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Original paper

Diamond alpha differentiability of interval-valued functions and its applicability to interval differential equations on time scales

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Abstract

Modelling phenomena with interval differential equations (IDEs) is an effective way to consider the uncertainties that are unavoidable when collecting data. Similarly to the theory of ordinary differential equations, IDEs have been parallelly investigated with the interval difference equations from the beginning. These two branches can be regarded as one when unifying continuous and discrete solution domains. A conspicuous advantage when merging these areas is that the proof of several analogous properties in both theories need not be repeated. The paper provides a common and efficient tool for studying IDEs not only with continuous or discrete solution domains but also with more general ones. We propose the diamond-α derivative for interval-valued functions (IVFs) on time scales with respect to the generalized Hukuhara difference. Differently from most of the studies on the derivatives of functions on time scales, using the language of epsilon-delta, the novel concept is naturally studied according to the limit of IVFs on time scales as in classical mathematics. A particular class of IDEs on time scales is then considered with respect to the diamond-α derivative. Numerical problems are elaborated to illustrate the necessity and efficiency of the latter.

Keywords: Generalized Hukuhara difference, time scales, dynamic derivatives, interval differential equations.

1 Introduction

The theory of differential equations plays the most important role in describing or modelling natural phenomena. Although differential equations provide exact models of nature, the obtained models are established with uncertain or vague information caused by imprecise measurements or indeterminacy of events in almost practical situations. To be able to describe nature with uncertainties, mathematicians have extended the theory to, for instance, the theory of random (stochastic) differential equations, the theory of fuzzy differential equations or the theory of IDEs. In this article, we focus on the latter topic.

1.1 Foundation of interval differential equations

As with the original theory of differential equations, the central notion for investigating IDEs is the derivative of IVFs. In the beginning, most of the IDEs are studied with respect to Moore's interval arithmetics [16]. However, those investigations face a shortcoming in that the difference between two intervals can not be interpreted by the subtraction [22]. To overcome this problem, the H-difference was introduced by M. Hukuhara in [11]. This concept ensures that the difference between two equal intervals is zero, i.e., $C \ominus L = 0$ iff $C = L$, a necessary condition for defining a natural distance between two intervals. However, the H-difference $C \ominus L$ does not exist in cases where the length of C is less than that of L. This is the reason why L. Stefanini and B. Bede introduced in [23] an extension of the H-difference that

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is well-defined for any two intervals. The gH-difference has made a significant step forward in studies of derivatives of IVFs and fuzzy-valued functions (FVFs). Using derivatives defined via this concept provides us preferable tools for investigating IDEs or further (fractional) fuzzy differential equations, see [2, 10, 13, 25].

1.2 Relevant literature review on time scale calculus with interval settings

A time scale is simply defined as a nonempty closed subset of the set $\mathbb{R}$ of real numbers. It is formed based on the backward and forward jump operators representing the steps from a point in it to its left and right closest points, respectively. Stefan Hilger first introduced this structure in his thesis in 1988. Time scale calculus is an extension of mathematical calculus where analytic concepts are investigated with respect to the jump operators. More precisely, all concepts in the calculus are split into the delta and nabla types corresponding to their formulation with respect to the forward or backward jump operators. On the other hand, time scale calculus can be considered as a unification of the discrete and continuous calculi (see in [5]). It provides a new perspective for establishing and studying real-world problems with general domains, not only restricted to intervals or discrete points but also extended to more general ones such as the sets of quantum numbers or the Cantor set. Notably, one can find several applications of the time scale theory in various fields of science, such as in pure and applied mathematics [6, 7], economics [8], dynamical systems [4] and so on.

Many properties of (interval) differential equations are hereditary to corresponding difference problems, while few results are entirely new in the latter. Studying (interval) differential equations on time scales embraces such discrepancies and helps avoid repeating analogous proofs twice. In this direction, very early on, S. Hong proposed in [9] the Hukuhara–Hilger derivative and studied the set-valued dynamic equations. Then, V. Lupulescu investigated in [15] the differentiability of IVFs on time scales with respect to the forward jump operator and the gH-difference. In recent investigations, huge works contributing to the time scale calculus with IVFs, or further, FVFs, have been conducted, see [12, 14, 24, 26, 27, 30] and elsewhere. These works focus on novel types of (partial) derivatives of IVFs as well as FVFs on time scales studied separately with respect to the jump operators and the applicability of them to IDEs or fuzzy differential equations.

1.3 Observations and contributions of the work

Studying IDEs on time scales focuses on the use of derivatives of IVFs on time scales for reformulating original continuous or discrete problems (ordinal differential (or difference) equations). To the best of our knowledge, the derivatives of interval or fuzzy-valued functions on time scales have been separately investigated with respect to the jump operators. These types of derivatives, in some cases (such as time scales with discrete points), can only describe the change of functions partially on the left or right side of considered points. With the desire to provide tools that can catch the change of functions on both sides of points in time scales, dynamic derivatives, which are combinations of the delta and nabla types, have been proposed in [20, 21] for the class of real-valued functions on time scales. Especially in [19], the authors directly defined a dynamic derivative of real-valued functions on time scales without referencing the delta and nabla derivatives. Motivated by this paper, we introduce a dynamic derivative called diamond-α derivative for IVFs on time scales. The goal is to provide a sophisticated tool for formulating IDEs that can model the uncertainties but be closer to the ideal statements of phenomena. The main contributions of the article can be briefly listed as follows.

- We introduce the limit of IVFs on time scales and consider its fundamental properties to apply to study the derivatives of IVFs on time scales.
- The diamond-α derivative of IVFs on time scales is introduced via the gH-difference and the proposed limit.
- Essential properties of the novel derivative are investigated clearly and systematically, highlighting the connection between it with the delta and nabla derivatives considered in [15].
- We demonstrate that the diamond-α derivative can be a better tool than the delta and nabla types for modelling IDEs on time scales. To do this, we consider the first-order ordinal linear problems of the following form:

$$u' = f(u, r), \quad r \in (a, b), \tag{1}$$

and its extensions to interval-valued problems on a time scale $\mathbb{T}$

$$D_{\mathbb{T}}(\mathscr{U}) = \mathscr{F}(\mathscr{U}, r), \quad r \in (a, b)_{\mathbb{T}}, \tag{2}$$

where $D_{\mathbb{T}}(\mathscr{U})$ denotes a derivative of type $D_{\mathbb{T}}$ of interval-valued function $\mathscr{U}$ on time scale $\mathbb{T}$. Considering (1) as an ideal model, we show that solutions of (2) when $D_{\mathbb{T}}$ is the diamond-α derivative can provide better approximations to the ideal solution (the solution of (1)) in comparisons with the cases where $D_{\mathbb{T}}$ is the delta or nabla derivatives.

1.4 Paper outline

Section 2 recalls essential points relating to time scale calculus and interval arithmetic. Section 3 introduces the novel limit of IVFs on time scales and its necessary properties for investigating the next section. In Section 4, we propose the diamond-α derivative of IVFs on time scales and elaborate on its properties. Section 5 is to demonstrate the outperformance of the proposed derivative in comparisons with the delta and nabla types in studies of IDEs on time scales. Finally, Section 6 is to conclude the main results of the paper and discuss about further research.

2 Preliminaries

In the sequel, let us denote by $\mathcal{I}$ the set of all nonempty closed intervals of $\mathbb{R}$, and $\bar{0}$ the set $\{0\}$.

2.1 Interval arithmetic

This subsection is to recall three arithmetic operations on intervals in $\mathcal{I}$, namely, Minkowski addition, scalar product, and generalized Hukuhara difference.

Definition 2.1. [1] *Let $C = [\underline{C}, \overline{C}], L = [\underline{L}, \overline{L}] \in \mathcal{I}$, and $r \in \mathbb{R}$. The addition $C \oplus L$ and scalar product $r \odot C$ are defined by*

$$C \oplus L = [\underline{C} + \underline{L}, \overline{C} + \overline{L}],$$

and

$$r \odot C = \begin{cases} [r\underline{C}, r\overline{C}], & r > 0 \\ \bar{0}, & r = 0 \\ [r\overline{C}, r\underline{C}], & r < 0, \end{cases}$$

respectively.

In the sequel, the scalar product $r \odot C$ will be denoted by rC whenever no confusion can appear.

Note that the equality $C \oplus (-1)C = \bar{0}$ is not true in general. To avoid this inefficiency, the Hukuhara difference of two intervals, say C and L, has been proposed in [11] as an interval $V = C \ominus L$ such that $C = L \oplus V$. With the Hukuhara difference, the equality $C \ominus C = \bar{0}$ is true for any $C \in \mathcal{I}$. However, the difference $C \ominus L$ does not exist for any $C, L \in \mathcal{I}$ (see in [17]). Therefore, it has been generalized as follows to overcome this situation.

Definition 2.2. [23] *Let $C, L, V \in \mathcal{I}$. The generalized Hukuhara difference of C and L (denoted by $C \ominus_{\mathrm{gH}} L$) is defined by*

$$C \ominus_{\mathrm{gH}} L = V \iff \begin{cases} \text{(I) } C = L \oplus V, \\ \text{(II) } L = C \oplus (-1)V. \end{cases} \tag{3}$$

Below, we provide essential properties of the generalized Hukuhara difference that need for our analysis.

Definition 2.3. *Let $C = [\underline{C}, \bar{C}] \in \mathcal{I}$. The length of C ($w(C)$, in abbreviation) is defined by $w(C) = \bar{C} - \underline{C}$.*

Proposition 2.4 (See [23]). *Let $C, L, V \in \mathcal{I}$ such that $C \ominus_{\mathrm{gH}} L = V$. Then,*

i) $C = L \oplus V$ *iff* $w(C) \geq w(L)$,

ii) $L = C \oplus (-1)V$ *iff* $w(C) < w(L)$.

Proposition 2.5. [23] *Let $C = [\underline{C}, \overline{C}]$ and $L = [\underline{L}, \overline{L}]$ be intervals in $\mathcal{I}$. Then,*

i) $C \ominus_{\mathrm{gH}} L = \left[\min\{\underline{C} - \underline{L}, \overline{C} - \overline{L}\}, \max\{\underline{C} - \underline{L}, \overline{C} - \overline{L}\}\right]$,

ii) $C \ominus_{\mathrm{gH}} L = [\underline{C} - \underline{L}, \overline{C} - \overline{L}]$, *if* $w(C) \geq w(L)$,

iii) $C \ominus_{\mathrm{gH}} L = [\overline{C} - \overline{L}, \underline{C} - \underline{L}]$, *if* $w(C) < w(L)$.

Proposition 2.6. [22] *Let $C, L \in \mathcal{I}$. Then,*

i) $C \ominus_{\mathrm{gH}} C = \bar{0}$, $C \ominus_{\mathrm{gH}} \bar{0} = C$, $\bar{0} \ominus_{\mathrm{gH}} C = -C$,

ii) $C \ominus_{\mathrm{gH}} L = (-L) \ominus_{\mathrm{gH}} (-C) = -(L \ominus_{\mathrm{gH}} C)$,

iii) $C \ominus_{\mathrm{gH}} (-L) = L \ominus_{\mathrm{gH}} (-C)$, $(-C) \ominus_{\mathrm{gH}} L = (-L) \ominus_{\mathrm{gH}} C$,

iv) $(C \oplus L) \ominus_{\mathrm{gH}} L = C$,

v) $(C \ominus_{\mathrm{gH}} L) \oplus L = C$ *if* $w(C) \geq w(L)$,

vi) $\lambda(C \ominus_{\mathrm{gH}} L) = \lambda C \ominus_{\mathrm{gH}} \lambda L$, *for any* $\lambda \in \mathbb{R}$.

Let $\mathfrak{D} : \mathcal{I} \times \mathcal{I} \to [0, \infty)$ be defined by

$$\mathfrak{D}(C, L) = \max\{|\underline{C} - \underline{L}|, |\overline{C} - \overline{L}|\}, \quad \text{for any } C, L \in \mathcal{I}.$$

This mapping is known as the Hausdorff distance in $\mathcal{I}$ and the metric space $(\mathcal{I}, \mathfrak{D})$ is complete, separable and locally compact [22, 23].

Proposition 2.7. [23] *Let* $C, L, V, N \in \mathcal{I}$ *and* $\lambda \in \mathbb{R}$. *Then, the following assertions are true.*

i) $\mathfrak{D}(C \oplus V, L \oplus V) = \mathfrak{D}(C, L)$,

ii) $\mathfrak{D}(\lambda C, \lambda L) = |\lambda| \mathfrak{D}(C, L)$,

iii) $\mathfrak{D}(C \oplus L, V \oplus N) \leq \mathfrak{D}(C, V) + \mathfrak{D}(L, N)$.

2.2 Time scale essentials

This subsection recalls fundamental concepts for defining derivatives of real-valued functions on time scales (see [5]).

A nonempty closed subset of $\mathbb{R}$ is defined as a *time scale on* $\mathbb{R}$. In the paper, it is denoted by $\mathbb{T}$ without additional explanations. Known that a time scale is introduced concerning two operators $\sigma(\cdot)$ and $\rho(\cdot)$, called the *forward* and *backward jump*, respectively. In particular,

$$\sigma(\bar{s}) = \inf\{r \in \mathbb{T} \mid r > \bar{s}\} \quad \text{and} \quad \rho(\bar{s}) = \sup\{r \in \mathbb{T} \mid r < \bar{s}\},$$

where we assume that $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$.

Definition 2.8. [5] *A point* $\bar{s} \in \mathbb{T}$ *is* right-scattered, left-scattered, right-dense, *and* left-dense *if* $\sigma(\bar{s}) > \bar{s}$, $\rho(\bar{s}) < \bar{s}$, $\sigma(\bar{s}) = \bar{s}$, *and* $\rho(\bar{s}) = \bar{s}$, *respectively.* $\mathbb{T}$ *is said to be* isolated *if all of its elements are both right-scattered and left-scattered.*

Denote by $[n, m]_{\mathbb{T}}$, $[n, m)_{\mathbb{T}}$, $(n, m)_{\mathbb{T}}$, and $(n, m]_{\mathbb{T}}$ intervals on $\mathbb{T}$, defined as the intersection of $[n, m]$, $[n, m)$, (n, m), and $(n, m]$ with $\mathbb{T}$, respectively.

Definition 2.9. [5] *Let* $\bar{s} \in \mathbb{T}$, *and* $\delta > 0$. *The interval* $(\bar{s} - \delta, \bar{s} + \delta)_{\mathbb{T}}$ *is called a neighborhood of* $\bar{s}$, *denoted by* $U_{\mathbb{T}}(\bar{s}, \delta)$. *In case* $\delta \geq \sigma(\bar{s}) - \bar{s}$ *and* $\delta \geq \bar{s} - \rho(\bar{s})$, *we call* $U_{\mathbb{T}}(\bar{s}, \delta)$ *a* proper neighborhood *of* $\bar{s}$.

Definition 2.10. *A sequence* $\{r_n\}_n \subset \mathbb{T}$ *converges to* $\bar{s} \in \mathbb{T}$, *denoted by* $r_n \to \bar{s}$, *if for every* $\epsilon > 0$, *there is an integer* $N > 0$ *such that* $r_n \in U_{\mathbb{T}}(\bar{s}, \epsilon)$ *for any* $n \geq N$.

Definition 2.11. *A sequence* $\{r_n\}_n \subset \mathbb{T}$ *converges to* $\bar{s}$ *from the left (or right) side, denoted by* $r_n \to \bar{s}^-$ *(or* $r_n \to \bar{s}^+$*) if, for every* $\epsilon > 0$, *there is an integer* $N > 0$ *such that* $r_n \leq \bar{s}$ *(or* $r_n \geq \bar{s}$*) and* $r_n \in U_{\mathbb{T}}(\bar{s}, \epsilon)$ *for any* $n \geq N$.

In the rest of this paper, let $\mathbb{T}^{\kappa}_{\kappa} = \mathbb{T}_{\kappa} \cap \mathbb{T}^{\kappa}$, where $\mathbb{T}^{\kappa} = \mathbb{T} \backslash \{\mathcal{E}\}$, if $\mathcal{E}$ is a left-scattered maximum of $\mathbb{T}$, otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$ and $\mathbb{T}_{\kappa} = \mathbb{T} \backslash \{e\}$, if e is a right-scattered minimum of $\mathbb{T}$, otherwise, $\mathbb{T}_{\kappa} = \mathbb{T}$. Moreover, for a given point $\bar{s} \in \mathbb{T}$, we define

$$\mu_{\bar{s}}(r) = \sigma(\bar{s}) - r \quad \text{and} \quad \eta_{\bar{s}}(r) = r - \rho(\bar{s}), \quad r \in \mathbb{T}.$$

Definition 2.12. [5] *Let* $\pi : \mathbb{T} \to \mathbb{R}$ *and* $\bar{s} \in \mathbb{T}^{\kappa}$. *The* delta derivative *of* π *at* $\bar{s}$, *if it exists, is a real number, denoted by* $\pi^{\Delta}(\bar{s})$, *satisfying that for every* $\epsilon > 0$, *there is* $\delta > 0$ *such that* $r \in U_{\mathbb{T}}(\bar{s}, \delta)$ *implies*

$$|\pi(\sigma(\bar{s})) - \pi(r) - \pi^{\Delta}(\bar{s})\mu_{\bar{s}}(r)| \leq \epsilon |\mu_{\bar{s}}(r)|.$$

Analogously, for $\bar{s} \in \mathbb{T}_{\kappa}$, *the* nabla derivative *of* π *at* $\bar{s}$, *if it exists, is a real number, denoted by* $\pi^{\nabla}(\bar{s})$, *satisfying that for every* $\epsilon > 0$, *there is* $\delta > 0$ *such that* $r \in U_{\mathbb{T}}(\bar{s}, \delta)$ *implies*

$$|\pi(r) - \pi(\rho(\bar{s})) - \pi^{\nabla}(\bar{s})\eta_{\bar{s}}(r)| \leq \epsilon |\eta_{\bar{s}}(r)|.$$

The function π *is delta (or nabla) differentiable at* $\bar{s}$ *if* $\pi^{\Delta}(\bar{s})$ *(or* $\pi^{\nabla}(\bar{s})$*) exists in* $\mathbb{R}$.

Definition 2.13. [19] *Let* $\pi : \mathbb{T} \to \mathbb{R}$, $\bar{s} \in \mathbb{T}^{\kappa}_{\kappa}$, *and* $\alpha \in [0,1]$. π *is said to be* diamond-α differentiable *at* $\bar{s}$ *if there is a real number, denoted by* $\pi^{\Diamond^{\alpha}}(\bar{s})$, *satisfying that for any* $\epsilon > 0$, *there is* $\delta > 0$ *such that* $r \in U_{\mathbb{T}}(\bar{s}, \delta)$ *implies*

$$|\alpha[\pi(\sigma(\bar{s})) - \pi(r)]\eta_{\bar{s}}(r) + (1-\alpha)[\pi(r) - \pi(\rho(\bar{s})]\mu_{\bar{s}}(r) - \pi^{\Diamond^{\alpha}}(\bar{s})\mu_{\bar{s}}(r)\eta_{\bar{s}}(r)| \leq \epsilon|\mu_{\bar{s}}(r)\eta_{\bar{s}}(r)|.$$

For further details on the diamond-α derivative of a real-valued function, we refer to [20, 21] and the references therein.

3 Limit of IVFs on time scales

Our principal goal in this section is to provide tools to define derivatives of IVFs on time scales naturally, as in classical mathematics. More precisely, we introduce the limit of IVFs on time scales and its related properties. This concept helps readers avoid using the language of (ϵ, δ), which can be complicated in some cases.

Definition 3.1. *Let* $\mathscr{F} : \mathbb{T} \to \mathcal{I}$ *and* $\bar{s} \in \mathbb{T}$. *An interval* $K \in \mathcal{I}$ *is the* $\mathbb{T}-$limit *of* $\mathscr{F}(r)$ *as* r *tends to* $\bar{s}$ *if, for every* $\{r_n\}_n \subset \mathbb{T}$, $r_n \neq \sigma(\bar{s}), r_n \neq \rho(\bar{s})$, *then* $r_n \to \bar{s}$ *implies* $\lim \mathfrak{D}(\mathscr{F}(r_n), K) = 0$. *Denote* $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}}} \mathscr{F}(r) = K$.

Note that the limit defined in Definition 3.1 extends the classical limit of real functions on $\mathbb{R}$. It differs from the limit of IVFs on time scales introduced by V. Lupulescu in [15].

Example 3.2. *Let* $\mathscr{F} : \mathbb{R} \to \mathcal{I}$ *be defined as follows:*

$$\mathscr{F}(r) = \begin{cases} [1,2], & r = 0, \\ \bar{0}, & r \neq 0. \end{cases}$$

Then, $\overset{\mathbb{T}}{\lim\limits_{r\to 0}} \mathscr{F}(r) = \bar{0}$ *but in the sense of* [15, Definition 6], *the limit of* $\mathscr{F}(r)$ *as* r *tends to* 0 *does not exist. Indeed, for any sequence* $\{r_n\}_n \subset \mathbb{R}$ *such that* $r_n \neq 0$, *and* $r_n \to 0$, *we see that* $\mathscr{F}(r_n) = \bar{0}$, *for any* $n \in \mathbb{N}^*$. *Therefore,* $\lim \mathfrak{D}(\mathscr{F}(r_n), \bar{0}) = \lim \mathfrak{D}(\bar{0}, \bar{0}) = 0$. *Namely,* $\overset{\mathbb{T}}{\lim\limits_{r\to 0}} \mathscr{F}(r) = \bar{0}$. *However, for any* $K \in \mathcal{I}$, *choose* ϵ *such that*

$$0 < \epsilon < \max\{\mathfrak{D}([1,2] \ominus_{gH} K, \bar{0}), \mathfrak{D}(\bar{0} \ominus_{gH} K, \bar{0})\},$$

then $\mathfrak{D}(\mathscr{F}(0) \ominus_{gH} K, \bar{0}) > \epsilon$, *in case* $K = \bar{0}$, *and in case* $K \neq \bar{0}$, $\mathfrak{D}(\mathscr{F}(r) \ominus_{gH} K, \bar{0}) > \epsilon$, *for any* $r \neq 0$. *Namely, for any* $\delta > 0$, *there exists* $r \in U_{\mathbb{T}}(0, \delta)$ *such that* $\mathfrak{D}(\mathscr{F}(r) \ominus_{gH} K, \bar{0}) > \epsilon$. *Therefore, the limit of* $\mathscr{F}(r)$ *as* r *tends to* 0 *in the sense of* [15, Definition 6] *does not exist.*

Theorem 3.3. *Let* $\mathscr{F} : \mathbb{T} \to \mathcal{I}$ *and* $\bar{s} \in \mathbb{T}$. *The limit of* $\mathscr{F}(r)$ *as* $r \to \bar{s}$, *if it exists, is unique.*

Proof. The proof is classical and directly follows from Definition 3.1 and the Hausdorff distance. □

Theorem 3.4. *Let* $\mathscr{F} : \mathbb{T} \to \mathcal{I}$, $\bar{s} \in \mathbb{T}$, *and* $I \in \mathcal{I}$. *Then,* $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}}} \mathscr{F}(r) = I$, *iff, for every* $\epsilon > 0$, *there is* $\delta > 0$ *such that* $\mathfrak{D}(\mathscr{F}(r), I) < \epsilon$, *for any* $r \in U_{\mathbb{T}}(\bar{s}, \delta)$, $r \neq \sigma(\bar{s})$, $r \neq \rho(\bar{s})$.

Proof. The proof is classical and based on the proof by contradiction. □

Theorem 3.5. *Let* $\mathscr{F} : \mathbb{T} \to \mathcal{I}$ *be an IVF such that* $\mathscr{F}(r) = [\pi(r), \theta(r)]$ *and* $\bar{s} \in \mathbb{T}$. *Then,* $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}}} \mathscr{F}(r)$ *exists, iff the limits* $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}}} \pi(r)$ *and* $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}}} \theta(r)$ *exist. Furthermore,*

$$\overset{\mathbb{T}}{\lim_{r\to\bar{s}}} \mathscr{F}(r) = \left[\overset{\mathbb{T}}{\lim_{r\to\bar{s}}} \pi(r), \overset{\mathbb{T}}{\lim_{r\to\bar{s}}} \theta(r)\right].$$

Proof. Let $\{r_n\}_n \subset \mathbb{T}$ be such that $r_n \neq \sigma(\bar{s}), r_n \neq \rho(\bar{s})$, and $r_n \to \bar{s}$. First, assume that $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}}} \mathscr{F}(r) = K = [\underline{K}, \overline{K}] \in \mathbb{R}$. It implies that $\lim \mathfrak{D}(\mathscr{F}(r_n), K) = 0$. It follows from the definition of Hausdorff distance that

$$\lim|\pi(r_n) - \underline{K}| = \lim|\theta(r_n) - \overline{K}| = 0.$$

Therefore, $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}}} \pi(r) = \underline{K}$ and $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}}} \theta(r) = \overline{K}$. The opposite implication is a direct consequence of the definition of Hausdorff distance. □

Definition 3.6. *Let $\mathscr{F}:\mathbb{T}\to\mathcal{I}$, $\bar{s}\in\mathbb{T}$, and $K\in\mathcal{I}$. Denote $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}^+}}\mathscr{F}(r)=K$ (or $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}^-}}\mathscr{F}(r)=K$), if for any $\{r_n\}_n\subset\mathbb{T}$ satisfying that $r_n\neq\sigma(\bar{s})$ and $r_n\to\bar{s}^+$ (or $r_n\neq\rho(\bar{s})$ and $r_n\to\bar{s}^-$), then $\lim\mathfrak{D}(\mathscr{F}(r_n),K)=0$.*

We obtain the following theorem as a direct consequence of Definitions 3.1 and 3.6.

Theorem 3.7. *Let $\mathscr{F}:\mathbb{T}\to\mathcal{I}$ and $\bar{s}\in\mathbb{T}$. It holds that*

i) *if $\bar{s}$ is a dense or isolated point, then*

$$\overset{\mathbb{T}}{\lim_{r\to\bar{s}}}\mathscr{F}(r)=K \textit{ iff } \overset{\mathbb{T}}{\lim_{r\to\bar{s}^-}}\mathscr{F}(r)=\overset{\mathbb{T}}{\lim_{r\to\bar{s}^+}}\mathscr{F}(r)=K.$$

ii) *if $\bar{s}$ is a left-scattered and right-dense point, then*

$$\overset{\mathbb{T}}{\lim_{r\to\bar{s}}}\mathscr{F}(r)=K \textit{ iff } \overset{\mathbb{T}}{\lim_{r\to\bar{s}^+}}\mathscr{F}(r)=K.$$

iii) *if $\bar{s}$ is a left-dense and right-scattered point, then*

$$\overset{\mathbb{T}}{\lim_{r\to\bar{s}}}\mathscr{F}(r)=K \textit{ iff } \overset{\mathbb{T}}{\lim_{r\to\bar{s}^-}}\mathscr{F}(r)=K.$$

Proof. i) Assume that $\bar{s}$ is an isolated point. It is clear that

$$\overset{\mathbb{T}}{\lim_{r\to\bar{s}}}\mathscr{F}(r)=K \text{ iff } \overset{\mathbb{T}}{\lim_{r\to\bar{s}^-}}\mathscr{F}(r)=\overset{\mathbb{T}}{\lim_{r\to\bar{s}^+}}\mathscr{F}(r)=K.$$

Furthermore, in this case, $K=\mathscr{F}(\bar{s})$. Let us assume that $\bar{s}$ is a dense point. If $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}}}\mathscr{F}(r)=K$, then it is trivial that $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}^-}}\mathscr{F}(r)=\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}^+}}\mathscr{F}(r)=K$. In the opposite implication, assume that $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}^-}}\mathscr{F}(r)=\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}^+}}\mathscr{F}(r)=K$, but $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}^-}}\mathscr{F}(r)\neq K$. Then, there exists $\{r_n\}_n\subset\mathbb{T}$, $r_n\neq\bar{s}$, satisfying that $r_n\to\bar{s}$ and $\lim\mathfrak{D}(\mathscr{F}(r_n),K)\neq 0$. As a result, there exists $\{r_{n_k}\}_k$ that is a subsequence of $\{r_n\}_n$ such that $r_{n_k}>\bar{s}$ or $r_{n_k}<\bar{s}$, $r_{n_k}\to\bar{s}$ and $\lim\mathfrak{D}(\mathscr{F}(r_{n_k}),K)\neq 0$. This is a contradiction with the assumption that $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}^-}}\mathscr{F}(r)=\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}^+}}\mathscr{F}(r)=K$.

ii) Suppose that $\bar{s}$ is left-scattered and right-dense. It is clear that

$$\overset{\mathbb{T}}{\lim_{r\to\bar{s}^+}}\mathscr{F}(r)=K \text{ if } \overset{\mathbb{T}}{\lim_{r\to\bar{s}}}\mathscr{F}(r)=K.$$

In other statement, suppose that $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}^+}}\mathscr{F}(r)=K$ and $\{r_n\}_n\subset\mathbb{T}$, $r_n\neq\bar{s}, r_n\neq\rho(\bar{s})$, be such that $r_n\to\bar{s}$. Since, $\bar{s}$ is a left-scattered point and $r_n\neq\bar{s}$, $\{r_n\}_n$ must tend to $\bar{s}$ from the right side. Consequently, $\lim\mathfrak{D}(\mathscr{F}(r_n),K)=0$. Therefore, we obtain that $\overset{\mathbb{T}}{\lim\limits_{r\to\bar{s}}}\mathscr{F}(r)=K$.

iii) The proof is similar to that of ii). □

4 Diamond-α derivative of IVFs on time scales

We propose the diamond-α derivative of IVFs on time scales according to the generalized Hukuhara difference. Let us assume $\alpha\in[0,1]$ if no additional explanation exists.

Definition 4.1. *Let $\mathscr{F}:\mathbb{T}\to\mathcal{I}$ and $\bar{s}\in\mathbb{T}^\kappa_\kappa$. The* generalized Hukuhara diamond-α derivative *(or $\Diamond^\alpha_{\rm gH}$-derivative, for short) of $\mathscr{F}$ at $\bar{s}$, if it exists, it is an interval, say $\mathscr{F}^{\Diamond^\alpha_{\rm gH}}(\bar{s})\in\mathcal{I}$, fulfilling that for every $\epsilon>0$, there is $\delta>0$ such that $r\in U_{\mathbb{T}}(\bar{s},\delta)$ implies*

$$\mathfrak{D}\Big(\alpha[\mathscr{F}(\sigma(\bar{s}))\ominus_{\rm gH}\mathscr{F}(r)]\eta_{\bar{s}}(r)\oplus(1-\alpha)[\mathscr{F}(r)\ominus_{\rm gH}\mathscr{F}(\rho(\bar{s}))]\mu_{\bar{s}}(r),\,\mathscr{F}^{\Diamond^\alpha_{\rm gH}}(\bar{s})\mu_{\bar{s}}(r)\eta_{\bar{s}}(r)\Big)\leq\epsilon|\mu_{\bar{s}}(r)\eta_{\bar{s}}(r)|. \tag{4}$$

If $\mathscr{F}$ has $\Diamond^\alpha_{\rm gH}$-derivative at $\bar{s}\in\mathbb{T}^\kappa_\kappa$, then we say that $\mathscr{F}$ is $\Diamond^\alpha_{\rm gH}$-differentiable *at $\bar{s}$.*

Remark 4.2. *The statement in (4) can be equivalently rewritten by*

$$\mathfrak{D}\left(\alpha\frac{\mathscr{F}(\sigma(\bar{s}))\ominus_{\mathrm{gH}}\mathscr{F}(r)}{\mu_{\bar{s}}(r)}\oplus(1-\alpha)\frac{\mathscr{F}(r)\ominus_{\mathrm{gH}}\mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(r)},\mathscr{F}^{\Diamond^{\alpha}_{\mathrm{gH}}}(\bar{s})\right)\leq\epsilon,$$

for all $r\in U_{\mathbb{T}}(\bar{s},\delta), r\neq\sigma(\bar{s})$, *and* $r\neq\rho(\bar{s})$. *Therefore, from Theorem 3.4, the* $\Diamond^{\alpha}_{\mathrm{gH}}$*-derivative* $\mathscr{F}^{\Diamond^{\alpha}_{\mathrm{gH}}}(\bar{s})$ *in Definition 4.1, if it exists, can be equivalently defined by*

$$\mathscr{F}^{\Diamond^{\alpha}_{\mathrm{gH}}}(\bar{s})=\overset{\mathbb{T}}{\lim_{r\to\bar{s}}}\left[\alpha\frac{\mathscr{F}(\sigma(\bar{s}))\ominus_{\mathrm{gH}}\mathscr{F}(r)}{\mu_{\bar{s}}(r)}\oplus(1-\alpha)\frac{\mathscr{F}(r)\ominus_{\mathrm{gH}}\mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(r)}\right].$$

We obtain the following theorem as a direct consequence of Remark 4.2 and Theorem 3.3.

Theorem 4.3. *Let* $\mathscr{F}:\mathbb{T}\to\mathcal{I}$ *and* $\bar{s}\in\mathbb{T}^{\kappa}_{\kappa}$. *The* $\Diamond^{\alpha}_{\mathrm{gH}}$*-derivative* $\mathscr{F}^{\Diamond^{\alpha}_{\mathrm{gH}}}(\bar{s})$, *if it exists, is unique.*

Remark 4.4. *The delta and nabla derivatives of real-valued functions on time scales have been extended to IVFs in* [15] *as follows.*

The generalized Hukuhara delta derivative *(or the* generalized Hukuhara nabla derivative*) of interval-valued function* $\mathscr{F}$ *at point* $\bar{s}\in\mathbb{T}^{\kappa}$ *(or* $\bar{s}\in\mathbb{T}_{\kappa}$*), if it exists, it is an interval* $\mathscr{F}^{\Delta_{\mathrm{gH}}}(\bar{s})$ *(or* $\mathscr{F}^{\nabla_{\mathrm{gH}}}(\bar{s})$*), fulfilling that for every* $\epsilon>0$, *there is* $\delta>0$, *such that* $r\in U_{\mathbb{T}}(\bar{s},\delta)$ *implies*

$$\mathfrak{D}\left(\mathscr{F}(\sigma(\bar{s}))\ominus_{\mathrm{gH}}\mathscr{F}(r),\mathscr{F}^{\Delta_{\mathrm{gH}}}(\bar{s})\mu_{\bar{s}}(r)\right)\leq\epsilon|\mu_{\bar{s}}(r)|,\tag{5}$$

$$\left(\mathit{or}\ \mathfrak{D}\left(\mathscr{F}(r)\ominus_{\mathrm{gH}}\mathscr{F}(\rho(\bar{s})),\mathscr{F}^{\nabla_{\mathrm{gH}}}(\bar{s})\eta_{\bar{s}}(r)\right)\leq\epsilon|\eta_{\bar{s}}(r)|\right).\tag{6}$$

For a glance, the $\Diamond^{\alpha}_{\mathrm{gH}}$-derivatives with $\alpha=1$ and $\alpha=0$ coincide with the extended delta and nabla types, respectively. However, this is not true in general. Indeed, let us consider time scale $\mathbb{T}=(-\infty,1]\cup\{2\}$ and a function $\mathscr{F}$ defined on $\mathbb{T}$ as follows:

$$\mathscr{F}(r)=\begin{cases}[1/2,1], & r<1,\\ [0,2], & r=1,\\ [0,1], & r=2.\end{cases}$$

We see that $r=1$ is a left-dense and right-scattered point and

$$\mathscr{F}^{\Diamond^{1}_{\mathrm{gH}}}(1)=\overset{\mathbb{T}}{\lim_{r\to1}}\frac{\mathscr{F}(\sigma(1))\ominus_{\mathrm{gH}}\mathscr{F}(r)}{\mu_1(r)}=\overset{\mathbb{T}}{\lim_{r\to1^-}}\frac{\mathscr{F}(2)\ominus_{\mathrm{gH}}\mathscr{F}(r)}{\mu_1(r)}=[-1/2,0].$$

However, this is not the delta derivative of $\mathscr{F}$ at $r=1$ since (5) is not satisfied with $\mathscr{F}^{\Delta_{\mathrm{gH}}}(1)=[-1/2,0]$. In fact, choose $\epsilon=1/4$, for any $\delta>0$, with $r=1\in U_{\mathbb{T}}(1,\delta)$, we have

$$\mathfrak{D}\left(\mathscr{F}(\sigma(1))\ominus_{\mathrm{gH}}\mathscr{F}(r),[-1/2,0]\mu_1(r)\right)=\mathfrak{D}\left([-1,0],[-1/2,0]\right)=\frac{1}{2}>\epsilon.$$

Remark 4.5. *As a straightforward consequence of (5) and (6), if interval-valued function* $\mathscr{F}$ *has the delta derivative* $\mathscr{F}^{\Delta_{\mathrm{gH}}}(\bar{s})$ *(or nabla derivative* $\mathscr{F}^{\nabla_{\mathrm{gH}}}(\bar{s})$*) at a point* $\bar{s}\in\mathbb{T}$, *then we obtain that*

$$\mathscr{F}^{\Delta_{\mathrm{gH}}}(\bar{s})=\overset{\mathbb{T}}{\lim_{r\to\bar{s}}}\frac{\mathscr{F}(\sigma(\bar{s}))\ominus_{\mathrm{gH}}\mathscr{F}(r)}{\mu_{\bar{s}}(r)}\ \left(\mathit{or}\ \mathscr{F}^{\nabla_{\mathrm{gH}}}(\bar{s})=\overset{\mathbb{T}}{\lim_{r\to\bar{s}}}\frac{\mathscr{F}(r)\ominus_{\mathrm{gH}}\mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(r)}\right).$$

Note that the opposite statement is not true in general.

Below, we provide essential properties of the diamond-α derivatives of IVFs on time scales and its relations to the delta and nabla types.

Theorem 4.6. *Let* $\mathscr{F}:\mathbb{T}\to\mathcal{I}$ *and a dense point* $\bar{s}\in\mathbb{T}^{\kappa}_{\kappa}$. *Then,* $\mathscr{F}$ *is* $\Diamond^{\alpha}_{\mathrm{gH}}$*-differentiable at* $\bar{s}$ *iff the following limit exists*

$$\overset{\mathbb{T}}{\lim_{r\to\bar{s}}}\frac{\mathscr{F}(\bar{s})\ominus_{\mathrm{gH}}\mathscr{F}(r)}{\bar{s}-r}.$$

Furthermore, if $\mathscr{F}$ *is* $\Diamond^{\alpha}_{\mathrm{gH}}$*-differentiable at* $\bar{s}$, *then*

$$\mathscr{F}^{\Diamond^{\alpha}_{\mathrm{gH}}}(\bar{s})=\overset{\mathbb{T}}{\lim_{r\to\bar{s}}}\frac{\mathscr{F}(\bar{s})\ominus_{\mathrm{gH}}\mathscr{F}(r)}{\bar{s}-r}.$$

Proof. Suppose that $\bar{s}$ is a dense point, namely, $\sigma(\bar{s}) = \rho(\bar{s}) = \bar{s}$. In this case, we obtain that

$$\alpha \frac{\mathscr{F}(\sigma(\bar{s})) \ominus_{\text{gH}} \mathscr{F}(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha) \frac{\mathscr{F}(r) \ominus_{\text{gH}} \mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(r)} = \alpha \frac{\mathscr{F}(\bar{s}) \ominus_{\text{gH}} \mathscr{F}(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha) \frac{\mathscr{F}(\bar{s}) \ominus_{\text{gH}} \mathscr{F}(r)}{\mu_{\bar{s}}(r)} = \frac{\mathscr{F}(\bar{s}) \ominus_{\text{gH}} \mathscr{F}(r)}{\mu_{\bar{s}}(r)}.$$

Therefore, from Remark 4.2, we obtain that $\mathscr{F}$ is $\Diamond^{\alpha}_{\text{gH}}$-differentiable at $\bar{s}$ if and only if the following limit exists,

$$\overset{\mathbb{T}}{\lim_{r \to \bar{s}}} \frac{\mathscr{F}(\bar{s}) \ominus_{\text{gH}} \mathscr{F}(r)}{\bar{s} - r}.$$

Furthermore, if $\mathscr{F}$ is $\Diamond^{\alpha}_{\text{gH}}$-differentiable at $\bar{s}$, then

$$\mathscr{F}^{\Diamond^{\alpha}_{\text{gH}}}(\bar{s}) = \overset{\mathbb{T}}{\lim_{r \to \bar{s}}} \frac{\mathscr{F}(\bar{s}) \ominus_{\text{gH}} \mathscr{F}(r)}{\bar{s} - r}.$$

□

Theorem 4.7. *Let $\mathscr{F} : \mathbb{T} \to \mathcal{I}$, and let $\bar{s} \in \mathbb{T}^{\kappa}_{\kappa}$ be an isolated point. Then, $\mathscr{F}$ is $\Diamond^{\alpha}_{\text{gH}}$-differentiable at $\bar{s}$ and*

$$\mathscr{F}^{\Diamond^{\alpha}_{\text{gH}}}(\bar{s}) = \alpha \frac{\mathscr{F}(\sigma(\bar{s})) \ominus_{\text{gH}} \mathscr{F}(\bar{s})}{\mu_{\bar{s}}(\bar{s})} \oplus (1-\alpha) \frac{\mathscr{F}(\bar{s}) \ominus_{\text{gH}} \mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(\bar{s})}.$$

Proof. Since $\bar{s}$ is an isolated point, we obtain that

$$\overset{\mathbb{T}}{\lim_{r \to \bar{s}}} \left[\alpha \frac{\mathscr{F}(\sigma(\bar{s})) \ominus_{\text{gH}} \mathscr{F}(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha) \frac{\mathscr{F}(r) \ominus_{\text{gH}} \mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(r)} \right] = \alpha \frac{\mathscr{F}(\sigma(\bar{s})) \ominus_{\text{gH}} \mathscr{F}(\bar{s})}{\mu_{\bar{s}}(\bar{s})} \oplus (1-\alpha) \frac{\mathscr{F}(\bar{s}) \ominus_{\text{gH}} \mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(\bar{s})}.$$

From Remark 4.2, we obtain that $\mathscr{F}$ is $\Diamond^{\alpha}_{\text{gH}}$-differentiable at $\bar{s}$ and

$$\mathscr{F}^{\Diamond^{\alpha}_{\text{gH}}}(\bar{s}) = \alpha \frac{\mathscr{F}(\sigma(\bar{s})) \ominus_{\text{gH}} \mathscr{F}(\bar{s})}{\mu_{\bar{s}}(\bar{s})} \oplus (1-\alpha) \frac{\mathscr{F}(\bar{s}) \ominus_{\text{gH}} \mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(\bar{s})}.$$

□

Theorem 4.8. *Let $\mathscr{F} : \mathbb{T} \to \mathcal{I}$, and $\bar{s} \in \mathbb{T}^{\kappa}_{\kappa}$. Assume that $\mathscr{F}$ is Δ_{gH}- and ∇_{gH}-differentiable at $\bar{s}$*[1]*. Then $\mathscr{F}$ is $\Diamond^{\alpha}_{\text{gH}}$-differentiable at $\bar{s}$ and*

$$\mathscr{F}^{\Diamond^{\alpha}_{\text{gH}}}(\bar{s}) = \alpha \mathscr{F}^{\Delta_{\text{gH}}}(\bar{s}) \oplus (1-\alpha) \mathscr{F}^{\nabla_{\text{gH}}}(\bar{s}).$$

Proof. Let $\epsilon > 0$ be arbitrary. Since $\mathscr{F}$ is Δ_{gH}- and ∇_{gH}-differentiable at $\bar{s} \in \mathbb{T}^{\kappa}_{\kappa}$, there is $\delta > 0$ such that $r \in U_{\mathbb{T}}(\bar{s}, \delta)$ implies

$$\mathfrak{D}\left(\mathscr{F}(\sigma(\bar{s})) \ominus_{\text{gH}} \mathscr{F}(r), \mathscr{F}^{\Delta_{\text{gH}}}(\bar{s}) \mu_{\bar{s}}(r) \right) \leq \frac{\epsilon}{2} |\mu_{\bar{s}}(r)|,$$

and

$$\mathfrak{D}\left(\mathscr{F}(r) \ominus_{\text{gH}} \mathscr{F}(\rho(\bar{s})), \mathscr{F}^{\nabla_{\text{gH}}}(\bar{s}) \eta_{\bar{s}}(r) \right) \leq \frac{\epsilon}{2} |\eta_{\bar{s}}(r)|.$$

It follows that

$$\mathfrak{D}\left(\alpha[\mathscr{F}(\sigma(\bar{s})) \ominus_{\text{gH}} \mathscr{F}(r)] \eta_{\bar{s}}(r), \alpha \mathscr{F}^{\Delta_{\text{gH}}}(\bar{s}) \mu_{\bar{s}}(r) \eta_{\bar{s}}(r) \right) \leq \frac{\epsilon \alpha}{2} |\mu_{\bar{s}}(r) \eta_{\bar{s}}(r)|,$$

and

$$\mathfrak{D}\left((1-\alpha)[\mathscr{F}(r) \ominus_{\text{gH}} \mathscr{F}(\rho(\bar{s}))] \mu_{\bar{s}}(r), (1-\alpha) \mathscr{F}^{\nabla_{\text{gH}}}(\bar{s}) \mu_{\bar{s}}(r) \eta_{\bar{s}}(r) \right) \leq \frac{\epsilon(1-\alpha)}{2} |\mu_{\bar{s}}(r) \eta_{\bar{s}}(r)|.$$

We obtain

$$\begin{aligned} &\mathfrak{D}\left(\alpha[\mathscr{F}(\sigma(\bar{s})) \ominus_{\text{gH}} \mathscr{F}(r)] \eta_{\bar{s}}(r) \oplus (1-\alpha)[\mathscr{F}(r) \ominus_{\text{gH}} \mathscr{F}(\rho(\bar{s}))] \mu_{\bar{s}}(r),\ [\alpha \mathscr{F}^{\Delta_{\text{gH}}}(\bar{s}) \oplus (1-\alpha) \mathscr{F}^{\Delta_{\text{gH}}}(\bar{s})] \mu_{\bar{s}}(r) \eta_{\bar{s}}(r) \right) \\ &\leq \mathfrak{D}\left(\alpha[\mathscr{F}(\sigma(\bar{s})) \ominus_{\text{gH}} \mathscr{F}(r)] \eta_{\bar{s}}(r), \alpha \mathscr{F}^{\Delta_{\text{gH}}}(\bar{s}) \mu_{\bar{s}}(r) \eta_{\bar{s}}(r) \right) + \mathfrak{D}\left((1-\alpha)[\mathscr{F}(r) \ominus_{\text{gH}} \mathscr{F}(\rho(\bar{s}))] \mu_{\bar{s}}(r), (1-\alpha) \mathscr{F}^{\nabla_{\text{gH}}}(\bar{s}) \mu_{\bar{s}}(r) \eta_{\bar{s}}(r) \right) \\ &\leq \frac{\epsilon \alpha}{2} |\mu_{\bar{s}}(r) \eta_{\bar{s}}(r)| + \frac{\epsilon(1-\alpha)}{2} |\mu_{\bar{s}}(r) \eta_{\bar{s}}(r)| \\ &\leq \epsilon |\mu_{\bar{s}}(r) \eta_{\bar{s}}(r)|, \end{aligned}$$

for any $r \in U_{\mathbb{T}}(\bar{s}, \delta)$. Therefore, $\mathscr{F}$ is $\Diamond^{\alpha}_{\text{gH}}$-differentiable at $\bar{s}$ and $\mathscr{F}^{\Diamond^{\alpha}_{\text{gH}}}(\bar{s}) = \alpha \mathscr{F}^{\Delta_{\text{gH}}}(\bar{s}) \oplus (1-\alpha) \mathscr{F}^{\nabla_{\text{gH}}}(\bar{s})$. □

[1] $\mathscr{F}$ has the generalized Hukuhara delta and nabla derivatives at $\bar{s}$.

Theorem 4.9. *Let $\mathscr{F} : \mathbb{T} \to \mathcal{I}$ be represented by $\mathscr{F}(r) = [\pi(r), \theta(r)]$ for all $r \in \mathbb{T}$, and let $\bar{s} \in \mathbb{T}^{\kappa}_{\kappa}$. Assume that $w(\mathscr{F})$ is monotone on a proper neighborhood of $\bar{s}$ and $\mathscr{F}$ is $\Diamond^{\alpha}_{\mathrm{gH}}$-differentiable at $\bar{s}$. Then, π and θ are $\Diamond^{\alpha}$-differentiable at $\bar{s}$. Furthermore,*

i) *if $w(\mathscr{F})$ is increasing on a proper neighborhood of $\bar{s}$, then*

$$\mathscr{F}^{\Diamond^{\alpha}_{\mathrm{gH}}}(\bar{s}) = \left[\pi^{\Diamond^{\alpha}}(\bar{s}), \theta^{\Diamond^{\alpha}}(\bar{s})\right],$$

ii) *if $w(\mathscr{F})$ is decreasing on a proper neighborhood of $\bar{s}$, then*

$$\mathscr{F}^{\Diamond^{\alpha}_{\mathrm{gH}}}(\bar{s}) = \left[\theta^{\Diamond^{\alpha}}(\bar{s}), \pi^{\Diamond^{\alpha}}(\bar{s})\right].$$

Proof. Let V be a proper neighborhood of $\bar{s}$ and $w(\mathscr{F})$ be monotone on V. *i)* We assume that $w(\mathscr{F})$ is increasing on V for any $r \in V$, $r \neq \sigma(\bar{s})$, and $r \neq \rho(\bar{s})$. Consider the following cases.

Case 1: $\rho(\bar{s}) < r < \sigma(\bar{s})$. It implies that $\eta_{\bar{s}}(r) > 0$ and $\mu_{\bar{s}}(r) > 0$. Take into account that $w(\mathscr{F})$ is increasing on V, we have $w(\mathscr{F}(\sigma(\bar{s}))) > w(\mathscr{F}(r))$ and $w(\mathscr{F}(r)) > w(\mathscr{F}(\rho(\bar{s})))$ for any $r \in V$. From Proposition 2.5, we obtain that

$$\begin{aligned}
&\alpha\frac{\mathscr{F}(\sigma(\bar{s})) \ominus_{\mathrm{gH}} \mathscr{F}(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha)\frac{\mathscr{F}(r) \ominus_{\mathrm{gH}} \mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(r)} \\
&= \frac{\alpha}{\mu_{\bar{s}}(r)}\left([\pi(\sigma(\bar{s})), \theta(\sigma(\bar{s}))] \ominus_{\mathrm{gH}} [\pi(r), \theta(r)]\right) \oplus \frac{1-\alpha}{\eta_{\bar{s}}(r)}\left([\pi(r), \theta(r)] \ominus_{\mathrm{gH}} [\pi(\rho(\bar{s})), \theta(\rho(\bar{s}))]\right) \\
&= \alpha\left[\frac{\pi(\sigma(\bar{s})) - \pi(r)}{\mu_{\bar{s}}(r)}, \frac{\theta(\sigma(\bar{s})) - \theta(r)}{\mu_{\bar{s}}(r)}\right] \oplus (1-\alpha)\left[\frac{\pi(r) - \pi(\rho(\bar{s}))}{\eta_{\bar{s}}(r)}, \frac{\theta(r) - \theta(\rho(\bar{s}))}{\eta_{\bar{s}}(r)}\right] \\
&= \left[\alpha\frac{\pi(\sigma(\bar{s})) - \pi(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha)\frac{\pi(r) - \pi(\rho(\bar{s}))}{\eta_{\bar{s}}(r)},\ \alpha\frac{\theta(\sigma(\bar{s})) - \theta(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha)\frac{\theta(r) - \theta(\rho(\bar{s}))}{\eta_{\bar{s}}(r)}\right].
\end{aligned}$$

Case 2: $r > \sigma(\bar{s})$. It implies that $\eta_{\bar{s}}(r) > 0$ and $\mu_{\bar{s}}(r) < 0$. Since $w(\mathscr{F})$ is increasing on V, we obtain that $w(\mathscr{F}(\sigma(\bar{s}))) < w(\mathscr{F}(r))$ and $w(\mathscr{F}(r)) > w(\mathscr{F}(\rho(\bar{s})))$ for any $r \in V$. Similarly, we have

$$\begin{aligned}
&\alpha\frac{\mathscr{F}(\sigma(\bar{s})) \ominus_{\mathrm{gH}} \mathscr{F}(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha)\frac{\mathscr{F}(r) \ominus_{\mathrm{gH}} \mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(r)} \\
&= \frac{\alpha}{\mu_{\bar{s}}(r)}[\theta(\sigma(\bar{s})) - \theta(r), \pi(\sigma(\bar{s})) - \pi(r)] \oplus \frac{1-\alpha}{\eta_{\bar{s}}(r)}[\pi(r) - \pi(\rho(\bar{s})), \theta(r) - \theta(\rho(\bar{s}))] \\
&= \left[\alpha\frac{\pi(\sigma(\bar{s})) - \pi(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha)\frac{\pi(r) - \pi(\rho(\bar{s}))}{\eta_{\bar{s}}(r)},\ \alpha\frac{\theta(\sigma(\bar{s})) - \theta(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha)\frac{\theta(r) - \theta(\rho(\bar{s}))}{\eta_{\bar{s}}(r)}\right].
\end{aligned}$$

Case 3: $r < \rho(\bar{s})$. It implies that $\eta_{\bar{s}}(r) < 0$ and $\mu_{\bar{s}}(r) > 0$. Since $w(\mathscr{F})$ is increasing on V, we obtain that $w(\mathscr{F}(\sigma(\bar{s}))) > w(\mathscr{F}(r))$ and $w(\mathscr{F}(r)) < w(\mathscr{F}(\rho(\bar{s})))$ for any $r \in V$. Similarly, we have

$$\begin{aligned}
&\alpha\frac{\mathscr{F}(\sigma(\bar{s})) \ominus_{\mathrm{gH}} \mathscr{F}(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha)\frac{\mathscr{F}(r) \ominus_{\mathrm{gH}} \mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(r)} \\
&= \frac{\alpha}{\mu_{\bar{s}}(r)}[\pi(\sigma(\bar{s})) - \pi(r), \theta(\sigma(\bar{s})) - \theta(r)] \oplus \frac{1-\alpha}{\eta_{\bar{s}}(r)}[\theta(r) - \theta(\rho(\bar{s})), \pi(r) - \pi(\rho(\bar{s}))] \\
&= \left[\alpha\frac{\pi(\sigma(\bar{s})) - \pi(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha)\frac{\pi(r) - \pi(\rho(\bar{s}))}{\eta_{\bar{s}}(r)},\ \alpha\frac{\theta(\sigma(\bar{s})) - \theta(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha)\frac{\theta(r) - \theta(\rho(\bar{s}))}{\eta_{\bar{s}}(r)}\right].
\end{aligned}$$

Furthermore, since $\mathscr{F}$ is $\Diamond^{\alpha}_{\mathrm{gH}}$-differentiable at $\bar{s}$, we derive

$$\overset{\mathbb{T}}{\lim_{r \to \bar{s}}}\left[\alpha\frac{\mathscr{F}(\sigma(\bar{s})) \ominus_{\mathrm{gH}} \mathscr{F}(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha)\frac{\mathscr{F}(r) \ominus_{\mathrm{gH}} \mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(r)}\right] = \mathscr{F}^{\Diamond^{\alpha}_{\mathrm{gH}}}(\bar{s}) \in \mathcal{I}.$$

It follows from Theorem 3.5 that

$$\overset{\mathbb{T}}{\lim_{r \to \bar{s}}}\left[\alpha\frac{\pi(\sigma(\bar{s})) - \pi(r)}{\mu_{\bar{s}}(r)} \oplus (1-\alpha)\frac{\pi(r) - \pi(\rho(\bar{s}))}{\eta_{\bar{s}}(r)}\right],$$

and

$$\lim_{r\to\bar{s}}^{\mathbb{T}}\left[\alpha\frac{\theta(\sigma(\bar{s}))-\theta(r)}{\mu_{\bar{s}}(r)}\oplus(1-\alpha)\frac{\theta(r)-\theta(\rho(\bar{s}))}{\eta_{\bar{s}}(r)}\right],$$

exist in $\mathbb{R}$. Thus, π and θ are $\Diamond^\alpha$-differentiable at $\bar{s}$. In addition, we have $\mathscr{F}^{\Diamond^\alpha_{\text{gH}}}(\bar{s})=[\pi^{\Diamond^\alpha}(\bar{s}),\theta^{\Diamond^\alpha}(\bar{s})]$. Similarly, we obtain *ii*). Namely, if $w(\mathscr{F})$ is decreasing on a proper neighborhood of $\bar{s}$, then π and θ are $\Diamond^\alpha$-differentiable at $\bar{s}$. Furthermore, we obtain that $\mathscr{F}^{\Diamond^\alpha_{\text{gH}}}(\bar{s})=[\theta^{\Diamond^\alpha}(\bar{s}),\pi^{\Diamond^\alpha}(\bar{s})]$. □

Definition 4.10. *Let $\mathscr{F}:\mathbb{T}\to\mathcal{I}$ be such that $\mathscr{F}(r)=[\pi(r),\theta(r)]$ for all $r\in\mathbb{T}$, and $\bar{s}\in\mathbb{T}^\kappa_\kappa$. Assume that $\mathscr{F}$ is $\Diamond^\alpha_{\text{gH}}$-differentiable at $\bar{s}$. Then, $\mathscr{F}$ is said to be*

i) $\Diamond^\alpha_{\text{gH},(\text{I})}$-differentiable *at $\bar{s}$ if*

$$\mathscr{F}^{\Diamond^\alpha_{\text{gH}}}(\bar{s})=[\pi^{\Diamond^\alpha}(\bar{s}),\theta^{\Diamond^\alpha}(\bar{s})],\tag{7}$$

ii) $\Diamond^\alpha_{\text{gH},(\text{II})}$-differentiable *at $\bar{s}$ if*

$$\mathscr{F}^{\Diamond^\alpha_{\text{gH}}}(\bar{s})=[\theta^{\Diamond^\alpha}(\bar{s}),\pi^{\Diamond^\alpha}(\bar{s})].\tag{8}$$

Example 4.11. *Consider time scale $\mathbb{T}=p\mathbb{Z}=\{pk\mid k\in\mathbb{Z}\}$, $p>0$, and $\mathscr{F}:\mathbb{T}\to\mathcal{I}$ be an IVF. By Theorem 4.7, $\mathscr{F}$ is $\Diamond^\alpha_{\text{gH}}$-differentiable on $\mathbb{T}$ and*

$$\mathscr{F}^{\Diamond^\alpha_{\text{gH}}}(\bar{s})=\alpha\frac{\mathscr{F}(\sigma(\bar{s}))\ominus_{\text{gH}}\mathscr{F}(\bar{s})}{\mu_{\bar{s}}(\bar{s})}\oplus(1-\alpha)\frac{\mathscr{F}(\bar{s})\ominus_{\text{gH}}\mathscr{F}(\rho(\bar{s}))}{\eta_{\bar{s}}(\bar{s})}\quad \text{for all } \bar{s}\in\mathbb{T}.$$

Choose $\alpha=0.5$ and assume that $w(\mathscr{F})$ is monotone. Then, we obtain

$$\begin{aligned}\mathscr{F}^{\Diamond^{0.5}_{\text{gH}}}(\bar{s})&=\frac{1}{2}\frac{\mathscr{F}(\bar{s}+p)\ominus_{\text{gH}}\mathscr{F}(\bar{s})}{p}\oplus\frac{1}{2}\frac{\mathscr{F}(\bar{s})\ominus_{\text{gH}}\mathscr{F}(\bar{s}-p)}{p}\\&=\frac{[\mathscr{F}(\bar{s}+p)\ominus_{\text{gH}}\mathscr{F}(\bar{s})]\oplus[\mathscr{F}(\bar{s})\ominus_{\text{gH}}\mathscr{F}(\bar{s}-p)]}{2p}\\&=\frac{1}{2p}\left[\mathscr{F}(\bar{s}+p)\ominus_{\text{gH}}\mathscr{F}(\bar{s}-p)\right].\end{aligned}$$

The $\Diamond^\alpha_{\text{gH}}$-derivative $\mathscr{F}^{\Diamond^{0.5}_{\text{gH}}}(\bar{s})$ in this case is known as the generalized p-symmetric difference *of $\mathscr{F}$ at $\bar{s}$ and denoted by $D^p_{\text{gH}}\mathscr{F}(\bar{s})$. In particular, let $\mathscr{F}:[0,\infty)_{p\mathbb{Z}}\to\mathcal{I}$ be defined by $\mathscr{F}(r)=[r^2,r^3+r]$ for all $r\in[0,\infty)_{p\mathbb{Z}}$. It is easy to see that $w(\mathscr{F}(r))=r^3-r^2+r$ is increasing on $[0,\infty)_{p\mathbb{Z}}$. Therefore, $\mathscr{F}$ is $\Diamond^\alpha_{\text{gH},(\text{I})}$-differentiable on $[0,\infty)_{p\mathbb{Z}}$. Furthermore, we obtain that*

$$\begin{aligned}\mathscr{F}^{\Diamond^{0.5}_{\text{gH}}}(r)&=D^p_{\text{gH}}\mathscr{F}(r)\\&=\left[\frac{1}{2p}((r+p)^2-(r-p)^2),\frac{1}{2p}((r+p)^3+(r+p)-(r-p)^3-(r-p))\right]\\&=[2r,3r^2+p^2+1].\end{aligned}$$

Example 4.12. *Let $\mathbb{T}=\left\{\sqrt{3n+2}\mid n\in\mathbb{N}\right\}$ and $F:\mathbb{T}\to\mathcal{I}$ be such that $\mathscr{F}(r)=\frac{1}{r^2}[1,2]$, for all $r\in\mathbb{T}$. For any $r=\sqrt{3n+2}\in\mathbb{T}$, for some $n\in\mathbb{N}$, from the definitions of forward and backward operators, one can see that*

$$\sigma(r)=\sqrt{r^2+3}\quad \text{for } n\in\mathbb{N},$$

and

$$\rho(r)=\sqrt{r^2-3}\quad \text{for } n\geq 2.$$

It means that every point $r=\sqrt{3n+2}$ is isolated for $n\in\mathbb{N}$, $n\geq 2$. According to Theorem 4.7, we obtain that $\mathscr{F}$ is $\Diamond^\alpha_{\text{gH}}$-differentiable at $r\in\mathbb{T}^\kappa_\kappa$. Moreover, $w(\mathscr{F}(r))=1/r^2$ is decreasing on $\mathbb{T}$. Therefore, we obtain that $\mathscr{F}$ is $\Diamond^\alpha_{\text{gH},(\text{II})}$-differentiable at $r\in\mathbb{T}^\kappa_\kappa$ and $\mathscr{F}^{\Diamond^\alpha_{\text{gH}}}(r)=[\theta^{\Diamond^\alpha}(r),\pi^{\Diamond^\alpha}(r)]$, where $\pi(r)=1/r^2$ and $\theta(r)=2/r^2$. We have

$$\begin{aligned}\pi^{\Diamond^\alpha}(r)&=\alpha\frac{\pi(\sqrt{r^2+3})-\pi(r)}{\sqrt{r^2+3}-r}+(1-\alpha)\frac{\pi(r)-\pi(\sqrt{r^2-3})}{r-\sqrt{r^2-3}}\\&=\frac{-3[\alpha(r^2-3)(r-\sqrt{r^2-3})+(1-\alpha)(r^2+3)(\sqrt{r^2+3}-r)]}{r^2(r^4-9)(\sqrt{r^2+3}-r)(r-\sqrt{r^2-3})},\end{aligned}$$

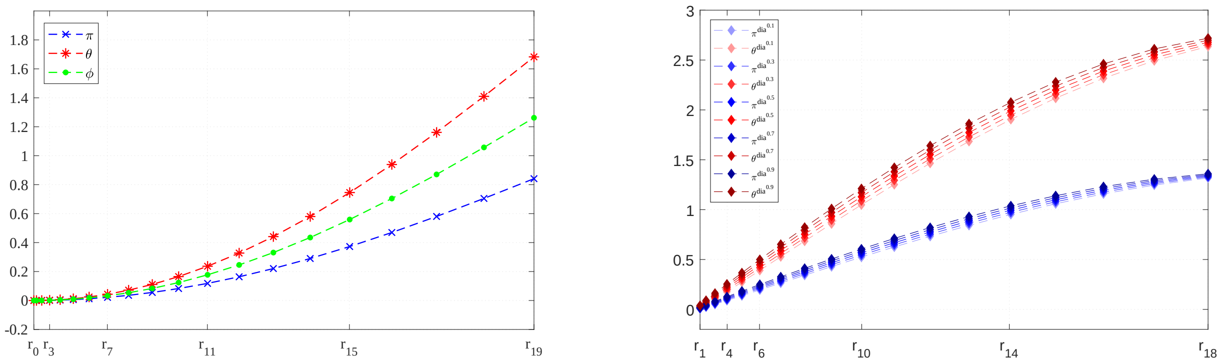
and $\theta^{\Diamond^\alpha}(r) = 2\pi^{\Diamond^\alpha}(r)$. *It follows that*

$$\mathscr{F}^{\Diamond^\alpha_{\text{gH}}}(r) = [2\pi^{\Diamond^\alpha}(r), \pi^{\Diamond^\alpha}(r)] = [-6,-3]\frac{[\alpha(r^2-3)(r-\sqrt{r^2-3}) + (1-\alpha)(r^2+3)(\sqrt{r^2+3}-r)]}{r^2(r^4-9)(\sqrt{r^2+3}-r)(r-\sqrt{r^2-3})}.$$

Example 4.13. *Let* $\{T_i\}_i$ *be a sequence of integers determined by* $T_0 = 0,\ T_i = T_{i-1} + i$, *for any* $i \geq 1$. *Choose* $N = 19$ *and let* $\mathbb{T}$ *be defined as follows:*

$$\mathbb{T} = \left\{ r_i \mid r_i = \frac{T_i}{T_N}, i = 0, \ldots, N \right\}.$$

Let $\mathscr{F} : \mathbb{T} \to \mathcal{I}$ *be defined by* $\mathscr{F}(r) = [\pi(r), \theta(r)] = [1,2]r\sin r$ *for any* $r \in \mathbb{T}$. *This function is considered as an extension of the function* $\phi(r) = \frac{3}{2}r\sin r = \frac{3}{2}[\pi(r) + \theta(r)]$ *restricted on* $\mathbb{T}$ *(see Figure 1a). Since* $\mathbb{T}$ *is a time scale of isolated points and* $w(\mathscr{F}) = |r\sin r|$ *is increasing on* $\mathbb{T}$, *it follows from Theorems 4.7 and 4.9 that* $\mathscr{F}$ *is* $\Diamond^\alpha_{\text{gH}}$*-differentiable on* $\mathbb{T}^\kappa_\kappa = \mathbb{T}\backslash\{r_0, r_N\}$ *and* $\mathscr{F}^{\Diamond^\alpha_{\text{gH}}}(r) = [\pi^{\Diamond^\alpha}(r), \theta^{\Diamond^\alpha}(r)]$. *The* $\Diamond^\alpha_{\text{gH}}$*-derivatives* $\mathscr{F}^{\Diamond^\alpha_{\text{gH}}}(r)$ *with different values of* α *are displayed in Figure 1b.*



(a) $\mathscr{F}(r)$ (b) $\mathscr{F}^{\Diamond^\alpha_{\text{gH}}}(r)$

Figure 1: Function $\mathscr{F}(r)$ and its diamond-α derivatives with various values of α.

5 IDEs on time scales with diamond-α derivatives

In this section, we consider IDEs of the following form

$$\begin{cases} D_{\mathbb{T}}\mathscr{U}(r) = \mathscr{F}(r, \mathscr{U}(r)), & r \in (a,b)_{\mathbb{T}} \subset \mathbb{T}^\kappa_\kappa, \\ \mathscr{U}(r_0) = \mathscr{U}(\sigma(r_0)) = \mathscr{U}^*, \end{cases} \tag{9}$$

where $D_{\mathbb{T}}\mathscr{U}$ is a derivative of $\mathscr{U}$ on $\mathbb{T}$, $\mathscr{F} : (a,b)_{\mathbb{T}} \times \mathcal{I} \to \mathcal{I}$ is represented by $\mathscr{F}(r,[\pi,\theta]) = [\underline{\mathscr{F}}(r,\pi,\theta), \overline{\mathscr{F}}(r,\pi,\theta)]$ where $\underline{\mathscr{F}}(r,\pi,\theta) = \zeta_1(r)\pi + \tau_1(r)\theta + \gamma_1(r)$, $\overline{\mathscr{F}}(r,\pi,\theta) = \zeta_2(r)\pi + \tau_2(r)\theta + \gamma_2(r)$, with $\zeta_i,\ \tau_i,\ \gamma_i,\ (i = 1,2)$, are real-valued functions such that $\underline{\mathscr{F}}(r,\pi,\theta) \leq \overline{\mathscr{F}}(r,\pi,\theta)$, $r_0 \in [a,b]_{\mathbb{T}}$, $\mathscr{U}^* = [\pi^*, \theta^*] \in \mathcal{I}$, and $\mathscr{U}(r)$ is an unknown IVF. It is known that this problem is an extension of the ordinary differential equation

$$\begin{cases} u'(r) = f(r, u(r)), & r \in (a,b), \\ u(r_0) = u^*, \end{cases} \tag{10}$$

with $f : (a,b) \times \mathbb{R} \to \mathbb{R}$ determined by $f(r,u) = \zeta(r)u + \gamma(r)$ with $\zeta,\ \gamma$ are real-valued functions on (a,b) and $u(r)$ is an unknown function. As mentioned in the introduction, the extension means that (10) is considered as a model of a phenomenon in an ideal environment while (9) is the model of the same phenomenon, however, with uncertainties that are unavoidable in real-world.

From the content of the paper, one can see that the derivative in Problem (9) can be one of the delta, nabla, or diamond-α types. In this section, we illustrate that the diamond-α derivative is the most efficient tool among the mentioned ones in formulating the IDEs as in (9). Namely, we numerically show that modelling the latter with the diamond-α derivative leads to solutions which are closer to the solutions of (10) (solutions of the same problem but with ideal assumptions). To illustrate the efficiency of the diamond-α derivative in formulating Problem (9), which is an extension of Problem (10), we consider Problem (9) with $D_{\mathbb{T}}\mathscr{U} \in \left\{\mathscr{U}^{\Delta_{\mathrm{gH}}}, \mathscr{U}^{\nabla_{\mathrm{gH}}}, \mathscr{U}^{\Diamond^{\alpha}_{\mathrm{gH}}}\right\}$. We assume that $\mathscr{U}(r) = [\pi(r), \theta(r)]$ for all $r \in (a,b)_{\mathbb{T}}$. Then, $\mathscr{F}(r, \mathscr{U}(r)) = [\underline{\mathscr{F}}(r, \pi(r), \theta(r)), \overline{\mathscr{F}}(r, \pi(r), \theta(r))]$, where $\underline{\mathscr{F}}, \overline{\mathscr{F}} : (a,b)_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Scenario 1. If $w(\mathscr{U})$ is increasing, then according to Theorem 4.9, we obtain $D_{\mathbb{T}}\mathscr{U}(r) = [D_{\mathbb{T}}\pi(r), D_{\mathbb{T}}\theta(r)]$, and therefore problem (9) can be expressed by the following problems

$$\begin{cases} D_{\mathbb{T}}\pi(r) = \underline{\mathscr{F}}(r, \pi(r), \theta(r)), & r \in (a,b)_{\mathbb{T}}, \\ D_{\mathbb{T}}\theta(r) = \overline{\mathscr{F}}(r, \pi(r), \theta(r)), & r \in (a,b)_{\mathbb{T}}, \\ \pi(r_0) = \pi(\sigma(r_0)) = \pi^*, \\ \theta(r_0) = \theta(\sigma(r_0)) = \theta^*. \end{cases} \tag{11}$$

Scenario 2. If $w(\mathscr{U})$ is decreasing, then according to Theorem 4.9, we obtain $D_{\mathbb{T}}\mathscr{U}(r) = [D_{\mathbb{T}}\theta(r), D_{\mathbb{T}}\pi(r)]$, and therefore problem (9) can be expressed by the following problems

$$\begin{cases} D_{\mathbb{T}}\theta(r) = \underline{\mathscr{F}}(r, \pi(r), \theta(r)), & r \in (a,b)_{\mathbb{T}}, \\ D_{\mathbb{T}}\pi(r) = \overline{\mathscr{F}}(r, \pi(r), \theta(r)), & r \in (a,b)_{\mathbb{T}}, \\ \pi(r_0) = \pi(\sigma(r_0)) = \pi^*, \\ \theta(r_0) = \theta(\sigma(r_0)) = \theta^*. \end{cases} \tag{12}$$

It is noticed that $\mathscr{U}(r) = [\pi(r), \theta(r)]$ is a solution to problem (9) if and only if $(\pi(r), \theta(r))$ is a solution to one of the problems (11) and (12). Furthermore, we restrict our consideration to an isolated time scale $\mathbb{T}$. Below, we provide a solution method to Problems (11) and (12).

Assume that $\mathbb{T} = \{r_0, r_1, \ldots, r_{N+1} \mid r_i < r_{i+1}, \forall i = \overline{0,N}\}$ with $\mathbb{T}^\kappa = \mathbb{T} \setminus \{r_{N+1}\}$, $\mathbb{T}_\kappa = \mathbb{T} \setminus \{r_0\}$, and $\mathbb{T}^\kappa_\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$. Since, $\mathbb{T}$ is an isolated time scale, according to Theorem 4.7, we obtain that

$$\begin{aligned} \pi^{\Delta}(r_i) &= \frac{\pi(r_{i+1}) - \pi(r_i)}{\mu_{r_i}(r_i)}, \quad i = \overline{1,N}, \\ \pi^{\nabla}(r_i) &= \frac{\pi(r_i) - \pi(r_{i-1})}{\eta_{r_i}(r_i)}, \quad i = \overline{1,N}, \\ \pi^{\Diamond^\alpha}(r_i) &= \alpha\frac{\pi(r_{i+1}) - \pi(r_i)}{\mu_{r_i}(r_i)} + (1-\alpha)\frac{\pi(r_i) - \pi(r_{i-1})}{\eta_{r_i}(r_i)}, \quad i = \overline{1,N}, \end{aligned}$$

which are linear combinations of $\pi(r_{i-1})$, $\pi(r_i)$, and $\pi(r_{i+1})$. Furthermore, we see that $\pi^{\Diamond^\alpha}(r_i)$ with $\alpha = 1$ and $\alpha = 0$ coincide with $\pi^{\Delta}(r_i)$ and $\pi^{\nabla}(r_i)$, respectively. Therefore, to formulate $D_{\mathbb{T}}\pi(r_i)$, it is sufficient to consider the $\pi^{\Diamond^\alpha}(r_i)$ only. Similarly, $D_{\mathbb{T}}\theta(r_i)$, $i = \overline{1,N}$, which are linear combinations of $\theta(r_{i-1})$, $\theta(r_i)$, and $\theta(r_{i+1})$, can be formulated by $\theta^{\Diamond^\alpha}(r_i)$. Moreover, from the assumptions on $\underline{\mathscr{F}}$ and $\overline{\mathscr{F}}$, we have

$$\underline{\mathscr{F}}(r_i, \pi(r_i), \theta(r_i)) = \zeta_1(r_i)\pi(r_i) + \tau_1(r_i)\theta(r_i) + \gamma_1(r_i),$$

and

$$\overline{\mathscr{F}}(r_i, \pi(r_i), \theta(r_i)) = \zeta_2(r_i)\pi(r_i) + \tau_2(r_i)\theta(r_i) + \gamma_2(r_i),$$

for any $i = \overline{1,N}$. Let us write for short $\pi_i := \pi(r_i)$ and $\theta_i := \theta(r_i)$ for all $i = \overline{0,N}$. Consequently, Problems (11) and (12) can be represented by

$$\begin{cases} \alpha\frac{\pi_{i+1}-\pi_i}{\mu_{r_i}(r_i)} + (1-\alpha)\frac{\pi_i - \pi_{i-1}}{\eta_{r_i}(r_i)} = \zeta_1(r_i)\pi_i + \tau_1(r_i)\theta_i + \gamma_1(r_i), & i = \overline{1,N}, \\ \alpha\frac{\theta_{i+1}-\theta_i}{\mu_{r_i}(r_i)} + (1-\alpha)\frac{\theta_i - \theta_{i-1}}{\eta_{r_i}(r_i)} = \zeta_2(r_i)\pi_i + \tau_2(r_i)\theta_i + \gamma_2(r_i), & i = \overline{1,N}, \\ \pi_0 = \pi_1 = \pi^*, \\ \theta_0 = \theta_1 = \theta^*, \end{cases} \tag{13}$$

and

$$\begin{cases} \alpha\frac{\theta_{i+1}-\theta_i}{\mu_{r_i}(r_i)} + (1-\alpha)\frac{\theta_i - \theta_{i-1}}{\eta_{r_i}(r_i)} = \zeta_1(r_i)\pi_i + \tau_1(r_i)\theta_i + \gamma_1(r_i), & i = \overline{1,N}, \\ \alpha\frac{\pi_{i+1}-\pi_i}{\mu_{r_i}(r_i)} + (1-\alpha)\frac{\pi_i - \pi_{i-1}}{\eta_{r_i}(r_i)} = \zeta_2(r_i)\pi_i + \tau_2(r_i)\theta_i + \gamma_2(r_i), & i = \overline{1,N}, \\ \pi_0 = \pi_1 = \pi^*, \\ \theta_0 = \theta_1 = \theta^*, \end{cases} \tag{14}$$

respectively. [2] It is easy to see that (13) and (14) are systems of linear equations of π_i and θ_i, $i = \overline{2, N}$. Solving them, we obtain solutions to Problems (11) and (12).

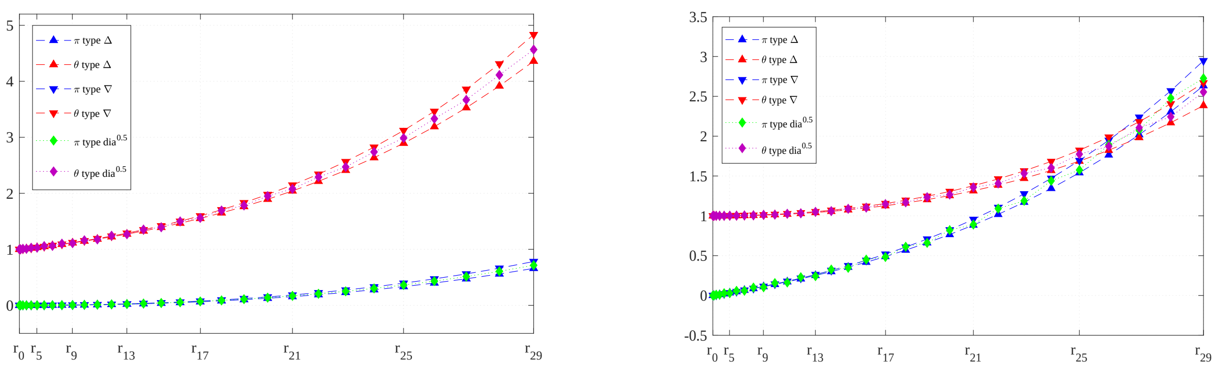
Example 5.1. *Consider the following problem*

$$\begin{cases} D_{\mathbb{T}}\mathscr{U}(r) = \mathscr{U}(r) \oplus [r, r^2 + 2r], & r \in \mathbb{T}^{\kappa}_{\kappa} \\ \mathscr{U}(r_0) = \mathscr{U}(\sigma(r_0)) = [0, 1], \end{cases} \tag{15}$$

where $\mathbb{T}$ is the time scale considered as in Example 4.13 with $N = 29$. This problem is an extension of the problem

$$\begin{cases} u'(r) = u(r) + \frac{1}{2}(r^2 + 3r), & r \in (0, 1) \\ u(0) = 1/2, \end{cases} \tag{16}$$

which has a unique solution $u(r) = 3e^r - \frac{1}{2}(r^2 + 5r + 5)$. Solving (15) with the solution method described above, we obtain solutions as in Figure 2 corresponding to the use of the delta, nabla, and diamond-α derivatives.



(a) Solutions obtained with (13)

(b) Solutions obtained with (14)

Figure 2: Solutions of Problem (15) when $D_{\mathbb{T}}\mathscr{U}$ is $\mathscr{U}^{\Delta_{\text{gH}}}$, $\mathscr{U}^{\nabla_{\text{gH}}}$, and $\mathscr{U}^{\Diamond^{0.5}_{\text{gH}}}$.

To evaluate the mentioned derivatives in formulating Problem (15), *we compare the obtained solutions with the solution of Problem* (16). *More precisely, assume that $\mathscr{U}(r) = [\pi(r), \theta(r)]$ is a solution of (15), we evaluate the approximation of $\frac{1}{2}[\pi(r) + \theta(r)]$ to $u(r)$ restricted on $\mathbb{T}$ (see Figure 3a) by RMSE (root mean square error) and relative error. These evaluations are provided in Table 1 and Figure 3b.*

From these approximations, it is clear that when using the diamond-α derivative in Problem (15), *one can obtain the best estimation of the ideal solution $u(r)$ restricted on $\mathbb{T}$. Namely, the proposed derivative is the best tool in comparison with the others for extending Problem* (16) *to an interval differential equation on a time scale as in* (15).

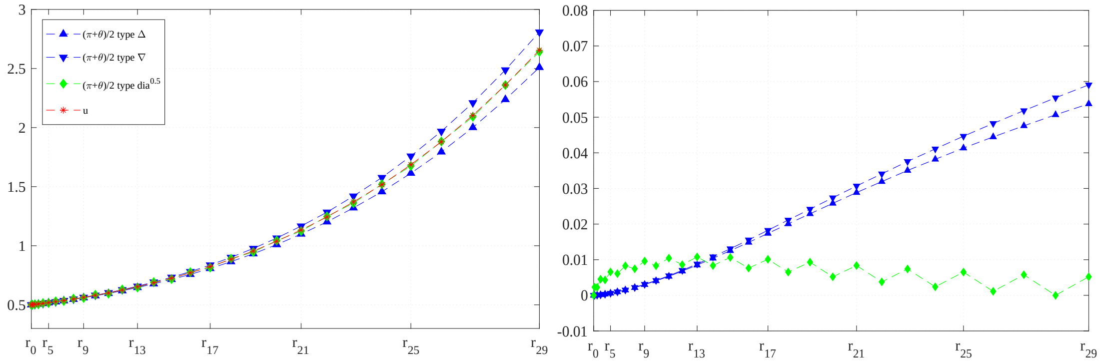
Table 1: Evaluation by RMSE the approximation of $u(r)$ on $\mathbb{T}$ by the solutions of Problem (15) with delta, nabla, and diamond-α derivatives.

Derivatives	Δ_{gH}	∇_{gH}	$\Diamond^{0.5}_{\text{gH}}$
RMSE	4.87×10^{-2}	5.0×10^{-2}	6.51×10^{-3}

6 Conclusions

We introduced the diamond-α derivative to IVFs on time scales via the generalized Hukuhara difference. Unlike most investigations on derivative types of functions on time scales, the proposed concept in this paper is naturally introduced based on the limit of IVFs. Consequently, essential properties of the derivative are verified using the latter's results. In

[2] Note that when formulating Problem (9) with the nabla derivative, since we fix the initial condition with the forward jump operator σ, systems (13) and (14) in this case are with $i = \overline{2, N}$.



(a) (b)

Figure 3: (a)-Approximations of $u(r)$ on $\mathbb{T}$ by the solutions of Problem (15) with delta, nabla, and diamond-α derivatives. (b)-Evaluation by relative error the approximation of $u(r)$ on $\mathbb{T}$ by the solutions of Problem (15) with delta (delta-dashed blue curve), nabla (nabla-dashed blue curve), and diamond-α (diamond-dashed green curve) derivatives.

addition, we consider its applications to formulating IDEs on time scales to answer why we need the novel derivative. We demonstrated that the diamond-α derivative is the best tool for this issue compared to the well-known derivatives, namely, the delta and nabla ones. This means that one can obtain from the solutions of IDEs establishing with the proposed derivative the best approximation of the solutions of the corresponding ODEs restricted on given time scales.

Recently, there have been many studies related to (fuzzy) interval differential-integral equations considering different types of domains, e.g., discrete, unbounded, or time scales domains, besides bounded intervals, as in classical studies [3, 18, 29, 28]. Because of the generality of time scale domains, these studies can theoretically be extended to time scales based on the calculus of (fuzzy-) interval-valued functions on the domains. For example, one can study the existence and stability of the solutions of (fuzzy) interval differential equations as in [28, 29]. However, also due to the generality, the properties achieved on time scales are still quite limited, and [18] is one of the few initial articles for this extension. Our work has not yet reached the studies of solution properties of (fuzzy) interval differential equations on time scales. However, by studying the diamond-α derivative to IVFs on time scales as naturally possible, the paper tries to simplify the way to deeper investigations on (fuzzy) interval differential equations on time scales. Moreover, the paper opens several research directions in the analysis of IVFs, focusing on integral types of IVFs and IDEs on time scales. For future works, we planned to elaborate on particular classes of IDEs with the diamond-α derivative at which the existence of solutions and the solution methods will be taken into account. Further, it is also interesting to study higher orders of the diamond-α derivative and its applications to the higher order of IDEs.

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