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Choquet integrals of fuzzifying measurable functions and their applications

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Abstract

This study introduces the Choquet integral of fuzzifying functions with respect to a fuzzy measure. To express various phenomena or ambiguous values in many applications may not be enough to show them as a function, in which case a fuzzifying function can be applied to achieve better expressions or flexibility for given function values. To apply Choquet integrals of fuzzifying functions, we consider Choquet integrals of interval-valued functions as an operator which are α-level functions of fuzzifying functions. In this study, we investigate some properties of Choquet integral of fuzzifying functions and present their applications. As part of this, a series of relevant examples and their subsequent applications are provided, along with the fuzzification of two integrands: A probability density function (PDF) and a utility function.

Keywords: Fuzzy sets, Choquet integral, fuzzifying function, probability density function, utility function.

1 Introduction

G. Choquet [3] first introduced the notion of the Choquet integral, which is a nonadditive integral related to a fuzzy measure. The fuzzy measure related integrals were first applied in the area of statistical mechanics and potential theory, as well as the criteria aggregation operators that are associated with fuzzy decision making (see [5, 7, 16, 17, 20, 21]). As Choquet integrals are applied to decision theory, there are various ways to construct measurements in the decision-making process by developing effective fuzzy measures. Recently, Wood and Jang [22, 23] as well as Kim and Jang [12] introduced that fuzzy measures reflect the diversities of the economic policies, while the integrand of Choquet integrals might be regarded as the one who will receive benefits from their policies. This integrand may not be enough to exactly express the benefit values for the desired policies, or the constructed function for expected utilities may contain some uncertainty by itself. To overcome these deficiencies, a fuzzifying function in the Choquet integral can be considered to provide additional flexibility to the integrands depending on the α-level.

In this study, to propose Choquet integrals for the fuzzifying functions, we apply the concepts of the interval-valued Choquet integral for the α-level functions $\widetilde{f}_\alpha, \alpha \in (0, 1]$ of a fuzzifying function $\widetilde{f}$. To do this, we assume that the α-level functions of a fuzzifying function consist of interval-valued measurable functions. Then, by using Choquet integrals of the α-level functions, we define Choquet integrals of fuzzifying functions as an operator from a set-valued function to another set-valued function [1, 2] In addition, as practical applications for this work, we present the Choquet fuzzifying probability (CFP) using Choquet integrals of probability density fuzzifying functions (PDFFs) and construct Choquet expected fuzzifying utility (CEFU) model. Although the presented results in the paper immediately comes from the well known properties of Choquet integrals and fuzzifying functions, the approach can provide a new insight into probability density functions (PDFs) and utility functions as an application of Choquet integrals.

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The remainder of the study is structured as follows. Section 1 briefly recalls very basic definitions and the properties of a fuzzy measure and the Choquet integral. In addition, we review the Choquet integrals of interval-valued measurable functions and their associated properties. Section 2 defines the Choquet integrals of fuzzifying functions by applying Choquet integrals, and investigates their subsequent properties. Section 3 presents two subsections with Choquet fuzzifying probability (CFP) and Choquet expected fuzzifying utility (CEFU) model. Section 4 concludes the study, and offers some remarks.

1.1 Fuzzy measure

Throughout this study, we denote $\mathbb{R}^+$ as the interval $[0,\infty)$, and U as a non-empty universal set in $\mathbb{R}^+$, Ω as a σ-algebra on U. A fuzzy set $\widetilde{A}$ in a family $\mathcal{F}(U)$ of all fuzzy subsets of U is defined by

$$\widetilde{A} = \{\, (x, \mathfrak{m}_{\widetilde{A}}(x)) \big|\, x \in U\},$$

where a function $\mathfrak{m}_{\widetilde{A}} : U \to [0,1]$ is a membership function of a fuzzy set $\widetilde{A}$ (see [6, 13, 14, 24] for details). In particular, a fuzzy set $\widetilde{A}$ is said to be normal if there exists $x_0 \in U$ such that $\mathfrak{m}_{\widetilde{A}}(x_0) = 1$.

We recall the following basic notions: fuzzy measure, its continuity, submodularity as well as the supermodularity of a fuzzy measure.

Definition 1.1. [8, 14] *Let (U,Ω) be a measurable space.*

(i) *An extended real-valued function $\mu : \Omega \to [0,\infty]$ is called a fuzzy measure if it satisfies*

  (a) $\mu(\emptyset) = 0$,

  (b) $\mu(A) \le \mu(B)$ *whenever* $A, B \in \Omega$ *and* $A \subset B$.

(ii) *A fuzzy measure μ is said to be continuous from below if for every $\{A_n\} \subseteq \Omega$ and $A \in \Omega$, $A_n \nearrow A$ implies $\mu(A_n) \nearrow \mu(A)$.*

(iii) *A fuzzy measure μ is said to be continuous from above if for every $\{A_n\} \subseteq \Omega$ and $A \in \Omega$ with $\mu(A_1) < \infty$, $A_n \searrow A$ implies $\mu(A_n) \searrow \mu(A)$.*

(iv) *A fuzzy measure μ is said to be continuous if it is both continuous from below and from above.*

(v) *A fuzzy measure μ is said to be submodular if*

$$\mu(A) + \mu(B) \ge \mu(A \cup B) + \mu(A \cap B), \quad \forall A, B \in \Omega.$$

(vi) *A fuzzy measure μ is said to be supermodular if*

$$\mu(A) + \mu(B) \le \mu(A \cup B) + \mu(A \cap B), \quad \forall A, B \in \Omega.$$

Note that a function $f : U \to \mathbb{R}^+$ is said to be measurable if $\{x \in U | f(x) > r\} \in \Omega$, for all $r \ge 0$, and the Choquet integral of a measurable function f with respect to a fuzzy measure μ is considered, as follows.

Definition 1.2. [16, 20] *Let $f : U \to \mathbb{R}^+$ be a measurable function on (U,Ω).*

(i) *The Choquet integral of a nonnegative measurable function f over $A \subseteq U$ with respect to a fuzzy measure μ is defined by*

$$(\mathcal{C})\int_A f \, d\mu = \int_0^\infty \mu\left(\left\{x \in U \big| f(x) \ge r\right\} \cap A\right) dr,$$

where the integral on the right-hand side is the Lebesgue integral.

(ii) *A measurable function f is said to be integrable if the Choquet integral of f is defined and its value is finite.*

We write $(\mathcal{C})\int f \, d\mu$ instead of $(\mathcal{C})\int_U f \, d\mu$ and recall that two functions f and g are comonotonic, in symbol, $f \sim g$ if and only if for all $x, x' \in U$

$$f(x) < f(x') \text{ implies } g(x) \le g(x').$$

Then we have the following properties about comonotonicity.

Theorem 1.3. [5, 16, 20] *Let f, g, and h be functions. Then*

(i) $f \sim f$.

(ii) $f \sim g$ *implies* $g \sim f$.

(iii) $f \sim a$, *for all* $a \in (0, \infty)$.

(iv) $f \sim g$ *and* $f \sim h$ *implies* $f \sim (g + h)$.

We now introduce several properties of the Choquet integrals that will be applied in the next section.

Theorem 1.4. [4, 16] *Let μ be a fuzzy measure on (U, Ω) and let f and g be measurable functions on U.*

(i) *If $f \leq g$, then $(\mathcal{C})\int f \, d\mu \leq (\mathcal{C})\int g \, d\mu$.*

(ii) *If $f \sim g$ and $a, b \in (0, \infty)$, then $(\mathcal{C})\int (af + bg) d\mu = a \, (\mathcal{C})\int f \, d\mu + b \, (\mathcal{C})\int g \, d\mu$.*

(iii) *If we define $(f \vee g)(x) = f(x) \vee g(x)$, for all $x \in U$, then it holds that*

$$(\mathcal{C})\int (f \vee g) \, d\mu \geq (\mathcal{C})\int f \, d\mu \vee (\mathcal{C})\int g \, d\mu.$$

(iv) *If we define $(f \wedge g)(x) = f(x) \wedge g(x)$, for all $x \in U$, then it holds that*

$$(\mathcal{C})\int (f \wedge g) \, d\mu \leq (\mathcal{C})\int f \, d\mu \wedge (\mathcal{C})\int g \, d\mu.$$

(v) *If μ is supermodular, then the Choquet integral with respect to μ is superadditive, i.e.,*

$$(\mathcal{C})\int (f + g) d\mu \geq (\mathcal{C})\int f d\mu + (\mathcal{C})\int g d\mu.$$

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The primary contribution of this work is to introduce the Choquet integral of a fuzzifying measurable function and to investigate their relevant properties by applying the concept [6] of the fuzzifying function. Thus, we next briefly recall the Choquet integral of interval-valued functions.

1.2 Choquet integrals of interval-valued functions

Let $I(\mathbb{R}^+)$ be the set of all closed intervals in $\mathbb{R}^+$ whose elements are described as

$$I(\mathbb{R}^+) := \left\{ [a^-, a^+] \mid 0 \leq a^- \leq a^+ < \infty \right\}.$$

In particular, we allow degenerate intervals $a = [a, a]$ for any $a \in \mathbb{R}^+$. Then the interval operators in $I(\mathbb{R}^+)$ are defined as follows.

Definition 1.5. [9, 10, 15] *For each $\bar{a} = [a^-, a^+], \bar{b} = [b^-, b^+] \in I(\mathbb{R}^+)$, the arithmetic, comparison, and logical operators are given by*

(i) $\bar{a} + \bar{b} = [a^- + b^-, a^+ + b^+]$,

(ii) $\bar{a}\bar{b} = [a^- b^-, a^+ b^+]$,

(iii) $\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]$,

(iv) $\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,

(v) $\bar{a} \leq \bar{b}$ *if and only if* $a^- \leq b^-$ *and* $a^+ \leq b^+$,

(vi) $\bar{a} < \bar{b}$ *if and only if* $\bar{a} \leq \bar{b}$ *and* $\bar{a} \neq \bar{b}$, *and*

(vii) $\bar{a} \subseteq \bar{b}$ *if and only if* $b^- \leq a^-$ *and* $a^+ \leq b^+$.

Utilizing the operators defined in Definition 1.5, we can establish the algebraic and comparison operations of fuzzy sets as follows.

Definition 1.6. [14] *Let* $\widetilde{A}$ *and* $\widetilde{B}$ *be fuzzy sets of a nonempty measurable space* U *in* $\mathbb{R}^+$ *and let all* α*-levels* $\widetilde{A}_\alpha$ *and* $\widetilde{B}_\alpha$ *be closed intervals in* $I(\mathbb{R}^+)$, *where the* α*-level* $\widetilde{A}_\alpha$ *of a fuzzy set* $\widetilde{A}$ *is defined by* $\widetilde{A}_\alpha = \left\{x \in U | \mathfrak{m}_{\widetilde{A}}(x) \geq \alpha\right\}$.

(i) *The algebraic sum* $\widetilde{A}+\widetilde{B}$ *of two fuzzy sets* $\widetilde{A}$ *and* $\widetilde{B}$ *of* U *is defined as* $(\widetilde{A}+\widetilde{B})_\alpha = \widetilde{A}_\alpha + \widetilde{B}_\alpha \quad \forall \alpha \in (0,1]$, *provided the sum* $\widetilde{A}_\alpha + \widetilde{B}_\alpha = \{x+y \in U | x \in \widetilde{A}_\alpha, y \in \widetilde{B}_\alpha\} \in I(\mathbb{R}^+)$.

(ii) *The algebraic product* $\widetilde{A}\widetilde{B}$ *of two fuzzy sets* $\widetilde{A}$ *and* $\widetilde{B}$ *of* U *is defined as* $(\widetilde{A}\widetilde{B})_\alpha = \widetilde{A}_\alpha \widetilde{B}_\alpha \quad \forall \alpha \in (0,1]$.

(iii) *Two fuzzy sets are equal if all* α*-levels are the same, i.e.,* $\widetilde{A} = \widetilde{B}$ *if and only if* $\widetilde{A}_\alpha = \widetilde{B}_\alpha \quad \forall \alpha \in (0,1]$.

(iv) *The maximum* $\widetilde{A} \vee \widetilde{B}$ *of two fuzzy sets* $\widetilde{A}$ *and* $\widetilde{B}$ *of* U *is defined as* $\widetilde{A} \vee \widetilde{B}$ *if and only if* $\widetilde{A}_\alpha \vee \widetilde{B}_\alpha \quad \forall \alpha \in (0,1]$.

(v) *The minimum* $\widetilde{A} \wedge \widetilde{B}$ *of two fuzzy sets* $\widetilde{A}$ *and* $\widetilde{B}$ *of* U *is defined as* $\widetilde{A} \wedge \widetilde{B}$ *if and only if* $\widetilde{A}_\alpha \wedge \widetilde{B}_\alpha \quad \forall \alpha \in (0,1]$.

Note that a property $\mathcal{P}$ is said to hold almost everywhere with respect to a measure μ (μ-*a.e.*) in U if there exists a set $N \in \Omega$ with $\mu(N) = 0$ and all $x \in U \backslash N$ has the property $\mathcal{P}$. Also we recall that a set-valued function F is a mapping from U to the power set of $\mathbb{R}^+$ and F is said to be measurable if its graph is measurable (see [26] for details).

Definition 1.7. [10, 25, 26] *Let* F *be a set-valued function and* $A \in \Omega$.

(i) *The Choquet integral of* F *on* A *is defined by* $(\mathcal{C})\int_A F \, d\mu = \left\{ (\mathcal{C})\int_A f \, d\mu \middle| f \in S(F) \right\}$, *where* $S(F)$ *is the family of all measurable selections of* F.

(ii) *A set-valued function* F *is said to be Choquet integrable on* A *if* $(\mathcal{C})\int_A F \, d\mu \neq \emptyset$, *which does not include* ∞.

For simplicity of notation, we will write $(\mathcal{C})\int F \, d\mu$ instead of $(\mathcal{C})\int_U F \, d\mu$ and note that obviously $(\mathcal{C})\int F \, d\mu$ may be empty.

Definition 1.8. [11, 26] *A set-valued function* F *is said to be Choquet integrably bounded if there is a Choquet integrable function* $g : U \to \mathbb{R}^+$ *such that*

$$\|F(x)\| = \sup_{r \in F(x)} \{r\} \leq g(x) \quad \text{for each } x \in U.$$

Theorem 1.9. [10, 26] *If* $F = [f^-, f^+]$ *is a measurable and Choquet integrably bounded interval-valued function, then we have*

$$(\mathcal{C})\int F \, d\mu = \left[(\mathcal{C})\int f^- \, d\mu, (\mathcal{C})\int f^+ \, d\mu \right].$$

We recall some properties of the Choquet integral of measurable interval-valued functions using the ordering of $I(\mathbb{R}^+)$ in Definition 1.5

Theorem 1.10. [9, 26] *If* F, F_1, *and* F_2 *are measurable interval-valued functions on* (U, Ω), *i.e.,* $F, F_1, F_2 \in I(\mathbb{R}^+)$, *then the followings hold true:*

(i) *If* $A, B \in \Omega$ *with* $A \subseteq B$, *then we have* $(\mathcal{C})\int_A F \, d\mu \leq (\mathcal{C})\int_B F \, d\mu$.

(ii) *If* $a \in \mathbb{R}^+$, *then we have* $(\mathcal{C})\int a \, F \, d\mu = a \, (\mathcal{C})\int F \, d\mu$.

(iii) *If* $F_1 \leq F_2$ *(i.e.,* $F_1(x) \leq F_2(x)$, μ*-a.e.), then* $(\mathcal{C})\int F_1 \, d\mu \leq (\mathcal{C})\int F_2 \, d\mu$.

2 Choquet integrals of fuzzifying functions

In this section, we introduce a Choquet integrable fuzzifying function with its properties by applying Choquet integrals to interval-valued fuzzifying functions.

The fuzzifying function $\widetilde{f}$ of a crisp function f is considered as a mapping from the same domain to a new range, i.e., $\widetilde{f} : U \to \mathcal{F}(\mathbb{R}^+)$ given by

$$\widetilde{f} = \{(\widetilde{f}_\alpha, \alpha) | \alpha \in (0,1]\} \text{ with } \widetilde{f}_\alpha = [f_\alpha^-, f_\alpha^+] \quad \forall \alpha \in (0,1] \text{ and } \widetilde{f}_1 = [f_1^-, f_1^+] = f.$$

For example, we can consider a fuzzifying function $\widetilde{f}$ from a function $f(x) = 4x^2,\ 0 \le x \le 1$ as following: If we define the fuzzifying function $\widetilde{f}(x) = \widetilde{4}x^2,\ 0 \le x \le 1$, where a fuzzy set $\widetilde{4}$ of 4 is given by the following membership function

$$\mathfrak{m}_{\widetilde{4}}(s) = \begin{cases} \frac{1}{2}s - 1 & \text{if } 2 \le s \le 4, \\ -s + 5 & \text{if } 4 < s \le 5. \end{cases}$$

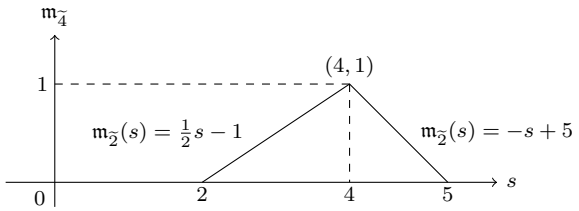

Figure 1: Membership function of a fuzzy set $\widetilde{4}$

Then, the $\alpha-$ cut $\widetilde{f}_\alpha$ is obtained by

$$\widetilde{f}_\alpha(x) = [f_\alpha^-(x), f_\alpha^+(x)] = [2(\alpha+1)x^2, (5-\alpha)x^2],$$

for $\alpha \in (0,1]$, and $\widetilde{f}_1(x) = 4x^2 = f(x)$.

We now propose the Choquet integral of a fuzzifying function, which is an operator from some set to another set.

Definition 2.1. *Let (U, Ω) be a measurable space and $f : U \to \mathbb{R}^+$ a measurable function.*

(i) *A fuzzifying function $\widetilde{f} : U \to \mathcal{F}(\mathbb{R}^+)$ is said to be measurable if f_α^- and f_α^+ are measurable functions for each $\alpha \in (0,1]$.*

(ii) *The Choquet integral of a measurable fuzzifying function $\widetilde{f}$ with respect to a fuzzy measure μ is given by*

$$(\mathcal{C})\int \widetilde{f}\ d\mu = \left\{ \left((\mathcal{C})\int \widetilde{f}_\alpha\ d\mu, \alpha \right) \middle| \alpha \in (0,1] \right\}.$$

(iii) *A measurable fuzzifying function $\widetilde{f}$ is said to be Choquet integrable if f_α^- and f_α^+ are Choquet integrable, i.e., $(\mathcal{C})\int f_\alpha^-\ d\mu$ and $(\mathcal{C})\int f_\alpha^+ d\mu$ are Choquet integrable for all $\alpha \in (0,1]$.*

(iv) *A measurable fuzzifying function $\widetilde{f}$ is said to be Choquet integrably bounded if there is a Choquet integrable function g such that for each $\alpha \in (0,1]$*

$$\|\widetilde{f}_\alpha(x)\| = \sup_{r \in \widetilde{f}_\alpha(x)} \{r\} \le g(x) \quad \forall x \in U.$$

The definition of the Choquet integral of a fuzzifying function produces a fuzzy set of interval-valued Choquet integrals as follows.

Theorem 2.2. *Let $f : U \to \mathbb{R}^+$ be a crisp function. If a measurable fuzzifying function $\widetilde{f}$ is Choquet integrable, then $\widetilde{f}_\alpha$ are Choquet integrable for all $\alpha \in (0,1]$.*

Proof. Since $\widetilde{f}$ is Choquet integrable, f_α^- and f_α^+ are Choquet integrable and $f_\alpha^- \le f_\alpha^+$ for all $\alpha \in (0,1]$. Thus, $(\mathcal{C})\int f_\alpha^-\ d\mu$ and $(\mathcal{C})\int f_\alpha^+\ d\mu$ exist and $(\mathcal{C})\int f_\alpha^-\ d\mu \le (\mathcal{C})\int f_\alpha^+\ d\mu$. So, $(\mathcal{C})\int \widetilde{f}_\alpha d\mu = \left[(\mathcal{C})\int f_\alpha^-(x)\ d\mu, (\mathcal{C})\int f_\alpha^+(x)\ d\mu\right] \neq \emptyset \quad \forall \alpha \in (0,1]$. Therefore, by Definition 1.7 (ii), $\widetilde{f}_\alpha$ are Choquet integrable $\forall \alpha \in (0,1]$. □

The following subsequent properties are established for the Choquet integral of a measurable fuzzifying function.

Theorem 2.3. *Let $f : U \to \mathbb{R}^+$ be a measurable function. Then the following properties are established.*

(i) *If a fuzzifying function $\widetilde{f}$ is Choquet integrably bounded, then*

$$(\mathcal{C})\int \widetilde{f}\ d\mu = \Big\{\Big(\Big[(\mathcal{C})\int f_\alpha^-(x)\ d\mu, (\mathcal{C})\int f_\alpha^+(x)\ d\mu\Big], \alpha\Big)\Big|\alpha \in (0,1]\Big\}.$$

(ii) *If a measurable fuzzifying function $\widetilde{f}$ is Choquet integrably bounded and $A, B \in \Omega$ with $A \subseteq B$, then $(\mathcal{C})\int_A \widetilde{f}d\mu \leq (\mathcal{C})\int_B \widetilde{f}d\mu$.*

(iii) *If fuzzifying functions $\widetilde{f}$ and $\widetilde{g}$ obtained from measurable functions $f : U \to \mathbb{R}^+$ and $g : U \to \mathbb{R}^+$ are Choquet integrably bounded satisfying $\widetilde{f} \leq \widetilde{g}$, i.e., $\widetilde{f}_\alpha \leq \widetilde{g}_\alpha$ for all $\alpha \in (0,1]$, then we have $(\mathcal{C})\int \widetilde{f}d\mu \leq (\mathcal{C})\int \widetilde{g}d\mu$.*

(iv) *If a fuzzifying $\widetilde{f}$ is Choquet integrable, then $(\mathcal{C})\int a\ \widetilde{f}\ d\mu = a\ (\mathcal{C})\int \widetilde{f}\ d\mu \quad \forall a \in \mathbb{R}^+$.*

Proof. (i) By Theorem 1.9, we have $(\mathcal{C})\int \widetilde{f}_\alpha\ d\mu = \Big[(\mathcal{C})\int f_\alpha^-(x)\ d\mu, (\mathcal{C})\int f_\alpha^+(x)\ d\mu\Big]$ for all $\alpha \in (0,1]$. Thus Definition 2.1 (ii) yields that

$$\begin{aligned}(\mathcal{C})\int \widetilde{f}\ d\mu =& \Big\{\Big((\mathcal{C})\int \widetilde{f}_\alpha(x)\ d\mu, \alpha\Big)\Big|\alpha \in (0,1]\Big\}\\ =& \Big\{\Big(\Big[(\mathcal{C})\int f_\alpha^-(x)\ d\mu, (\mathcal{C})\int f_\alpha^+(x)\ d\mu\Big], \alpha\Big)\Big|\alpha \in (0,1]\Big\},\end{aligned}$$

which shows the first assertion.

(ii) Since $\widetilde{f}$ is Choquet integrably bounded by Definition 2.1(iv), $\widetilde{f}_\alpha = [f_\alpha^-, f_\alpha^+]$ are Choquet integrably bounded for all $\alpha \in (0,1]$. By Theorem 1.10 (i), we get $(\mathcal{C})\int_A \widetilde{f}_\alpha d\mu \leq (\mathcal{C})\int_B \widetilde{f}_\alpha\ d\mu$ for all $\alpha \in (0,1]$. Therefore, Definition 2.1 (ii) provides that

$$(\mathcal{C})\int_A \widetilde{f}d\mu = \Big\{\Big((\mathcal{C})\int_A \widetilde{f}_\alpha d\mu, \alpha\Big)\Big|\alpha \in (0,1]\Big\} \leq \Big\{\Big((\mathcal{C})\int_B \widetilde{f}_\alpha d\mu, \alpha\Big)\Big|\alpha \in (0,1]\Big\} = (\mathcal{C})\int_B \widetilde{f}d\mu.$$

Similarly, one can show (iii) and (iv) and we omit the proofs. □

Definition 2.4. *Let f and g be crisp functions. Then two Choquet integrable fuzzifying functions $\widetilde{f}$ and $\widetilde{g}$ are comonotonic, in symbol, $\widetilde{f} \sim \widetilde{g}$, if and only if $\widetilde{f}_\alpha \sim \widetilde{g}_\alpha$, i.e., $f_\alpha^- \sim g_\alpha^-$ and $f_\alpha^+ \sim g_\alpha^+$ $\forall \alpha \in (0,1]$.*

From Definition 2.4 with the similar techniques of the proof in Theorem 2.3, we obtain the linear property of the Choquet integrable fuzzifying functions.

Theorem 2.5. *Let f and g be crisp Choquet integrable functions. If $\widetilde{f}$ and $\widetilde{g}$ are Choquet integrable fuzzifying functions satisfying $\widetilde{f} \sim \widetilde{g}$, then*

$$(\mathcal{C})\int \Big(a\widetilde{f} + b\widetilde{g}\Big)\ d\mu = a\ (\mathcal{C})\int \widetilde{f}d\mu + b\ (\mathcal{C})\int \widetilde{g}\ d\mu \quad \forall a, b \in \mathbb{R}^+.$$

We next introduce the minimum and maximum operations of Choquet integrable fuzzifying functions.

Definition 2.6. *Let $\widetilde{f}$ and $\widetilde{g}$ be Choquet integrable fuzzifying functions. Then we define $\widetilde{f} \vee \widetilde{g}$ by $(\widetilde{f} \vee \widetilde{g})(x) = \widetilde{f}(x) \vee \widetilde{g}(x)$, i.e., $(\widetilde{f} \vee \widetilde{g})_\alpha(x) = \widetilde{f}_\alpha(x) \vee \widetilde{g}_\alpha(x)$ $\forall x \in \mathbb{R}^+$ and $\forall \alpha \in (0,1]$. Similarly, we define $\widetilde{f} \wedge \widetilde{g}$ by $(\widetilde{f} \wedge \widetilde{g})(x) = \widetilde{f}(x) \wedge \widetilde{g}(x)$.*

Definition 2.6 immediately provides the following results, so we omit the proof.

Theorem 2.7. *If $\widetilde{f}$ and $\widetilde{g}$ are Choquet integrable fuzzifying functions, then*

$$(\mathcal{C})\int \widetilde{f} \vee \widetilde{g}\ d\mu \geq (\mathcal{C})\int \widetilde{f}\ d\mu \vee (\mathcal{C})\int \widetilde{g}\ d\mu, \quad (\mathcal{C})\int \widetilde{f} \wedge \widetilde{g}\ d\mu \leq (\mathcal{C})\int \widetilde{f}\ d\mu \wedge (\mathcal{C})\int \widetilde{g}\ d\mu.$$

3 Applications: CFP and CEFU model

3.1 Choquet fuzzifying probability (CFP)

In this subsection, we apply fuzzifying functions to a probability density function (PDF) with the Choquet integral and present their properties with a computational example.

Let us consider a continuous random variable X for a sample space S in $\mathbb{R}^+$ and a Choquet integrable PDF $f_X : S \to \mathbb{R}^+$ satisfying the followings:

(i) f_X is positive everywhere in the support S, i.e., $f_X(x) > 0 \quad \forall x \in S$.

(ii) The Choquet integration of f_X satisfies $(\mathcal{C})\int_S f_X(x)d\mu(x) = 1$.

If $f_X(x)$ is the PDF of X at x, then the Choquet probability P that X belongs to some event E is given by the Choquet integral of $f_X(x)$ over E with respect to a fuzzy measure μ, i.e.,

$$P(E) = P(X \in E) = (\mathcal{C})\int_E f_X(x)d\mu. \tag{1}$$

By applying the PDF to the Choquet integral, we define a probability density fuzzifying function (PDFF) $\widetilde{f}_X$ as $\widetilde{f}_X : S \to \mathcal{F}(\mathbb{R}^+)$ satisfying the following conditions:

(i) $\widetilde{f}_X > 0 \quad \forall x \in S,$ i.e., $\mathfrak{m}_{\widetilde{f}_X}(x) > 0 \quad \forall x \in S$.

(ii) $(\mathcal{C})\int_S \widetilde{f}_X(x)d\mu(x) = \widetilde{1}$, where $\widetilde{1}$ is a convex fuzzy set of 1 with $\mathfrak{m}_{\widetilde{1}}(1) = 1$.

We note that if $\widetilde{f}_X$ is a PDFF of X, then the CFP $\widetilde{P}$ that X belongs to some event E is given by the Choquet integral of $\widetilde{f}_X$ with respect to a fuzzy measure μ over E, i.e.,

$$\widetilde{P}(E) = \widetilde{P}(X \in E) = (\mathcal{C})\int_E \widetilde{f}_X(x)d\mu. \tag{2}$$

Then, we introduce some properties of the CFP $\widetilde{P}(X)$ without proof, which are the immediate results from (2).

Theorem 3.1. *Let X be a continuous random variable with a PDFF $\widetilde{f}_X$ and let $\widetilde{P}$ be the CFP given in* (2)*. Then*

(i) $0 \le \widetilde{P}(E) \le 1$ *for each event E of S.*

(ii) $\widetilde{P}(S) = \widetilde{1}$.

Theorem 3.2. *Let $\{E_1, E_2, \cdots\}$ be a sequence of events in S with $E_i \cap E_j = \emptyset$ for any $i \neq j$ and let $\widetilde{P}$ be the CFP given in* (2)*.*

(i) *If a fuzzy measure μ is supermodular, then $\widetilde{P}(\bigcup_{n=1}^N E_n) \ge \sum_{n=1}^N \widetilde{P}(E_n)$ for any $N \ge 1$.*

(ii) *If a fuzzy measure μ is submodular, then $\widetilde{P}(\bigcup_{n=1}^N E_n) \le \sum_{n=1}^N \widetilde{P}(E_n)$ for any $N \ge 1$.*

Example 3.3. *Suppose that $\mathfrak{m}$ is the Lebesgue measure and $\mu = \mathfrak{m}^2$ is a fuzzy measure. Let S be a sample space in $\mathbb{R}^+$ and let $X : S \to [0,1]$ be a random variable with the PDF $f_X(x) = 2x$, $0 < x < 1$. Then, the PDFF is considered as $\widetilde{f}_X(x) = \widetilde{2}x$, $0 < x < 1$, where $\widetilde{2}$ is a fuzzy set on $\mathbb{R}$ given by* $\mathfrak{m}_{\widetilde{2}}(s) = \begin{cases} s-1 & \text{if } 1 \le s < 2, \\ -\frac{1}{2}s + 2 & \text{if } 1 < s \le 4. \end{cases}$

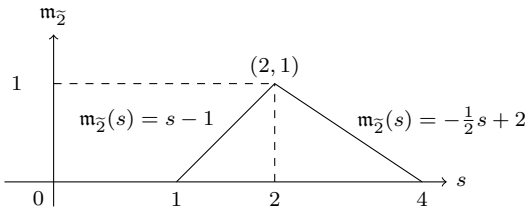

Figure 2: Membership function of PDFF $\mathfrak{m}_{\widetilde{2}}$

Thus, the CFP $\widetilde{P}_\alpha([0,\frac{1}{2}])$ *can be computed at* α*-level for all* $\alpha \in (0,1]$ *as follows:*

$$\widetilde{P}_\alpha\Big(\Big[0,\frac{1}{2}\Big]\Big) = (\mathcal{C})\int_{[0,\frac{1}{2}]} (\widetilde{2})_\alpha x \; d\mu = \Big[(\mathcal{C})\int_{[0,\frac{1}{2}]} (\alpha+1)x \; d\mu, (\mathcal{C})\int_{[0,\frac{1}{2}]} (4-2\alpha)x \; d\mu\Big] = \Big[\frac{\alpha+1}{24}, \; \frac{2-\alpha}{12}\Big].$$

Thus, $\widetilde{P}([0,\frac{1}{2}])$ *is given by* $\widetilde{P}\Big(\Big[0,\frac{1}{2}\Big]\Big) = \Big\{\Big(\Big[\frac{\alpha+1}{24}, \; \frac{2-\alpha}{12}\Big], \alpha\Big)\Big| \alpha \in (0,1]\Big\}$*. Therefore, the membership function of* $\widetilde{P}([0,\frac{1}{2}])$ *can be expressed as* $\mathfrak{m}_{\widetilde{P}([0,\frac{1}{2}])}(s) = \begin{cases} 24s-1 & \text{if } \frac{1}{24} \le s \le \frac{1}{12}, \\ 2-12s & \text{if } \frac{1}{12} < s \le \frac{1}{6}, \end{cases}$ *which shows that* $\widetilde{P}([0,\frac{1}{2}])$ *is a fuzzy set* $\widetilde{\frac{1}{12}}$.

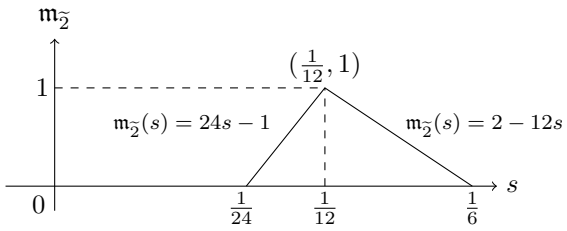

Figure 3: Membership function of PDFF $\mathfrak{m}_{\widetilde{2}}$

3.2 Choquet expected fuzzifying utility (CEFU) model

In this subsection, we apply the Choquet integral of a fuzzifying utility function in decision-making under uncertainty as follows. Let $\mathcal{L}$ be the set of Choquet integrable functions $f : S \to \mathbb{R}^+$ for a sample space S and $(\mathcal{C})\int \mathrm{u}(f)d\mu > 0$.

Definition 3.4. [18, 19] *Let* $\mathrm{u} : \mathbb{R}^+ \to \mathbb{R}^+$ *be a utility function and* $f \in \mathcal{L}$*. Then, the Choquet expected utility model stimulates the decision maker to rank act* f *with the help of* u*. In detail, the ranking* $C_{\mathrm{u},\mu}$ *of act* f *is performed through*

$$C_{\mathrm{u},\mu}(f) = (\mathcal{C})\int \mathrm{u}(f)d\mu,$$

where μ *is a fuzzy measure. Denote* $f \succ_{\mathrm{u}} g$ *when* $C_{\mathrm{u},\mu}(f) > C_{\mathrm{u},\mu}(g)$ *for two acts* f *and* g *in* $\mathcal{L}$*.*

Applying Theorem 1.9 to Definition 3.4, we propose a new model, called the CEFU model, which provides more flexible ranking information to the decision maker to rank an act function f with the help of a fuzzifying utility function $\widetilde{\mathrm{u}}$.

Definition 3.5. 1. *For a utility function* $\mathrm{u} : \mathbb{R}^+ \to \mathbb{R}^+$*, we define the fuzzifying utility function* $\widetilde{\mathrm{u}}$ *of* u *by*

$$\widetilde{\mathrm{u}} : \mathbb{R}^+ \to \mathcal{F}(\mathbb{R}^+) \text{ via } \widetilde{\mathrm{u}} = \big\{(\widetilde{\mathrm{u}}_\alpha, \alpha) \mid \alpha \in (0,1], \quad \widetilde{\mathrm{u}}_\alpha = [\mathrm{u}_\alpha^-, \mathrm{u}_\alpha^+]\big\} \text{ and } \widetilde{\mathrm{u}}_1 = \mathrm{u}.$$

2. *A fuzzifying utility function* $\widetilde{\mathrm{u}}$ *is continuous if and only if for each* $\alpha \in (0,1]$, $\widetilde{\mathrm{u}}_\alpha = [\mathrm{u}_\alpha^-, \mathrm{u}_\alpha^+]$ *if and only if for each sequence* $\{x_n\}$ *with* $\lim_{n\to\infty} x_n = x$*,*

$$d_{\mathcal{H}} - \lim_{n\to\infty} \widetilde{\mathrm{u}}_\alpha(x_n) = \widetilde{\mathrm{u}}_\alpha(x),$$

where $d_{\mathcal{H}}$ *is the Haussdorff measure on* $\mathbb{R}^+$*.*

3. *We say that* $\widetilde{\mathrm{u}}$ *is strictly increasing when* $\widetilde{\mathrm{u}}_\alpha = [\mathrm{u}_\alpha^-, \mathrm{u}_\alpha^+]$ *is strictly increasing for each* $\alpha \in (0,1]$*, i.e.,* $\widetilde{\mathrm{u}}_\alpha(x) \prec_* \widetilde{\mathrm{u}}_\alpha(x')$ *whenever* $x < x'$*, which means* $\mathrm{u}_\alpha^-(x) < \mathrm{u}_\alpha^-(x')$ *and* $\mathrm{u}_\alpha^+(x) < \mathrm{u}_\alpha^+(x')$ *for each* $\alpha \in (0,1]$ *whenever* $x < x'$*. Thus, we say that* $\widetilde{\mathrm{u}}_\alpha$ *is strictly increasing for* $\alpha \in (0,1]$ *if and only if* u_α^- *and* u_α^+ *are strictly increasing.*

Remark 3.6. 1. *We recall that* $d_{\mathcal{H}} - \lim_{n\to\infty} \widetilde{\mathrm{u}}_\alpha(x_n) = \widetilde{\mathrm{u}}_\alpha(x)$ *if and only if* $\lim_{n\to\infty} \mathrm{u}_\alpha^-(x_n) = \mathrm{u}_\alpha^-(x)$ *and* $\lim_{n\to\infty} \mathrm{u}_\alpha^+(x_n) = \mathrm{u}_\alpha^+(x)$*. Thus, for each* $\alpha \in (0,1]$, $\widetilde{\mathrm{u}}_\alpha$ *is continuous if and only if* u_α^- *and* u_α^+ *are continuous.*

2. *If* $\mathrm{u} : \mathbb{R}^+ \to \mathbb{R}^+$ *is a continuous and strictly increasing utility function, then the fuzzifying utility function* $\widetilde{\mathrm{u}}$ *is also a continuous and strictly increasing function.*

The CEFU model provides a fuzzy set containing rank information that can help a decision maker ranks act f with variant fuzzifying ways depending on the preferred policies. From the model, the fuzzifying ranking $C_{\widetilde{\mathrm{u}},\mu}$ of acts f is performed through

$$C_{\widetilde{\mathrm{u}},\mu}(f) = (\mathcal{C})\int \widetilde{\mathrm{u}}(f)d\mu,$$

i.e., $f \prec_{\widetilde{\mathrm{u}}} g \Leftrightarrow C_{\widetilde{\mathrm{u}},\mu}(f) \prec_* C_{\widetilde{\mathrm{u}},\mu}(g)$, where μ is a fuzzy measure and $C_{\widetilde{\mathrm{u}},\mu}(f) \prec_* C_{\widetilde{\mathrm{u}},\mu}(g)$ if and only if for each $\alpha \in (0,1]$, $C_{\widetilde{\mathrm{u}}_\alpha,\mu}(f) \prec_* C_{\widetilde{\mathrm{u}}_\alpha,\mu}(g)$, which means $C_{\mathrm{u}^-_\alpha,\mu}(f) < C_{\mathrm{u}^-_\alpha,\mu}(g)$ and $C_{\mathrm{u}^+_\alpha,\mu}(f) < C_{\mathrm{u}^+_\alpha,\mu}(g)$ for each $\alpha \in (0,1]$.

Definition 3.7. (i) *A function* $\mathrm{u} : \mathbb{R}^+ \to \mathbb{R}^+$ *is said to be linear if for any* $x_1, x_2 \in \mathbb{R}^+$,

$$\mathrm{u}(tx_1 + (1-t)x_2) = t\mathrm{u}(x_1) + (1-t)u(x_2) \quad \forall t \in [0,1].$$

(ii) *A fuzzifying utility function* $\widetilde{\mathrm{u}} : \mathbb{R}^+ \to \mathcal{F}(\mathbb{R}^+)$ *is said to be linear if for all* $\alpha \in (0,1]$, $\widetilde{\mathrm{u}} = [\mathrm{u}^-_\alpha, \mathrm{u}^+_\alpha]$ *is linear, i.e.,* u^-_α *and* u^+_α *are linear.*

We now investigate properties of the preference relation for the CEFU model in the axioms [19]. Let UF be the set of all utility functions $\mathrm{u} : \mathbb{R}^+ \to \mathbb{R}^+$ which are continuous and strictly increasing.

Theorem 3.8. *For all* $\mathrm{u} \in$ *UF, the relation* $\succ_{\mathrm{u}}$ *satisfies the following axioms:*

(i) Weak order: *(a) For all* f *and* g *in* $\mathcal{L}$ *with* $f \neq g$ μ*-a.e.,* $f \succ_{\mathrm{u}} g$ *or* $g \succ_{\mathrm{u}} f$*. (b) For all* f, g *and* h *in* $\mathcal{L}$*, if* $f \succ_{\mathrm{u}} g$ *and* $g \succ_{\mathrm{u}} h$*, then* $f \succ_{\mathrm{u}} h$*.*

(ii) Independence: *For all* $f, g,$ *and* h *in* $\mathcal{L}$ *with* $f \sim h$ *and* $g \sim h$*: If* $f \succ_{\mathrm{u}} g$ *for a linear utility* u*, then* $sf+(1-s)h \succ_{\mathrm{u}} sg + (1-s)h$ *for all* $s \in (0,1)$*.*

(iii) Continuity: *For all* $f, g,$ *and* h *in* $\mathcal{L}$ *with* $f \sim h$ *and* $g \sim h$ *and a linear utility* u*: If* $f \succ_{\mathrm{u}} g$ *and* $g \succ_{\mathrm{u}} h$*, then there are* s *and* t *in* $(0,1)$ *such that* $(sf + (1-s)h) \succ_{\mathrm{u}} g$ *and* $g \succ_{\mathrm{u}} (tf + (1-t)h)$*.*

(iv) Monotonicity: *For all* f *and* g *in* $\mathcal{L}$*: If* $f(x) \geq g(x)$ *for all* $x \in \mathbb{R}^+$ *with* $f \neq g$ μ*-a.e., then* $f \succ_{\mathrm{u}} g$*.*

Proof. (i) Since $f \neq g$ μ-*a.e.* and u is strictly increasing, $\mathrm{u}(f) \neq u(g)$ μ-*a.e.*, so that $C_{\mathrm{u},\mu}(f) < C_{\mathrm{u},\mu}(g)$ or $C_{\mathrm{u},\mu}(f) > C_{\mathrm{u},\mu}(g)$. Thus, $f \succ_{\mathrm{u}} g$ or $f \prec_{\mathrm{u}} g$, which proves part (a). The second part (b) directly follows from Definition 3.5.

(ii) Let $f \sim h$ and $g \sim h$, then $sf \sim (1-s)h$ and $sg \sim (1-s)h$. Thus, $\mathrm{u}(sf) \sim \mathrm{u}((1-s)h)$ and $\mathrm{u}(sg) \sim \mathrm{u}((1-s)h)$ from a strictly increasing u. Using the linearity of u and $C_{\mathrm{u},\mu}(f) \succ_* C_{\mathrm{u},\mu}(g)$ for $f \succ_{\mathrm{u}} g$, we have

$$\begin{aligned}(\mathcal{C})\int \mathrm{u}(sf + (1-s)h)d\mu &= s(\mathcal{C})\int \mathrm{u}(f)d\mu + (1-s)(\mathcal{C})\int \mathrm{u}(h)d\mu \\ &\geq s(\mathcal{C})\int \mathrm{u}(g)d\mu + (1-s)(\mathcal{C})\int \mathrm{u}(g)d\mu \\ &= (\mathcal{C})\int \mathrm{u}\left(sg + (1-s)g\right)d\mu,\end{aligned}$$

which proves that $sf + (1-s)h \succ_{\mathrm{u}} sg + (1-s)h$ for all $s \in (0,1)$.

(iii) Since $f \succ_{\mathrm{u}} g$ and $g \succ_{\mathrm{u}} h$ for a linear $\mathrm{u} \in$ UF, we have $C_{\mathrm{u},\mu}(f) > C_{\mathrm{u},\mu}(g) > C_{\mathrm{u},\mu}(h)$. Thus, there exist $s, t \in (0,1)$ such that $sC_{\mathrm{u},\mu}(f) + (1-s)C_{\mathrm{u},\mu}(h) > C_{\mathrm{u},\mu}(g)$ and $C_{\mathrm{u},\mu}(g) > tC_{\mathrm{u},\mu}(f) + (1-t)C_{\mathrm{u},\mu}(h)$. Using the linearity of u, we have

$$\begin{aligned}C_{\mathrm{u},\mu}(sf + (1-s)h)d\mu = (\mathcal{C})\int \mathrm{u}(sf + (1-s)h)d\mu &= (\mathcal{C})\int (s\mathrm{u}(f) + (1-s)\mathrm{u}(h))d\mu \\ &= sC_{\mathrm{u},\mu}(f) + (1-s)C_{\mathrm{u},\mu}(h) > C_{\mathrm{u},\mu}(g),\end{aligned}$$

which shows $sf + (1-s)g \succ_{\mathrm{u}} g$. Similarly, $g \succ_{\mathrm{u}} (tf + (1-t)h$ can be obtained.

(iv) Since u is strictly increasing, $\mathrm{u}(f) \neq \mathrm{u}(g)$ μ-*a.e.* and $\mathrm{u}(f(x)) \geq \mathrm{u}(g(x))$ for all $x \in \mathbb{R}^+$, so that $\mathrm{u}(f) > \mathrm{u}(g)$ μ-*a.e.* and $C_{\mathrm{u},\mu}(f) > C_{\mathrm{u},\mu}(g)$, which shows $f \succ_{\mathrm{u}} g$.

□

Let FUF be the set of all fuzzifying utility functions $\widetilde{\mathrm{u}} : \mathbb{R}^+ \to \mathcal{F}(\mathbb{R}^+)$ that are continuous and strictly increasing with $\widetilde{\mathrm{u}}_1 = \mathrm{u}$ where $\mathrm{u} \in \mathrm{UF}$. The relation $\prec_{\widetilde{\mathrm{u}}}$ provides the following properties and we omit the proof since it is easily established by Theorem 3.8.

Theorem 3.9. *If $\widetilde{\mathrm{u}} \in FUF$, then the relation $\prec_{\widetilde{\mathrm{u}}}$ satisfies the following axioms:*

(i) Weak order: *(a) For all f and g in $\mathcal{L}$, $f \succ_{\widetilde{\mathrm{u}}_1} g$ or $g \succ_{\widetilde{\mathrm{u}}_1} f$. (b) For all f, g and h in $\mathcal{L}$, if $f \succ_{\widetilde{\mathrm{u}}} g$ and $g \succ_{\widetilde{\mathrm{u}}} h$, then $f \succ_{\widetilde{\mathrm{u}}} h$.*

(ii) Independence: *For all f, g, and h in $\mathcal{L}$ with $f \sim h$ and $g \sim h$ and for all $s \in (0,1)$ and a linear utility $\widetilde{\mathrm{u}}$: If $f \succ_{\widetilde{\mathrm{u}}} g$, then $sf + (1-s)h \succ_{\widetilde{\mathrm{u}}} sg + (1-s)h$.*

(iii) Continuity: *For all f, g, and h in $\mathcal{L}$ with $f \sim h$ and $g \sim h$ and a linear utility $\widetilde{\mathrm{u}}$: If $f \succ_{\widetilde{\mathrm{u}}} g$ and $g \succ_{\widetilde{\mathrm{u}}} h$, then there are s_1, s_2 and t_1, t_2 in $[0,1]$ such that for all $\alpha \in (0,1]$,*

$$[C_{\mathrm{u}^-_\alpha,\mu}(s_1 f + (1-s_1)h), C_{\mathrm{u}^+_\alpha,\mu}(s_1 f + (1-s_1)h)] > C_{\widetilde{\mathrm{u}}_\alpha,\mu}(g),$$

and

$$C_{\widetilde{\mathrm{u}}_\alpha,\mu}(g) > [C_{\mathrm{u}^-_\alpha,\mu}(s_1 f + (1-s_1)h), C_{\mathrm{u}^+_\alpha,\mu}(s_1 f + (1-s_1)h)].$$

(iv) Monotonicity: *For all f and g in $\mathcal{L}$: If $f(x) \geq g(x)$ for all $x \in \mathbb{R}^+$ with $f \neq g$ μ-a.e., then $f \succ_{\widetilde{\mathrm{u}}} g$.*

Example 3.10. *In* [22, 23]*, the authors examined these respective trading relationships between Republic of Korea (KR) and four trading partners, USA (US), New Zealand (NZ), India (IN), and Turkey (TR) (i.e., KR–US, KR–NZ, KR–IN, and KR–TR) by incorporating a clearly defined set of Harmonized System (HS) product code categories for years* $2010, 2011, 2012$, *and* 2013*:* (c_1) *Live animals; animal products,* (c_2) *Meat and edible meat offal,* (c_3) *Fish and crustaceans, mollusks and other aquatic invertebrates,* (c_4) *Dairy produce; birds' eggs; natural honey; edible products of animal origin, not elsewhere specified or included,* (c_5) *Products of animal origin, not elsewhere specified or included. Denote that* t_i *: year,* $v(\mathrm{t})$ *: trading value taken from* [22, 23]*,* $\mathrm{u}(v(\mathrm{t}))$ *: the utility of* $v(\mathrm{t})$*, and* $\mathcal{U}_{CEF}(\mathrm{u}(v(\mathrm{t}))$ *: the Choquet expected utility function of* u *from* $v(\mathrm{t})$*. The discrete Choquet integral of f with respect to a fuzzy measure μ is considered, as follows:*

$$(\mathcal{C})\int f d\mu = \sum_{i=1}^{n} f(x_{(i)})\Big[\mu(E^{(i)}) - \mu(E^{(i+1)})\Big], \quad E^{(i)} = \{x_{(i)}, x_{(i+1)}, \cdots, x_{(n)}\}, E^{(n+1)} = \emptyset,$$

where $(\cdot)$ indicates a permutation on $\{1, 2, 3, \cdots, n\}$ satisfying $f(x_{(1)}) \leq f(x_{(2)}) \leq \cdots \leq f(x_{(n)})$. Then, by considering f in Theorem 2.3 to be u *and x to be $v_{(i)} := v(\mathrm{t}_{(i)})$ for $i = 1, 2, 3, 4$,*

$$(\mathcal{C})\int \widetilde{\mathrm{u}}(v)\ d\mu = \Big\{\Big(\Big[(\mathcal{C})\int \mathrm{u}^-_\alpha(v)\ d\mu,\ (\mathcal{C})\int \mathrm{u}^+_\alpha(v)\ d\mu\Big], \alpha\Big)\Big|\alpha \in (0,1]\Big\}. \tag{3}$$

Define a fuzzy measure μ on $\mathbb{R}^+$ by

$$\mu(E^{(5)}) = 0, \quad \mu(E^{(4)}) = \mu(\{v_{(4)}\}) = 0.1, \quad \mu(E^{(3)}) = \mu(\{v_{(3)}, v_{(4)}\}) = 0.3,$$
$$\mu(E^{(2)}) = \mu(\{v_{(2)}, v_{(3)}, v_{(4)}\}) = 0.6, \quad \mu(E^{(1)}) = \mu(\{v_{(1)}, v_{(2)}, v_{(3)}, v_{(4)}\}) = 1,$$

where $v_{(i)}$ represent the trading values at year $\mathrm{t}_{(i)}$ in US dollars. We consider the scaled utility defined as $\mathrm{u}(v_{(i)}) = \sqrt{\frac{v_{(i)}}{\max\{v_{(i)}\}}}$, where $\max\{v_{(i)}\}$ represents the maximum value of trading values $v(\mathrm{t})$ for the four countries, which is equal to $\$100,141,401$. Using formula (3), $\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v))$ can be computed for $\widetilde{\mathrm{u}}(v(\mathrm{t})) = \widetilde{1}\mathrm{u}(v(\mathrm{t}))$, where $\widetilde{1}$ is a subset of $\mathbb{R}^+$ given by

$$\mathfrak{m}_{\widetilde{1}}(s) = \begin{cases} s & \text{if } 0 \leq s \leq 1, \\ -\frac{1}{2}s + \frac{3}{2} & \text{if } 1 < s \leq 3. \end{cases}$$

Note that $\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v))$ for trading values is a fuzzy set representing South Korea's trading relationship with each partner country for the years $2010, 2011, 2012$*, and* 2013*. In fact,*

$$\mathcal{U}_{CEF}\left(\widetilde{\mathrm{u}}(v(\mathrm{t}))\right) = \Big\{\left(\left[\mathcal{U}_{CEF}(\mathrm{u}^-_\alpha(v(\mathrm{t}))),\ \mathcal{U}_{CEF}(\mathrm{u}^+_\alpha(v(\mathrm{t})))\right], \alpha\right)\Big|\alpha \in (0,1]\Big\},$$

where the Choquet expected utilities of u_α^- *and* u_α^+ *for* $v(\mathrm{t})$ *are respectively given by*

$$\begin{aligned}\mathcal{U}_{CEF}(\mathrm{u}_\alpha^-(v(\mathrm{t}))) &= \sum_{i=1}^{4} \mathrm{u}_\alpha^-(v_{(i)})[\mu(E^{(i)}) - \mu(E^{(i+1)})] \\ &= 0.4\mathrm{u}_\alpha^-(v_{(1)}) + 0.3\mathrm{u}_\alpha^-(v_{(2)}) + 0.2\mathrm{u}_\alpha^-(v_{(3)}) + 0.1\mathrm{u}_\alpha^-(v_{(4)}), \\ \mathcal{U}_{CEF}(\mathrm{u}_\alpha^+(v(\mathrm{t}))) &= \sum_{i=1}^{4} \mathrm{u}_\alpha^+(v_{(i)})[\mu(E^{(i)}) - \mu(E^{(i+1)})] \\ &= 0.4\mathrm{u}_\alpha^+(v_{(1)}) + 0.3\mathrm{u}_\alpha^+(v_{(2)}) + 0.2\mathrm{u}_\alpha^+(v_{(3)}) + 0.1\mathrm{u}_\alpha^+(v_{(4)}).\end{aligned}$$

Categories	$\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v))$ of KR–US	$\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v))$ of KR–NZ
c_1	$[0.0566\alpha,\ 0.1699 - 0.1133\alpha]$	$[0.0066\alpha,\ 0.0198 - 0.0132\alpha]$
c_2	$[0.0670\alpha,\ 0.2010 - 0.1340\alpha]$	$[0.0,\ 0.0 - 0.0]$
c_3	$[0.9389\alpha,\ 2.8168 - 0.8778\alpha]$	$[0.8590\alpha,\ 2.5770 - 1.7180\alpha]$
c_4	$[0.2064\alpha,\ 0.6191 - 0.4127\alpha]$	$[0.0394\alpha,\ 0.1181 - 0.0787\alpha]$
c_5	$[0.0530\alpha,\ 0.1591 - 0.1061\alpha]$	$[0.0156\alpha,\ 0.0468 - 0.0312\alpha]$
Categories	$\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v))$ of KR–IN	$\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v))$ of KR–TR
c_1	$[0.0030\alpha,\ 0.0089 - 0.0059\alpha]$	$[0.0029\alpha,\ 0.0087 - 0.0058\alpha]$
c_2	$[0.0325\alpha,\ 0.0974 - 0.0650\alpha]$	$[0.0,\ 0.0]$
c_3	$[0.0063\alpha,\ 0.0190 - 0.0127\alpha]$	$[0.0609\alpha,\ 0.1826 - 0.1217\alpha]$
c_4	$[0.0079\alpha,\ 0.0238 - 0.0158\alpha]$	$[0.0,\ 0.0]$
c_5	$[0.0,\ 0.0]$	$[0.0,\ 0.0]$

Table 1: $\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v))$ of KR–US, KR–NZ, KR–IN and KR–TR

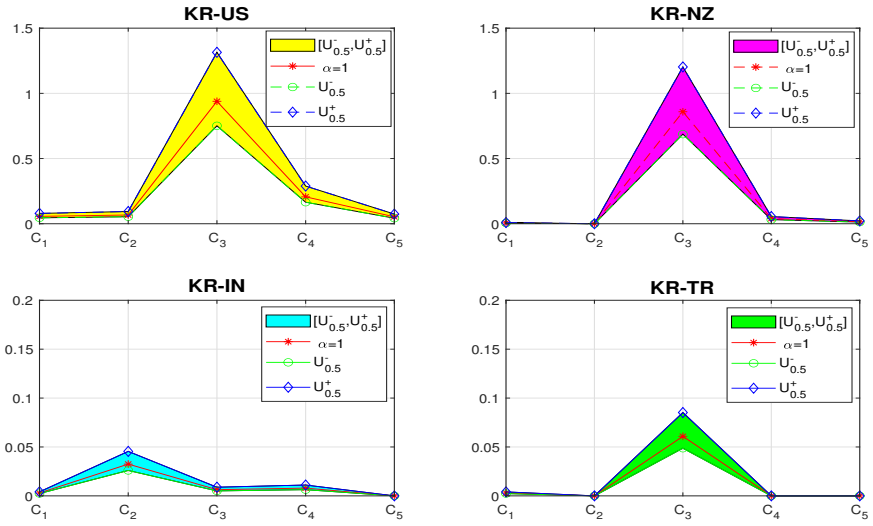

Figure 4: $[\mathcal{U}_\alpha^-, \mathcal{U}_\alpha^+] := [\mathcal{U}_{CEF}(\mathrm{u}_\alpha^-(v(\mathrm{t}))),\ \mathcal{U}_{CEF}(\mathrm{u}_\alpha^+(v(\mathrm{t})))]$ at $\alpha = \frac{1}{2}$ of each category

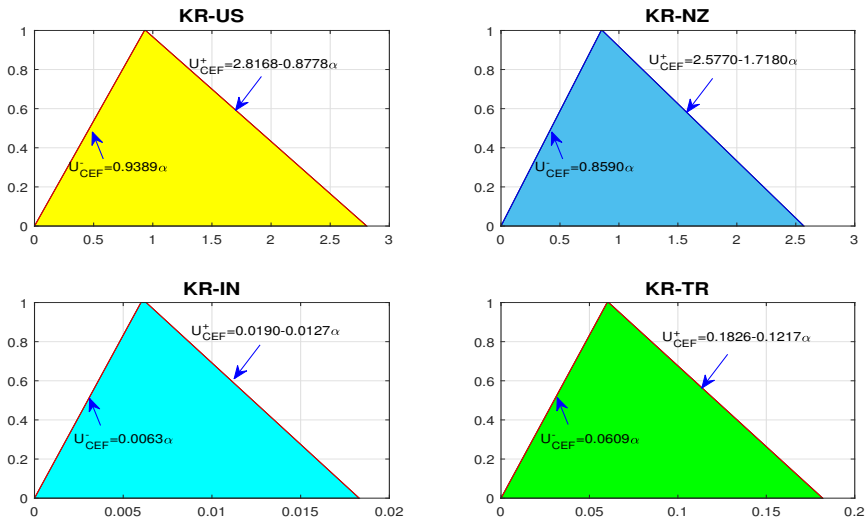

Figure 5: CEFU $\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v))$ for category c_3

For further examinations of the CEFU, we display the values of $\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v(\mathrm{t})))$ *at* α*-levels with* $\alpha = 0.5$ *and* $\alpha = 1$ *for each trading partner country in Figure 4. As is known, the Choquet expected fuzzifying utility at* α*-level with* $\alpha = 0.5$ *is represented by the following interval values*

$$\mathcal{U}_{CEF}\left(\widetilde{\mathrm{u}}_{0.5}(v(\mathrm{t}))\right) = \left[\mathcal{U}_{CEF}(\mathrm{u}^{-}_{0.5}(v(\mathrm{t}))),\ \mathcal{U}_{CEF}(\mathrm{u}^{+}_{0.5}(v(\mathrm{t})))\right],$$

which are expressed by colored areas with respect to the categories in the figures. As seen in Fig. 4, the widest range of $\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}_{0.5}(v(\mathrm{t}))$ *for KR–US, KR–NZ, and KR–TR occurs in category 3, while that for KR–IN appears in category 2. Also, we display the* $\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v(\mathrm{t})))$ *of each country, for the fixed category* c_3 *in Fig. 5. Note that from* (3), $\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v))$ *are represented by interval-valued normal fuzzy sets of the Choquet expected utility of* $\mathrm{u}(v)$, (see [23]).

The Choquet expected utilities for each category are computed, and the values are shown in Fig. 6 (top), and the percentages of the utilities from KR–NZ, KR–IN, KR–TR trading values are investigated, in comparison to the utilities of KR–US trading values, which are shown in Fig. 6 (bottom). Note that these values correspond to those of the Choquet expected fuzzifying utilities at α*-level with* $\alpha = 1$*. For example, the Choquet expected utilities of KR–NZ, KR–IN, and KR–TR are shown as 91%, 1%, and 6%, respectively, compared to that of KR–US for category 3.*

We next compute the interval-valued percentages of $\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}_{0.8}(v(\mathrm{t})))$ *at* α*-level with* $\alpha = 0.8$ *with respect to those of KR–US trading values, which are formulated by*

$$[\mathcal{U}^{-}_{0.8}, \mathcal{U}^{+}_{0.8}] := \left[\mathcal{U}_{CEF}(\mathrm{u}^{-}_{0.8}(v(\mathrm{t}))),\ \mathcal{U}_{CEF}(\mathrm{u}^{+}_{0.8}(v(\mathrm{t})))\right] \Big/ \left[\mathcal{U}^{-}_{KR-US},\ \mathcal{U}^{+}_{KR-US}\right] \times 100(\%),$$

where $\left[\mathcal{U}^{-}_{KR-US},\ \mathcal{U}^{+}_{KR-US}\right]$ *is the CEFU of KR–US trading values at* α*-level with* $\alpha = 0.8$*. Here, the division of interval is computed by*

$$[a^{-}, a^{+}]/[b^{-}, b^{+}] = \left[\frac{a^{-}}{b^{-}} \wedge \frac{a^{-}}{b^{+}} \wedge \frac{a^{+}}{b^{-}} \wedge \frac{a^{+}}{b^{+}}, \frac{a^{-}}{b^{-}} \vee \frac{a^{-}}{b^{+}} \vee \frac{a^{+}}{b^{-}} \vee \frac{a^{+}}{b^{+}}\right],\ b^{-} > 0.$$

The computed results are plotted in Fig. 6 (right) with each bar charts containing values of $\mathcal{U}^{-}_{0.8}$,$\mathcal{U}^{+}_{0.8}$*, and Ave* $:= \frac{1}{2}(\mathcal{U}^{-}_{0.8} + \mathcal{U}^{+}_{0.8})$ *for each category. For example, the Choquet expected fuzzifying utilities of KR–NZ and KR–TR trading values for category 3 provide interval-valued percentages* $[52\%, 160\%]$ *and* $[4\%, 11\%]$ *in comparison to those of the KR–US trading values. As seen in Fig. 6 (right), these interpretations of* $\mathcal{U}_{CEF}(\widetilde{\mathrm{u}}(v))$ *can provide more flexibility for a decision maker depending on the type of trade policies that are being implemented between South Korea and other trading partner countries by imposing different* α*-levels. Thus, the fuzzifying utility can be developed by preferred policies to improve the effect on the measurement of policy preferences.*

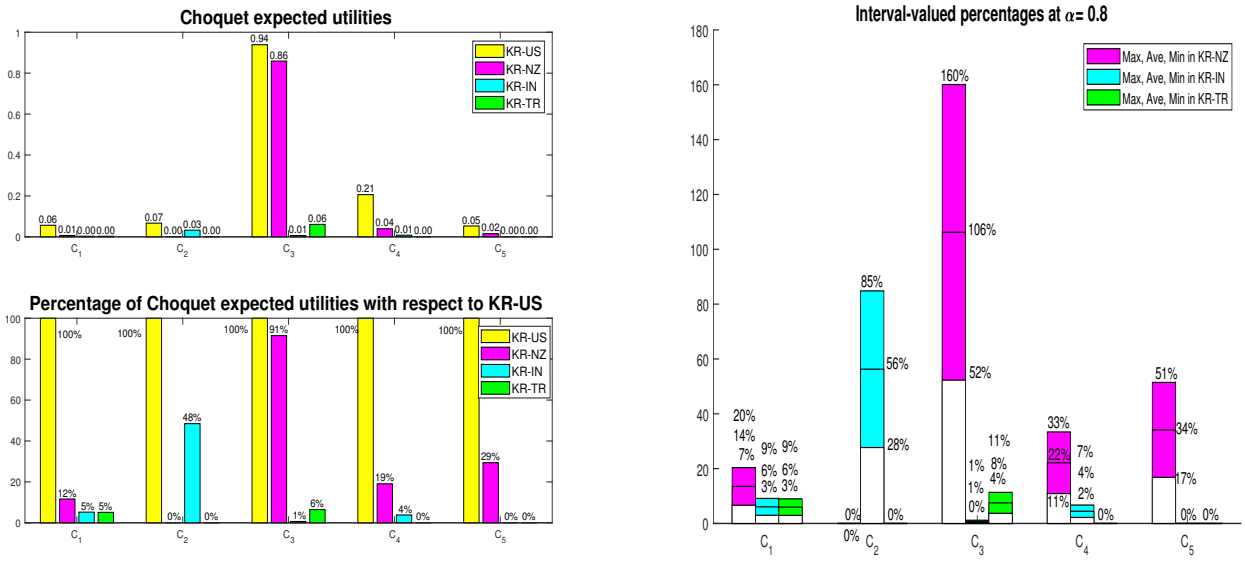

Figure 6: Choquet expected utilities (left-top) and percentage with respect to the values of KR–US (left-bottom) and Interval-valued percentages $[\mathcal{U}^{-}_{0.8}, \mathcal{U}^{+}_{0.8}]$ of $\mathcal{U}_{CEF}$ with respect to the values of KR–US at α-level with $\alpha = 0.8$ (right)

4 Conclusion

In this paper, we introduced the Choquet integral of fuzzifying functions with respect to a fuzzy measure by applying interval-valued Choquet integrals in Definition 2.1. The Choquet integrals of fuzzifying functions are considered as the set of the Choquet integral at α-level intervals for all $\alpha's$ between 0 and 1. We presented some basic properties of the proposed Choquet integral of fuzzifying functions in Theorems 2.3, 2.5 and 2.7. As examples, by applying a fuzzifying function to a PDF and a utility function we have presented the CFP and CEFUs in Examples 3.3 and 3.10. In addition, we have shown their properties in Theorems 3.8 and 3.9 and a practical application to obtain various α-levels for the CEFU in Fig. 6. These integrals can provide insights into the diversities of the uncertainty surrounding utility functions as well as the uncertainty of measures that are being used to drive a country's economic and trade policy agenda given today's increasingly uncertain global landscape. Moreover, our findings can also be used to assist decision-making processes by providing additional flexibility through the construction of more meaningful fuzzification of integrands.

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