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Fuzzy sets on uniform spaces

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Abstract

Given a uniform space $(X,\mathcal{U})$, we introduce, from the uniformity $\mathcal{U}$, some uniformities on the set $\mathcal{F}(X)$ of all normal, upper semicontinuous with compact support fuzzy sets on X: the Skorokhod uniformity, the level-wise uniformity, the endograph uniformity and the sendograph uniformity. For metric spaces we prove that these uniformities coincide with the uniformities induced by the Skorokhod metric (the level-wise metric, the endograph metric and the sendograph metric, respectively). We study completeness of this class of uniform spaces.

Keywords: Fuzzy set, uniform (metric) space, hyperspace, Vietoris topology, Skorokhod metric, level-wise metric, endograph metric, sendograph metric, completeness.

1 Introduction

For a topological space (X,τ), we consider the set $\mathcal{F}(X)$ of all normal, upper semicontinuous with compact support fuzzy sets on (X,τ). The case where (X,τ) is the topological space (X,τ_d) induced by a metric space (X,d), the set $\mathcal{F}(X)$ has received a great deal of attention in the literature when it is endowed with several metrics defined from the metric d (specially, the level-wise metric, the Skorokhod metric, the endograph metric and the sendograph metric). Among others, the interesting reader can consult [3, 9, 10, 15, 22, 24, 25]. These classes of metric spaces also have a wide range of applications to dynamical systems [12, 18], differential equations [16, 21], function spaces [5, 14], fixed point theory [2, 7], etc.

This kind of constructions have been rarely considered outside the metric context. The aim of this paper is to generalize the notions of the above metrics to the setting of uniform spaces. To be precise, from a uniform space $(X,\mathcal{U})$, we introduce several uniformities on $\mathcal{F}(X)$, which are related to the level-wise metric, Skorokhod metric, endograph metric and sendograph metric, respectively. Although uniform spaces allow us to work with ideas and procedures closer to metric spaces than to topological spaces, it is worth noting that our results can be framed in a broad class of topological spaces, the so-called Tychonoff spaces: indeed, the topology of a space (X,τ) can be induced by a uniformity on the set X if and only if (X,τ) is a Tychonoff space.

This paper is organized as follows. In Section 2 we present some basic notions and facts on uniform spaces and hyperspaces, and a brief discussion on fuzzy sets. Two technical useful lemmas on uniform spaces, Lemma 2.10 and Lemma 2.11, are proved . Given a uniform space $(X,\mathcal{U})$, Sections 3 and 4 are devoted to introduce some uniformities on $\mathcal{F}(X)$ from the uniformity $\mathcal{U}$ and to study their relationship to the uniformities induced by the Skorokhod metric (the level-wise metric, the endogrpah metric and the sendograph metric, respectively). In Section 5 we analyze completeness of the uniform spaces we are previously introduced.

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2 Preliminaries

One of the central concepts of this paper is the notion of *a uniform space*. From our point of view, the most efficient approach to uniform spaces is by way of *entourages of the diagonal*. We begin by introducing some notation. Let X be a set and let A and B be subsets by of $X \times X$, i.e., relations on the set X. The inverse relation of A will be denoted by A^{-1} , and the composition of A and B will be denoted by $A \circ B$. Thus, we have

$$A^{-1} = \{(x, y) \in X \times X : (y, x) \in A\},$$

and

$$A \circ B = \{(x, y) \in X \times X \,:\, \text{there exists } z \in X \text{ such that } (x, z) \in A \text{ and } (z, y) \in B\}.$$

The symbol A^2 stands for $A \circ A$ and Δ_X for the diagonal of X, that is, the subset $\{(x, x) : x \in X\}$ of $X \times X$. Every set $A \subseteq X \times X$ that contains Δ_X and satisfies the condition $A = A^{-1}$ is called an *entourage of the diagonal*. We will denote by $\mathcal{D}_X$ the family of all entourages of the diagonal of X.

Definition 2.1. *A* uniformity *on a non-empty set X is a subfamily $\mathcal{U}$ of $\mathcal{D}_X$ which satisfies the following conditions:*

(U1) *If $A \in \mathcal{U}$ and $A \subseteq B \in \mathcal{D}_X$, then $B \in \mathcal{U}$.*

(U2) *If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$.*

(U3) *For every $A \in \mathcal{U}$, there exists $B \in \mathcal{U}$ such that $B^2 \subseteq A$.*

(U4) $\bigcap_{A \in \mathcal{U}} A = \Delta_X$.

A *uniform space* is a pair $(X, \mathcal{U})$ consisting of a set X and a uniformity $\mathcal{U}$ on the set X. Let $(X, \mathcal{U})$ be a uniform space. A family $\mathcal{B} \subseteq \mathcal{U}$ is called a base for the uniformity $\mathcal{U}$ if for every $A \in \mathcal{U}$, there exists $B \in \mathcal{B}$ such that $B \subseteq A$. The following result is well known and easy to prove.

Proposition 2.2. *Let X be a non-empty set. A non-empty family $\mathcal{B}$ of subsets of $X \times X$ is a base for some uniformity on X if and only if it satisfies the following properties:*

(BS1) *For any $A, B \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $C \subset A \cap B$.*

(BS2) *For every $A \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $B^{-1} \subseteq A$.*

(BS3) *For every $A \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $B^2 \subseteq A$.*

(BS3) $\bigcap_{A \in \mathcal{B}} A = \Delta_X$.

As usual, a set X equipped with a topology τ is called a *topological space* and it will be denoted by (X, τ). It is a well-known fact that every uniformity $\mathcal{U}$ on a set X induces a topology $\tau_{\mathcal{U}}$ on X. To be precise, the topology $\tau_{\mathcal{U}}$ is the family $\{V \subseteq X \,:\, \text{for every } x \in V, \text{ there exists } U \in \mathcal{U} \text{ such that } U(x) \subseteq V\}$, where $U(x) = \{y \in X \,:\, (x, y) \in U\}$. In this case, the topological space $(X, \tau_{\mathcal{U}})$ is a Tychonoff space (for the details we refer to the reader to Chapter 8 of the classic text [4]).

Example 2.3. [4] *A topological group G is a group with a topology such that the function $\mu : G \times G \to G$ defined by $\mu(x, y) = xy^{-1}$ is continuous. If G is a Hausdorff topological group with identity e, denote by $\mathcal{N}(e)$ to the family of open neighborhoods of e. For each $U \in \mathcal{N}(e)$, we put $L_U = \{(x, y) \in G \times G : y \in xU\}$. Then the family $\{L_U : U \in \mathcal{N}(e)\}$ is a base for a uniformity $\mathcal{L}$ on G such a uniformity is called the left uniformity. Moreover, the topology induced by $\mathcal{L}$ on G coincides with the original topology on G and $L_U(x) = xU$ for each $U \in \mathcal{N}(e)$ and all $x \in G$.*

We turn to a brief discussion of the hyperspaces that we will consider in this paper. Given a topological space (X, τ), the symbols $\mathcal{C}(X)$ and $\mathcal{K}(X)$ denote, respectively, the hyperspaces defined by

$$\mathcal{C}(X) = \{E \subseteq X \,:\, E \text{ is closed and non-empty}\},$$

$$\mathcal{K}(X) = \{E \in \mathcal{C}(X) \,:\, E \text{ is compact}\}.$$

Thus, in the case of a uniform space $(X, \mathcal{U})$, $\mathcal{C}(X)$ (respectively, $\mathcal{K}(X)$) denotes the hyperspace of all non-empty closed (respectively, non-empty compact) subsets of $(X, \tau_{\mathcal{U}})$. We will see that $\mathcal{C}(X)$ and $\mathcal{K}(X)$ can be endowed with a natural uniformity in this situation.

Let $(X,\mathcal{U})$ be a uniform space. For each $U \in \mathcal{U}$ and each $A \subset X$, let us define $U(A) = \bigcup_{x\in A} U(x)$. Now, for each $U \in \mathcal{U}$ consider the families

$$\mathcal{C}[U] = \{(A, B) \in \mathcal{C}(X) \times \mathcal{C}(X) : A \subseteq U(B),\ B \subseteq U(A)\},$$
$$\mathcal{K}[U] = \{(A, B) \in \mathcal{K}(X) \times \mathcal{K}(X) \,:\, A \subseteq U(B),\ B \subseteq U(A)\}.$$

Among the most interesting results in the theory of hyperspaces are the following well-known results.

Proposition 2.4. [19] *If $(X,\mathcal{U})$ is a uniform space, then $\{\mathcal{K}[U] : U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{K}(\mathcal{U})$ on $\mathcal{K}(X)$.*

A remarkable result by Michael [19] allows us to describe the topology induced by the uniformity $\mathcal{K}(\mathcal{U})$. Let us recall that, for any topological space (X,τ), the topology τ induces a topology τ_V on $\mathcal{C}(X)$, the so-called *Vietoris topology*, a base for τ_V is the family of all sets of the form

$$\mathcal{V}\langle V_1, V_2, \dots, V_k\rangle = \left\{ B \in \mathcal{C}(X) \,:\, B \subset \bigcup_{i=1}^{k} V_i \text{ and } B \cap V_i \neq \emptyset \text{ for } i = 1, 2, \dots, k \right\},$$

where $V_1, V_2, \dots, V_n$ is a finite sequence of non-empty open sets of X.

Theorem 2.5. [19] *If $(X,\mathcal{U})$ is a uniform space, then the topology induced by $\mathcal{K}(\mathcal{U})$ on $\mathcal{K}(X)$ coincides with the Vietoris topology induced by $\tau_{\mathcal{U}}$ on $\mathcal{K}(X)$.*

Allowing for the previous result, if no confusion can arise, $\mathcal{K}(X)$ will be denote the hyperspace of all non-empty compact subsets of $(X,\tau_{\mathcal{U}})$ equipped with the Vietoris topology induced by $\tau_{\mathcal{U}}$. For the hyperspace $\mathcal{C}(X)$ we have the following.

Proposition 2.6. [19] *If $(X,\mathcal{U})$ is a uniform space, then $\{\mathcal{C}[U] : U \in \mathcal{U}\}$ is a base for a uniformity $\mathcal{C}(\mathcal{U})$ on $\mathcal{C}(X)$.*

It is worth mentioning that a metric space (X,d) has a compatible uniformity defined in a natural way. Indeed, it is routine to verify that the family $\mathcal{U}_d$ defined as the family of all sets

$$V_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\},$$

for each $\varepsilon > 0$, is a base for a compatible uniformity on (X,d).

Given an algebraic group G and $A, B \subseteq G$, we put $AB = \{ab : a \in A,\ b \in B\}$.

Example 2.7. *Let G be a Hausdorff topological group. For each $U \in \mathcal{N}(e)$, we consider L_U as in Example 2.3. Then we have the following:*

$$\mathcal{K}[L_U] = \{(A, B) \in \mathcal{K}(G) \times \mathcal{K}(G) \,:\, A \subseteq BU,\ B \subseteq AU\},$$
$$\mathcal{C}[L_U] = \{(A, B) \in \mathcal{C}(G) \times \mathcal{C}(G) : A \subseteq BU,\ B \subseteq AU\}.$$

We now turn to a brief discussion of fuzzy sets. A fuzzy set u on a topological space (X,τ) is a function $u\colon X \to \mathbb{I}$, where $\mathbb{I}$ stands for the closed unit interval $[0,1]$. The *support* of u, denoted by u_0, is the set $\overline{\{x \in X : u(x) > 0\}}$.

Define the so-called *α-level* set of a fuzzy set as $u_\alpha = \{x \in X : u(x) \geq \alpha\}$ for each $\alpha \in (0,1]$. A fuzzy set u is said to be *normal* if u_1 is non-empty. By $\mathcal{F}(X)$ we denote the family of all upper semicontinuous normal fuzzy sets on (X,τ) with compact support. Let us note that if $u \in \mathcal{F}(X)$, then u_α is compact for every $\alpha \in \mathbb{I}$ and $u_0 = \overline{\bigcup\{u_\alpha : \alpha \in (0,1]\}}$.

Given a metric space (X,d), it is possible to define a metric d_∞ on $\mathcal{F}(X)$ from the Hausdorff metric d_H on $\mathcal{K}(X)$ as

$$d_\infty(u, v) = \sup\{d_H(u_\alpha, v_\alpha) : \alpha \in \mathbb{I}\}.$$

The metric d_∞ is called the *level-wise metric*. The symbol τ_∞ stands for the topology induced by d_∞.

Another interesting metric on $\mathcal{F}(X)$ is the *Skorokhod metric*. Denote by $\mathbb{T}$ the family of all strictly increasing homeomorphisms from $\mathbb{I}$ onto itself. If $t \in \mathbb{T}$, we define the following:

$$\|t\| = \sup\{|t(\alpha) - \alpha| : \alpha \in \mathbb{I}\}.$$

Given a metric space (X,d), $u \in \mathcal{F}(X)$ and $t \in \mathbb{T}$, we have that $t \circ u \in \mathcal{F}(X)$. For short, we denote $t \circ u$ by tu. According to [13], the *Skorokhod metric* is defined as

$$d_0(u, v) = \inf\{\epsilon : \exists t \in \mathbb{T} \text{ such that } \|t\| \leq \epsilon \text{ and } d_\infty(u, tv) \leq \epsilon\},$$

for all $u, v \in \mathcal{F}(X)$. It is clear that $d_0(u, v) \leq d_\infty(u, v)$ for each $u, v \in \mathcal{F}(X)$. Thus the topology τ_0 induced by d_0 is weaker than the topology τ_∞ induced by d_∞, i.e., $\tau_0 \subseteq \tau_\infty$. For $u \in \mathcal{F}(X)$ and $\epsilon > 0$, the symbol $B_0(u,\epsilon)$ denotes the open ball, in $\mathcal{F}_0(X) = (\mathcal{F}(X), d_0)$, with center at u and radius ϵ.

An important tool when we are working with fuzzy sets is the next representation theorem.

Theorem 2.8. [13, Proposition 4.9] *Let (X,τ) be a Hausdorff space and $u \in \mathcal{F}(X)$. If $L\colon \mathbb{I} \to (\mathcal{K}(X),\tau_V)$ is the function defined by $L(\alpha) = u_\alpha$ for all $\alpha \in \mathbb{I}$, then the following hold:*

i) *L is left continuous on $(0,1]$;*

ii) $\lim_{\lambda\to\alpha^+} L(\lambda) = \overline{\bigcup_{\beta>\alpha} u_\beta}$ *and* $\lim_{\lambda\to\alpha^+} L(\lambda) \subseteq u_\alpha$ *for each $\alpha \in [0,1)$;*

iii) *L is right continuous at 0.*

Conversely, for any decreasing family $\{u_\alpha : \alpha \in \mathbb{I}\} \subseteq \mathcal{K}(X)$ satisfying i)–iii)*, there exists a unique $w \in \mathcal{F}(X)$ such that $w_\alpha = u_\alpha$ for every $\alpha \in \mathbb{I}$.*

Put $u_{\alpha^+} = \lim_{\lambda\to\alpha^+} L(\lambda)$. Then from ii) of previous proposition it follows that L is right continuous at α if and only if $u_{\alpha^+} = u_\alpha$. The following fact is easy to show.

Proposition 2.9. *Let $(X,\mathcal{U})$ be a uniform space. If $W \in \mathcal{U}$, $A,B,C,F,G,H \in \mathcal{K}(X)$ and $A \subseteq B \subseteq C$, then we have the following:*

i) *If $(A,C) \in \mathcal{K}[W]$, then $(A,B) \in \mathcal{K}[W]$.*

ii) *If $(A,C) \in \mathcal{K}[W]$, then $(B,C) \in \mathcal{K}[W]$.*

iii) *If $(A,F) \in \mathcal{K}[W]$ and $(G,H) \in \mathcal{K}[W]$, then $(A\cup G, F\cup H) \in \mathcal{K}[W]$.*

Given a uniform space $(X,\mathcal{U})$, $\mathcal{F}(X)$ denotes the family of all upper semicontinuous normal fuzzy sets on $(X,\tau_{\mathcal{U}})$ with compact support. The next two facts will be helpful in the sequel.

Proposition 2.10. *Let $(X,\mathcal{U})$ be a uniform space. If $u \in \mathcal{F}(X)$, then, for each $W \in \mathcal{U}$, there exists a sequence $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k = 1$ such that $(u_{\alpha_{i-1}^+}, u_{\alpha_i}) \in \mathcal{K}[W]$ for each $i = 1,2,\ldots,k$.*

Proof. We say that a number $\rho \in (0,1]$ has the property $P(u,W)$ if there exists a sequence $0 = \beta_0 < \beta_1 < \cdots < \beta_m = \rho$ such that $(u_{\beta_{i-1}^+}, u_{\beta_i}) \in \mathcal{K}[W]$ for each $i = 1,2,\ldots,m$. Define the set $Y = \{\rho \in (0,1] : \rho \text{ has the property } P(u,W)\}$. Remark that $u_{0^+} = u_0$ and observe that, by virtue of Theorem 2.5 and the right continuity of $L : \mathbb{I} \to (\mathcal{K}(X),\tau_V)$ at $\alpha = 0$, we can choose $\epsilon > 0$ such that $u_\gamma = L(\gamma) \in \mathcal{K}[W](u_0)$ for each $\gamma \in [0,\epsilon]$. This fact easily implies that $\epsilon \in Y$, so Y is non-empty. Let λ be the supremum of all the numbers in Y. From Theorem 2.5 and the left continuity of L at λ, it is easy to see that λ also has property $P(u,W)$, thus $\lambda \in Y$. We will prove that $\lambda = 1$. If $\lambda < 1$, then from Theorem 2.5 and ii) of Theorem 2.8 it follows that there exists $\delta > 0$ such that $u_\gamma = L(\gamma) \in \mathcal{K}[W](u_{\lambda^+})$ for every $\gamma \in (\lambda, \lambda+\delta] \subseteq (\lambda, 1]$. This implies that $\lambda+\delta$ has property $P(u,W)$ and $\lambda+\delta \in Y$. This contradiction allows us to conclude that $\lambda = 1$, which finishes the proof. ☐

Proposition 2.11. *Let $(X,\mathcal{U})$ be a uniform space. If $u \in \mathcal{F}(X)$, then for each $W \in \mathcal{U}$ and $\delta > 0$, there exists a sequence $0 = \beta_0 < \beta_1 < \cdots < \beta_n = 1$ such that $\beta_i - \beta_{i-1} < \delta$ and $(u_{\beta_{i-1}^+}, u_{\beta_i}) \in \mathcal{K}[W]$ for each $i = 1,2,\ldots,n$.*

Proof. From Proposition 2.10, there is a sequence $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k = 1$ such that $(u_{\alpha_{i-1}^+}, u_{\alpha_i}) \in \mathcal{K}[W]$ for each $i = 1,2,\ldots,k$. Take $m \in \mathbb{N}$ such that $1/m < \delta$ and observe that the set $\{\alpha_0,\alpha_1,\ldots,\alpha_k\} \cup \{\frac{i}{m} : i = 1,2,\ldots,m-1\}$ is a refinement of $\{\alpha_0,\alpha_1,\ldots,\alpha_k\}$. So by reordering, we obtain a sequence $0 = \beta_0 < \beta_1 < \cdots < \beta_n = 1$ which has the property $\beta_i - \beta_{i-1} \leq \frac{1}{m} < \delta$ for each $i = 1,2,\ldots,n$. If $\alpha_{i-1} \leq \beta_{j-1} < \beta_j \leq \alpha_i$ for some $i = 1,2,\ldots,k$, then $u_{\alpha_i} \subseteq u_{\beta_j} \subseteq u_{\beta_{j-1}^+} \subseteq u_{\alpha_{i-1}^+}$. Since $(u_{\alpha_{i-1}^+}, u_{\alpha_i}) \in \mathcal{K}[W]$, we conclude that $(u_{\beta_{j-1}^+}, u_{\beta_j}) \in \mathcal{K}[W]$. Therefore, $(u_{\beta_{j-1}^+}, u_{\beta_j}) \in \mathcal{K}[W]$ for each $j = 1,2,\ldots,n$. ☐

3 The Skorokhod and level-wise uniformities

The goal of this section is to introduce two uniformities closely related with the Skorokhod metric d_0 and level-wise metric d_∞. In this context, this allows us to work with Tychonoff spaces instead of metric spaces. We begin by some notation and two propositions.

Let $(X,\mathcal{U})$ be a uniform space. If $U \in \mathcal{U}$ and $\epsilon > 0$, we consider the set

$$G[U,\epsilon] = \{(u,v) \in \mathcal{F}(X)\times\mathcal{F}(X) : \exists\, t \in \mathbb{T} \text{ with } \|t\| < \epsilon \text{ and } (u_\alpha, v_{t(\alpha)}) \in \mathcal{K}[U],\ \forall \alpha \in \mathbb{I}\}.$$

Proposition 3.1. *If $(X,\mathcal{U})$ is a uniform space, then $\{G[U,\epsilon] : U \in \mathcal{U}$ and $\epsilon > 0\}$ is a base for a uniformity on $\mathcal{F}(X)$.*

Proof. We have to prove (BS1)–(BS4) of Proposition 2.2.

(BS1) Let us consider two positive real numbers ϵ_1 and ϵ_2, then define $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. For a given pair $W_1, W_2 \in \mathcal{U}$, put $W = W_1 \cap W_2 \in \mathcal{U}$. It is easy to see that $G[W,\epsilon] \subseteq G[W_i, \epsilon_i]$ for $i = 1,2$. Thus $G[W,\epsilon] \subseteq G[W_1,\epsilon_1] \cap G[W_2,\epsilon_2]$.

(BS2) Given $W \in \mathcal{U}$ and $\epsilon > 0$, let (v,u) be an element of $(G[W,\epsilon])^{-1}$. Note that $(u,v) \in G[W,\epsilon]$. So there exists $t \in \mathbb{T}$ such that $\|t\| < \epsilon$, $u_\alpha \in \mathcal{K}[W](v_{t(\alpha)})$ and $v_{t(\alpha)} \in \mathcal{K}[W](u_\alpha)$ for each $\alpha \in \mathbb{I}$. It is easy to see that $\|t^{-1}\| = \|t\|$. Now, if $\alpha \in \mathbb{I}$, we define $\beta = t(\alpha)$. Observe that $u_{t^{-1}(\beta)} \in \mathcal{K}[W](v_\beta)$ and $v_\beta \in \mathcal{K}[W](u_{t^{-1}(\beta)})$ for each $\beta \in \mathbb{I}$. Thus $(v,u) \in G[W,\epsilon]$. This proves that $(G[W,\epsilon])^{-1} = G[W,\epsilon]$.

(BS3) Take $U \in \mathcal{U}$ and $\epsilon > 0$. We can choose $W \in \mathcal{U}$ such that $W^2 \subseteq U$. Take $(u,v) \in (G[W,\epsilon/2])^2$. Select $z \in \mathcal{F}(X)$ for which $(u,z) \in G[W,\epsilon/2]$ and $(z,v) \in G[W,\epsilon/2]$. Then there exist $t_1, t_2 \in \mathbb{T}$ such that $|t_i(\alpha) - \alpha| < \epsilon/2$ for $i = 1,2$ and each $\alpha \in \mathbb{I}$. Moreover, for each $\alpha \in \mathbb{I}$, we have that

i) $u_\alpha \in \mathcal{K}[W](z_{t_1(\alpha)})$.

ii) $z_{t_1(\alpha)} \in \mathcal{K}[W](u_\alpha)$.

iii) $z_\alpha \in \mathcal{K}[W](v_{t_2(\alpha)})$.

iv) $v_{t_2(\alpha)} \in \mathcal{K}[W](z_\alpha)$.

Observe that $\|t_2 \circ t_1\| < \epsilon$, because, for each $\alpha \in \mathbb{I}$, we have:

$$|t_2(t_1(\alpha)) - \alpha| \leq |t_2(t_1(\alpha)) - t_1(\alpha)| + |t_1(\alpha) - \alpha| < \epsilon/2 + \epsilon/2 = \epsilon.$$

We are going to prove that $(u,v) \in G[W^2,\epsilon]$. Take $\beta \in \mathbb{I}$. By iii), we have that $z_{t_1(\beta)} \in \mathcal{K}[W](v_{t_2(t_1(\beta))})$. The latter fact and i) imply that $u_\beta \in \mathcal{K}[W^2](v_{t_2(t_1(\beta))})$. On the other hand, by ii) and iv, we conclude that $v_{t_2(t_1(\beta))} \in \mathcal{K}[W^2](u_\beta)$ and, consequently, we have that $(u,v) \in G[W^2,\epsilon]$. Therefore, we have just proved that $(G[W,\epsilon/2])^2 \subseteq G[W^2,\epsilon] \subseteq G[U,\epsilon]$.

(BS4) Take $W \in \mathcal{U}$ and $\epsilon > 0$. If t is the identity map on $\mathbb{T}$, then it is evident that $\|t\| = 0$ and $u_\alpha = u_{t(\alpha)}$ for every $\alpha \in \mathbb{I}$, hence $(u,u) \in G[W,\epsilon]$. Therefore, we have proved that $\Delta_{\mathcal{F}(X)} \subseteq G[W,\epsilon]$.

Suppose now that $(u,v) \in G[W,\epsilon]$ for every $W \in \mathcal{U}$ and $\epsilon > 0$. Fix W and, for each $n \in \mathbb{N}$, choose $t_n \in \mathbb{T}$ such that $\|t_n\| < 1/n$, $u_\alpha \in \mathcal{K}[W](v_{t_n(\alpha)})$ and $v_{t_n(\alpha)} \in \mathcal{K}[W](u_\alpha)$ for each $\alpha \in \mathbb{I}$. Since $t(0) = 0$ for each $t \in \mathbb{T}$, we have that $u_0 \in \mathcal{K}[W](v_{t_n(0)}) = \mathcal{K}[W](v_0)$ and $v_0 = v_{t_n(0)} \in \mathcal{K}[W](u_0)$ which implies that $(u_0, v_0) \in \mathcal{K}[W]$. The latter fact is also valid for $\alpha = 1$. Now fix a number $\alpha_0 \in (0,1)$ and assume that the set $\mathcal{S} = \{n \in \mathbb{N} : t_n(\alpha_0) > \alpha_0\}$ is infinite. Take $m \in \mathcal{S}$. Since $v_{t_m(\alpha_0)} \subseteq v_{\alpha_0}$, we have $u_{\alpha_0} \in \mathcal{K}[W](v_{t_m(\alpha_0)}) \subseteq \mathcal{K}[W](v_{\alpha_0})$. It is easy to see that $1/n \geq \alpha_0 - t_n^{-1}(\alpha_0) > 0$ and $v_{\alpha_0} \in \mathcal{K}[W](u_{t_n^{-1}(\alpha_0)})$ whenever $n \in \mathcal{S}$, since $\|t_n^{-1}\| < 1/n$ for each $n \in \mathbb{N}$. So there is no loss of generality in assuming that $\{t_n^{-1}(\alpha_0) : n \in \mathcal{S}\}$ is an increasing sequence which converges to α_0. Theorems 2.5 and 2.8 tell us that the map, from $\mathbb{I}$ to $\mathcal{K}(X)$ defined trough $\gamma \to u_\gamma$ is left-continuous on $(0,1]$ when we consider $\mathcal{K}(X)$ equipped with the Vietoris topology induced by $\tau_{\mathcal{U}}$. Therefore, $v_{\alpha_0} \in \mathcal{K}[W](\bigcap\{u_{t_n^{-1}(\alpha_0)} : n \in \mathcal{S}\}) = \mathcal{K}[W](u_{\alpha_0})$, which implies that $(u_{\alpha_0}, v_{\alpha_0}) \in \mathcal{K}[W]$. In the case that the set $\mathcal{S}' = \{n \in \mathbb{N} : t_n(\alpha_0) < \alpha_0\}$ is infinite, using $\{t_n(\alpha_0) : n \in \mathcal{S}'\}$ and an argument similar to the previous one, we can show that $(u_{\alpha_0}, v_{\alpha_0}) \in \mathcal{K}[W]$.

Next, to finish the proof we suppose that $\mathcal{S}$ and $\mathcal{S}'$ are finite. Then exists $N \in \mathbb{N}$ such that $t_n(\alpha_0) = \alpha_0$ for each $n \geq N$. We can conclude that $(u_{\alpha_0}, v_{\alpha_0}) \in \mathcal{K}[W]$. Thus, we have proved that $(u_\beta, v_\beta) \in \mathcal{K}[W]$ for each $W \in \mathcal{U}$ and every $\beta \in [0,1]$. By Proposition 2.4 it follows that $u_\beta = v_\beta$ for each $\beta \in \mathbb{I}$, which implies that $u = v$. Therefore, $(u,v) \in \Delta_{\mathcal{F}(X)}$ and $\Delta_{\mathcal{F}(X)} = \bigcap\{G[U,\epsilon] : U \in \mathcal{U},\ \epsilon > 0\}$. □

The uniformity on $\mathcal{F}(X)$ defined in the previous proposition will be denoted by $\mathcal{U}_0$ and it is called the *Skorokhod uniformity.*

Example 3.2. *Let G be a Hausdorff topological group. For each $U \in \mathcal{N}(e)$, we consider L_U as in Example 2.3 and $\epsilon > 0$. Then we have the following:*

$$G[L_U,\epsilon] = \{(u,v) \in \mathcal{F}(G) \times \mathcal{F}(G) : \exists\, t \in \mathbb{T} \text{ s.t. } \|t\| < \epsilon,\ u_\alpha \subseteq v_{t(\alpha)}U \text{ and } v_{t(\alpha)} \subseteq u_\alpha U\ \forall \alpha \in \mathbb{I}\}.$$

We turn now to the definition of the *level-wise uniformity.* Given a uniform space $(X,\mathcal{U})$ and $U \in \mathcal{U}$, we define the set:

$$\mathcal{F}[U] = \{(u,v) \in \mathcal{F}(X) \times \mathcal{F}(X) : (u_\alpha, v_\alpha) \in \mathcal{K}[U], \text{ for all } \alpha \in \mathbb{I}\}.$$

If every $t \in \mathbb{T}$ taken in the proof of Proposition 3.1 is replaced by the identity map, we can show the following.

Proposition 3.3. *If $(X,\mathcal{U})$ is a uniform space, then $\{\mathcal{F}[U] : U \in \mathcal{U}\}$ is a base for a uniformity on $\mathcal{F}(X)$.*

We will denote by $\mathcal{U}_\infty$ the uniformity on $\mathcal{F}(X)$ defined in the previous proposition. It is called the *leve-wise uniformity.* The next fact is straightforward.

Example 3.4. *Let G be a Hausdorff topological group. For each $U \in \mathcal{N}(e)$, we consider L_U as in Example 2.3. Then we have the following:*

$$\mathcal{F}[L_U] = \{(u,v) \in \mathcal{F}(G) \times \mathcal{F}(G) : u_\alpha \subseteq v_{t(\alpha)}U \text{ and } v_{t(\alpha)} \subseteq u_\alpha U \text{ for all } \alpha \in \mathbb{I}\}.$$

Proposition 3.5. *Let $(X,\mathcal{U})$ be a uniform space. Then the uniformity $\mathcal{U}_\infty$ is stronger than $\mathcal{U}_0$ and also the topology induced on $\mathcal{F}(X)$ by $\mathcal{U}_\infty$ is stronger than the topology induced by $\mathcal{U}_0$.*

When we are working with the previous uniformities, a natural question arises: what is the relationship between these uniformities and the uniformities induced from metric spaces? We will address this problem in the sequel. Recall that a metric space (X,d) has a natural uniformity $\mathcal{U}_d$ a base of which is the family $\{U_\epsilon : \epsilon > 0\}$, where $U_\epsilon = \{(x,y) \in X \times X : d(x,y) < \epsilon\}$. Moreover, if $A,B \in \mathcal{K}(X)$, the Hasudorff distance between A and B is defined as $d_H(A,B) = \max\{d(A,B), d(B,A)\}$, where $d(A,B) = \sup\{d(a,B) : a \in A\}$ and $d(a,B) = \inf\{d(a,b) : b \in B\}$. The next result is related to our question above.

Proposition 3.6. *If (X,d) is a metric space, then the uniformity $\mathcal{U}_{d_0}$ on $\mathcal{F}(X)$ coincides with the uniformity $(\mathcal{U}_d)_0$.*

Proof. The uniformity induced by the metric d_0 is defined through the base $\{W_\delta : \delta > 0\}$, where $W_\delta = \{(u,v) \in \mathcal{F}(X) \times \mathcal{F}(X) : d_0(u,v) < \delta\}$. Pick $\delta > 0$ and $\epsilon \in (0,\delta)$. Let us show that $G[U_\epsilon,\epsilon] \subseteq W_\delta$. Take $(u,v) \in G[U_\epsilon,\epsilon]$. Then exists $t \in \mathbb{T}$ with $\|t\| < \epsilon$ such that $u_\alpha \in \mathcal{K}[U_\epsilon](v_{t(\alpha)})$ and $v_{t(\alpha)} \in \mathcal{K}[U_\epsilon](u_\alpha)$ for each $\alpha \in \mathbb{I}$. Fix now a number $\alpha \in \mathbb{I}$. Then, for each $x \in u_\alpha$ there exists $y \in v_{t(\alpha)}$ with $d(x,y) < \epsilon$. Thus, $d(x, v_{t(\alpha)}) < \epsilon$ for each $x \in u_\alpha$. This fact and compactness of u_α show that $d(u_\alpha, v_{t(\alpha)}) < \epsilon$. The proof of $d(v_{t(\alpha)}, u_\alpha) < \epsilon$ is analogous. Then $d_H(u_\alpha, v_{t(\alpha)}) < \epsilon$ for each $\alpha \in \mathbb{I}$, which implies that $d_0(u,v) \le \epsilon < \delta$ and $\|t\| < \delta$. This proves that $G[U_\epsilon,\epsilon] \subseteq W_\delta$. Therefore, $\mathcal{U}_{d_0} \subseteq (\mathcal{U}_d)_0$.

Take $\epsilon > 0$ and $(u,v) \in W_\epsilon$. Then there exists $t \in \mathbb{T}$ such that $\|t\| < \epsilon$ and $d_H(u_\beta, v_{t(\beta)}) < \epsilon$ for each $\beta \in \mathbb{I}$. We will prove that $(u,v) \in G[U_\epsilon,\epsilon]$. Fix $\alpha \in \mathbb{I}$ and notice that for each $x \in u_\alpha$ there exists $z \in v_{t(\alpha)}$ such that $d(x,z) < \epsilon$, this fact implies that $x \in U_\epsilon(z) \subseteq U_\epsilon(v_{t(\alpha)})$. Thus $u_\alpha \in \mathcal{K}[U_\epsilon](v_{t(\alpha)})$. The proof of $v_{t(\alpha)} \in \mathcal{K}[U_\epsilon](u_\alpha)$ is analogous. This proves that $(u,v) \in G[U_\epsilon,\epsilon]$. Therefore, $W_\epsilon \subseteq G[U_\epsilon,\epsilon]$ and $(\mathcal{U}_d)_0 \subseteq \mathcal{U}_{d_0}$. We have thus proved that $\mathcal{U}_{d_0} = (\mathcal{U}_d)_0$. □

If every $t \in \mathbb{T}$ taken in the previous proof is replaced by the identity map, we can show the following.

Proposition 3.7. *If (X,d) is a metric space and $\mathcal{U}_d$ is the natural uniformity induced by d, then the uniformity induced by d_∞ on $\mathcal{F}(X)$ is the uniformity $(\mathcal{U}_d)_\infty$.*

As a consequence of the two previous results we get

Proposition 3.8. *If (X,d) is a metric space, then the topologies induced on $\mathcal{F}(X)$ by d_∞ and d_0 coincide with the topologies induced by $(\mathcal{U}_d)_\infty$ and $(\mathcal{U}_d)_0$, respectively.*

For a given uniform space $(X,\mathcal{U})$, we are going to introduce a uniformity on $\mathcal{F}(X)$ which is equivalent to $\mathcal{U}_0$. For each $t \in \mathbb{T}$, we define

$$||t||_1 = \sup\left\{\left|\log\left(\frac{t(\beta)-t(\alpha)}{\beta-\alpha}\right)\right| : 0 \le \alpha < \beta \le 1\right\}.$$

It easy to see that for each $s,t \in \mathbb{T}$, we have the following:

i) $\|t\|_1 = \|t^{-1}\|_1$,

ii) $\|s \circ t\|_1 = \|st\|_1 \le \|s\|_1 + \|t\|_1$.

Let (X,d) be a metric space. According to [10], the *enhanced-type Skorokhod metric* is defined as

$$d_1(u,v) = \inf\{\epsilon : \exists t \in \mathbb{T} \text{ such that } \|t\|_1 \le \epsilon \text{ and } d_\infty(u,tv) \le \epsilon\},$$

for all $u,v \in \mathcal{F}(X)$.

For a uniform space $(X,\mathcal{U})$, $U \in \mathcal{U}$ and $\epsilon > 0$, we define

$$G_1[U,\epsilon] = \{(u,v) \in \mathcal{F}(X) \times \mathcal{F}(X) : \exists t \in \mathbb{T} \text{ with } \|t\|_1 < \epsilon \text{ and } (u_\alpha, v_{t(\alpha)}) \in \mathcal{K}[U], \forall \alpha \in \mathbb{I}\}.$$

We can argue as in the proof of Proposition 3.1 to show the following.

Proposition 3.9. *If $(X,\mathcal{U})$ is a uniform space, then $\{G_1[U,\epsilon] : U \in \mathcal{U}$ and $\epsilon > 0\}$ is a base for a uniformity $\mathcal{U}_1$ on $\mathcal{F}(X)$.*

The uniformity $\mathcal{U}_1$ mentioned above is called the *Skorokhod-like uniformity.* We can argue as in Proposition 3.6 to show the following result.

Proposition 3.10. *If (X,d) is a metric space, then the uniformity $\mathcal{U}_{d_1}$ on $\mathcal{F}(X)$ coincides with the uniformity $(\mathcal{U}_d)_1$.*

Example 3.11. *Let G be a Hausdorff topological group. For each $U \in \mathcal{N}(e)$, we consider L_U as in Example 2.3 and $\epsilon > 0$. Then we have the following:*

$$G_1[L_U,\epsilon] = \{(u,v) \in \mathcal{F}(G)\times\mathcal{F}(G) : \exists\, t \in \mathbb{T} \text{ s.t. } \|t\|_1 < \epsilon,\ u_\alpha \subseteq v_{t(\alpha)}U \text{ and } v_{t(\alpha)} \subseteq u_\alpha U\ \forall \alpha \in \mathbb{I}\}.$$

If (X,d) is a complete metric space, then $(\mathcal{F}(X),d_1)$ is compete (see [15]). We do not know if the latter result can be extended to the Skorokhod-like uniformity.

Problem 3.12. *Let $(X,\mathcal{U})$ be a complete uniform space. Is $(\mathcal{F}(X),\mathcal{U}_1)$ complete?*

Denote by τ_1 the topology on $\mathcal{F}(X)$ induced by the uniformity $\mathcal{U}_1$.

Proposition 3.13. *Let $(X,\mathcal{U})$ be a uniform space. Then $\tau_0 = \tau_1$.*

Proof. Take $u \in \mathcal{F}(X)$, $W \in \mathcal{U}$ and $\epsilon > 0$. If $\epsilon_1 = \min\{1/8, \epsilon/4\}$, then we will prove that $G_1[W,\epsilon_1](u) \subseteq G[W,\epsilon](u)$. To see this, choose $v \in G_1[W,\epsilon_1](u)$. Then there exists $t \in \mathbb{T}$ such that $\|t\|_1 < \epsilon_1$, $u_\alpha \in \mathcal{K}[W](v_{t(\alpha)})$ and $v_{t(\alpha)} \in \mathcal{K}[W](u_\alpha)$ for any $\alpha \in \mathbb{I}$. Since $\|t\|_1 < \epsilon_1 < 1/4$, [15, Lemma 3.5] implies that $\|t\| \le 2\|t\|_1 < 2\epsilon_1 \le \epsilon/2$. Thus, $\|t\| < \epsilon$. This proves that $v \in G[W,\epsilon](u)$, which implies that $G_1[W,\epsilon_1](u) \subseteq G[W,\epsilon](u)$. Therefore, $\tau_0 \subseteq \tau_1$.

In order to prove that $\tau_1 \subseteq \tau_0$, we take $u \in \mathcal{F}(X)$, $U \in \mathcal{U}$ and $\epsilon > 0$. Choose $W \in \mathcal{U}$ such that $W^2 \subseteq U$ and $\delta \in (0,1/4)$ with $5\delta < \epsilon$. It follows from Proposition 2.11 that we can find a sequence $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n = 1$ such that $\alpha_i - \alpha_{i-1} < \delta$, $u^+_{\alpha_{i-1}} \subseteq W(u_{\alpha_i})$ and $u_{\alpha_i} \subseteq W(u^+_{\alpha_{i-1}})$ for each $i = 1,2,\ldots,n$. Define $\lambda = \min\{\frac{\alpha_i-\alpha_{i-1}}{2} : i = 1,2,\ldots,n\}$. We will prove that $G[W,\lambda^2](u) \subseteq G_1[U,\epsilon](u)$. For this, let v be an element of $G[W,\lambda^2](u)$. Then there exists $s \in \mathbb{T}$ with $\|s\| = \sup\{|s(\alpha)-\alpha| : \alpha \in \mathbb{I}\} < \lambda^2$, $u_\alpha \in \mathcal{K}[W](v_{s(\alpha)})$ and $v_{s(\alpha)} \in \mathcal{K}[W](u_\alpha)$ for any $\alpha \in \mathbb{I}$. It follows from the definition of λ that $\lambda < \alpha_i - \alpha_{i-1}$ for every $i = 1,2,\ldots,n$. Then we can find $t \in \mathbb{T}$ which is linear on $[\alpha_{i-1},\alpha_i]$ and $t(\alpha_i) = s(\alpha_i)$ for each $i = 0,1,2,\ldots,n$. Since $\|t\|_1 < 4\lambda < \epsilon$, [6, Lemma 3.7] implies that if $\alpha \in (0,1]$, then α and $s^{-1}(t(\alpha))$ belong to $(\alpha_{i-1},\alpha_i]$ for some $i = 1,2,\ldots,n$. Hence $u_\alpha \in \mathcal{K}[W](u_{s^{-1}(t(\alpha))})$ and $u_{s^{-1}(t(\alpha))} \in \mathcal{K}[W](u_\alpha)$ for every $\alpha \in \mathbb{I}$. Moreover, we have that $u_{s^{-1}(\alpha)} \in \mathcal{K}[W](v_\alpha)$ and $v_\alpha \in \mathcal{K}[W](u_{s^{-1}(\alpha)})$ for each $\alpha \in \mathbb{I}$. Since t is a homeomorphism, we can put $u_{s^{-1}(t(\alpha))} \in \mathcal{K}[W](v_{t(\alpha)})$ and $v_{t(\alpha)} \in \mathcal{K}[W](u_{s^{-1}(t(\alpha))})$. Consequently, we have the following:

$$u_\alpha \in \mathcal{K}[W](u_{s^{-1}(t(\alpha))}) \subseteq \mathcal{K}[W^2](v_{t(\alpha)}) \subseteq \mathcal{K}[U](v_{t(\alpha)}),$$

$$v_{t(\alpha)} \in \mathcal{K}[W](u_{s^{-1}(t(\alpha))}) \subseteq \mathcal{K}[W^2](u_\alpha) \subseteq \mathcal{K}[U](u_\alpha).$$

Thus $v \in G_1[U,\epsilon](u)$, which implies that $G[W,\lambda^2](u) \subseteq G_1[U,\epsilon](u)$. Therefore, $\tau_1 \subseteq \tau_0$. This finishes the proof. □

4 The sendograph and endograph uniformities

Let (X,U) be a uniform space. Denote by $\mathcal{F}^*(X)$ the family of fuzzy sets u on $(X,\mathcal{U})$ satisfying the following conditions:

i) u is upper semicontinuous.

ii) $u_\alpha \in \mathcal{K}(X)$ for every $\alpha \in (0,1]$.

iii) $u_0 = \overline{\bigcup\{u_\alpha : \alpha \in (0,1]\}}$.

Observe that $\mathcal{F}(X) \subseteq \mathcal{F}^*(X)$. The following fact will be helpful in Theorem 5.6, its proof is similar to the proof of Theorem 2.8 and the reference within.

Theorem 4.1. *Let X be a Hausdorff space and $u \in \mathcal{F}^*(X)$. If $L\colon (0,1] \to (\mathcal{K}(X),\tau_V)$ is defined by $L(\alpha) = u_\alpha$ for all $\alpha \in (0,1]$, then L is left-continuous on $(0,1]$.*

Conversely, if $\{u_\alpha : \alpha \in (0,1]\} \subseteq \mathcal{K}(X)$ is a decreasing family such that the function $G\colon (0,1] \to (\mathcal{K}(X),\tau_V)$ defined by $G(\alpha) = u_\alpha$ is left-continuous, then there exists a unique $w \in \mathcal{F}^*(X)$ such that $w_\alpha = u_\alpha$ for every $\alpha \in (0,1]$.

Let (X,τ) be a topological space. If $u \in \mathcal{F}^*(X)$, then the *endograph* of u is defined as $end(u) = \{(x,\alpha) \in X \times \mathbb{I} : u(x) \geq \alpha\}$. Notice that $end(u) \in \mathcal{C}(X \times \mathbb{I})$. Consider the uniformity $\mathcal{U}_{\mathbb{I}}$ defined on $\mathbb{I}$ by means of the base $\{V_\epsilon : \epsilon > 0\}$, where $V_\epsilon = \{(\alpha,\beta) \in \mathbb{I} \times \mathbb{I} : |\alpha - \beta| < \epsilon\}$. Let $(X,\mathcal{U})$ be a uniform space. Given $U \in \mathcal{U}$ and $\epsilon > 0$, we define the following sets:

$$E[U,\epsilon] = \{(u,v) \in \mathcal{F}^*(X) \times \mathcal{F}^*(X) : (end(u), end(v)) \in \mathcal{C}[U \times V_\epsilon]\}.$$

It follows from Proposition 2.6 that the family $\{E[U,\epsilon] : U \in \mathcal{U},\, \epsilon > 0\}$ is base for a uniformity $\mathcal{U}_E$ on $\mathcal{F}^*(X)$. The uniformity $\mathcal{U}_E$ is called the *endograph uniformity.* Note that we also can consider the uniform space $(\mathcal{F}(X),\mathcal{U}_E)$.

Example 4.2. *Let G be a Hausdorff topological group and $\epsilon > 0$. For each $U \in \mathcal{N}(e)$, we consider L_U as in Example 2.3. Then we have the following:*

$$E[L_U,\epsilon] = \{(u,v) \in \mathcal{F}^*(G) \times \mathcal{F}^*(G) : (end(u), end(v)) \in \mathcal{C}[L_U \times V_\epsilon]\}.$$

If $u \in \mathcal{F}(X)$, the *sendograph* of u is defined by $send(u) = end(u) \cap (u_0 \times \mathbb{I})$. Observe that $send(u) \in \mathcal{K}(X \times \mathbb{I})$. Let $(X,\mathcal{U})$ be a uniform space. Given $U \in \mathcal{U}$ and $\epsilon > 0$, we define the following sets:

$$S[U,\epsilon] = \{(u,v) \in \mathcal{F}(X) \times \mathcal{F}(X) : (send(u), send(v)) \in \mathcal{K}[U \times V_\epsilon]\}.$$

By Proposition 2.4, the family $\{S[U,\epsilon] : U \in \mathcal{U},\, \epsilon > 0\}$ is base for a uniformity $\mathcal{U}_S$ on $\mathcal{F}(X)$. The uniformity $\mathcal{U}_S$ is called the *sendograph uniformity.*

Example 4.3. *Let G be a Hausdorff topological group and $\epsilon > 0$. For each $U \in \mathcal{N}(e)$, we consider L_U as in Example 2.3. Then we have the following:*

$$S[L_U,\epsilon] = \{(u,v) \in \mathcal{F}(G) \times \mathcal{F}(G) : (send(u), send(v)) \in \mathcal{K}[L_U \times V_\epsilon]\}.$$

Consider now a metric space (X,d). Define the metric $\overline{d}$ on $X \times \mathbb{I}$ as follows:

$$\overline{d}((x,a),(y,b)) = \max\{d(x,y), |a-b|\}.$$

The *endograph metric* d_E on $\mathcal{F}(X)$ is the Hausdorff distance $\overline{d}_H$ (with respect to $X \times \mathbb{I}$) between $end(u)$ and $end(v)$ for each $u,v \in \mathcal{F}(X)$. The *sendograph metric* d_S on $\mathcal{F}(X)$ is the Hausdorff metric $\overline{d}_H$ (on $\mathcal{K}(X \times \mathbb{I})$) between the non-empty compact subsets $send(u)$ and $send(v)$ for every $u,v \in \mathcal{F}(X)$ (see [17]). It is a well known fact that $d_E \leq d_S \leq d_\infty$. We can argue as in previous section to show the following.

Proposition 4.4. *Let (X,d) be a metric space. Then $\mathcal{U}_{d_S} = (\mathcal{U}_d)_S$ and $\mathcal{U}_{d_E} = (\mathcal{U}_d)_E$.*

Proposition 4.5. *Let $(X,\mathcal{U})$ be a uniform space. Then we have that $\mathcal{U}_E \subseteq \mathcal{U}_S \subseteq \mathcal{U}_0$, where $\mathcal{U}_E$ is considered on $\mathcal{F}(X)$.*

Proof. Let us show that $\mathcal{U}_S \subseteq \mathcal{U}_0$. Take $U \in \mathcal{U}$ and $\epsilon > 0$. Fix $(u,v) \in G[U,\epsilon]$. Then there exists $t \in \mathbb{T}$ such that $\|t\| < \epsilon$, $u_\alpha \in \mathcal{K}[U](v_{t(\alpha)})$ and $v_{t(\alpha)} \in \mathcal{K}[U](u_\alpha)$ for each $\alpha \in \mathbb{I}$. Since $u_\alpha \times \{\alpha\} \subseteq send(u)$ and $[v]_{t(\alpha)} \times \{t(\alpha)\} \subseteq send(v)$, we have that $u_\alpha \in \mathcal{K}[U](v_{t(\alpha)})$ implies that $u_\alpha \times \{\alpha\} \subseteq [U \times V_\epsilon]([v]_{t(\alpha)} \times \{t(\alpha)\}) \subseteq [U \times V_\epsilon](send(v))$ for each $\alpha \in \mathbb{I}$. Thus, $send(u) \subseteq [U \times V_\epsilon](send(v))$.

Since $t^{-1} \in T$ and $\|t^{-1}\| = \|t\|$, for a given $\beta \in \mathbb{I}$, we have that $v_\beta \in \mathcal{K}[U](u_{t^{-1}(\beta)})$. Thus, we can argue as above to show that $send(v) \subseteq [U \times V_\epsilon](send(u))$. Hence $(u,v) \in S[U,\epsilon]$. Therefore, $G[U,\epsilon] \subseteq S[U,\epsilon]$, which implies that $\mathcal{U}_S \subseteq \mathcal{U}_0$.

In order to show that $\mathcal{U}_E \subseteq \mathcal{U}_S$, choose $W \in \mathcal{U}$ and $\epsilon > 0$. Note that if $u,v \in \mathcal{F}(X)$ satisfy $(send(u), send(v)) \in \mathcal{K}[W \times V_\epsilon]$, then $(end(u), end(v)) \in \mathcal{C}[W \times V_\epsilon]$. □

Corollary 4.6. *Let $(X,\mathcal{U})$ be a uniform space. Suppose that τ_S, τ_E and τ_0 are the topologies on $\mathcal{F}(X)$ induced by the uniformities $\mathcal{U}_S$, $\mathcal{U}_E$ and $\mathcal{U}_0$, respectively. Then $\tau_E \subseteq \tau_S \subseteq \tau_0$.*

As a consequence of Proposition 4.4 and Corollary 4.6, we have the following.

Corollary 4.7. [10] *Let (X,d) be a metric space. Suppose that τ_E, τ_S and τ_0 are the topologies on $\mathcal{F}(X)$ induced by the endograph metric d_E, the sendograph metric d_S and the Skorokhod metric d_0, respectively. Then $\tau_E \subseteq \tau_S \subseteq \tau_0$.*

5 Completeness

Let $(X,\mathcal{U})$ be a uniform space. A filter $\mathcal{F}$ on X is said to be a *Cauchy filter* on $(X,\mathcal{U})$, if for each $U \in \mathcal{U}$, there exists $F \in \mathcal{F}$ such that $F \times F \subseteq U$. The uniform space $(X,\mathcal{U})$ is called *complete* if every Cauchy filter on $(X,\mathcal{U})$ converges.

Theorem 5.1. *If $(X,\mathcal{U})$ is a complete uniform space, then $(\mathcal{F}(X),\mathcal{U}_\infty)$ is complete.*

Proof. Consider a Cauchy filter $\mathcal{F}$ on $(\mathcal{F}(X),\mathcal{U}_\infty)$. For each $\alpha \in \mathbb{I}$ and $F \in \mathcal{F}$, we define $F_\alpha = \{u_\alpha : u \in F\}$ and $\mathcal{F}_\alpha = \{F_\alpha : F \in \mathcal{F}\}$. Since $\mathcal{F}$ is a Cauchy filter on $(\mathcal{F}(X),\mathcal{U}_\infty)$, we have that $\mathcal{F}_\alpha$ is a Cauchy filter on $(\mathcal{K}(X),\mathcal{K}(\mathcal{U}))$ for each $\alpha \in \mathbb{I}$. According to [20], $(\mathcal{K}(X),\mathcal{K}(\mathcal{U}))$ is a complete uniform space. Hence $\mathcal{F}_\alpha$ converges to $u_\alpha \in \mathcal{K}(X)$ for each $\alpha \in \mathbb{I}$.

Claim 1: If $\alpha,\beta \in \mathbb{I}$ with $\alpha \le \beta$, then $u_\beta \subseteq u_\alpha$.

For each $V \in \mathcal{U}$, we can choose $W \in \mathcal{U}$ such that $W^2 \subseteq V$. Since $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ converge to u_α and u_β, respectively, we conclude that $\mathcal{K}[W](u_\alpha) \in \mathcal{F}_\alpha$ and $\mathcal{K}[W](u_\beta) \in \mathcal{F}_\beta$. Hence there exist $P,Q \in \mathcal{F}$ such that $P_\alpha = K[W](u_\alpha)$ and $Q_\beta = \mathcal{K}[W](u_\beta)$. Put $F = (P \cap Q) \in \mathcal{F}$ and take $v \in F$. Then $v_\alpha \in \mathcal{K}[W](u_\alpha)$ and $v_\beta \in \mathcal{K}[W](u_\beta)$. Therefore,

$$u_\beta \subseteq W(v_\beta) \subseteq W(v_\alpha) \subseteq W(W(u_\alpha)) \subseteq W^2(u_\alpha) \subseteq V(u_\alpha).$$

Since $u_\beta \subseteq V(u_\alpha)$ for each $V \in \mathcal{U}$, we have that $u_\beta \subseteq u_\alpha$.

Claim 2: Fix $\alpha \in (0,1]$. If an increasing sequence $(\alpha_n)_n$ in $[0,1]$ converges to α, then $\bigcap_{n\in\mathbb{N}} u_{\alpha_n} = u_\alpha$.

By Claim 1, $u_\alpha \subseteq \bigcap_{n\in\mathbb{N}} u_{\alpha_n}$. Now take $V \in \mathcal{U}$ and choose $W \in \mathcal{U}$ such that $W^5 \subseteq V$. Since $(\alpha_n)_n$ converges to α, there exists $n \in \mathbb{N}$ such that $u_{\alpha_n} \in \mathcal{K}[W](u_\alpha)$. Hence

$$u_{\alpha_n} \subseteq W(u_\alpha). \tag{5.1}$$

Since $\mathcal{F}_\alpha$ converges to u_α, we have that $\mathcal{K}[W](u_\alpha) \in \mathcal{F}_\alpha$. So $\mathcal{K}[W](u_\alpha) = P_\alpha$ for some $P \in \mathcal{F}$. Similarly, $\mathcal{K}[W](u_{\alpha_n}) = Q_{\alpha_n}$ for some $Q \in \mathcal{F}$. Put $F = (P \cap Q)$ and take $v \in F$. Hence $v_{\alpha_n} \in Q_{\alpha_n} = \mathcal{K}[W](u_{\alpha_n})$. Thus,

$$v_{\alpha_n} \subseteq W(u_{\alpha_n}) \quad \textit{and} \quad u_{\alpha_n} \subseteq W(v_{\alpha_n}). \tag{5.2}$$

Similarly, $v_\alpha \in P_\alpha = \mathcal{K}[W](u_\alpha)$. Hence,

$$v_\alpha \subseteq W(u_\alpha) \quad \textit{and} \quad u_\alpha \subseteq W(v_\alpha). \tag{5.3}$$

Therefore, by (5.2), (5.1) and (5.3), we have that

$$v_{\alpha_n} \subseteq W(u_{\alpha_n}) \subseteq W^2(u_\alpha) \subseteq W^3(v_\alpha). \tag{5.4}$$

Finally, take $x \in \bigcap_{n\in\mathbb{N}} u_{\alpha_n}$. Applying (5.2), (5.4) and (5.3), we infer that

$$x \in u_{\alpha_n} \subseteq W(v_{\alpha_n}) \subseteq W^4(v_\alpha) \subseteq W^5(u_\alpha) \subseteq V(u_\alpha).$$

Since $V \in \mathcal{U}$ is arbitrary, we have that $\bigcap_{n\in\mathbb{N}} u_{\alpha_n} = u_\alpha$.

We can use a similar argument to show that if $(\alpha_n)_n$ is a decreasing sequence in $(0,1]$ which converges to 0, then $\overline{\bigcup_{n\in\mathbb{N}} u_{\alpha_n}} = u_0$. Therefore, the family $\{u_\alpha : \alpha \in [0,1]\}$ determines a unique fuzzy set $u \in \mathcal{F}(X)$. Let us show that the filter $\mathcal{F}$ converges to u. To see this, Take $V \in \mathcal{U}$ and $W \in \mathcal{U}$ such that $W^2 \subseteq V$. Since $\mathcal{F}$ is a Cauchy filter, there is $P \in \mathcal{F}$ such that $P \times P \subseteq \mathcal{F}[W]$. We claim that $P \subseteq \mathcal{F}[V](u)$. Indeed, take $v \in P$ and $\alpha \in [0,1]$. Since $\mathcal{F}_\alpha$ converges to u_α, we have that $\mathcal{K}[W](u_\alpha) = Q_\alpha$ for some $Q \in \mathcal{F}$. If $z \in P \cap Q$, then $(z,v) \in \mathcal{F}[W]$, $z_\alpha \subseteq W(u_\alpha)$ and $u_\alpha \subseteq W(z_\alpha)$. Hence,

$$u_\alpha \subseteq W(z_\alpha) \subseteq W^2(v_\alpha) \subseteq V(v_\alpha),$$
$$v_\alpha \subseteq W(z_\alpha) \subseteq W^2(u_\alpha) \subseteq V(u_\alpha).$$

We have thus proved that $u_\alpha \subseteq V(v_\alpha)$ and $v_\alpha \subseteq V(u_\alpha)$ for each $\alpha \in [0,1]$. So $v \in \mathcal{F}[V](u)$. Hence $P \subseteq \mathcal{F}[V](u)$. This implies that $\mathcal{F}[V](u) \in \mathcal{F}$ for each $V \in \mathcal{U}$. Thus, $\mathcal{F}$ converges to u and $(\mathcal{F}(X),\mathcal{U}_\infty)$ is a complete uniform space. □

Let (X,d) be a metric space. According to [4, Proposition 8.3.5], if $\mathcal{U}_d$ is the uniformity on X induced by the metric d, then the uniform space $(X,\mathcal{U}_d)$ is complete if and only if the metric space (X,d) is complete. The latter result, Proposition 3.6 and Theorem 5.1 imply the following.

Corollary 5.2. *Let (X,d) be a metric space. Then the metric space $(\mathcal{F}(X),d_\infty)$ is complete.*

We now study completeness on the sendograph uniformity.

Theorem 5.3. *If $(X,\mathcal{U})$ is a complete uniform space, then so is $(\mathcal{F}(X),\mathcal{U}_S)$.*

Proof. According to [20], the uniform space $(\mathcal{K}(X\times\mathbb{I}),\mathcal{K}(\mathcal{U}\times\mathcal{U}_{\mathbb{I}}))$ is complete. Let us show that $S=\{send(u):u\in\mathcal{F}(X)\}$ is closed in $(\mathcal{K}(X\times\mathbb{I}),\mathcal{K}(\mathcal{U}\times\mathcal{U}_{\mathbb{I}}))$. For every $C\in\mathcal{K}(X\times\mathbb{I})$, we put $C_\alpha=\{x\in X:(x,\alpha)\in C\}$ for each $\alpha\in[0,1]$. Consider $C\in\mathcal{K}(X\times\mathbb{I})$ such that $C\in\overline{S}$.
Claim I: $C_\alpha\neq\emptyset$ for each $\alpha\in[0,1]$.

Suppose the contrary, we claim that there exists $\alpha\in\mathbb{I}$ such that $C_\alpha=\emptyset$. Let $p\colon X\times\mathbb{I}\to\mathbb{I}$ the projection map onto the second factor. Then $p(C)$ is a compact subset of $\mathbb{I}$ and $\alpha\notin p(C)$. Hence there exists $\epsilon>0$ such that $(\alpha-\epsilon,\alpha+\epsilon)\cap p(C)=\emptyset$. Therefore, $\mathcal{K}[U\times V_\epsilon](C)\cap S=\emptyset$ for each $U\in\mathcal{U}$. This contradiction shows Claim I.
Claim II: $C_\alpha\in\mathcal{K}(X)$ for each $\alpha\in\mathbb{I}$.

This follows from the fact that $(X\times\{\alpha\})\cap C$ is closed in C.
Claim III: $C_\alpha\subseteq C_\beta$ for each $0\leq\beta<\alpha\leq 1$.

Pick $x\in C_\alpha$. By definition, $(x,\alpha)\in C$. Let us prove that $(x,\beta)\in\overline{C}=C$. Take $U\in\mathcal{U}$ and $\epsilon\in(0,\alpha-\beta)$. By hypothesis, there exists $u\in\mathcal{F}(X)$ such that $(C,send(u))\in\mathcal{K}[U\times V_\epsilon]$. Then, we can choose $(y,\gamma)\in send(u)$ with $(y,\gamma)\in[U\times V_\epsilon](x,\alpha)$. This means that $(x,y)\in U$ and $(\alpha,\gamma)\in V_\epsilon$. Since $\epsilon<\alpha-\beta$, we have that $\beta<\gamma$. Therefore, $y\in u_\gamma\subseteq u_\beta$ so that $(y,\beta)\in send(u)$. Once again, $(C,send(u))\in\mathcal{K}[U\times V_\epsilon]$ implies that we can take $(z,\theta)\in C$ such that $(z,\theta)\in[U\times V_\epsilon](y,\beta)$. So $(y,z)\in U$ and $(\beta,\theta)\in V_\epsilon$. Thus, $(x,z)\in U^2$ and $(\beta,\theta)\in V_\epsilon$. It follows that $(z,\theta)\in[U^2\times V_\epsilon](x,\beta)$. Therefore, $C\cap[U^2\times V_\epsilon](x,\beta)\neq\emptyset$ for each $U\in\mathcal{U}$ and every $\epsilon\in(0,\alpha-\beta)$. So $(x,\beta)\in\overline{C}=C$, whence $x\in C_\beta$. This proves Claim III.
Claim IV: $C_\alpha=\bigcap_{n\in\mathbb{N}}C_{\alpha_n}$ for every increasing sequence $(\alpha_n)_n$ in $\mathbb{I}$, which converges to $\alpha\in(0,1]$.

By Claim III, $C_\alpha\subseteq\bigcap_{n\in\mathbb{N}}C_{\alpha_n}$. Take $x\in\bigcap_{n\in\mathbb{N}}C_{\alpha_n}$. Hence $(x,\alpha_n)\in C$ for every $n\in\mathbb{N}$. Since $(\alpha_n)_n$ converges to α and C is closed, $(x,\alpha_n)_n$ converges to $(x,\alpha)\in C$. Hence $x\in C_\alpha$.
Claim V: $C_0=\overline{\bigcup_{\alpha\in(0,1]}C_\alpha}$.

By Claim II, $A=\overline{\bigcup_{\alpha\in(0,1]}C_\alpha}\subseteq C_0$. If there exist $x\in C_0\setminus A$, then we can choose $U\in\mathcal{U}$ such that $U(x)\cap A=\emptyset$. Hence $\mathcal{K}[U\times V_1](C)\cap S=\emptyset$. This contradiction shows Claim V.

It follows from Claims I-V that there is $c\in\mathcal{F}(X)$ such that $send(c)=C$. Therefore, $S=\{send(u):u\in\mathcal{F}(X)\}$ is closed in $(\mathcal{K}(X\times\mathbb{I}),\mathcal{K}(\mathcal{U}\times\mathcal{U}_{\mathbb{I}}))$. Thus, the uniform space $(\mathcal{F}(X),\mathcal{U}_S)$ is complete. □

Once again, [4, Proposition 8.3.5], Proposition 4.4 and Theorem 5.3 imply the following.

Corollary 5.4. *Let (X,d) be a metric space. Then the metric space $(\mathcal{F}(X),d_S)$ is complete.*

Since $\mathcal{K}(X\times[\alpha,1])$ is closed in $\mathcal{K}(X\times\mathbb{I})$ for each $\alpha\in[0,1]$, we can argue as in the proof of Theorem 5.3 to show the next result.

Lemma 5.5. *Let $(X,\mathcal{U})$ be a uniform space and $\alpha\in(0,1)$. If $end_\alpha(u)=end(u)\cap(X\times[\alpha,1])$ for each $u\in\mathcal{F}^*(X)$, then $E_\alpha=\{end_\alpha(u):u\in\mathcal{F}^*(X)\}$ is closed in the uniform space $(\mathcal{K}(X\times\mathbb{I}),\mathcal{K}(\mathcal{U}\times\mathcal{U}_{\mathbb{I}}))$.*

Theorem 5.6. *If $(X,\mathcal{U})$ is a uniform space, then $E=\{end(u):u\in\mathcal{F}^*(X)\}$ is closed in the uniform space $(\mathcal{C}(X\times\mathbb{I}),\mathcal{C}(\mathcal{U}\times\mathcal{U}_{\mathbb{I}}))$.*

Proof. Select $C\in(\mathcal{C}(X\times\mathbb{I}),\mathcal{C}(\mathcal{U}\times\mathcal{U}_{\mathbb{I}}))$ such that $C\in\overline{E}$. Fix $\alpha\in(0,1)$ and observe that $C\cap(X\times[\alpha,1])\in\overline{E_\alpha}=E_\alpha$. Then, if $C_\alpha=\{x\in X:(x,\alpha)\in C\}$ for each $\alpha\in[0,1]$, Lemma 5.5 implies claims **I-IV**.
Claim I: $C_\alpha\neq\emptyset$ for each $\alpha\in(0,1]$.
Claim II: $C_\alpha\in\mathcal{K}(X)$ for each $\alpha\in(0,1]$.
Claim III: $C_\alpha\subseteq C_\beta$ for each $0<\beta<\alpha\leq 1$.
Claim IV: $C_\alpha=\bigcap_{n\in\mathbb{N}}C_{\alpha_n}$ for every increasing sequence $(\alpha_n)_n$ in $\mathbb{I}$ which converges to $\alpha\in(0,1]$.
Claim V: $C_0=X$.

Let us show **Claim V**. Suppose the contrary, then there exists $x\in X$ such that $(x,0)\notin C$. Since $C\in(\mathcal{C}(X\times\mathbb{I}),\mathcal{C}(\mathcal{U}\times\mathcal{U}_{\mathbb{I}}))$, there is a symmetric entourage $U\in\mathcal{U}$ and $\epsilon>0$ such that $[U\times V_\epsilon](x,0)\cap C=\emptyset$. Then $\mathcal{C}[U\times V_\epsilon](C)\cap E=\emptyset$. This contradiction proves **Claim V**.

From Claims **I-V** and Theorem 4.1 it follows that there exists a unique $u\in\mathcal{F}^*(X)$ such that $end(u)=C$. Hence $E=\{end(u):u\in\mathcal{F}^*(X)\}$ is closed in $(\mathcal{C}(X\times\mathbb{I}),\mathcal{C}(\mathcal{U}\times\mathcal{U}_{\mathbb{I}}))$. □

It is worth to mention that the topology on $\mathcal{K}(X)$ induced by the uniform space $(\mathcal{C}(X),\mathcal{C}(\mathcal{U}))$ is the Vietoris topology. According to [11], a uniform space $(X,\mathcal{U})$ is *supercomplete* if the uniform space $(\mathcal{C}(X),\mathcal{C}(\mathcal{U}))$ is complete. Recall that $(\mathcal{K}(X),\mathcal{K}(\mathcal{U}))$ is a closed subspace of the uniform space $(\mathcal{C}(X),\mathcal{C}(\mathcal{U}))$. The latter notions permit us to study the completeness of $(\mathcal{F}^*(X),\mathcal{U}_E)$.

Theorem 5.7. *If $(X,\mathcal{U})$ is a supercomplete uniform space, then $(\mathcal{F}^*(X),\mathcal{U}_E)$ is complete.*

Proof. According to [1, Corollary 2.6], the product $X \times Y$ of a supercomplete space and a Hausdorff compact space Y is supercomplete. So $(X \times \mathbb{I}, \mathcal{U} \times \mathcal{U}_{\mathbb{I}})$ is supercomplete. Hence $(\mathcal{C}(X \times \mathbb{I}), \mathcal{C}(\mathcal{U} \times \mathcal{U}_{\mathbb{I}}))$ is complete. By Theorem 5.6, $E = \{end(u) : u \in \mathcal{F}^*(X)\}$ is closed in $(\mathcal{C}(X \times \mathbb{I}), \mathcal{C}(\mathcal{U} \times \mathcal{U}_{\mathbb{I}}))$ and, consequently, $(\mathcal{F}^*(X),\mathcal{U}_E)$ is complete. □

Every complete metric space is supercomplete [11]. A paracompact Čech complete space is supercomplete, hence every Čech complete topological group is supercomplete [1]. We also have that if X is a Hausdorff compact space, then the free Abelian topological group $A(X)$ over X is supercomplete [23]. Moreover, countable products of paracompact C-scattered spaces are also supercomplete [8]. (Recall that a Tychonoff space (X,τ) is C-scattered if each of its non-empty closed subspaces contains a compact set with nonempty relative interior). C-scattered spaces play an important role in the framework of supercomplete spaces. Indeed, a supercomplete space $(X,\mathcal{U})$ has the property that its product with every supercomplete space is again supercomplete if and only if X is C-scattered.

Corollary 5.8. *If a uniform space $(X,\mathcal{U})$ satisfies one of the following conditions:*

a) *$(X,\mathcal{U})$ is a complete metric space.*

b) *$(X,\mathcal{U})$ is a paracompact Čech complete space.*

c) *$(X,\mathcal{U})$ is a Čech complete topological group.*

d) *$(X,\mathcal{U})$ is the free Abelian topological group $A(Y)$ over a compact Hausdorff space Y.*

e) *$(X,\mathcal{U})$ is a countable product of paracompact C-scattered spaces.*

Then $(\mathcal{F}^*(X),\mathcal{U}_E)$ is complete.

Corollary 5.9. *Let (X,d) be a metric space. Then the metric space $(\mathcal{F}^*(X),d_E)$ is complete.*

Corollary 5.10. *If $(X,\mathcal{U})$ is a supercomplete uniform space, then $(\mathcal{F}^*(X),\mathcal{U}_E)$ is the completion of $(\mathcal{F}(X),\mathcal{U}_E)$.*

Proof. It remains to show that $\mathcal{F}(X)$ is dense in $(\mathcal{F}^*(X),\mathcal{U}_E)$. Take $u \in \mathcal{F}^*(X)$, $U \in \mathcal{U}$ and $\epsilon > 0$. Let us show that $E[U,\epsilon](u) \cap \mathcal{F}(X) \neq \emptyset$. Fix $\alpha_0 \in (0,1)$ such that $\alpha_0 < \epsilon$. We define $v \in \mathcal{F}(X)$ as follows: $[v]_\alpha = [u]_{\alpha_0}$ if $\alpha \in [0,\alpha_0]$ and $[v]_\alpha = [u]_\alpha$ if $\alpha \in (\alpha_0,1]$. Note that $end(v) \subseteq end(u)$. Hence $end(v) \subseteq [U \times V_\epsilon](end(u))$.

Let us show that $end(u) \subseteq [U \times V_\epsilon](end(v))$. Take $(x,\beta) \in end(u)$. If $\beta = 0$, then $(x,\beta) \in end(v)$. By the construction of v, we have that if $\beta \in (\alpha_0,1]$, then $(x,\beta) \in end(v)$. We now suppose that $\beta \in (0,\alpha_0]$. Hence $((x,\beta),(x,0)) \in U \times V_\epsilon$. Therefore, $end(u) \subseteq [U \times V_\epsilon](end(v))$. We have thus proved that $(end(u), end(v)) \in \mathcal{C}[U \times V_\epsilon]$. So $v \in E[U,\epsilon](u) \cap \mathcal{F}(X)$. This finishes the proof. □

As a consequence of the previous results, we have the next corollary.

Corollary 5.11. *Suposse that $(X,\mathcal{U})$ satisfies one of the following conditions:*

a) *$(X,\mathcal{U})$ is a complete metric space.*

b) *$(X,\mathcal{U})$ is a paracompact Čech complete space.*

c) *$(X,\mathcal{U})$ is a Čech complete topological group.*

d) *$(X,\mathcal{U})$ is the free Abelian topological group $A(Y)$ over a compact Hausdorff space Y.*

e) *$(X,\mathcal{U})$ is a countable product of paracompact C-scattered spaces.*

Then $(\mathcal{F}^*(X),\mathcal{U}_E)$ is the completion of $(\mathcal{F}(X),\mathcal{U}_E)$.

Corollary 5.12. *Let (X,d) be a metric space. Then the metric space $(\mathcal{F}^*(X),d_E)$ is the completion of $(\mathcal{F}(X),d_E)$.*

We do not know if the condition "supercomplete" in Theorem 5.7 can be replaced by "complete".

Problem 5.13. *Let $(X,\mathcal{U})$ be a complete uniform space. Is $(\mathcal{F}^*(X),\mathcal{U}_E)$ complete?*

6 Conclusions

In this paper, we introduced several uniformities on the set $\mathcal{F}(X)$ of all normal, upper semicontinuous with compact support fuzzy sets on X. We study completeness on this uniform spaces. Such uniformities allow us to study some dynamical properties on $\mathcal{F}(X)$.

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