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Original paper

S-generalized distances with respect to ordinal sums

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Abstract

In this paper, the class of S-generalized distances such that the involved t-conorms S are ordinal sums is discussed. It is shown that these S-generalized distances can be thought of as families of generalized distances with respect to some Archimedean t-conorms. We also deal with the S-generalized distance aggregations, which merge a family of S_i-generalized distances into a new S-generalized distance.

Keywords: t-conorm, S-generalized distance, ordinal sum, aggregation function.

1 Introduction

Generalized distances were presented as an extension of ordinary, boolean, and probabilistic metrics [7]. They are functions that assign to each pair from a given universal set a value in an ordered and commutative semigroup. Generalized distance is called S-generalized distance when it works with semigroups $([0,1], S, \leq, 0)$ and the operation S is a triangular conorm. S-generalized distances such as ultrametric are widely applied in computer science. The ultrametric has been introduced in [18] to achieve a wider framework for applications and potential connections to domain theory. And since the idea of bisimulation [12, 17] is one of concurrency theory's most significant contributions to computer science, ultrametric was utilized to measure the similarity of states in a (nondeterministic) fuzzy transition system (FTS) [3, 6, 20]. So it is very interesting to study the S-generalized distance.

Aggregation function is a technique of combining a collection of data from several sources into a representative one value. Aggregation functions have a wide range of applications in probability, statistics, decision theory and computer science. Many techniques of aggregation impose a restriction when selecting the most suitable aggregation function for the problem under consideration. In general, this restriction requires that aggregated results have similar properties to inputs. An example of this kind of situation matches the case in which a collection of metrics (distances) must be amalgamated into a new one. Since the concept of metric plays a central role as measurement tool in applied research, many authors have studied in depth how a collection of metrics can be combined into a single one by means of aggregation function. In fact, in 1981 Borsík and Doboš profoundly studied the general problem of merging a collection of distances (not necessarily finite) into a single one [2]. Recently, Pradera et al. have provided, in the spirit of Borsík and Doboš, a general solution to the problem of merging data represented by means of a finite family of generalized distances and pseudometrics [13, 14]. Several general techniques for merging a finite number of distances into another one have been studied by Casasnovas and Rosselló in [4]. In recent years, many scholars have began to study the problem of aggregation of fuzzy binary relations and S-generalized distances [16].

The idea of ordinal sums has its roots in extension of algebraic structures, namely of posets and lattices [1, 21], and of semigroups [5]. The S-generalized distance with respect to ordinal sums is initially taken into consideration in this paper, then followed by a study of the S-generalized distance aggregation function, which merge a family of

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S_i-generalized distances into a new S-generalized distance. Furthermore, we focus on a t-conorm S which is an ordinal sum of continuous Archimedaen t-conorms S_i.

The paper is organized as follows. First, we will review the fundamental definitions of the t-conorms and S-generalized distances that will be used later. In Section 3, we characterize S-generalized distances with respect to ordinal sums. In Section 4, we characterize the S-generalized distance aggregation function. Finally, we conclude the article with a brief summary.

2 Preliminaries

Let's review some basic definitions that will be applied further.

Definition 2.1. [9] *A t-conorm is a two place function $S : [0,1]^2 \to [0,1]$, such that for all $x, y, z \in [0,1]$ the following conditions are satisfied:*
$(S1)$ $S(x,y) = S(y,x)$,
$(S2)$ $S(S(x,y), z) = S(x, S(y,z))$,
$(S3)$ $S(x,y) \le S(x,z)$ *whenever* $y \le z$,
$(S4)$ $S(x,0) = x$.

The following are the four basic t-conorms S_M, S_P, S_L, and S_D given by

$$\begin{aligned} S_M(x,y) &= max(x,y), \\ S_P(x,y) &= x + y - x \cdot y, \\ S_L(x,y) &= min(x+y,1), \\ S_D(x,y) &= \begin{cases} 1 & (x,y) \in]0,1]^2, \\ max(x,y) & otherwise. \end{cases} \end{aligned}$$

Remark 2.2. [9] *The associativity $(S2)$ allows us to extend each t-conorm S in a unique way to an n-ary operation in the usual way by induction, defining for each n-tuple $(x_1, x_2, ..., x_n) \in [0,1]^n$*

$$\mathop{S}_{i=1}^{n} x_i = S\left(\mathop{S}_{i=1}^{n-1} x_i, x_n\right) = S(x_1, x_2, ..., x_n).$$

If, in particular, we have $x_1 = x_2 = ... = x_n = x$, we shall briefly write $x_S^{(n)} = S(x, x, ..., x)$, and $x_S^{(0)} = 0$ and $x_S^{(1)} = x$ for each $x \in [0,1]$.

Remark 2.3. *For any t-conorm S, we have that $S_M \le S \le S_D$, i.e., $S_M(x,y) \le S(x,y) \le S_D(x,y)$, for all $(x,y) \in [0,1]^2$.*

Definition 2.4. [9] *A t-conorm S is called Archimedean if for each $(x,y) \in]0,1[^2$ there is an $n \in N$ with $x_S^{(n)} > y$.*

The t-conorms mentioned so far, except $S = S_M$, are all examples of Archimedean t-conorms. As we will explain soon, all continuous t-conorms fall into one of the three following categories: the maximum t-conorm, Archimedean t-conorms, ordinal sums.

Remark 2.5. *The t-conorm S_D is Archimedean, but it is not continuous. In the rest of paper, we focus on continuous (Archimedean) t-conorms.*

Theorem 2.6. [9] *A continuous t-conorm S is an Archimedean t-conorm if and only if there exists an increasing and continuous function* s $: [0,1] \to [0,+\infty]$ *with $s(0) = 0$ such that for all $a, b \in [0,1]$ it is*

$$S(a,b) = s^{-1}(\min(s(1), s(a) + s(b))). \tag{1}$$

Function s is called an additive generator of S, and is uniquely determined up to a positive multiplicative constant.

The class of continuous Archimedean t-conorms may be further divided into two different subclasses:

Remark 2.7. [8]

(i) *The subclass of Archimedean t-conorms that are strictly increasing in* $[0,1[^2$, *are called strict t-conorms. Their additive generators verify* $s(1)=\infty$, *thus allowing to write* S *in terms of* s *and its ordinary inverse, that is,* $S(a,b)=s^{-1}(s(a)+s(b))$. *A typical example of strict t-conorm is* $S_P(x,y)=x+y-xy$. *All strict t-conorms are mutually isomorphic, i.e., if* S_1, S_2 *are strict t-conorms, then there exists a bijection* $\varphi:[0,1]\to[0,1]$ *such that*

$$\varphi^{-1}\left(S_1\left(\varphi\left(x\right),\varphi\left(y\right)\right)\right)=S_2\left(x,y\right). \tag{2}$$

T-conorm $S:[0,1]^2\to[0,1]$ *is strict if and only if it is isomorphic to the t-conorm* S_P.

(ii) *The subclass of Archimedean t-conorms that are not strictly increasing, are called non-strict or nilpotent t-conorms. In this cases* $s(1)$ *is finite, and the function* $\hat{s}=s/s(1):[0,1]\to[0,1]$, *called the normed additive generator of* S, *is uniquely determined. This allows to express* S *in terms of* $\hat{s}$ *by means of* $S(a,b)=\hat{s}^{-1}\left(\min\left(1,\hat{s}(a)+\hat{s}(b)\right)\right)$. *The Lukasiewicz t-conorm,* $S_L(x,y)=\min(1,x+y)$, *is a widely used nilpotent t-conorm. All nilpotent t-conorms are mutually isomorphic, i.e., if* S_1, S_2 *are nilpotent t-conorms, then there exists a bijection* $\varphi:[0,1]\to[0,1]$ *such that*

$$\varphi^{-1}\left(S_1\left(\varphi\left(x\right),\varphi\left(y\right)\right)\right)=S_2\left(x,y\right). \tag{3}$$

T-conorm $S:[0,1]^2\to[0,1]$ *is nilpotent if and only if it is isomorphic to the t-conorm* S_L.

Theorem 2.8. [9] *Let* $(S_k)_{k\in K}$ *be a family of t-conorms and* $(]a_k,b_k[)_{k\in K}$ *be a family of nonempty, pairwise disjoint open subintervals of* $[0,1]$ *with finite or countable index set* K. *Then the function* $S:[0,1]^2\to[0,1]$ *defined by*

$$S(x,y)=\begin{cases} a_k+(b_k-a_k)S_k\left(\frac{x-a_k}{b_k-a_k},\frac{y-a_k}{b_k-a_k}\right) & \text{if } (x,y)\in[a_k,b_k]^2,\\ max(x,y) & \text{otherwise,}\end{cases}$$

is a t-conorm which is called the ordinal sum of summands $\langle a_k,b_k,S_k\rangle$, $k\in K$, *and we denote it by*

$$S=\left(\langle a_k,b_k,S_k\rangle\right)_{k\in K}. \tag{4}$$

Remark 2.9. *Note that if* $K=\emptyset$, *then* $S=S_M$. *That* $K=\{1\}$ *is a singleton, and then, if* $]a_0,b_0[=]0,1[$, *it holds* $S=S_0$ *for any t-conorm* S_0 *(this is so-called trivial ordinal sum representation of* S_0.*)*

Theorem 2.10. [9] S *is a continuous t-conorm, if and only if there exists a family of continuous, increasing mappings* $s_k:[a_k,b_k]\to[0,\infty]$ *such that* $s_k(a_k)=0$, *satisfying the following equation:*

$$S(a,b)=\begin{cases} s_k^{-1}\left(min\left(s_k(b_k),s_k(a)+s_k(b)\right)\right) & \text{if } a,b\in]a_k,b_k[,\\ max(a,b) & \text{otherwise,}\end{cases} \tag{5}$$

where $]a_k,b_k[_{k\in K}$ *is a family of nonempty, pairwise disjoint open subintervals of* $[0,1]$.

Continuous t-conorm S in Theorem 2.10 can be achieved as an ordinal sum by considering a family of increasing bijective functions $\varphi_k:[0,1]\to[a_k,b_k]$ and the associated Archimedean t-conorms S_k which are induced on the unit interval by the additive generators $s_k\circ\varphi_k:[0,1]\to[0,\infty]$. The standard choice for the functions φ_k is

$$\varphi_k(a)=a_k+a(b_k-a_k), \tag{6}$$

although others could be considered.

Additionally for simplicity's sake, we will focus solely on a certain class of ordinal sums, specifically those satisfying:

$$0=a_0<a_1<...<a_n,\ \text{and}\ b_i=a_{i+1},\ b_n=1. \tag{7}$$

Next, we introduce the definition of pseudometric, which is closely related to S-generalized distance.

Definition 2.11. [13] *Let* X *be a non-empty set. A function* $d:X\times X\to[0,\infty)$ *is said to be a pseudometric (pseudo-distance) on* X *if it verifies the following properties for any* $x,y,z\in X$:
(i) $d(x,x)=0$,
(ii) $d(x,y)=d(y,x)$,
(iii) $d(x,z)\le d(x,y)+d(y,z)$.

If condition (i) is replaced by the stronger one "$d(x,y)=0$*" if and only if "*$x=y$*" then* d *is said to be a metric.*

According to the above definition, we know that pseudometrics can give values in the whole non-negative real line. Nevertheless, in some cases it is necessary to work with bounded pseudometrics, that is, operators whose range is included in a closed real interval $[0,1]$. One example of this situation is given by the S-generalized distance. (However, S-generalized distance is not a pseudometric, in general.)

Definition 2.12. *Let X be a set, S be a t-conorm. A function $d : X \times X \to [0,1]$ is said to be an S-generalized distance on X if it verifies the following properties for any x, y, z in X:*
(i) $d(x,x) = 0$,
(ii) $d(x,y) = d(y,x)$,
(iii) $d(x,z) \le S(d(x,y), d(y,z))$.

Remark 2.13. *When t-conorm $S = S_M$, the S-generalized distance is also called a ultrametric.*

One of the purposes in this paper is to find out which aggregation operators allow us to combine a collection of S_i-generalized distances defined on the same universe ($[0,1]$) into a single one. Next, we introduce the definition of S-generalized distance aggregation function.

Definition 2.14. *A function $F : [0,1]^n \to [0,1]$ is said to aggregate a family of S_i-generalized distances $(i = 1, ..., n)$ into an S-generalized distance if $F(d_1, ..., d_n)$ is an S-generalized distance on X for any set X and any collection of S_i-generalized distances $d_1, ..., d_n$ on X, where $F(d_1, ..., d_n)$ is given by $F(d_1, ..., d_n)(x,y) = F(d_1(x,y), ..., d_n(x,y))$, $x, y \in X$.*

The study of S-generalized distance aggregation functions in Definition 2.14 focuses on two cases: on the one hand, S and S_i are both ordinal sum forms, and they have the same family of subintervals $]a_k, b_k[$; on the other hand, S and S_i are both continuous Archimedean.

The aggregation problem of ultrametric is introduced in literature [15], and the appropriate characterization and conclusion of pseudometric aggregation is provided in [10, 11, 13]. In the next section, we will study the S-generalized distances with respect to ordinal sums.

3 A characterization theorem

Theorem 3.1. *A function d on a set X is an S-generalized distance w.r.t. an ordinal sum $S = (\langle a_k, b_k, S_k\rangle)_{k\in K}$, if and only if, there exists a family $\{d_k\}_{k\in K}$ of S_k-generalized distances w.r.t. continuous Archimedean t-conorms S_k and a family of increasing bijective functions $\varphi_k : [0,1] \to [a_k, b_k]$ such that $d = \sup_{k\in K} e_k$ with*

$$e_k(x,y) = \begin{cases} \varphi_k \circ d_k(x,y) & \text{if } d_k(x,y) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Let's make a few comments before proceeding on to the proof.

With the exception of the pairings $d_k(x,y) = 0$, which remain unchanged, the blocks e_k from which d is generated are created by compressing the range of each d_k (often $[0,1]$) into narrower slices and then stacking them on top of one another.

Proof. ($\Rightarrow$) The function d_k are defined by

$$d_k(x,y) = \begin{cases} 0 & \text{if } d(x,y) \le a_k, \\ \varphi_k^{-1} \circ d(x,y) & \text{if } a_k < d(x,y) < b_k, \\ 1 & \text{if } d(x,y) \ge b_k, \end{cases} \tag{9}$$

for all $x, y \in X$.

First, we need to prove that these are S_k-generalized distances w.r.t. the Archimedean t-conorms S_k obtained from the additive generators $s_k \circ \varphi_k : [0,1] \to [0,\infty]$.

Both (i) and (ii) are straightforward in Definition 2.12. To prove that d_k is an S_k-generalized distance, we only need to prove that d_k satisfies (iii). For any given three elements x, y and z, there are six different cases to be considered:

(i) If $d_k(x,y) = 1$, then $S_k(d_k(x,y), d_k(y,z)) = 1 \ge d_k(x,z)$, so that (iii) of Definition 2.12 is satisfied.

(ii) If $0 \le d_k(x,y) < 1$ and $d_k(y,z) = 1$, then the proof is similar to (i).

(iii) If $d_k(x, y) = 0$ and $0 < d_k(y, z) < 1$, then $d(x, y) \leq a_k,\ a_k < d(y, z) < b_k$ and the ordinal sum S reduces to S_M in that particular case, which means

$$S(d(x, y), d(y, z)) = S_M(d(x, y), d(y, z)) = d(y, z) \geq d(x, z). \tag{10}$$

The last inequality is due to (iii) in the Definition 2.12.

According to Equation (9), we have $d_k(y, z) \geq d_k(x, z)$, and

$$S_k(d_k(x, y), d_k(y, z)) \geq d_k(y, z) \geq d_k(x, z), \tag{11}$$

which prove (iii) of Definition 2.12.

(iv) If $0 < d_k(x, y) < 1$ and $d_k(y, z) = 0$, then the proof is similar to (iii).

(v) If $0 < d_k(x, y), d_k(y, z) < 1$, then

$$\begin{aligned} S_k\left(d_k(x, y), d_k(y, z)\right) &= S_k(\varphi_k^{-1} \circ d(x, y), \varphi_k^{-1} \circ d(y, z)) \\ &= S_k\left(\frac{d\left(x, y\right) - a_k}{b_k - a_k}, \frac{d\left(y, z\right) - a_k}{b_k - a_k}\right) \\ &= \varphi_k^{-1} \circ S(d(x, y), d(y, z)) \\ &\geq \varphi_k^{-1} \circ d(x, z). \end{aligned} \tag{12}$$

i) If $0 \leq d(x, z) \leq b_k$, then $S_k\left(d_k(x, y), d_k(y, z)\right) \geq \varphi_k^{-1} \circ d(x, z) \geq d_k(x, z)$.

ii) $d(x, z) > b_k$ is impossible, since d is an S-generalized distance and $a_k < d(x, y),\ d(y, z) < b_k$. Therefore,

$$b_k \geq S(d(x, y), d(y, z)) \geq d(x, z) > b_k.$$

This is a contradiction of that. So, we have $d(x, z) \leq b_k$, which prove (iii) of Definition 2.12.

(vi) If $d_k(x, y) = d_k(y, z) = 0$, then $d(x, y) \leq a_k,\ d(y, z) \leq a_k$, we have

$$a_k \geq S(d(x, y), d(y, z)) \geq d(x, z).$$

As a consequence, $d_k(x, z) = 0,\ S_k(d_k(x, y), d_k(y, z)) \geq d_k(x, z)$.

Finally, we need to show that $d = \sup_{k \in K} e_k$. To this end, let us express e_k in terms of d instead of d_k:

(i) If $a_k < d(x, y) < b_k$, then $d_k(x, y) = \varphi_k^{-1} \circ d(x, y) > 0$ and $e_k(x, y) = \varphi_k \circ d_k(x, y) = d(x, y)$.

(ii) If $d(x, y) \leq a_k$, then $d_k(x, y) = 0$ and $e_k(x, y) = 0$.

(iii) If $d(x, y) \geq b_k$, then $d_k(x, z) = 1$ and $e_k(x, y) = b_k$.

Now, suppose $a_k \leq d(x, y) < b_k$ for a certain pair (x, y). For any i and j such that $i < k < j$ we shall have that $e_i(x, y) = b_i$, $e_k(x, y) = d(x, y)$ and $e_j(x, y) = 0$, and therefore $d(x, y) = \sup_{k \in K} e_k(x, y)$.

Note that nothing changes if k corresponds to the first or the last interval. Also, note that the conditions $a_k \leq d(x, y) < b_k$ account for all the possible values of $d(x, y)$ except $d(x, y) = 0$ because we are assuming $b_k = a_{k+1}$ in which case $e_k(x, y) = 0$.

($\Leftarrow$) Consider t-conorm S which is the ordinal sum of Archimedean t-conorms S_k with the system of intervals $]a_k, b_k[$. There exists a family $\{d_k\}_{k \in K}$ of S_k-generalized distances w.r.t. continuous Archimedean t-conorms S_k for e_k as given by Equation (8). It has to be proved that $d = \sup_{k \in K} e_k$ is an S-generalized distance with respect to $S = (\langle a_k, b_k, S_k \rangle)_{k \in K}$. As usually, both (i) and (ii) are straightforward in Definition 2.12, so we turn our attention to (iii) in Definition 2.12.

In order to move on, let's first state a few useful facts.

(i) $e_k(x, y) \leq b_k \leq a_j \leq e_j(z, t)$, if $k < j$, except $e_j(z, t) = 0$.

(ii) $d(x, y) = 0$ if and only if $d_k(x, y) = 0$ for all $k \in K$.

(iii) If $d(x, y) > 0$ then $d(x, y) = e_k(x, y)$, if and only if $d_k(x, y) \in]a_k, b_k]$ and $d_i(x, y) = 0$ for all $i > k$.

(iv) If $d(x,z)=0$ then $d_k(x,z)=0$ for all $k \in K$, thus this possibility should be excluded.

(v) If $d(x,y)=e_i(x,y)$, $d(y,z)=e_j(y,z)$ and $d(x,z)=e_k(x,z)$ then $k \leq max\{i,j\}$, for any $x,y,z \in X$.

For the last fact above, we have $e_k(x,y)=0$ and $e_k(y,z)=0$, if $k > max\{i,j\}$. Then $d_k(x,z)=0$, which is against $d(x,z)=e_k(x,z)$.

Now we can proceed to prove (iii) in Definition 2.12 with respect to $S=(\langle a_k,b_k,S_k\rangle)_{k\in K}$, that is, to prove that $S(d(x,y),d(y,z)) \geq d(x,z)$ for all $x,y,z \in X$. Let us suppose $d(x,y)=e_i(x,y)$, $d(y,z)=e_j(y,z)$ and $d(x,z)=e_k(x,z)$. From the facts above it follows that nine possible cases deserve attention, which are

$$i<j=k, \quad i=k<j, \quad i=j=k;$$

$$i=k>j, \quad i=j>k, \quad i>j>k;$$

$$i>j=k, \quad i<k<j, \quad i>k>j.$$

We only prove the first four cases, the rest cases are similar to (ii).

(i) $i<j=k$:

$$\begin{aligned} S(d(x,y),d(y,z)) &= max(d(x,y),d(y,z)) \\ &= max(e_i(x,y),e_j(y,z)) \\ &= e_j(y,z). \end{aligned} \tag{13}$$

On the other hand, from $S_k(d_k(x,y),d_k(y,z)) \geq d_k(x,z)$ and $d_k(x,y)=0$ it follows that $d(y,z) \geq d(x,z)$ and thus $S(d(x,y),d(y,z)) \geq d(x,z)$.

(ii) $i=k<j$:

$$\begin{aligned} S(d(x,y),d(y,z)) &= max(d(x,y),d(y,z)) \\ &= max(e_i(x,y),e_j(y,z)) \\ &= e_j(y,z) \\ &\geq e_k(x,z) \\ &= d(x,z). \end{aligned} \tag{14}$$

(iii) $i=k=j$:

$$\begin{aligned} S(d(x,y),d(y,z)) &= S(e_i(x,y),e_i(y,z)) \\ &= s_i^{-1}(s_i \circ e_i(x,y) + s_i \circ e_i(y,z)) \\ &= s_i^{-1}(s_i \circ \varphi_i \circ d_i(x,y) + s_i \circ \varphi_i \circ d_i(y,z)) \\ &= \varphi_i \circ {\varphi_i}^{-1} \circ s_i^{-1}(s_i \circ \varphi_i \circ d_i(x,y) + s_i \circ \varphi_i \circ d_i(y,z)) \\ &= \varphi_i \circ S_i(d_i(x,y),d_i(y,z)) \\ &\geq \varphi_i \circ d_i(x,z) = e_i(x,z) = d(x,z). \end{aligned} \tag{15}$$

(iv) $i=k>j$. This proof is similar to (i).

This brings the proof to a conclusion. □

Remark 3.2. *The ordinal sum $S=(\langle a_k,b_k,S_k\rangle)_{k\in K}$ is Archimedean if Card(K)=1, $a_k=0$, $b_k=1$ and S_k is Archimedean. In this case $d=d_k$.*

Next, let's illustrate Theorem 3.1 with an example.

Example 3.3. *Consider the ordinal sum $S=(\langle a_k,b_k,S_k\rangle)_{k\in K=\{0,1\}}=(\langle 0,0.5,S_L\rangle, \langle 0.5,1,S_L\rangle)$ of two summands (see Figure 1).*

Let d and d_k be S and S_k generalized distances defined in $X=\{x,y,z\}$, respectively. For $\varphi_k(a)=a_k+a(b_k-a_k)$ with $k=0,1$, we have

(i) *If $d(x,y)=0$, then $d(x,y) \leq a_0$, $d(x,y) \leq a_1$, we have*

$$d_0(x,y)=0, \quad e_0(x,y)=0; \quad d_1(x,y)=0, \quad e_1(x,y)=0.$$

Therefore, $d(x,y)=\sup_{k\in K} e_k(x,y)=0$.

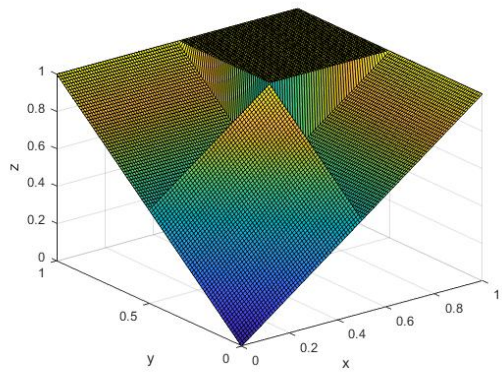

Figure 1: $S = (\langle 0, 0.5, S_L\rangle, \langle 0.5, 1, S_L\rangle)$

(ii) *If* $d(x,y) \in]0, 0.5[$, *then* $0 = a_0 < d(x,y) < b_0 = 0.5$, $d(x,y) < a_1 = 0.5$, *we have*

$$d_0(x,y) = \varphi_0^{-1} \circ d(x,y), \quad e_0(x,y) = d(x,y); \quad d_1(x,y) = 0, \quad e_1(x,y) = 0.$$

Therefore, $d(x,y) = \sup_{k \in K} e_k(x,y) = e_0(x,y) = d(x,y)$.

(iii) *If* $d(x,y) = 0.5$, *then* $d(x,y) \geq b_0$, $d(x,y) \leq a_1$, *we have*

$$d_0(x,y) = 1, \quad e_0(x,y) = 0.5; \quad d_1(x,y) = 0, \quad e_1(x,y) = 0.$$

Therefore, $d(x,y) = \sup_{k \in K} e_k(x,y) = e_0(x,y) = 0.5$.

(iv) *If* $d(x,y) \in]0.5, 1[$, *then* $d(x,y) > b_0$, $0.5 = a_1 < d(x,y) < b_1 = 1$, *we have*

$$d_0(x,y) = 1, \quad e_0(x,y) = \varphi_0(1) = 0.5; \quad d_1(x,y) = \varphi_1^{-1} \circ d(x,y), \quad e_1(x,y) = d(x,y).$$

Therefore, $d(x,y) = \sup_{k \in K} e_k(x,y) = e_1(x,y) = d(x,y)$.

(v) *If* $d(x,y) = 1$, *then* $d(x,y) \geq b_0$, $d(x,y) \geq b_1$, *we have*

$$d_0(x,y) = 1, \quad e_0(x,y) = 0.5; \quad d_1(x,y) = 1, \quad e_1(x,y) = 1.$$

Therefore, $d(x,y) = \sup_{k \in K} e_k(x,y) = e_1(x,y) = 1$.

Next, we consider the problem of merging S-generalized distances.

4 Aggregation of S-generalized distances

In this section, we primarily focus on aggregating S-generalized distances. Before delving into the topic, let's review the key concepts related to metric aggregation functions.

Definition 4.1. [11] *Let* $n \in N$. *A function* $F : [0,\infty)^n \to [0,\infty)$ *is called an n-metric aggregation function provided that, for any nonempty set* X *and any collection of metrics* $\{d_1, ..., d_n\}$ *on* X, *the function* $F(d_1, ..., d_n) : X^2 \to [0,\infty)$ *is a metric, where* $F(d_1, ..., d_n)(x,y) = F(d_1(x,y), ..., d_n(x,y))$ *for all* $x, y \in X$.

Following [2], fixed $n \in N$, we consider the set $[0,\infty)^n$ ordered by the point wise order relation $\preceq$, i.e., $\bar{a} \preceq \bar{b} \Leftrightarrow a_i \leq b_i$ for all $i = 1, ..., n$, where $\leq$ denotes the usual order on $[0,\infty)$.

Remark 4.2. [11] *We will say that a function* $F : [0,\infty)^n \to [0,\infty)$ *is amenable provided that* $F\left(\bar{a}\right) = 0 \Leftrightarrow a_1 = a_2 = ... = a_n = 0$.

Definition 4.3. [11] *A function* $F:[0,\infty)^n \to [0,\infty)$ *will be said to be subadditive if* $F\left(\bar{a}+\bar{b}\right) \le F\left(\bar{a}\right)+F\left(\bar{b}\right)$ *for all* $\bar{a},\bar{b} \in [0,\infty)^n$*. If* $\bar{a},\bar{b} \in (0,\infty)^n$*, then* F *is said to be positive subadditive. Moreover, a function* $F:[0,\infty)^n \to [0,\infty)$ *will be said to be monotone provided that* $F\left(\bar{a}\right) \le F\left(\bar{b}\right)$ *for all* $\bar{a},\bar{b} \in [0,\infty)^n$ *whenever* $\bar{a} \preceq \bar{b}$*. If* $\bar{a},\bar{b} \in (0,\infty)^n$*, then* F *is said to be positive monotone.*

Proposition 4.4. [11] *Let* $n \in N$*. A function* $F:[0,\infty)^n \to [0,\infty)$ *be a monotone, subadditive and amenable function. Then* F *is an* n*-metric aggregation function.*

Proposition 4.5. [11] *Let* $n \in N$*, and let* $F:[0,\infty)^n \to [0,\infty)$ *be a function which is positive monotone, positive subadditive, and in addition, it satisfies the following conditions:*

(i) $F(0,\dots,0)=0$,

(ii) *If* $F\left(\bar{a}\right)=0$*, then* $\min\{a_1,\dots,a_n\}=0$.

Then F *is an* n*-metric aggregation function.*

Next, we characterize the S-generalized distance aggregation function and first introduce the concept of extended domination.

Definition 4.6. [19] *Let* A *and* $A_1,\dots,A_n$ *be non-decreasing binary functions defined in* X^2*,* B *be a non-decreasing* n*-ary function defined in* X^n*. We say that* A *extensively dominates* B *with respect to* $(A_1,\dots,A_n)$*, if for all* $\bar{a},\bar{b} \in [0,1]^n$*,*

$$A\left(B\left(\bar{a}\right),B\left(\bar{b}\right)\right) \ge B\left(A_1\left(a_1,b_1\right),\dots,A_n\left(a_n,b_n\right)\right).$$

Theorem 4.7. [19] *Let* S *be a* t*-conorm and* d_i*,* $i=1,\dots,n$*, be a collection of* S_i*-generalized distances. If a function* $F:[0,1]^n \to [0,1]$ *is non-decreasing, amenable and* S *extensively dominates* F *with respect to* $(S_1,\dots,S_n)$*, i.e.,*

$$S\left(F\left(\bar{a}\right),F\left(\bar{b}\right)\right) \ge F\left(S_1\left(a_1,b_1\right),\dots,S_n\left(a_n,b_n\right)\right),$$

for all $\bar{a},\bar{b} \in [0,1]^n$*, then* $F(d_1,\dots,d_n)$ *is an* S*-generalized distance.*

In the following, we provide a characterization of those functions that are able to merge a collection of S-generalized distances into a new one whenever all involved t-conorms are continuous Archimedean.

The following theorem first addresses the situation in which all of the related t-conorms are nilpotent:

Theorem 4.8. [15] *An operator* $F:[0,1]^n \to [0,1]$ *aggregates* n S_i*-generalized distances* $d_1,\dots,d_n$ *into an* S*-generalized distance* d*, with both* S *and* S_i*,* $i=1,\dots,n$*, nilpotent continuous Archimedean* t*-conorms, if and only if*

$$F=\hat{s}^{-1}\circ G\circ\left(\hat{s}_1\times\dots\times\hat{s}_n\right), \tag{16}$$

where $\hat{s}$ *and* $\hat{s}_i$ *are the normed additive generators of* S *and* S_i*, respectively, and* $G:[0,1]^n \to [0,1]$ *is an operator that aggregates pseudometrics.*

Likewise, the aggregate of generalized distances defined with respect to strict continuous Archimedean t-conorms yields the following result.

Theorem 4.9. [15] *An operator* $F:[0,1]^n \to [0,1]$ *aggregates* n S_i*-generalized distances* $d_1,\dots,d_n$ *into an* S*-generalized distance* d*, with both* S *and* S_i*,* $i=1,\dots,n$*, strict continuous Archimedean* t*-conorms, if and only if*

$$F=s^{-1}\circ G\circ\left(s_1\times\dots\times s_n\right), \tag{17}$$

where s *and* s_i *are additive generators of* S *and* S_i*, respectively, and* $G:(R^+)^n \to R^+$ *is an operator that aggregates pseudometrics.*

After establishing the theoretical foundations of these two theorems, we need to explore the practical application of these concepts. To do this, let's consider an example to investigate the relationship between the functions F and G under various conditions.

Example 4.10. *Let $S^{'}$, S_1 and S_2 be continuous Archimedean t-conorms, and let d_1 and d_2 be generalized distances with respect to S_1 and S_2, respectively. The function F is an $S^{'}$-generalized distance aggregation function. The function G is an operator used to aggregate pseudometrics. Here are some possible choices:*

(i) *Suppose $S^{'}$, S_1 and S_2 be nilpotent t-conorms isomorphic with S_L by $\varphi, \varphi_1, \varphi_2$. $s(x) = x$ is normed additive generator of S_L. According to Remark 2.7, we have*

$$s^{'}(x) = s(\varphi(x)) = \varphi(x)\,,\ s_1(x) = s(\varphi_1(x)) = \varphi_1(x)\,,\ s_2(x) = s(\varphi_2(x)) = \varphi_2(x)\,.$$

According to Theorem 4.8, we have

$$\begin{aligned} F(d_1,d_2) &= \left(s^{'}\right)^{-1} \circ G \circ (s_1 \times s_2)(d_1,d_2) \\ &= \varphi^{-1} s^{-1} \circ G(s\circ\varphi_1(d_1), s\circ\varphi_2(d_2)) \\ &= \varphi^{-1} \circ G(\varphi_1(d_1), \varphi_2(d_2))\,. \end{aligned}$$

– *If $\varphi(x) = x,\ \varphi_1(x) = \varphi_2(x) = x^p,\ G(x,y) = w_1 \cdot x^{\frac{1}{p}} + w_2 \cdot y^{\frac{1}{p}}$, where $w_1, w_2 \geq 0,\ w_1 + w_2 = 1,\ p \geq 1$, then*

$$\begin{aligned} F(d_1,d_2) &= \varphi^{-1} \circ G(d_1^p, d_2^p) \\ &= \varphi^{-1} \circ \left(w_1 (d_1^p)^{\frac{1}{p}} + w_2 (d_2^p)^{\frac{1}{p}}\right) \\ &= w_1 d_1 + w_2 d_2, \end{aligned} \tag{18}$$

for any $x, y \in [0,1]$. (According to the third section of [13], the function G is a pseudometric aggregation function.)

– *If $\varphi(x) = \varphi_1(x) = \varphi_2(x) = x^p,\ G(x,y) = w_1 x + w_2 y$, where $w_1, w_2 \geq 0,\ w_1 + w_2 = 1,\ p \geq 1$, then*

$$\begin{aligned} F(d_1,d_2) &= \varphi^{-1} \circ G(d_1^p, d_2^p) \\ &= \varphi^{-1} \circ (w_1 \cdot d_1^p + w_2 \cdot d_2^p) \\ &= \sqrt[p]{w_1 \cdot d_1^p + w_2 \cdot d_2^p}, \end{aligned}$$

for any $x, y \in [0,1]$.

- *If $p = 1$, then this result is consistent with the result of Equation (18);*
- *If $p = \infty$, then $F(d_1,d_2) = \max(d_1,d_2)$.*

(ii) *Suppose $S^{'}$, S_1 and S_2 are strict t-conorms isomorphic with S_P by $\varphi, \varphi_1, \varphi_2$. The additive generator of S_P is $s(x) = -\ln(1-x)$, its inverse is $s^{-1}(x) = 1 - e^{-x}$, according to Remark 2.7, we have*

$$\begin{aligned} s^{'}(x) &= s(\varphi(x)) = -\ln(1-\varphi(x))\,, \\ s_1(x) &= s(\varphi_1(x)) = -\ln(1-\varphi_1(x))\,, \\ s_2(x) &= s(\varphi_2(x)) = -\ln(1-\varphi_2(x))\,. \end{aligned}$$

Suppose $G(x,y) = w_1 x + w_2 y,\ w_1, w_2 \geq 0$, we have

$$\begin{aligned} F(d_1,d_2) &= \left(s^{'}\right)^{-1} \circ G \circ (s_1 \times s_2)(d_1,d_2) \\ &= \left(s^{'}\right)^{-1} \circ (w_1 \cdot s_1(d_1) + w_2 \cdot s_2(d_2)) \\ &= \varphi^{-1} s^{-1} \circ (w_1 \cdot s\circ\varphi_1(d_1) + w_2 \cdot s\circ\varphi_2(d_2)) \\ &= \varphi^{-1} \circ \left[1 - (1-\varphi_1(d_1))^{w_1} \cdot (1-\varphi_2(d_2))^{w_2}\right]. \end{aligned}$$

(iii) *Suppose $S^{'}$, S_1 and S_2 are strict t-conorms S_P, but their additive generators all differ from S_P by positive multiplicative constants.*
If $G(x,y) = w_1 x + w_2 y,\ \ w_1, w_2 \geq 0,\ \ c \geq 1,\ \ c_i \geq 1,\ \ i = 1,2.$
Suppose $s(x) = -c\ln(1-x)$, $s_1(x) = -c_1 \ln(1-x)$, $s_2(x) = -c_2 \ln(1-x)$, then

$$\begin{aligned} F(d_1,d_2) &= s^{-1} \circ G \circ (s_1 \times s_2)(d_1,d_2) \\ &= 1 - e^{-\frac{w_1 \cdot (-c_1 \cdot \ln(1-d_1)) + w_2 \cdot (-c_2 \cdot \ln(1-d_2))}{c}} \\ &= 1 - (1-d_1)^{\frac{w_1 \cdot c_1}{c}} (1-d_2)^{\frac{w_2 \cdot c_2}{c}}. \end{aligned}$$

When $w_1 \cdot c_1 = w_2 \cdot c_2 = c$, we have $F(d_1,d_2) = S_P(d_1,d_2) = d_1 + d_2 - d_1 \cdot d_2$.

In Theorem 4.8 and Theorem 4.9, the aggregated t-conorms S_i are of the same type, either nilpotent or strict. In the next theorem, we study the case where S_i contains both strict and nilpotent when S is strict.

Theorem 4.11. *Let d_i be S_i-generalized distances defined on X, d be an S-generalized distance defined on X, where S is strict and S_i, $i = 1, ..., n$, are continuous Archimedean t-conorms. If function $F : [0,1]^n \to [0,1]$ is defined by*

$$F(d_1, ..., d_n)(x, y) = s^{-1}\left(\sum_{i=1}^{n} w_i \cdot s_i(d_i(x, y))\right), \tag{19}$$

for all $x, y \in X$, where s and s_i are the additive generators of S and S_i, respectively, $w_i \geq 0$, $\sum_{i=1}^{n} w_i \leq 1$, then F aggregates n S_i-generalized distances $d_1, ..., d_n$ into an S-generalized distance d.

Proof. Both (i) and (ii) are straightforward in Definition 2.12, to prove that d is an S-generalized distance, we only need to prove that d satisfies (iii), $\forall x, y, z \in X$, we have

$$F(d_1, ..., d_n)(x, y) = s^{-1}\left(\sum_{i=1}^{n} w_i \cdot s_i \circ d_i(x, y)\right),$$

$$F(d_1, ..., d_n)(y, z) = s^{-1}\left(\sum_{i=1}^{n} w_i \cdot s_i \circ d_i(y, z)\right),$$

$$F(d_1, ..., d_n)(x, z) = s^{-1}\left(\sum_{i=1}^{n} w_i \cdot s_i \circ d_i(x, z)\right).$$

(i) Supoose $\exists j \in \{1, ..., n\}$ such that S_j is strict. Let d_j be an S_j-generalized distance. We have

$$\begin{aligned} S_j(d_j(x, y), d_j(y, z)) &= s_j^{-1}(s_j \circ d_j(x, y) + s_j \circ d_j(y, z)) \\ &\geq d_j(x, z). \end{aligned} \tag{20}$$

That is

$$s_j \circ d_j(x, y) + s_j \circ d_j(y, z) \geq s_j \circ d_j(x, z). \tag{21}$$

(ii) Supoose $\exists k \in \{1, ..., n\}$ such that S_k is nilpotent. Let d_k be an S_k-generalized distance. We have

$$\begin{aligned} S_k(d_k(x, y), d_k(y, z)) &= s_k^{-1}(\min(s_k(1), s_k \circ d_k(x, y) + s_k \circ d_k(y, z))) \\ &\geq d_k(x, z). \end{aligned}$$

That is

$$s_k \circ d_k(x, y) + s_k \circ d_k(y, z) \geq s_k \circ d_k(x, z).$$

Based on the above conclusions, we have

$$\begin{aligned} S(F(d_1, ..., d_n)(x, y), F(d_1, ..., d_n)(y, z)) &= s^{-1}(s(F(d_1, ..., d_n)(x, y)) + s(F(d_1, ..., d_n)(y, z))) \\ &= s^{-1}\left(\sum_{i=1}^{n} w_i \cdot s_i \circ d_i(x, y) + \sum_{i=1}^{n} w_i \cdot s_i \circ d_i(y, z)\right) \\ &= s^{-1}\left(\sum_{i=1}^{n} w_i(s_i \circ d_i(x, y) + s_i \circ d_i(y, z))\right) \\ &\geq s^{-1}\left(\sum_{i=1}^{n} w_i \cdot s_i \circ d_i(x, z)\right) \\ &= F(d_1, ..., d_n)(x, z). \end{aligned} \tag{22}$$

The inequality is owing to the monotonicity of s and s^{-1}. This brings the proof to a conclusion. □

Next, we consider the aggregation problem of S-generalized distances, when $S \geq S_i$.

Proposition 4.12. [19] *Let S be a t-conorm and d_i, $i = 1, ..., n$, be a collection of S_i-generalized distances. If $S \geq S_i$ for all $i = 1, ..., n$, then $S(d_1, ..., d_n)$ is an S-generalized distance.*

Example 4.13. *Let d_1, d_2 be S_i-generalized distances defined on set $X = \{x, y, z\}$ where $S_1 = S_M$, $S_2 = S_L$. Let*

$$d_1(x,y) = 0.2, \quad d_1(y,z) = 0.3, \quad d_1(x,z) = 0.3.$$

$$d_2(x,y) = 0.2, \quad d_2(y,z) = 0.3, \quad d_2(x,z) = 0.5.$$

Obviously, d_i is the S_i-generalized distance for $i = 1, 2$. Since,

$$S_L(S_L(d_1, d_2)(x,y), S_L(d_1, d_2)(y,z)) \geq S_L(d_1, d_2)(x,z).$$

Similar results hold for other cases. Therefore, $S_L(d_1, d_2)$ is an S_L-generalized distance.

Below we consider the case where S and S_i are ordinal sums and have the same family of subintervals $]a_k, b_k[$.

Theorem 4.14. *Let $S = (\langle a_k, b_k, S_k \rangle)_{k \in K}$, $S_i = \left(\left\langle a_k, b_k, S_k^{(i)} \right\rangle\right)_{k \in K}$, and d_i be S_i-generalized distance defined on set X for $i = 1, ..., n$, finite index set K. If $F = S_M$ and $S_k \geq S_k^{(i)}$, $k \in K$, then F aggregates $d_1, ..., d_n$ into an S-generalized distance d.*

Proof. Both (i) and (ii) are straightforward in Definition 2.12. To prove that d is an S-generalized distance, we only need to prove that d satisfies (iii). It is enough to observe that $S_k \geq S_k^{(i)}$ implies $S \geq S_i$ for all $i \in 1, ..., n$. If

$$d(x,z) = S_M(d_1(x,z), ..., d_n(x,z)) = d_j(x,z),$$

then, we have

$$\begin{aligned} S(d(x,y), d(y,z)) &= S(F(d_1, ..., d_n)(x,y), F(d_1, ..., d_n)(y,z)) \\ &= S(S_M((d_1(x,y), ..., d_n(x,y)), S_M((d_1(y,z), ..., d_n(y,z))) \\ &\geq S(d_j(x,y), d_j(y,z)) \\ &\geq S_j(d_j(x,y), d_j(y,z)) \\ &\geq d_j(x,z) \\ &= S_M((d_1(x,z), ..., d_n(x,z)) \\ &= F(d_1, ..., d_n)(x,z) \\ &= d(x,z). \end{aligned} \tag{23}$$

This brings the proof to a conclusion. □

Remark 4.15. *Let $S = (\langle a_k, b_k, S_k \rangle)_{k \in K}$, $S_i = \left(\left\langle a_k, b_k, S_k^{(i)} \right\rangle\right)_{k \in K}$, and $d_1, ..., d_n$ be S_i-generalized distances defined on set X for $i = 1, ..., n$. K is a finite index set.*

(i) *If $F = S_M$ and $S_k < S_k^{(i)}$, $k \in K$, then $F(d_1, \ldots, d_n)$ may be not an S-generalized distance.*

(ii) *If $F(x_1, ..., x_n) = \sum_{i=1}^{n} w_i x_i$, $\sum_{i=1}^{n} w_i \leq 1$ and $S_k > S_k^{(i)}$, $k \in K$, then $F(d_1, \ldots, d_n)$ also may be not an S-generalized distance.*

Example 4.16. *Suppose $S = (\langle 0, 0.5, S_{k_1} \rangle, \langle 0.5, 1, S_{k_2} \rangle)$, $S_i = \left(\left\langle 0, 0.5, S_{k_1}^{(i)} \right\rangle, \left\langle 0.5, 1, S_{k_2}^{(i)} \right\rangle\right)$, and d_i is the S_i-generalized distance defined on set $X = \{x, y, z\}$, $i = 1, 2$. Let $S_{L^1}(x,y) = min(1, x + y + 2\sqrt{xy})$, $x, y \in [0,1]$. Indeed, S_{L^1} is a t-conorm which is φ-isomorphic to S_L, where $\varphi(x) = \sqrt{x}$, $x \in [0,1]$. Obviously, $S_{L^1} > S_L$.*

(i) *If $F = S_M$, $S_{k_1} = S_{k_2} = S_L$, $S_{k_1}^{(i)} = S_{k_2}^{(i)} = S_{L^1}$, $i = 1, 2$, and d_1, d_2 are defined as follows:*

$$d_1(x,y) = d_2(x,y) = 0.2, \ d_1(y,z) = d_2(y,z) = 0.2, \ d_1(x,z) = d_2(x,z) = 0.5,$$

for $x, y, z \in X$, then

$$S(F(d_1, d_2)(x,y), F(d_1, d_2)(y,z)) = 0.4 < 0.5 = F(d_1, d_2)(x,z).$$

So, $F(d_1, d_2)$ is not an S-generalized distance.

(ii) *If* $F(x,y) = 0.12x + 0.6y$, $S_{k_1} = S_{k_2} = S_{L^1}$, $S_{k_1}^{(i)} = S_{k_2}^{(i)} = S_L$, $i = 1, 2$, *and* d_1, d_2 *are defined as follows:*

$$d_1(x,y) = 0.6, \quad d_1(y,z) = 0.7, \quad d_1(x,z) = 0.8,$$

$$d_2(x,y) = 0.7, \quad d_2(y,z) = 0.7, \quad d_2(x,z) = 0.8,$$

for $x, y, z \in X$ *, then*

$$S\left(F\left(d_1, d_2\right)(x,y), F\left(d_1, d_2\right)(y,z)\right) = 0.504 \leq 0.576 = F\left(d_1, d_2\right)(x,z).$$

So, $F(d_1, d_2)$ *is not an* S*-generalized distance.*

5 Conclusions

All continuous t-conorms fall into one of the three categories: the maximum t-conorm, Archimedean t-conorms and ordinal sums. Many academics have investigated the corresponding S-generalized distance aggregation function when S is either the maximum or Archimedean. With respect to ordinal sums form of t-conorm, we have provided a first insight into the class of S-generalized distances. Both theoretically and practically, this is a crucial step. Then, the characterization of S-generalized distance aggregation function of continuous Archimedean forms is considered, and their isomorphism is analyzed, and finally the problem of S-generalized distances of ordinal sums is studied. As future work, we will discuss whether some concrete families of aggregation functions could be applied as S-generalized distances.

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