition 3.3. *Let $(P, \leq_1)$ and $(Q, \leq_2)$ be two posets with $P \cap Q = \emptyset$. Assume that $g : P \to Q$ is a decreasing operation. Then the g-sum $(P \oplus_g Q, \preceq)$ is a chain if and only if both P and Q are chains.*

Proof. Suppose that $(P \oplus_g Q, \preceq)$ is a chain. If $x, y \in P$, then $x \preceq y$ or $y \preceq x$ and hence $x \leq_1 y$ or $y \leq_1 x$ by (2), implying that P is a chain. Similarly, we can show that Q is a chain.

Conversely, suppose that both P and Q are chains. Corollary 3.2 yields that $(P \oplus_g Q, \preceq)$ is a poset.

- If $x, y \in P$ or $x, y \in Q$, then, by (2), we have that $x \preceq y$ or $y \preceq x$.
- If $x \in P$ and $y \in Q$, then either $y <_2 g(x)$ or $g(x) \leq_2 y$ in Q because Q is a chain. Further, $y <_2 g(x)$ implies $x \preceq y$ and $g(x) \leq_2 y$ yields $y \preceq x$.
- The proof for the case $x \in Q$ and $y \in P$ is similar to that for $x \in P$ and $y \in Q$.

Summarizing for any $x, y \in P \cup Q$, we have $x \preceq y$ or $y \preceq x$, implying that $(P \oplus_g Q, \preceq)$ is a chain. □ □

Due to the associativity of the linear sum $\oplus$, n-ary form of $\oplus$ is uniquely given and thus it can be extended to work on any finite posets. However, extending the g-sum into n-ary form is much complex. We illustrate it by taking 3-ary form as an example:

- For $(P \oplus_g Q) \oplus_h R$, g is a mapping of P into Q, and h is a mapping of $P \cup Q$ into R.
- For $P \oplus_i (Q \oplus_j R)$, i is a mapping of P into $Q \cup R$, and j is a mapping of Q into R.

Let $(P, \leq_1)$ and $(Q, \leq_2)$ be two posets with $P \cap Q = \emptyset$ and $g : P \to Q$ be a decreasing operation. The *g-sum* $P \oplus^g Q$ is defined by taking the following relation on $P \cup Q$: $x \ll y$ if and only if

$$\begin{array}{ll} & x, y \in P \text{ and } x \leq_1 y, \\ \text{or} & x, y \in Q \text{ and } y \leq_2 x, \\ \text{or} & x \in P, y \in Q \text{ and } y \leq_2 g(x), \\ \text{or} & x \in Q, y \in P \text{ and } g(y) <_2 x. \end{array} \tag{3}$$

Similar to $P \oplus_g Q$, the following result holds:

Theorem 3.4. *Let $(P, \leq_1)$ and $(Q, \leq_2)$ be two posets with $P \cap Q = \emptyset$. Assume that $g : P \to Q$ is a decreasing operation. If there is no $x, y \in P$ such that $x \| y$ and $g(x) <_2 g(y)$, then the g-sum $(P \oplus^g Q, \ll)$ is a poset.*

Corollary 3.5. *Let $(P, \leq_1)$ be a chain and let $(Q, \leq_2)$ be a poset with $P \cap Q = \emptyset$. Assume that $g : P \to Q$ is a decreasing operation. Then the g-sum $(P \oplus^g Q, \ll)$ is a poset.*

Proposition 3.6. *Let $(P, \leq_1)$ and $(Q, \leq_2)$ be two posets with $P \cap Q = \emptyset$. Assume that $g : P \to Q$ is a decreasing operation. Then the g-sum $(P \oplus^g Q, \ll)$ is a chain if and only if both P and Q are chains.*

Remark 3.7. *Let $(P, \leq_1)$ and $(Q, \leq_2)$ be two posets with $P \cap Q = \emptyset$. Assume that $g : P \to Q$ is a decreasing operation. By the definitions, it is easy to see that $\preceq \neq \ll$ if $x \in P$, $y \in Q$ and $y = g(x)$, and $\preceq = \ll$ otherwise.*

4 Application to construct left- and right-continuous idempotent uninorms

In this section, the bounded chain L is supposed to be the unit interval $[0, 1]$ and $e \in]0, 1[$.

4.1 A new perspective on describing the structure of idempotent unonorms

The left- and right-continuous idempotent uninorms were characterized by De Baets [6] in terms of characterizing function g defined by

$$g(x) = \begin{cases} \sup\{y \mid U(x, y) = x\} & \text{if } x \leq e, \\ \inf\{y \mid U(x, y) = x\} & \text{if } x \geq e. \end{cases}$$

In the result of De Baets, the key point is to transform the commutativity of U into some kind of symmetry of its graph $\{(x, g(x)) \mid x \in [0, 1]\}$ w.r.t. the diagonal $\{(x, x) \mid x \in [0, 1]\}$. Here, we provide another perspective to avoid such a complicated symmetry: *For an idempotent uninorm U, we have* (i) *$U(x, y) \in \{x, y\}$ for all $x, y \in [0, 1]$;* (ii) *$U(x, y) = \min(x, y)$ for all $x, y \in [0, e]$ and* (iii) *$U(x, y) = \max(x, y)$ for all $x, y \in [e, 1]$. Due to the commutativity, it is sufficient to describe its structure on $[0, e[\times]e, 1]$. On the contrary, for any function $\widetilde{U} : [0, e[\times]e, 1] \to [0, 1]$ that is increasing and satisfies $\widetilde{U}(x, y) \in \{x, y\}$, then the following operation is a uninorm:*

$$U^*(x, y) = \begin{cases} \min(x, y) & \textit{if } (x, y) \in [0, e]^2, \\ \max(x, y) & \textit{if } (x, y) \in [e, 1]^2, \\ \widetilde{U}(x, y) & \textit{if } (x, y) \in [0, e[\times]e, 1], \\ \widetilde{U}(y, x) & \textit{if } (x, y) \in]e, 1] \times [0, e[. \end{cases} \tag{4}$$

Clearly, U^* is commutative and increasing, and has e as neutral element. Proposition 2 [8] says that a locally internal, commutative and monotonic operation is associative. Thus, U^* is a uninorm. Similar to the work of De Baets, we can describe the function $\widetilde{U} : [0, e[\times]e, 1] \to [0, 1]$ that is increasing and satisfies $\widetilde{U}(x, y) \in \{x, y\}$ in terms of the decreasing function $g : [0, e] \to [e, 1]$ defined as $g(x) = \sup\{y \mid \widetilde{U}(x, y) = x\}$. Plainly,

$$\widetilde{U}(x, y) = \begin{cases} x & \textit{if } g(x) < y, \\ y & \textit{if } g(x) > y, \\ x \textit{ or } y & \textit{if } g(x) = y. \end{cases} \tag{5}$$

On the contrary, for any decreasing function $g : [0, e] \to [e, 1]$, $\widetilde{U} : [0, e[\times]e, 1] \to [0, 1]$ defined by (5) *is increasing and satisfies $\widetilde{U}(x, y) \in \{x, y\}$. Therefore, idempotent uninorms can describe in terms of the decreasing function $g : [0, e] \to [e, 1]$. In particular, in the case of left-continuous idempotent uninorms, $\widetilde{U}(x, y) = x$ whenever $g(x) = y$ and in the case of right-continuous idempotent uninorms, $\widetilde{U}(x, y) = y$ whenever $g(x) = y$.*

4.2 Constructing left- and right-continuous idempotent uninorms

Let $P = [0, e]$, $Q =]e, 1]$ and $g : [0, e] \to [e, 1]$ be a decreasing function with $g(e) = e$. Proposition 3.3 says that the g-sum $(P \oplus_g Q, \preceq)$ is a chain. For all $x \in Q$, we have $g(e) = e < x$ and further $x \prec e$. Thus, $\preceq \in \mathcal{L}_e$, i.e., $\preceq$ satisfies the conditions (i) and (ii) of Theorem 1.1. Hence, the meet operation $\curlywedge$ of $\preceq$ given as follows

$$x \curlywedge y = \begin{cases} x \wedge y & \text{if } x, y \in [0, e], \\ & \quad \text{or } x \in [0, e[, y \in]e, 1], y < g(x), \\ & \quad \text{or } x \in]e, 1], y \in [0, e[, g(y) \leq x, \\ x \vee y & \text{otherwise} \end{cases}$$

is a uninorm. In particular, $\curlywedge$ is left-continuous.

Let $P = [0, e[$, $Q = [e, 1]$ and $g : [0, e] \to [e, 1]$ be a decreasing function with $g(e) = e$. Proposition 3.6 says that the g-sum $(P \oplus^g Q, \ll)$ is a chain. For all $x \in P$, we have $e = g(e) \leq g(x)$ and further $x \ll e$. Thus, $\ll \in \mathcal{L}_e$. Hence, the meet operation $\bar{\wedge}$ of $\ll$ given as follows

$$x \,\bar{\wedge}\, y = \begin{cases} x \wedge y & \text{if } x, y \in [0, e], \\ & \quad \text{or } x \in [0, e[, y \in]e, 1], y \leq g(x), \\ & \quad \text{or } x \in]e, 1], y \in [0, e[, g(y) < x, \\ x \vee y & \text{otherwise} \end{cases}$$

is a uninorm. In particular, $\bar{\wedge}$ is right-continuous.

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