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Original paper

Applying the lexicographic maximum solution of min-product fuzzy relational inequalities for finding the optimal pricing with a fixed priority in a supply chain system

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Abstract

Fuzzy relational inequalities composed by the min-product operation are established to model the optimal pricing with fixed priority in a single product supply chain system. The solution algorithm has been proposed for solving such an optimization problem and finding the optimal solution (is called lexicographic maximum solution). In this study, a novel approach is proposed to finding the optimal pricing with fixed priority in a single product supply chain system. This approach is based on new properties of solution set in a min-product fuzzy relational inequality. These new properties allow us directly determine the optimal value of variable without many duplicate checks in the solution procedure. A numerical example is provided to illustrate the procedure.

Keywords: Fuzzy relational inequalities, min-product operation, lexicographic maximum solution, optimal pricing.

1 Introduction

In the literature, a system of fuzzy relational equations usually formulates in a matrix form as follows:

$$A \circ x^T = b^T,$$

where $A = [a_{ij}]_{m\times n}$, $x = (x_j)_{1\times n}$ and $b = (b_i)_{1\times m}$ are all defined over [0, 1]. The operation "$\circ$" represents a well-defined algebraic composition for matrix multiplication.

When the operation "$\circ$" represents max-min or max-product compositions to $A \circ x^T = b^T$, then fuzzy relational equations are respectively expressed as

$$\max\{\min_{j\in\mathcal{J}}\{a_{ij}, x_j\}\} = b_i, \forall\ i \in \mathcal{I} \text{ and } \max_{j\in\mathcal{J}}\{a_{ij}x_j\} = b_i, \forall\ i \in \mathcal{I},$$

where index sets $\mathcal{I} = \{1, 2, \cdots, m\}$ and $\mathcal{J} = \{1, 2, \cdots, n\}$.

Various methods have been proposed to explore the solution sets of fuzzy relational equations with max-min or max-product composition [21, 23, 24, 25, 30]. They are special cases of the max-triangular-norm (max-t-norm). It is well-known that the solution set of fuzzy relational equations with continuous max-t-norm composition can be

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completely determined by the maximum solution and a finite number of minimal solutions [4, 14, 28]. Finding all minimal solutions of fuzzy relational equations was found to be closely associated with the covering problem, which is NP-hard problem [15]. However, the solvability of fuzzy relational equations with max-t-norm composition has also been considered [2, 6, 22]. Lin et al. [18] showed that fuzzy relational equations with the max-product, the max-continuous Archimedean t-norm and the max-arithmetic mean compositions are essentially equivalent to the covering problem. These compositions are called max-continuous u-norm.

After the different fuzzy relational equations were proposed, Fang and Li [5] investigated the linear programming problem subject to a system of fuzzy relational equations as following form:

$$\begin{aligned} \text{Minimize} \qquad & Z(x) = \sum_{j=1}^{n} c_j x_j, \\ \text{subject to} \qquad & A \circ x^T = b^T, \end{aligned}$$

where $c_j \in R$ is the cost coefficient associated with variable x_j, operation "$\circ$" represents the max-product composition. Shivanian and Khorram [27] studied the optimization of a linear objective function subject to fuzzy relation inequality constraints with max-product composition. They proposed some simplification operations to accelerate the resolution of the problem. After then, various optimization problems subject to a system of unipolar fuzzy relational equations or inequalities with different compositions were proposed [8, 9, 10, 11, 12, 15, 26, 35].

A system of bipolar fuzzy relational equations with max-min composition proposed by Freson et al. [7] can be formulated in the matrix form as follows:

$$A^+ \circ x^T \vee A^- \circ \overline{x}^T = b^T,$$

where $x = (x_j)_{1\times n}$, $\overline{x} = (\overline{x}_j)_{1\times n}$, $A^+ = [a_{ij}^+]_{m\times n}$, $A^- = [a_{ij}^-]_{m\times n}$ and $b = (b_i)_{1\times m}$ are all defined over [0, 1]. The notation "$\vee$" denotes max operation and the operation "$\circ$" represents the max-min composition. $\overline{x}_j = 1 - x_j$ denotes the bipolar character.

The linear optimization problem subjected to bipolar fuzzy relational equations with max-min composition proposed by Freson et al. [7] can be formulated as follows:

$$\begin{aligned} \text{Minimize} \qquad & Z(x) = \sum_{j=1}^{n} c_j x_j, \\ \text{subject to} \qquad & A^+ \circ x^T \vee A^- \circ \overline{x}^T = b^T, \end{aligned}$$

where $c_j \in R$ is the cost coefficient associated with variable x_j, operation "$\circ$" represents the max-min composition. Since then, different optimization problems subject to bipolar fuzzy relational equations have been continuously proposed [1, 16, 20, 29].

As the above mentioned, most of fuzzy relational optimization problems had a constraint framework composed of different max-t-norm compositions, for example, max-min, max-product, max-Lukasiewicz, and max-Archimedean t-norm, etc. Recently, the constraint part of fuzzy relational optimization problems is no longer limited to the composition of max-t-norm, for example, addition-min and min-product compositions.

Li and Yang [17] first introduced a fuzzy relational inequality model with an addition-min composition to study data transmission in a BitTorrent-like peer-to-peer file-sharing system as follows:

$$\sum_{j=1}^{n} \min\{a_{ij}, x_j\} \geq b_i, \forall\, i \in \mathcal{I} \quad \text{or} \quad \sum_{j=1}^{n} (a_{ij} \wedge x_j) \geq b_i, \forall\, i \in \mathcal{I},$$

where a_{ij} and x_j are defined over [0, 1] for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$, $b = (b_i)_{i\in\mathcal{I}}$ is an m-dimensional vector with $b_i \in (0, \infty)$ and the term "$\wedge$" denotes "min" operator.

After the addition-min fuzzy relational inequalities were proposed, the fuzzy relational optimization problems involving different objective functions subject to a system of addition-min composition were successively proposed. Let $Z(x)$ denote a general function to reflect the managerial consideration (for instance, cost) associated with the transmission levels $x = (x_j)_{1\times n}$. The following optimization model could provide an optimal transmission levels with the function $Z(x)$.

$$\begin{aligned} \text{Minimize} \qquad & Z(x), \\ \text{subject to} \qquad & \sum_{j=1}^{n} \min\{a_{ij}, x_j\} \geq b_i, \forall\, i \in \mathcal{I}. \end{aligned}$$

Note that the objective function $Z(x)$ may be quite general to reflect the managerial consideration. The problem with a linear cost objective function $Z(x) = \sum_{j\in J} c_j x_j$ was studied by Yang [31], Guu and Wu [13], where $c_j > 0$ is the cost coefficients associated with variable x_j. To control the network congestion, Yang et al. [34] considered the min-max programming problem to minimize the objective function $Z(x) = \max\{x_1, x_2, \cdots, x_n\}$ subject to addition-min fuzzy relational inequalities, which one can view it as an indicator for congestion caused by the transmission x. Chiu et al. [3] proved that when the problem is feasible, there will always be an optimal solution with the same value for all variables. Based on this result, the min-max programming problem can be simplified into a single-variable optimization model.

Recently, Zhou et al. [36] first proposed the min-product fuzzy relational inequalities and used it to model the pricing relation in a supply chain system. Lin et al. [19] studied a maximin programming problem subject to the min-product fuzzy relational inequalities. An algorithm was proposed to obtain the optimal solution based on the quasi-maximal matrix and the corresponding index set. Yang [32] investigated the resolution method and structure of the complete solution set to min-product fuzzy relational inequalities. The concept of strong solution was further defined to represent a feasible pricing scheduling for all the suppliers and retailers. Based on the relevant results of solution set and strong solution, a detailed algorithm is also provided for finding all the strong solutions to a system of min-product fuzzy relational inequalities.

Let us consider that there are n suppliers denoted by $S_j, j \in \mathcal{J} = \{1, 2, \cdots, n\}$. They supply a single type of commodities to m retailers denoted by $R_i, i \in \mathcal{I} = \{1, 2, \cdots, m\}$. Assume that the commodity price from the jth supplier S_j to the retailers is $x_j > 0$, while its cost is $\underline{x}_j > 0, \forall j \in \mathcal{J}$. Basically, the supplying price x_j of S_j should be bigger than or equal to the cost price $\underline{x}_j$, i.e., $x_j \geq \underline{x}_j$. Considering the transportation expense and the profit of the retailer, the selling price of the commodity which is supplied by S_j and sold by R_i (at the ith market), denoted by p_{ij} (retail price of the commodity), should be bigger than the prime price x_j. Since $p_{ij} > x_j$, there exists some $a_{ij} > 1$ such that $p_{ij} = a_{ij} x_j, \forall\, i \in \mathcal{I}, j \in \mathcal{J}$.

In addition, in order to make the commodity available for sale in the ith market, the price of such commodity should be less than or equal to the highest acceptable price, denoted by $b_i > 0, \forall\, i \in \mathcal{I}$. Combining the price limitation of the commodity to make them sellable, it turns out to be

$$\min_{j\in\mathcal{J}}\{a_{ij}x_j\} = \min\{a_{i1}x_1, a_{i2}x_2, \cdots, a_{in}x_n\} \leq b_i, \forall\, i \in \mathcal{I}.$$

Furthermore, in order to ensure that all suppliers participate in the supply chain, the price of the commodity x_j offered by the supplier S_j needs to meet at least one market, i.e., $x_j \leq \overline{x}_j = \max_{i\in\mathcal{I}}\{\frac{b_i}{a_{ij}}\}$.

Since all the variables and parameters in the above system are bounded, they could be normalized into the unit interval [0, 1]. After normalization, the above system could be viewed as a system of min-product fuzzy relational inequalities as follows:

$$\begin{aligned} g_i(x) = \min_{j\in\mathcal{J}}\{a_{ij}x_j\} \leq b_i, \forall\, i \in \mathcal{I}, \\ \underline{x}_j \leq x_j \leq \overline{x}_j, \forall j \in \mathcal{J}, \end{aligned} \tag{1}$$

where a_{ij}, x_j, and b_i are defined over [0, 1] with $\overline{x}_j = \max_{i\in\mathcal{I}}\{\frac{b_i}{a_{ij}}\}$ and $a_{ij} \neq 0, a_{ij} \geq b_i$, for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$.

Each solution $x = (x_1, x_2, \cdots, x_n)$ of system (1) accurately represents a feasible pricing scheduling in the supply chain. The jth component x_j is the price on which the jth supplier S_j provides its local commodity to the retailers. The bigger the value of x_j is, the higher profit the supplier S_j will get. Consequently, in order to maximize the profit of the suppliers, we should maximize all the prices, $x_1, x_2, \cdots, x_n$.

In most cases, it is not possible to maximize all the prices simultaneously. Hence, consider maximizing $x_1, x_2, \cdots, x_n$ under some fixed priority grade, denoted by $x_1 \to x_2 \to \cdots \to x_n$, is a feasible option. Namely, the optimization objective with fixed priority grade considers that the first objective is to maximize x_1, and the second objective is to maximize x_2, and so on, until the last objective is to maximize x_n. Without loss of generality, we only discuss the optimal solution with the natural priority grade $x_1 \to x_2 \to \cdots \to x_n$. Such an optimal solution is called *the lexicographic maximum solution.*

Integrating the various factors involved in the above commodity pricing problems, the lexicographic maximum solution of the min-product fuzzy relational inequalities for finding optimal pricing with fixed priority in a single product supply chain system, which first proposed by Zhou et al. [36], can be expressed as follows:

$$\begin{aligned} \text{Maximize} \quad & x_1 \to x_2 \to \cdots \to x_n, \\ \text{subject to} \quad & g_i(x) = \min_{j\in\mathcal{J}}\{a_{ij}x_j\} \leq b_i, \forall\, i \in \mathcal{I}, \\ & \underline{x}_j \leq x_j \leq \overline{x}_j, \forall j \in \mathcal{J}, \end{aligned} \tag{2}$$

where a_{ij}, x_j, and b_i are defined over [0, 1] with $\overline{x}_j = \max_{i\in\mathcal{I}}\{\frac{b_i}{a_{ij}}\}$ and $a_{ij} \neq 0, a_{ij} \geq b_i$, for all $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Basically, $\underline{x}_j$ and $\overline{x}_j$ are regarded as the lower limit value and upper limit value of the variable x_j, respectively.

Zhou et al. [36] proposed an algorithm for solving the lexicographic maximum solution to problem (2). This algorithm first uses the lower limit value $\underline{x} = (\underline{x}_j)_{j\in\mathcal{J}}$ of x to check whether the problem (2) has a feasible solution, i.e., $g_i(\underline{x}) \leq b_i, \forall\ i \in \mathcal{I}$. When the problem has a feasible solution, the algorithm replaces the lower limit value of the variable with the possible maximum value 1, so as to check whether it is still a feasible solution of the problem (2) according to the priority grade of objective function. That is, a vector $y^1 = (1, \underline{x}_2, \cdots, \underline{x}_n)$ with the value $x_1 = 1$ is used to determine whether it is also a feasible solution in the first iteration of algorithm. If it is, then the optimal value of variable x_1 is obtained from the upper limit value $\overline{x}_1$. Otherwise, the algorithm finds the maximum value of x_1 as the optimal value from the remaining constraints that are not satisfied by y^1.

After finding the optimal value x_1^* for x_1, the solution procedure iterates x_1 using the variable x_2 to find the optimal value of x_2. Similar to the first iteration, the vector $y^2 = (x_1^*, 1, \underline{x}_3, \cdots, \underline{x}_n)$ with the value $x_2 = 1$ is used to check whether it is a feasible solution to the problem (2). In the next iterations, the proposed algorithm sequentially checks the vectors y^3, y^4 in turn, and so on until the last vector y^n to find the optimal value of all variables.

After analysis of the algorithm proposed by Zhou et al. [36], one can see that each time the solution of one variable is modified, all constraints in problem (2) must be checked again and again. As a result, some repeated inspection procedures are generated. Essentially, after determining the optimal value x_j^*, once $a_{ij}x_j^* \geq b_i$ holds, the ith constraint in problem (2) does not need be checked again because it has already been satisfied. In order to reduce the amount of inspection procedures to quickly obtain the optimal solution, we propose a new approach to finding the optimal pricing for problem (2) in this paper. This new approach is based on the concept that the properties of the solution set in the min-product fuzzy relational inequality can be used to directly determine the optimal value of variables without duplicate checks during the solution procedure.

This paper is organized as follows. In Section 2, some definitions and preliminary properties of the min-product fuzzy relational inequalities are presented. In Section 3, the new property and solution algorithm of the lexicographic maximum solution of problem (2) are introduced. In Section 4, a numerical example is provided to illustrate how the proposed algorithm finds the lexicographic maximum solution step by step. The conclusion is in Section 5.

2 Properties of min-product fuzzy relational inequalities

In this section, some definitions and existing preliminary properties of min-product fuzzy relational inequalities in system (1) are presented first. Then, we discuss some new theoretical properties of problem (2). On the basis of these new properties, the solution procedure of min-product fuzzy relational inequalities is proposed to find the lexicographic maximum solution to problem (2).

Definition 2.1. *Let $x = (x_j)_{j\in\mathcal{J}}$. The solution set of system (1) is denoted by*

$$X(A, b) := \{x \in [0, 1]^n | g_i(x) = \min_{j\in\mathcal{J}}\{a_{ij}x_j\} \leq b_i, \forall\ i \in \mathcal{I}, \underline{x}_j \leq x_j \leq \overline{x}_j, \forall j \in \mathcal{J}\}.$$

Follows from Definition 2.1, it is obvious that the system (1) is consistent, $X(A, b) \neq \emptyset$, if and only if $\underline{x} = (\underline{x}_j)_{1\times n}$ is one of its solution(s), i.e., it holds that $g_i(\underline{x}) = \min_{j\in\mathcal{J}}\{a_{ij}\underline{x}_j\} \leq b_i, \underline{x}_j \leq \overline{x}_j, \forall\ i \in \mathcal{I}, j \in \mathcal{J}$.

Definition 2.2. *Let $x^1 = (x_j^1)_{i\in\mathcal{J}}$ and $x^2 = (x_j^2)_{j\in\mathcal{J}}$ be two vectors. For any vector x^1 and x^2, $x^1 \leq x^2$ if and only if $x_j^1 \leq x_j^2$, for all $j \in \mathcal{J}$.*

Definition 2.3. *A solution $\hat{x} \in X(A, b)$ is the maximal solution if $x \leq \hat{x}$ indicates $x = \hat{x}$ for any $x \in X(A, b)$. A solution $\check{x} \in X(A, b)$ is a minimum solution if and only if $\check{x} \leq x$, for all $x \in X(A, b)$.*

When the system (1) is consistent, the lower limit value $\underline{x}$ is always its minimum solution, i.e., $\check{x} = \underline{x}$, according to Definition 2.3.

Note that $a_{ij} \neq 0$ and $a_{ij} \geq b_i$ for all $i \in \mathcal{I}, j \in \mathcal{J}$ in system (1). To further explore the properties of the solution set $X(A, b)$ of the min-product fuzzy relational inequality, the matrix $D = (d_{ij})_{m\times n}$ is defined as follows, where

$$d_{ij} = \begin{cases} \frac{b_i}{a_{ij}}, & \text{if } \underline{x}_j \leq \frac{b_i}{a_{ij}}, \\ \times, & \text{if } \underline{x}_j > \frac{b_i}{a_{ij}}. \end{cases} \tag{3}$$

Basically, the matrix D of (3) covers a lot of information about the properties of the solution set in system (1), including whether the solution set $X(A, b)$ is an empty set, which inequalities are satisfied by the variables, and which variables can meet the inequalities, and so on.

Theorem 2.4. [36] *System (1) is consistent if and only if there exists at least one element greater than* 0 *in each row and each column of matrix* D*.*

Theorem 2.4 can be used to check whether the min-product fuzzy relational inequality of (1) has a solution, as shown in Example 2.5 below.

Example 2.5. *Considering the optimal pricing problem of a supply chain system with four suppliers and four markets, the min-product fuzzy relational inequality of the system (1) is as follows:*

$$\begin{aligned}
&g_1(x) = \min\{0.8x_1, 0.6x_2, 0.9x_3, 0.7x_4\} \leq 0.4,\\
&g_2(x) = \min\{0.9x_1, 0.8x_2, 0.6x_3, 0.7x_4\} \leq 0.6,\\
&g_3(x) = \min\{0.8x_1, 0.7x_2, 0.5x_3, 0.6x_4\} \leq 0.45,\\
&g_4(x) = \min\{0.5x_1, 0.6x_2, 0.6x_3, 0.7x_4\} \leq 0.4,\\
&0.6 \leq x_1 \leq 0.8, \quad 0.7 \leq x_2 \leq 0.75, \quad 0.5 \leq x_3 \leq 1.0, \quad 0.6 \leq x_4 \leq 0.857.
\end{aligned}$$

The following matrix D *can be yielded by (3).*

$$D = (d_{ij})_{4\times 4} = \begin{bmatrix} \times & \times & \times & \times \\ \frac{2}{3} & 0.75 & 0.75 & \frac{6}{7} \\ \times & \times & 0.9 & 0.75 \\ 0.8 & \times & \frac{2}{3} & \times \end{bmatrix}. \tag{4}$$

Obviously, the first row of matrix D *in (4) has no element greater than* 0*, so that Example 2.5 is inconsistent according to Theorem 2.4. This result indicates that the product cost (lower limit value)* $\underline{x}_j, j = \{1,2,3,4\}$ *proposed by each supplier makes the market mechanism* $a_{1j}\underline{x}_j$ *higher than the highest acceptable price* b_1 *in the first market, i.e.,* $a_{1j}\underline{x}_j > b_1, j = \{1,2,3,4\}$.

Definition 2.6. [33] *Let* $x = (x_1, x_2, \cdots, x_n)$*,* $y = (y_1, y_2, \cdots, y_n)$ *and* $x, y \in X = [0,1]^n$*. We denote* $x \prec y$*, if there exists some* $j \in \mathcal{J}$ *such that* $x_k = y_k$ *for all* $k \leq j-1$ *and* $x_j < y_j$*. Moreover,* $x \preceq y$ *means either* $x \prec y$ *or* $x = y$ *holds.*

The total order "$\preceq$*" defined above is called the lexicographic order. The lexicographic order "*$\preceq$*" is equivalent to the following another statement by Definition 2.6.* $x \preceq y$ *if and only if it holds that*
(1). $x_1 \leq y_1$*; and*
(2). *If* $x_1 = y_1$*, then* $x_2 \leq y_2$*; and*
⋮
(n). *If* $x_1 = y_1, \cdots$ *and* $x_{n-1} = y_{n-1}$*, then* $x_n \leq y_n$*.*

Definition 2.7. *A solution* $x^* \in X(A,b)$ *is said to be a lexicographic maximum solution of system (1), if* $x \preceq x^*$ *holds, for all* $x \in X(A,b)$*.*

Theorem 2.8. [33]

(i) *The lexicographic maximum solution exists if and only if the system (1) is consistent.*

(ii) *If the system (1) has a lexicographic maximum solution, then the lexicographic maximum solution should be unique.*

(iii) *If the system (1) is consistent, then the lexicographic maximum solution is one of its maximal solution(s).*

Theorem 2.8 shows that the lexicographic maximum solution is one of the maximal solutions of system (1). Therefore, the unique lexicographic maximum solution in problem (2) can be obtained by pairwise comparison of maximal solutions. However, solving all the maximal solutions of system (1) is not easy work because it is a NP-hard problem. In the following, we will discuss some theoretical properties of the lexicographic maximum solution to problem (2). On the basis of these properties, the solution procedure of min-product fuzzy relational inequalities is proposed for finding the lexicographic maximum solution to problem (2), without using pairwise comparison of maximal solutions.

3 Resolution algorithm for the lexicographic maximum solution

Zhou et al. [36] proposed an algorithm for solving the lexicographic maximum solution of problem (2). However, the proposed procedure is necessary to check each variable one by one in the system of min-product fuzzy relational inequality. In other words, based on the objective function with fixed priority, *Maximize* $x_1 \to x_2 \to \cdots \to x_n$, in problem (2), their algorithm first maximizes the variable x_1 and then continues to maximize x_2. By analogy, the last variable x_n will eventually be maximized.

Essentially, it is not necessary to solve the problem (2) in such a sequential order because both the objective function and the constraint part of the min-product fuzzy relational inequality are not continuous functions. In addition, the matrix $D = (d_{ij})_{m\times n}$ in (3) already provides enough information for solving the lexicographic maximum solution of problem (2).

Definition 3.1. *For the matrix D in (3), let index sets $I_j(D) := \{i \in \mathcal{I} | d_{ij} > 0\}, j \in \mathcal{J}$ and $J_i(D) := \{j \in \mathcal{J} | d_{ij} > 0\}, i \in \mathcal{I}$. For a given value x_j, let $I_j(x_j, D) := \{i \in \mathcal{I} | d_{ij} > 0$ and $a_{ij}x_j \le b_i\}, j \in \mathcal{J}$.*

According to Definition 3.1, the relationship between the index sets $I_j(D)$ and $I_j(x_j, D)$ is obviously shown in Remark 3.2.

Remark 3.2. *For any $j \in \mathcal{J}$, $I_j(x_j, D) \subseteq I_j(D)$ holds to the matrix D in (3).*

Essentially, the index set $I_j(D)$ indicates those inequalities that can be satisfied by the variable x_j. The index set $I_j(x_j, D)$ shows those inequalities that can be satisfied by the variable x_j when the value of x_j is given. Echoing the result of Example 2.5, the matrix D contains $d_{21} = \frac{2}{3}$ and $d_{41} = 0.8$ in (4), i.e., $I_1(D) = \{2, 4\}$; it follows that the variable x_1 has the opportunity to be selected to satisfy the second and fourth inequalities. Moreover, $I_1(\frac{2}{3}, D) = \{2, 4\}$ denotes that the second and fourth inequalities are satisfied simultaneously when $x_1 = \frac{2}{3}$ is given, since there is

$$d_{21} = \frac{2}{3} > 0 \text{ and } a_{21}x_1 = 0.9 \times \frac{2}{3} = 0.6 \le b_2 = 0.6,$$
$$d_{41} = 0.8 > 0 \text{ and } a_{41}x_1 = 0.5 \times \frac{2}{3} = \frac{1}{3} \le b_4 = 0.4.$$

$I_1(0.8, D) = \{4\}$ means that only the fourth inequality is satisfied by $x_1 = 0.8$ since

$$d_{21} = \frac{2}{3} > 0 \text{ and } a_{21}x_1 = 0.9 \times 0.8 = 0.72 > b_2 = 0.6,$$
$$d_{41} = 0.8 > 0 \text{ and } a_{41}x_1 = 0.5 \times 0.8 = 0.4 \le b_4 = 0.4.$$

Clearly, according to Remark 3.2, $I_1(\frac{2}{3}, D) = \{2, 4\} = I_1(D)$ and $I_1(0.8, D) = \{4\} \subseteq I_1(D)$ hold.

The index set $J_i(D)$ defined in Definition 3.1 indicates the possible variables of x that may be selected to satisfy the ith inequality of problem (2). For the matrix D of Example 2.5 in (4), $J_3(D) = \{3, 4\}$ means that the variables x_3 and x_4 can be selected to satisfy the third inequality because $d_{33} = 0.9 > 0$ and $d_{34} = 0.75 > 0$. However, $J_1(D) = \emptyset$ implies that no variable can be selected to satisfy the first inequality because there doesn't exist $d_{1j} > 0, j \in \mathcal{J} = \{1, 2, 3, 4\}$. Therefore, the solution set of Example 2.5 is empty.

3.1 Some properties for the lexicographic maximum solution of problem (2)

These three indices $I_j(D), I_j(x_j, D)$ and $J_i(D)$, defined in Definition 3.1, contain some properties of finding the lexicographic maximum solution to problem (2).

Lemma 3.3. *If $I_j(D) \ne \emptyset$, for any $j \in \mathcal{J}$, then there exists $x_j^\sharp = \min_{i\in I_j(D)}\{d_{ij}\}$, such that $a_{ij}x_j^\sharp \le b_i, \forall\, i \in I_j(D) = I_j(x_j^\sharp, D)$ and $I_j(x_j, D) \subseteq I_j(x_j^\sharp, D)$ if $x_j^\sharp \le x_j$.*

Proof. For the matrix D in (3), there are

$$d_{ij} = \begin{cases} \frac{b_i}{a_{ij}}, & \text{if } \underline{x}_j \le \frac{b_i}{a_{ij}}, \\ \times, & \text{if } \underline{x}_j > \frac{b_i}{a_{ij}}, \end{cases}$$

and $I_j(D) = \{i \in \mathcal{I} | d_{ij} > 0\}, \forall j \in \mathcal{J}$. Since $x_j^\sharp = \min_{i\in I_j(D)}\{d_{ij}\}$, it leads to

$$a_{ij}x_j^\sharp = a_{ij} \min_{i\in I_j(D)}\{d_{ij}\} = a_{ij} \min_{i\in I_j(D)}\{\frac{b_i}{a_{ij}}\} \le a_{ij} \cdot \frac{b_i}{a_{ij}} = b_i, \ \forall\, i \in I_j(D).$$

Hence, we have $a_{ij}x_j^\sharp \le b_i, \forall\, i \in I_j(D) = I_j(x_j^\sharp, D)$.

In addition, $I_j(x_j, D) := \{i \in \mathcal{I} | d_{ij} > 0$ and $a_{ij}x_j \le b_i\}, \forall j \in \mathcal{J}$ is defined in Definition 3.1. If $x_j^\sharp \le x_j$, then it implies that $a_{ij}x_j^\sharp \le a_{ij}x_j \le b_i$, for any $i \in I_j(x_j, D)$. Therefore, we have $I_j(x_j, D) \subseteq I_j(x_j^\sharp, D)$, if $x_j^\sharp \le x_j$. □

Lemma 3.4. *Let $x^* = (x_j^*)_{j\in\mathcal{J}} \in X(A,b) \neq \emptyset$. If $r, s \in \mathcal{J}, r < s$ and $I_r(D) \subseteq I_s(D)$, then there exists $x_r^* = \overline{x}_r$ such that $x^* \in X(A,b)$.*

Proof. Since $x^* = (x_1^*, \cdots, x_r^*, \cdots, x_s^*, \cdots, x_n^*) \in X(A,b)$. It means that

$$\min_{j\in\mathcal{J}}\{a_{ij}x_j^*\} = \wedge_{j\in\mathcal{J}, j\neq r,s}\{a_{ij}x_j^*\} \wedge a_{ir}x_r^* \wedge a_{is}x_s^* \leq b_i, \forall\ i \in \mathcal{I} \text{ and } \underline{x}_j \leq x_j^* \leq \overline{x}_j, \forall j \in \mathcal{J}.$$

Obviously, it contains $\wedge_{j\in\mathcal{J}, j\neq r,s}\{a_{ij}x_j^*\} \wedge a_{ir}x_r^* \wedge a_{is}x_s^* \leq b_i, \forall\ i \in I_r(D)$. In this case, there are two scenarios where the relationship between $a_{ir}x_r^*$ and $a_{is}x_s^*$ needs to be discussed, as follows:

Case 1. $\wedge_{j\in\mathcal{J}, j\neq r,s}\{a_{ij}x_j^*\} \wedge a_{is}x_s^* \leq a_{ir}x_r^*, \forall\ i \in I_r(D)$.

Since $x_r^* \leq \overline{x}_r$, the following inequality holds

$$\begin{aligned} \wedge_{j\in\mathcal{J}}\{a_{ij}x_j^*\} &= (\wedge_{j\in\mathcal{J}, j\neq r,s}\{a_{ij}x_j^*\} \wedge a_{is}x_s^*) \wedge a_{ir}x_r^* \\ &= (\wedge_{j\in\mathcal{J}, j\neq r,s}\{a_{ij}x_j^*\} \wedge a_{is}x_s^*) \wedge a_{ir}\overline{x}_r \leq b_i, i \in I_r(D). \end{aligned}$$

Hence, there exists $(x_1^*, \cdots, \overline{x}_r, \cdots, x_s^*, \cdots, x_n^*) \in X(A,b)$.

Case 2. $\wedge_{j\in\mathcal{J}, j\neq r,s}\{a_{ij}x_j^*\} \wedge a_{is}x_s^* > a_{ir}x_r^*, \exists\ i \in I_r(D)$.

According to Lemma 3.3, since $I_s(D) \neq \emptyset, s \in \mathcal{J}$, then there exists $x_s^\sharp = \min_{i\in I_s(D)}\{d_{is}\}$, such that $a_{is}x_s^\sharp \leq b_i, \forall\ i \in I_s(D)$. In addition, since $I_r(D) \subseteq I_s(D)$, there is $a_{ir}\overline{x}_r \wedge a_{is}x_s^\sharp \leq b_i, \forall\ i \in I_r(D)$. This result implies that $(\wedge_{j\in\mathcal{J}, j\neq r,s}\{a_{ij}x_j^*\} \wedge a_{is}x_s^\sharp) \wedge a_{ir}\overline{x}_r \leq b_i, i \in I_r(D)$. Hence, there exists $(x_1^*, \cdots, \overline{x}_r, \cdots, x_s^\sharp, \cdots, x_n^*) \in X(A,b)$. □

Theorem 3.5. *Let $x^* = (x_j^*)_{j\in\mathcal{J}}$ be the lexicographic maximum solution of problem (2).*

(i) *If there exist $r, s \in \mathcal{J}$ such that $r < s$ and $I_r(D) \subseteq I_s(D)$, then $x_r^* = \overline{x}_r$ holds.*

(ii) *If there exists $r \in \mathcal{J}$ such that $I_r(D) \subseteq \cup_{s\in\mathcal{J}, s>r}I_s(D)$, then $x_r^* = \overline{x}_r$ holds.*

Proof. (i) Since $x^* = (x_j^*)_{j\in\mathcal{J}}$ is the lexicographic maximum solution of problem (2), if $x_r^* = \overline{x}_r$, then the proof is done. Suppose to the contrary that $x^* = (x_1^*, \cdots, x_{r-1}^*, x_r^*, \cdots, x_s^*, \cdots, x_n^*)$ is a lexicographic maximum solution of problem (2) but $x_r^* < \overline{x}_r$. Since there exists $r, s \in \mathcal{I}, r < s$ and $I_r(D) \subseteq I_s(D)$, we can find another solution containing $x_r^* = \overline{x}_r$ by Lemma 3.4, such as $y^* = (x_1^*, \cdots, x_{r-1}^*, \overline{x}_r, \cdots, x_s^*, \cdots, x_n^*)$ or $y^* = (x_1^*, \cdots, x_{r-1}^*, \overline{x}_r, \cdots, x_s^\sharp, \cdots, x_n^*)$. In this case, the result of $x^* \prec y^*$ is generated according to Definition 2.6. This result of assumption contradicts the fact that x^* is the lexicographic maximum solution of problem (2). Hence, $x_r^* = \overline{x}_r$ holds.

(ii) This is an extension of Theorem 3.5(i). Its proof is similar to the argument of Theorem 3.5(i).

□

Theorem 3.5(i) shows that when the set of inequalities satisfied by the preceding variable x_r is a subset of subsequent variable x_s, its value can be directly obtained by the upper bound for the lexicographic maximum solution of problem (2). Theorem 3.5(ii) is derived from Theorem 3.5(i). Therefore, Theorem 3.5(ii) can be extended to fit the union of more variables.

It is worth noting that if the property of Theorem 3.5 is satisfied in the process of finding the lexicographic maximum solution of problem (2), the value of x_r^* can be obtained directly from the upper bound $\overline{x}_r$ without checking any constraints. Moreover, the row corresponding to $x_r^* = \overline{x}_r$ in the matrix D can be deleted because the corresponding inequality is satisfied. The corresponding column of the matrix D associated with the variable x_r can also be deleted because the value of the variable x_r has been determined.

In order to explore the possible value of each variable of the lexicographic maximum solution $x^* = (x_j^*)_{j\in\mathcal{J}}$ in problem (2) with the priority grade $x_1 \to x_2 \to \cdots \to x_n$, we start from analyzing the relationship between the variable x_1 and the set $I_1(D)$ to get the relevant property and obtain the value of x_1^*. After determining the value of the variable x_1, the subsequent variable x_2 can inherit the relevant property because it has the highest priority, and so on.

For finding the value of each variable of the lexicographic maximum solution directly, we define the index sets

$$I_1^{(1)}(D) = \{i \in I_1(D) | i \notin \cup_{j\in\mathcal{J}-\{1\}}I_j(D)\} \text{ and } I_1^{(2)}(D) = \{i \in I_1(D) | i \in \cup_{j\in\mathcal{J}-\{1\}}I_j(D)\}.$$

Theorem 3.6. *Let $x^* = (x_j^*)_{j\in\mathcal{J}}$ be the lexicographic maximum solution of problem (2). In the matrix D of (3),*

(i) *if there exist $i \in I_1(D)$ such that $i \notin \cup_{j\in\mathcal{J}-\{1\}}I_j(D)$, then $x_1^* = \min_{i\in I_1(D)}\{d_{i1}\}$ holds.*

(ii) *if there exists $i \in I_1(D) = I_1^{(1)}(D) \cup I_1^{(2)}(D)$ where $I_1^{(1)}(D) \cap I_1^{(2)}(D) = \emptyset$, then $x_1^* = \min_{i\in I_1^{(1)}(D)}\{d_{i1}\}$ holds.*

Proof. (i) $i \in I_1(D)$ and $i \notin \cup_{j\in\mathcal{J}-\{1\}} I_j(D)$ are equivalent to $i \in I_1(D)$ and $J_i(D) = \{1\}$. This case indicates that the ith inequality, for all $i \in I_1(D)$, can be only satisfied by the variable x_1. Because $x^* = (x_j^*)_{j\in\mathcal{J}}$ is the lexicographic maximum solution of problem (2), there is $a_{i1}x_1^* \le b_i$ for the ith inequality such that $x_1^* \le \frac{b_i}{a_{i1}} = d_{i1}$, for all $i \in I_1(D)$. Therefore, it is concluded that $x_1^* = \min_{i\in I_1(D)}\{d_{i1}\}$.

(ii) Since $x^* = (x_j^*)_{j\in\mathcal{J}}$ is the lexicographic maximum solution in problem (2), we have

$$\min_{j\in\mathcal{J}}\{a_{ij}x_j^*\} = \wedge_{j\in\mathcal{J}}\{a_{ij}x_j^*\} \le b_i, \forall\, i \in \mathcal{I}.$$

Obviously, it contains the ith inequality for $i \in I_1(D) = I_1^{(1)}(D) \cup I_1^{(2)}(D)$ such that

$$a_{i1}x_1^* \le b_i, i \in I_1^{(1)}(D) = \{i \in I_1(D) | i \notin \cup_{j\in\mathcal{J}-\{1\}} I_j(D)\}, \tag{5}$$

and

$$a_{i1}x_1^* \wedge (\wedge_{j\in\mathcal{J}-\{1\}}\{a_{ij}x_j^*\}) \le b_i, i \in I_1^{(2)}(D) = \{i \in I_1(D) | i \in \cup_{j\in\mathcal{J}-\{1\}} I_j(D)\}. \tag{6}$$

According to the above Theorem 3.6(i), solving the value of the variable x_1^* in (5) can get $x_1^* = \min_{i\in I_1^{(1)}(D)}\{d_{i1}\}$. Moreover, according to Theorem 3.5(ii), there is $x_1^* = \overline{x}_1$ since $i \in I_1^{(2)}(D) \subseteq \cup_{j\in\mathcal{J}-\{1\}} I_j(D)$ in (6). Therefore, the value of the variable x_1^* for the ith inequality $i \in I_1(D)$ can be obtained by combining the results of (5) and (6) as follows:

$$x_1^* = \min\{\min_{i\in I_1^{(1)}(D)}\{d_{i1}\}, \overline{x}_1\} = \min_{i\in I_1^{(1)}(D)}\{d_{i1}\}.$$

□

Let us look closely at Theorem 3.6(i). Essentially, the cases of $i \in I_1(D)$ and $i \notin \cup_{j\in\mathcal{J}-\{1\}} I_j(D)$ also imply that $J_i(D) = \{1\}, i \in I_1(D)$ in the matrix D of (3). Let us concern on the number of inequalities that the variable x_1 can satisfy, which is the cardinality $n(I_1(D))$ of the set $I_1(D)$. The index set $I_1(D) = \{i\}$ with the cardinality $n(I_1(D)) = 1$ and $J_i(D) = \{1\}$ is one of the special cases of Theorem 3.6(i). This special case means that the ith inequality can only be satisfied by the variable x_1, and the variable x_1 only satisfies the ith inequality. The variable x_1 is a singleton. Hence, the value of the variable x_1^* can be directly assigned as $x_1^* = d_{i1} = \overline{x}_1$ in the process of finding the lexicographic maximum solution $x^* = (x_j^*)_{j\in\mathcal{J}}$ of problem (2). The ith inequality and variable x_1 do not need to be checked repeatedly, i.e., the ith row and the corresponding column of variable x_1 in the matrix D can be deleted.

Let us focus on the other cases of Theorem 3.6(i), the index set $i \in I_1(D)$ with the cardinality $n(I_1(D)) \ge 2$. Since $J_i(D) = \{1\}$ and $i \in I_1(D)$, we have $x_1^* = \min_{i\in I_1(D)}\{d_{i1}\}$ according to Theorem 3.6(i) such that $a_{i1}x_1^* \le b_i$, for all $i \in I_1(D)$. In other words, the ith inequality has been satisfied by x_1^*, for all $i \in I_1(D)$. Hence, after using Theorem 3.6(i) to find the lexicographic maximum solution of problem (2), the rows of $i \in I_1(D)$ and the corresponding column of the variable x_1 can be deleted from the matrix D.

Note that in the process of finding the lexicographic maximum solution of problem (2), if the variable x_1 meets Theorem 3.6(ii), the ith row, $i \in I_1^{(1)}(D)$, can deleted from the matrix D because the corresponding ith inequality of $i \in I_1^{(1)}(D)$ can be satisfied by $x_1^* = \min_{i\in I_1^{(1)}(D)}\{d_{i1}\}$, as shown in Theorem 3.6(i). For the same reason, the ith row, $i \in I_1^{(2)}(D)$ with $d_{i1} \ge \min_{i\in I_1^{(1)}(D)}\{d_{i1}\} = x_1^*$ can be also deleted from the matrix D. However, the ith row, $i \in I_1^{(2)}(D)$ with $d_{i1} < \min_{i\in I_1^{(1)}(D)}\{d_{i1}\} = x_1^*$ cannot be deleted from the matrix D because the corresponding ith inequality is not yet satisfied by x_1^*.

3.2 Solution algorithm of problem (2)

According to the properties of Theorems 3.5 and 3.6 above, a solution procedure for solving the min-product fuzzy relational inequalities is proposed to find the lexicographic maximum solution of problem (2). The solution procedure is summarized as the following algorithm.

Algorithm: for finding the lexicographic maximum solution of problem (2)

Step 1. Compute the matrix D according to (3), and generate the index sets $I_j(D)$ and $J_i(D), \forall\, i \in \mathcal{I}, j \in \mathcal{J}$ by Definition 3.1.

Step 2. Check the consistency of problem (2) by Theorem 2.4. If it is consistent, then go to Step 3. Otherwise, problem (2) is inconsistent and has no lexicographic maximum solution.

Step 3. Compute the upper limit value $\overline{x} = (\overline{x}_j)_{j\in\mathcal{J}}, \overline{x}_j = \max_{i\in I_j(D)}\{d_{ij}\}$ and let $x^* = (x_j^*)_{j\in\mathcal{J}}$.

Step 4. First apply Theorem 3.5, and then apply Theorem 3.6 to determine the values of variable x_j^*. Delete the corresponding column of variable x_j^* and rows which satisfied by x_j^* in the matrix D. Denote the remaining sub-matrix by D again and generate new index sets $I_j(D)$ and $J_i(D)$. Iterate Step 4 until no remaining rows are left in D. If there are undetermined variables, assign the upper limit value to them.

Step 5. Generate the lexicographic maximum solution of problem (2), $x^* = (x_j^*)_{j\in\mathcal{J}}$.

Let us concern the computational complexity of the proposed algorithm for finding the lexicographic maximum solution of problem (2). Problem (2) has n variables and m inequalities. Clearly, the most operations come from applying Theorems 3.5 and 3.6 in Step 4 to generate the value of each variable in the algorithm. The main operation comes from the comparison of elements in n sets, each set with m elements in the worst case. It requires nm^2 operations. Therefore, the computational complexity of the proposed algorithm is $O(nm^2)$.

Theorem 3.7. *The vector $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ obtained from the proposed algorithm is the lexicographic maximum solution of the problem (2).*

Proof. The problem (2) is consistent if and only if the cardinalities $n(I_j(D)) \geq 1$ and $n(J_i(D)) \geq 1$ for all $i \in \mathcal{I}, j \in \mathcal{J}$ by Theorem 2.4. The values of each variable in the vector x^* are fully generated in Step 4. The algorithm first applies Theorem 3.5, and then applies Theorem 3.6 to determine the value of variable. Since the property of Theorem 3.6 is based on the highest priority variable (x_1 at the beginning), the following proof shows how the algorithm can determine the value of the variable x_1 by applying Theorems 3.5 and 3.6.

Case 1. $n(I_1(D)) = 1$.

Subcase 1-1. If $\{i\} = I_1(D)$ and $i \in \cup_{j\in\mathcal{J}-\{1\}} I_j(D)$ exist, i.e., $I_1(D) \subseteq I_j(D), 1 < j$, then the algorithm generates $x_1^* = d_{i1} = \overline{x}_1$ by Theorem 3.5(i).

Subcase 1-2. If $\{i\} = I_1(D)$ and $i \notin \cup_{j\in\mathcal{J}-\{1\}} I_j(D)$ exist, i.e., x_1 is a singleton, then the algorithm generates $x_1^* = d_{i1} = \overline{x}_1$ by Theorem 3.6(i).

Case 2. $n(I_1(D)) \geq 2$.

Subcase 2-1. If $i \in I_1(D)$ and $i \in \cup_{j\in\mathcal{J}-\{1\}} I_j(D)$ exist, i.e., $I_1(D) \subseteq \cup_{j\in\mathcal{J}-\{1\}} I_j(D)$, then the algorithm generates $x_1^* = \overline{x}_1$ by Theorem 3.5(ii).

Subcase 2-2. If $i \in I_1(D)$ and all $i \notin \cup_{j\in\mathcal{J}-\{1\}} I_j(D)$ exist, then the algorithm generates $x_1^* = \min_{i\in I_1(D)}\{d_{i1}\}$ by Theorem 3.6(i).

Subcase 2-3. If $i \in I_1(D)$ and some $i \notin \cup_{j\in\mathcal{J}-\{1\}} I_j(D)$ exist, then there is

$$i \in I_1(D) = I_1^{(1)}(D) \cup I_1^{(2)}(D) \text{ with } I_1^{(1)}(D) \cap I_1^{(2)}(D) = \emptyset,$$

and

$$I_1^{(1)}(D) = \{i \in I_1(D) | i \notin \cup_{j\in\mathcal{J}-\{1\}} I_j(D)\}, I_1^{(2)}(D) = \{i \in I_1(D) | i \in \cup_{j\in\mathcal{J}-\{1\}} I_j(D)\}.$$

The algorithm generates $x_1^* = \min_{i\in I_1^{(1)}(D)}\{d_{i1}\}$ by Theorem 3.6(ii).

Essentially, the value of each variable x_j^*, for all $j \in \mathcal{J}$, can be fully generated in Step 4 by Theorems 3.5 and 3.6. Furthermore, Theorem 2.8(ii) shows that the lexicographic maximum solution is unique if the problem (2) is consistent. Hence, the vector $x^* = (x_1^*, x_2^*, \cdots, x_n^*)$ obtained from the proposed algorithm is the lexicographic maximum solution of the problem (2). □

3.3 A numerical example

A numerical example is provided to illustrate how the proposed algorithm finds the optimal pricing with fixed priority in the single product supply chain system of problem (2). For easy reference, the following example is taken from Zhou et al. [36]. This example is solved by the proposed algorithm that allows finding the lexicographic maximum solution of problem (2) quickly, without having to duplicate check the constraints.

Example 3.8. *Find the lexicographic maximum solution for the following problem.*

$$\begin{aligned} \text{Minimize} \quad & x_1 \to x_2 \to x_3 \to x_4 \to x_5 \to x_6 \to x_7 \to x_8 \\ \text{subject to} \quad & g_1(x) = \min\{0.7x_1, 0.85x_2, 0.81x_3, 0.8x_4, 0.95x_5, 0.9x_6, 0.7x_7, 0.8x_8\} \le 0.64, \\ & g_2(x) = \min\{0.85x_1, 0.8x_2, 0.75x_3, 0.85x_4, 0.95x_5, 0.9x_6, 0.95x_7, 0.85x_8\} \le 0.60, \\ & g_3(x) = \min\{0.7x_1, 0.7x_2, 0.75x_3, 0.85x_4, 0.9x_5, 0.82x_6, 0.88x_7, 0.8x_8\} \le 0.56, \\ & g_4(x) = \min\{0.9x_1, 0.8x_2, 0.76x_3, 0.8x_4, 0.8x_5, 0.9x_6, 0.95x_7, 0.85x_8\} \le 0.60, \\ & g_5(x) = \min\{0.8x_1, 0.65x_2, 0.75x_3, 0.8x_4, 0.85x_5, 0.82x_6, 0.9x_7, 0.74x_8\} \le 0.55, \\ & g_6(x) = \min\{0.7x_1, 0.85x_2, 0.8x_3, 0.9x_4, 0.9x_5, 0.75x_6, 0.75x_7, 0.7x_8\} \le 0.63, \\ & x = (x_1, x_2, \cdots, x_8) \ge (0.75, 0.8, 0.78, 0.72, 0.65, 0.7, 0.65, 0.75) = \underline{x}, \\ & x \le \overline{x}. \end{aligned}$$

Step 1. *Compute the matrix D according to (3), and generate the index sets $I_j(D)$ and $J_i(D), \forall\, i \in \mathcal{I} = \{1, 2, \cdots, 6\}, j \in \mathcal{J} = \{1, 2, \cdots, 8\}$ by Definition 3.1, as shown below:*

$$D = (d_{ij})_{6\times 8} = \begin{array}{c} \textit{variable} \to \\ g_1(x) \\ g_2(x) \\ g_3(x) \\ g_4(x) \\ g_5(x) \\ g_6(x) \end{array} \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \end{array} \\ \begin{bmatrix} 0.9143 & \times & 0.7901 & 0.8 & 0.6737 & 0.7111 & 0.9143 & 0.8 \\ \times & \times & 0.8 & \times & \times & \times & \times & \times \\ 0.8 & 0.8 & \times & \times & \times & \times & \times & \times \\ \times & \times & 0.7895 & 0.75 & 0.75 & \times & \times & \times \\ \times & 0.8462 & \times & \times & \times & \times & \times & \times \\ 0.9 & \times & 0.7875 & \times & 0.7 & 0.84 & 0.84 & 0.9 \end{bmatrix} \end{array}, \tag{7}$$

$$I_1(D) = \{1, 3, 6\}, I_2(D) = \{3, 5\}, I_3(D) = \{1, 2, 4, 6\}, I_4(D) = \{1, 4\},$$
$$I_5(D) = \{1, 4, 6\}, I_6(D) = \{1, 6\}, I_7(D) = \{1, 6\}, I_8(D) = \{1, 6\},$$

and

$$J_1(D) = \{1, 3, 4, 5, 6, 7, 8\}, J_2(D) = \{3\}, J_3(D) = \{1, 2\},$$
$$J_4(D) = \{3, 4, 5\}, J_5(D) = \{2\}, J_6(D) = \{1, 3, 5, 6, 7, 8\}.$$

Step 2. *Check the consistency of problem (2) by Theorem 2.4.*

Since the cardinalities $n(I_j(D)) \ge 1$ and $n(J_i(D)) \ge 1$ for all $i \in \mathcal{I} = \{1, 2, \cdots, 6\}, j \in \mathcal{J} = \{1, 2, \cdots, 8\}$, Example 3.8 is consistent. Go to Step 3.

Step 3. *Compute the upper limit value $\overline{x} = (\overline{x}_j)_{j\in\mathcal{J}}, \overline{x}_j = \max_{i\in I_j(D)}\{d_{ij}\}$ and let $x^* = (x_j^*)_{j\in\mathcal{J}}$.*

They are $\overline{x} = (0.9143, 0.8462, 0.8, 0.8, 0.75, 0.84, 0.9143, 0.9)$.

Step 4. *First apply Theorem 3.5, and then apply Theorem 3.6 to determine the values of variable x_j^*.*

For the current matrix D in (7), since

$$I_1(D) = \{1, 3, 6\} \subseteq I_2(D) \cup I_3(D) = \{1, 2, 3, 4, 5, 6\},$$

$$I_4(D) = \{1, 4\} \subseteq I_5(D) = \{1, 4, 6\}, \text{ and } I_6(D) = I_7(D) \subseteq I_8(D) = \{1, 6\},$$

according to Theorem 3.5, we have

$$x_1^* = \overline{x}_1 = 0.9143, x_4^* = \overline{x}_4 = 0.8, x_6^* = \overline{x}_6 = 0.84, \text{ and } x_7^* = \overline{x}_7 = 0.9143.$$

Columns 1, 4, 6 and 7, and the rows 1 and 6, i.e., rows of $g_1(x)$ and $g_6(x)$, corresponding to $x_1^* = \overline{x}_1, x_4^* = \overline{x}_4, x_6^* = \overline{x}_6$ and $x_7^* = \overline{x}_7$ can be deleted from matrix D. After deletion, the remaining matrix D becomes

$$D = \begin{array}{c} \textit{variable} \to \\ g_2(x) \\ g_3(x) \\ g_4(x) \\ g_5(x) \end{array} \begin{array}{c} \begin{array}{cccc} x_2 & x_3 & x_5 & x_8 \end{array} \\ \begin{bmatrix} \times & 0.8 & \times & \times \\ 0.8 & \times & \times & \times \\ \times & 0.7895 & 0.75 & \times \\ 0.8462 & \times & \times & \times \end{bmatrix} \end{array}, \tag{8}$$

and

$$I_2(D) = \{3, 5\}, I_3(D) = \{2, 4\}, I_5(D) = \{4\}, I_8(D) = \emptyset.$$

For the current matrix D in (8), since $I_2(D) = \{3, 5\}$ *and* $I_3(D) \cup I_5(D) \cup I_8(D) = \{2, 4\}$*, such that* $i \in I_2(D)$ *and all* $i \notin I_3(D) \cup I_5(D) \cup I_8(D)$ *hold, according to Theorem 3.6(i), we have*

$$x_2^* = \min_{i \in I_2(D)} \{d_{i2}\} = \min\{d_{32}, d_{52}\} = \min\{0.8, 0.8462\} = 0.8.$$

Moreover, $I_3(D) = \{2, 4\}, I_5(D) = \{4\}$ *and* $I_8(D) = \emptyset$ *such that* $2 \in I_3(D), 2 \notin I_5(D) \cup I_8(D)$*, i.e.,* $I_3^{(1)}(D) = \{2\}$*; and* $4 \in I_3(D), 4 \in I_5(D) \cup I_8(D)$*, i.e.,* $I_3^{(2)}(D) = \{4\}$ *hold, according to Theorem 3.6(ii), we have*

$$x_3^* = \min_{i \in I_3^{(1)}(D)} \{d_{i3}\} = \min\{d_{23}\} = \min\{0.8\} = 0.8.$$

Columns 2 and 3 of corresponding to x_2 *and* x_3*, and the rows of* $i \in I_2(D) = \{3, 5\}, i \in I_3^{(1)}(D) = \{2\}$*, i.e.,* $g_3(x), g_5(x)$ *and* $g_2(x)$*, can be deleted from matrix D. After deletion, the remaining matrix D becomes*

$$\begin{array}{ccc} & \textit{variable} \rightarrow & x_5 \quad x_8 \\ D = & g_4(x) & \begin{bmatrix} 0.75 & \times \end{bmatrix}, \end{array} \tag{9}$$

and

$$I_5(D) = \{4\}, I_8(D) = \emptyset.$$

For the current matrix D in (9), it is clear that x_5 *is a singleton since* $I_5(D) = \{4\}$ *and* $J_4(D) = \{5\}$*. According to Theorem 3.6(i), we have* $x_5^* = d_{45} = 0.75$*. The row of* $i \in I_5(D) = \{4\}$ *can be deleted from matrix D. After deletion, no remaining rows are left in matrix D. We have* $x_8^* = \overline{x}_8 = 0.9$*, since it is only undetermined variable.*

Step 5. *Generate the lexicographic maximum solution of Example 3.8,* $x^* = (x_j^*)_{j \in \mathcal{J}}$*.*

The proposed algorithm obtains the lexicographic maximum solution for Example 3.8 as

$$x^* = (0.9143, 0.8, 0.8, 0.8, 0.75, 0.84, 0.9143, 0.9).$$

4 Conclusions

In the literature, the min-product fuzzy relational inequalities has been proposed to model optimal pricing with fixed priority in a single product supply chain system. In order to maximize the price of commodities from suppliers with priority $x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_n$, the lexicographic maximum solution is utilized to find the optimal solution for such a supply chain system. We found that the feature matrix D obtained from the solution set of the problem contains enough information to directly find the lexicographic maximum solution of the problem. Based on this feature matrix D, we first present some new properties of the lexicographic maximum solution. These new properties then allow us to propose a new algorithm that determines the optimal solution for each variable quickly, without using it to check the feasibility of the problem. How the proposed algorithm can be used to quickly find the lexicographic maximum solution to the problem can be illustrated by numerical examples.

References

[1] S. Aliannezhadi, A. A. Molai, *Geometric programming with a single-term exponent subject to bipolar max-product fuzzy relation equation constraints*, Fuzzy Sets and Systems, **397** (2020), 61-83. https://doi.org/10.1016/j.fss.2019.08.012

[2] Z. L. Chai, *Solving fuzzy relation equations with max-continuous t-norm composition graphically*, 2015 12th International Conference on Fuzzy Systems and Knowledge Discovery (FSKD). https://doi.org/10.1109/fskd.2015.7381937

[3] Y. L. Chiu, S. M. Guu, J. Yu, Y. K. Wu, *A single-variable method for solving min-max programming problem with addition-min fuzzy relational inequalities*, Fuzzy Optimization and Decision Making, **18** (2019), 433-449. https://doi.org/10.1007/s10700-019-09305-9

[4] A. Di Nola, S. Sessa, W. Pedrycz, E. Sanchez, *Fuzzy relation equations and their applications in knowledge engineering*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1989. https://doi.org/10.1007/978-94-017-1650-5

[5] S. C. Fang, G. Li, *Solving fuzzy relation equations with a linear objective function*, Fuzzy Sets and Systems, **103** (1999), 107-113.https://doi.org/10.1016/S0165-0114(97)00184-X

[6] M. J. Fernández, P. Gil, *Some specific types of fuzzy relation equations*, Information Sciences, **164** (2004), 189-195. https://doi.org/10.1016/j.ins.2003.05.007

[7] S. Freson, B. De Baets, H. De Meyer, *Linear optimization with bipolar max-min constraints*, Information Sciences, **234** (2013), 3-15. https://doi.org/10.1016/j.ins.2011.06.009

[8] A. Ghodousian, *An algorithm for solving linear optimization problems subjected to the intersection of two fuzzy relational inequalities defined by Frank family of t-norms*, International Journal in Foundations of Computer Science and Technology, **8**(3) (2018), 1-20.

[9] A. Ghodousian, *Optimization of linear problems subjected to the intersection of two fuzzy relational inequalities defined by Dubois-Prade family of t-norms*, Information Sciences, **503** (2019), 291-306. https://doi.org/10.1016/j.ins.2019.06.058

[10] A. Ghodousian, E. Khorram, *Linear optimization with an arbitrary fuzzy relational inequality*, Fuzzy Sets and Systems, **206** (2012), 89-102. https://doi.org/10.1016/j.fss.2012.04.009

[11] A. Ghodousian, M. Raeisian Parvari, *A modfied PSO algorithm for linear optimization problem subject to the generalized fuzzy relational inequalities with fuzzy constraints (FRI-FC)*, Information Sciences, **418** (2017), 317-345. https://doi.org/10.1016/j.ins.2017.07.032

[12] F. F. Guo, L. P. Pang, D. Meng, Z. Q. Xia, *An algorithm for solving optimization problems with fuzzy relational inequality constraints*, Information Sciences, **252** (2013), 20-31. https://doi.org/10.1016/j.ins.2011.09.030

[13] S. M. Guu, Y. K. Wu, *A linear programming approach for minimizing a linear function subject to fuzzy relational inequalities with addition-min composition*, IEEE Transactions on Fuzzy Systems, **25**(4) (2017), 985-992. https://doi.org/10.1109/TFUZZ.2016.2593496

[14] G. J. Klir, B. Yuan, *Fuzzy sets and fuzzy logic theory and applications*, Prentice Hall PTR, Pearson Education, Inc., 1995.

[15] P. Li, S. C. Fang, *On the resolution and optimization of a system of fuzzy relational equations with sup-T composition*, Fuzzy Optimization and Decision Making, **7** (2008), 169-214. https://doi.org/10.1007/s10700-008-9029-y

[16] P. Li, Y. Liu, *Linear optimization with bipolar fuzzy relational equation constraints using the Łukasiewicz triangular norm*, Soft Computing, **18** (2014), 1399-1404. https://doi.org/10.1007/s00500-013-1152-1

[17] J. X. Li, S. J. Yang, *Fuzzy relation inequalities about the data transmission mechanism in Bit-Torrent-like Peer-to-Peer file sharing systems*, 2012 9th International Conference on Fuzzy systems and Knowledge Discovery (FSKD), (2012), 452-456. DOI:10.1109/fskd.2012.6233956. https://doi.org/10.1109/FSKD.2012.6233956

[18] J. L. Lin, Y. K. Wu, S. M. Guu, *On fuzzy relational equations and the covering problem*, Information Sciences, **181** (2011), 2951-2963. https://doi.org/10.1016/j.ins.2011.03.004

[19] H. T. Lin, X. B. Yang, H. M. Guo, C. F. Zheng, X. P. Yang, *Maximin optimization problem subject to min-product fuzzy relation inequalities with application in supply and demand scheme*, Complexity, (2019), 13 pages. https://doi.org/10.1155/2019/4960638

[20] C. C. Liu, Y. Y. Lur, Y. K. Wu, *Linear optimization of bipolar fuzzy relational equations with max-Łukasiewicz composition*, Information Sciences, **360** (2016), 149-162. https://doi.org/10.1016/j.ins.2016.04.041

[21] J. Loetamonphong, S. C. Fang, *An efficient solution procedure for fuzzy relation equations with max-product composition*, IEEE Transactions on Fuzzy Systems, **7**(4) (1999), 441-445. https://doi.org/10.1109/91.784204

[22] A. A. Molai, E. Khorram, *An algorithm for solving fuzzy relation equations with max-T composition operator*, Information Sciences, **178** (2008), 1293-1308. https://doi.org/10.1016/j.ins.2007.10.010

[23] K. Peeva, *Resolution of fuzzy relational equations-method, algorithm and software with applications*, Information Sciences, **234** (2013), 44-63. https://doi.org/10.1016/j.ins.2011.04.011

[24] B. S. Shieh, *Solutions of fuzzy relation equations based on continuous t-norms*, Information Sciences, **177** (2007), 4208-4215. https://doi.org/10.1016/j.ins.2007.04.006

[25] B. S. Shieh, *New resolution of finite fuzzy relation equations with max-min composition*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **16**(1) (2008), 19-33. https://doi.org/10.1142/S0218488508005029

[26] B. S. Shieh, *Minimizing a linear objective function under a fuzzy max-t-norm relation equation constraint*, Information Sciences, **181** (2011), 832-841. https://doi.org/10.1016/j.ins.2010.10.024

[27] E. Shivania, E. Khorram, *Optimization of linear objective function subject to fuzzy relation inequalities constraints with max-product composition*, Iranian Journal of Fuzzy Systems, **7**(3) (2010), 51-71. https://doi.org/10.22111/IJFS.2010.189

[28] G. B. Stamou, S. G. Tzafestas, *Resolution of composite fuzzy relation equations based on Archimedean triangular norms*, Fuzzy Sets and Systems, **120** (2001), 395-407. https://doi.org/10.1016/S0165-0114(99)00117-7

[29] V. L. Tiwari, A. Thapar, *Covering problem for solutions of max-Archimedean bipolar fuzzy relation equations*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **28**(4) (2020), 613-634. https://doi.org/10.1142/S0218488520500269

[30] Y. K. Wu, S. M. Guu, *An efficient procedure for solving a fuzzy relational equation with max-Archimedean t-norm composition*, IEEE Transactions on Fuzzy Systems, **16**(1) (2008), 73-84. https://doi.org/10.1109/TFUZZ.2007.902018

[31] S. J. Yang, *An algorithm for minimizing a linear objective function subject to the fuzzy relation inequalities with addition-min composition*, Fuzzy Sets and Systems, **255** (2014), 41-51. https://doi.org/10.1016/j.fss.2014.04.007

[32] X. P. Yang, *Solutions and strong solutions of min-product fuzzy relation inequalities with application in supply chain*, Fuzzy Sets and Systems, **384** (2020), 54-74. https://doi.org/10.1016/j.fss.2019.04.029

[33] X. P. Yang, G. Z. Zheng, X. G. Zhou, B. Y. Cao, *Lexicography minimum solution of fuzzy relation inequalities: Applied to optimal control in P2P file sharing system*, International Journal of Machine Learning and Cybernetics, **8**(5) (2017), 1555-1563. https://doi.org/10.1007/s13042-016-0527-x

[34] X. P. Yang, X. G. Zhou, B. Y. Cao, *Min-max programming problem subject to addition-min fuzzy relation inequalities*, IEEE Transactions on Fuzzy Systems, **24**(1) (2016), 111-119. https://doi.org/10.1109/TFUZZ.2015.2428716

[35] X. G. Zhou, X. P. Yang, B. Y. Cao, *Posynomial geometric programming problem subject to max-min fuzzy relation equations*, Information Sciences, **328** (2016), 15-25. https://doi.org/10.1016/j.ins.2015.07.058

[36] X. G. Zhou, X. G. Zhong, H. T. Lin, Z. J. Qin, X. P. Yang, *Lexicographic maximum solution of min-product fuzzy relation inequalities for modeling the optimal pricing with fixed priority grade in supply chain*, IEEE Access, **6** (2018), 71306-71316. DOI:10.1109/ACCESS.2018.2878748. https://doi.org/10.1109/ACCESS.2018.2878748