Iranian Journal of Fuzzy Systems

Volume 21, Number 1, (2024), pp. 51-64

Original paper

Idempotent semi-t-operators on bounded lattices

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Abstract

As a generalization of nullnorms, semi-t-operators are interesting both in theory and practical applications. In this paper, we investigate some properties of idempotent semi-t-operators on bounded lattices. Furthermore, we propose two construction methods of idempotent semi-t-operators on bounded lattices containing only two different elements which are incomparable with a but comparable with b. These methods are the generalization of several known ones in the literature.

Keywords: Semi-t-operators, bounded lattices, idempotent.

1 Introduction

The concepts of t-norms and t-conorms on the unit interval were introduced by Menger [22], they originate from the idea of developing classical triangle inequality into metric spaces to probabilistic metric spaces. Uninorm [7, 8, 18, 16], nullnorm [9, 17] and t-operators [20, 21] as a unification of t-norms and t-conorms on $[0, 1]$ developed rapidly. They have been used successfully in fuzzy logic, fuzzy system modeling, expert systems, and so on [13, 19, 31, 32]. Nullnorms and t-operators are identically discussed in [21]. In view of these, operators on bounded lattices have a wider application both in theory and application compared with the real interval $[0, 1]$, they have been paid much attention [2, 11, 14, 17, 26, 34]. Drygas [10] introduced semi-t-operators by weakening assumptions of the concept of t-operators (nullnorm). Because semi-t-operators are used in fuzzy logic to maintain as many logical properties as possible, many researchers have been working on research about logical properties of semi-t-operators [10, 15, 23, 24, 29, 33].

Idempotent operators are special cases of their corresponding operators. Since they have some good properties, many authors have been interested in exploring new construction methods [3, 5, 6, 12, 28], and characterizing them [9, 25, 30, 34]. Authors [6] studied and discussed the existence of idempotent nullnorms on bounded lattices, they also proved idempotent nullnorms may not always exist on arbitrary bounded lattice. Furthermore, they gave a construction method of idempotent nullnorms on a bounded lattice that there is only one element incomparable with zero element. Çaylı [4] proposed two construction methods of idempotent nullnorms on a bounded lattice whose elements incomparable with zero element are also incomparable with each other. Çaylı [3] introduced two new construction methods of idempotent nullnorms on a bounded lattice with different elements compared with zero element and comparable with each other. Wu et al. [30] obtained an equivalent characterization for the existence of idempotent nullnorms on any bounded lattice containing only two different elements which are incomparable with zero element soon afterward. At almost the same time, Zhang et al.[34] characterized idempotent nullnorms in terms of particular common solutions to two equations related to underlying meet and join operations.

Fang and Hu [12] first introduced the notion of semi-t-operators on bounded lattices, they presented several construction methods of semi-t-operators by pseudo-t-norms and pseudo-t-conorms. They also gave the least semi-t-operator and the greatest semi-t-operator on bounded lattices, unfortunately, they are wrong. Subsequently, Wang et al. [27] modified the least semi-t-operator and the greatest semi-t-operator on bounded lattices, meanwhile, they also proposed

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Received: August 2022; Revised: January 2023; Accepted: March 2023.

https://doi.org/10.22111/IJFS.2023.43078.7564

new construction methods of semi-t-operators. Wang et al. [28] pointed out that not all bounded lattices having idempotent semi-t-operators and presented some construction methods of idempotent semi-t-operators on bounded lattices that there is only one element in I_a^b (I_b^a or $I_{a,b}$). Motivated by this and the works on idempotent nullnorms, we explore construction methods of idempotent semi-t-operators on bounded lattices possessing only two elements in I_a^b (I_b^a or $I_{a,b}$), where the two elements in I_a^b (I_b^a) are incomparable with each other.

The rest of this paper is arranged into three sections. In Section 2, we recall some definitions, notations, and conclusions about operators on bounded lattices. In Section 3, we present some basic properties of idempotent semi-t-operators on bounded lattices with some constraints. In Section 4, we introduce two construction methods of idempotent semi-t-operators on bounded lattices containing only two distinct elements in I_a^b (I_b^a) which are incomparable to each other. Furthermore, we give two examples to demonstrate that the conditions in the methods can't be absent. Finally, some conclusions and two questions are added.

2 Preliminaries

In this section, we review some definitions and consequences related to semi-t-operators on a bounded lattice. We always denote an order as $\leq$. If every pair of elements in an ordered set has an infimum and a supremum, we call the ordered set a lattice (details see [1]). If a lattice L has the top and bottom elements, which are written as 1 and 0, respectively, that is, there exist two elements $0, 1 \in L$ such that $0 \leq x \leq 1$, $\forall x \in L$, we call it a bounded lattice.

For any elements $c, d \in L$, if c and d are incomparable, this we denote by $c \parallel d$, if c and d are comparable, this we denote by $c \nparallel d$. If $c \leq d$, this the sub-interval $[c, d]$ of L is denoted by $[c, d] = \{x \in L \mid c \leq x \leq d\}$. Similarly, we denote

$$(c, d] = \{x \in L \mid c < x \leq d\},\ [c, d) = \{x \in L \mid c \leq x < d\},\ (c, d) = \{x \in \mathbb{L} \mid c < x < d\}.$$

For the convenience, we use the notation $I_a^b = \{x \mid x \parallel a \ \ and \ \ x \nparallel b\}$ to express the set whose elements are incomparable with a but comparable with b, use the notation $I_b^a = \{x \mid x \parallel b \ \ and \ \ x \nparallel a\}$ to express the set whose elements are incomparable with b but comparable with a, and $I_{a,b} = \{x \mid x \parallel a \ \ and \ \ x \parallel b\}$ to represent the set whose elements are incomparable with a and b.

Definition 2.1. [1] *Let L be a lattice, it is distributive if one (or, equivalently, both) of the distributive laws holds, for all $x, y, z \in L$*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Definition 2.2. [12, 35] *If a binary non-decreasing function $T : L^2 \to L$ ($S : L^2 \to L$) is associative, and it has a neutral element 1 (0), that is, $T(1, x) = T(x, 1) = x$ ($S(0, x) = S(x, 0) = x$) $\forall x \in L$, then it is called a pseudo-t-norm (pseudo-t-conorm).*

We use the notation $\mathcal{F}(X_1, X_2)$ to denote the family of all surjective mappings of X_1 to X_2.

Definition 2.3. [12] *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice with $a, b \in L \setminus \{0, 1\}$. If a binary operation $P : L^2 \to L$ is associative, increasing and satisfies $P(0, 0) = 0$, $P(1, 1) = 1$ and such that the functions $P_0 \in \mathcal{F}(L, [0, a] \setminus I_b^a)$, $P_1 \in \mathcal{F}(L, [b, 1] \setminus I_a^b)$, $P^0 \in \mathcal{F}(L, [0, b] \setminus I_a^b)$, $P^1 \in \mathcal{F}(L, [a, 1] \setminus I_b^a)$, where $P_0(x) = P(0, x)$, $P_1(x) = P(1, x)$, $P^0(x) = P(x, 0)$, $P^1(x) = P(x, 1)$ and $P(0, 1) = a$, $P(1, 0) = b$ for $a, b \in L$, then we call it a semi-t-operator on L.*

Definition 2.4. [28] *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice with $a, b \in L \setminus \{0, 1\}$ and P be a semi-t-operator with $P(0, 1) = a$ and $P(1, 0) = b$. P is called an idempotent semi-t-operator if $P(x, x) = x$ for all $x \in L$.*

Proposition 2.5. [12] *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $a, b \in L \setminus \{0, 1\}$. If P is a semi-t-operator satisfying $P(0, 1) = a$ and $P(1, 0) = b$, then*
(1) *when $a \leq b$*
  (i) *$P(x, y) = x$ for $(x, y) \in [a, b] \times L$.*
  (ii) *$P|_{[0,a]^2} : [0, a]^2 \to [0, a]$ is a pseudo-t-conorm on $[0, a]$.*
  (iii) *$P|_{[b,1]^2} : [b, 1]^2 \to [b, 1]$ is a pseudo-t-norm on $[b, 1]$.*
(2) *when $b \leq a$*
  (i) *$P(x, y) = y$ for $(x, y) \in L \times [b, a]$.*
  (ii) *$P|_{[0,b]^2} : [0, b]^2 \to [0, b]$ is a pseudo-t-conorm on $[0, b]$.*
  (iii) *$P|_{[a,1]^2} : [a, 1]^2 \to [a, 1]$ is a pseudo-t-norm on $[a, 1]$.*

Proposition 2.6. [27] *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $a, b \in L \setminus \{0, 1\}$.*
(1) *When $a \leq b$*
 (i) *If $x \in I_a^b$, then $x < b$.*
 (ii) *If $x \in I_b^a$, then $x > a$.*
(2) *When $b \leq a$*
 (i) *If $x \in I_b^a$, then $x < a$.*
 (ii) *If $x \in I_a^b$, then $x > b$.*

Proposition 2.7. [12] *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $a, b \in L \setminus \{0, 1\}$, P be a semi-t-operator on L with $P(0, 1) = a$ and $P(1, 0) = b$. Then*
 (i) *$P(x, y) = a$ for $(x, y) \in [0, a] \times [a, 1]$.*
 (ii) *$P(x, y) = b$ for $(x, y) \in [b, 1] \times [0, b]$.*
 (iii) *$P(x, y) \leq x \wedge y$ for $(x, y) \in [a, 1] \times [b, 1]$.*
 (iv) *$x \vee y \leq P(x, y)$ for $(x, y) \in [0, b] \times [0, a]$.*
 (v) *$P(x, y) = x$ for all $(x, y) \in [a, b] \times L$ if $a \leq b$.*
 (vi) *$P(x, y) = y$ for all $(x, y) \in L \times [b, a]$ if $b \leq a$.*

Proposition 2.8. [28] *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and $a, b \in L \setminus \{0, 1\}$, P be a semi-t-operator on L with $P(0, 1) = a$ and $P(1, 0) = b$.*
 I. *If $a \leq b$, then*
 (1) *$P(x, y) = a$ for all $(x, y) \in [0, a] \times I_b^a$.*
 (2) *$(x \wedge a) \vee (y \wedge a) \leq P(x, y) \leq x \vee a$ for all $(x, y) \in I_a^b \times [0, a] \cup I_a^b \times I_a^b \cup I_a^b \times I_{a,b}$.*
 (3) *$a \leq P(x, y) \leq x \vee a$ for all $(x, y) \in I_a^b \times [a, b] \cup I_a^b \times [b, 1] \cup I_a^b \times I_b^a$.*
 (4) *$x \wedge b \leq P(x, y) \leq b$ for all $(x, y) \in I_b^a \times [0, a] \cup I_b^a \times [a, b] \cup I_b^a \times I_a^b$.*
 (5) *$x \wedge b \leq P(x, y) \leq (x \vee b) \wedge (y \vee b)$ for all $(x, y) \in I_b^a \times [b, 1] \cup I_b^a \times I_b^a \cup I_b^a \times I_{a,b}$.*
 (6) *$(x \wedge a) \vee (y \wedge a) \leq P(x, y) \leq b$ for all $(x, y) \in I_{a,b} \times [0, a] \cup I_{a,b} \times I_a^b$.*
 (7) *$(x \wedge a) \vee (y \wedge a) \leq P(x, y) \leq a$ for all $(x, y) \in [0, a] \times I_a^b \cup [0, a] \times I_{a,b}$.*
 (8) *$b \leq P(x, y) \leq (x \vee b) \wedge (y \vee b)$ for all $(x, y) \in [b, 1] \times I_b^a \cup [b, 1] \times I_{a,b}$.*
 (9) *$P(x, y) = b$ for all $(x, y) \in [b, 1] \times I_a^b$.*
 II. *If $b \leq a$, then*
 (1) *$P(x, y) = b$ for all $(x, y) \in I_a^b \times [0, b]$.*
 (2) *$(x \wedge b) \vee (y \wedge b) \leq P(x, y) \leq y \vee b$ for all $(x, y) \in [0, b] \times I_b^a \cup I_b^a \times I_b^a \cup I_{a,b} \times I_b^a$.*
 (3) *$b \leq P(x, y) \leq y \vee b$ for all $(x, y) \in [b, a] \times I_b^a \cup [a, 1] \times I_b^a \cup I_a^b \times I_b^a$.*
 (4) *$y \wedge a \leq P(x, y) \leq a$ for all $(x, y) \in [0, b] \times I_a^b \cup [b, a] \times I_a^b \cup I_b^a \times I_a^b$.*
 (5) *$y \wedge a \leq P(x, y) \leq (x \vee a) \wedge (y \vee a)$ for all $(x, y) \in [a, 1] \times I_a^b \cup I_a^b \times I_a^b \cup I_{a,b} \times I_a^b$.*
 (6) *$(x \wedge b) \vee (y \wedge b) \leq P(x, y) \leq a$ for all $(x, y) \in [0, b] \times I_{a,b} \cup I_b^a \times I_{a,b}$.*
 (7) *$(x \wedge b) \vee (y \wedge b) \leq P(x, y) \leq b$ for all $(x, y) \in I_b^a \times [0, b] \cup I_{a,b} \times [0, b]$.*
 (8) *$a \leq P(x, y) \leq (x \vee a) \wedge (y \vee a)$ for all $(x, y) \in I_a^b \times [a, 1] \cup I_{a,b} \times [a, 1]$.*
 (9) *$P(x, y) = a$ for all $(x, y) \in I_b^a \times [a, 1]$.*

Theorem 2.9. [28] *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $a, b \in L \setminus \{0, 1\}$, $I_a^b = \emptyset$, $I_b^a = \emptyset$ and $I_{a,b} = \emptyset$. A binary operation P is a semi-t-operator on L with $P(0, 1) = a$ and $P(1, 0) = b$ if and only if there exist a pseudo-t-norm T and a pseudo-t-conorm S such that*

$$P(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0,a]^2, \\ T(x,y) & \text{if } (x,y) \in [b,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1], \\ x & \text{if } (x,y) \in [a,b] \times L, \\ b & \text{if } (x,y) \in [b,1] \times [0,b], \end{cases}$$

when $a \leq b$ and

$$P(x,y) = \begin{cases} S(x,y) & \text{if } (x,y) \in [0,a]^2, \\ T(x,y) & \text{if } (x,y) \in [b,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1], \\ y & \text{if } (x,y) \in L \times [a,b], \\ b & \text{if } (x,y) \in [b,1] \times [0,b], \end{cases}$$

when $b \leq a$.

Lemma 2.10. [28] *Let* $(L, \wedge, \vee, 0, 1)$ *be a bounded lattice,* $a, b \in L \setminus \{0, 1\}$ *with* $b \leq a$*. Then a function* $P : L^2 \to L$ *is a semi-t-operator on* L *with* $a = P(0, 1)$ *and* $b = P(1, 0)$ *if and only if a function* $Q : L^2 \to L$ *given by* $Q(x, y) = P(y, x)$ *for all* $(x, y) \in L$ *is a semi-t-operator on* L *with* $b = Q(0, 1)$ *and* $a = Q(1, 0)$*.*

3 Properties of idempotent semi-t-operators

In this section, we investigate some properties of idempotent semi-t-operators on bounded lattices. We only list the results about idempotent semi-t-operators with $a, b \in L \setminus \{0, 1\}$ such that $a < b$. By Lemma 2.10, we can immediately get the corresponding results of the case $b \leq a$, and for simplicity, we will omit the details.

Proposition 3.1. *Let* $(L, \wedge, \vee, 0, 1)$ *be a bounded lattice,* $a, b \in L \setminus \{0, 1\}$ *and* $a < b$*,* P *be an idempotent semi-t-operator on* L *with* $P(0, 1) = a$ *and* $P(1, 0) = b$*. Then we can obtain the following results:*

(1) *If* $(x, y) \in [0, a]^2$*, then* $P(x, y) = x \vee y$*. If* $(x, y) \in [b, 1]^2$*, then* $P(x, y) = x \wedge y$*.*

(2) $P(x, 1) = x \vee a$ *and* $P(0, x) = x \wedge a$ *for all* $x \in I_a^b$*.*

(3) $P(x, 0) = x \wedge b$*,* $P(1, x) = x \vee b$ *for all* $x \in I_b^a$*.*

(4) $P(0, x) = x \wedge a$ *and* $P(1, x) = x \vee b$ *for all* $x \in I_{a,b}$*.*

(5) $P(x, y) = x \vee (y \wedge a)$*,* $\forall (x, y) \in [0, a] \times \{I_a^b \cup I_{a,b}\}$*.*

(6) $P(x, y) = x \wedge (y \vee b)$*,* $\forall (x, y) \in [b, 1] \times \{I_b^a \cup I_{a,b}\}$*.*

(7) $P(x, y) \in (a, b]$ *for all* $x \in I_a^b$ *and* $y \in [b, 1]$*.*

(8) $P(x, y) \in (a, b)$ *for all* $x \in I_b^a$ *and* $y \in [0, a]$*.*

Proof. (1) From Proposition 2.5 and the fact that the only idempotent pseudo-t-conorm on $[0, a]$ is defined as $S_\vee(x, y) = x \vee y$ for all $x, y \in [0, a]$ and the only idempotent pseudo-t-norm on $[b, 1]$ is defined as $T_\wedge(x, y) = x \wedge y$ for all $x, y \in [b, 1]$, we can get the proof.

(2) From Proposition 2.8 I (3), we get that $a \leq P(x, 1) \leq x \vee a$ for all $x \in I_a^b$. Since P is idempotent and monotone, then $x = P(x, x) \leq P(x, 1)$. Hence $P(x, 1) = x \vee a$ for all $x \in I_a^b$. $P(0, x) = x \wedge a$ is proved in the similar way.

(3) From Proposition 2.8 I (4), we have that $x \wedge b \leq P(x, 0) \leq b$ for all $x \in I_b^a$. Since P is idempotent and monotone, then $P(x, 0) \leq P(x, x) = x$. Hence $P(x, 0) = x \wedge b$ for all $x \in I_b^a$. $P(1, x) = x \vee b$ is proved in the similar way.

(4) It can be proved similarly as (2) and (3) by Proposition 2.8 I (7) and (8).

(5) By associativity of P, Proposition 3.1 (1), (2), (4), it follows that $P(x, y) = P(P(x, 0), y) = P(x, P(0, y)) = P(x, y \wedge a) = x \vee (y \wedge a)$, $\forall (x, y) \in [0, a] \times \{I_a^b \cup I_{a,b}\}$.

(6) By associativity of P, Proposition 3.1 (1), (3), (4), it yields that $P(x, y) = P(P(x, 1), y) = P(x, P(1, y)) = P(x, y \vee b) = x \wedge (y \vee b)$, $\forall (x, y) \in [b, 1] \times \{I_b^a \cup I_{a,b}\}$.

(7) Let $x \in I_a^b$ and $y \in [b, 1]$. Suppose that $P(x, y) \leq a$. Since P is idempotent, then for all $(x, x) \in I_a^b \times I_a^b$, $P(x, x) = x \leq P(x, y) \leq a$, this is a contradiction. From monotonicity of P, Proposition 2.6 (1) and Proposition 3.1 (2), we have $P(x, y) \leq P(x, 1) = x \vee a \leq b$. Therefore, $P(x, y) \in (a, b]$ for all $x \in I_a^b$ and $y \in [b, 1]$.

(8) Let $x \in I_b^a$ and $y \in [0, a]$. Suppose that $P(x, y) \geq b$. Since P is idempotent, then for all $(x, x) \in I_b^a \times I_b^a$, $P(x, x) = x > P(x, y) \geq b$, this is a contradiction. By monotonicity of P, Proposition 2.8 I (1), we have $P(x, y) > P(0, y) = a$. Therefore, $P(x, y) \in (a, b)$ for all $x \in I_b^a$ and $y \in [0, a]$. □

Proposition 3.2. *Let* $(L, \wedge, \vee, 0, 1)$ *be a bounded lattice,* $a, b \in L \setminus \{0, 1\}$ *and* $a < b$*,* P *be an idempotent semi-t-operator on* L *with* $P(0, 1) = a$ *and* $P(1, 0) = b$*.*

(1) *If* $I_b^a = \emptyset$ *and* $I_{a,b} = \emptyset$*, then* $P(x, y) = a$ *or* $P(x, y) = b$ *or* $P(x, y) = (x \wedge a) \vee (y \wedge a)$ *or* $P(x, y) \in I_a^b$ *for all* $x, y \in I_a^b$*.*

(2) *If* $I_b^a = \emptyset$ *and* $I_{a,b} = \emptyset$*, then* $P(x, y) \in [a, b]$ *if and only if* $(x \wedge a) \vee (y \wedge a) = a$ *for all* $x, y \in I_a^b$*.*

(3) *If* $I_a^b = \emptyset$ *and* $I_{a,b} = \emptyset$*, then* $P(x, y) = a$ *or* $P(x, y) = b$ *or* $P(x, y) = (x \vee b) \wedge (y \vee b)$ *or* $P(x, y) \in I_b^a$ *for all* $x, y \in I_b^a$*.*

(4) *If* $I_a^b = \emptyset$ *and* $I_{a,b} = \emptyset$*, then* $P(x, y) \in [a, b]$ *if and only if* $(x \vee b) \wedge (y \vee b) = b$ *for all* $x, y \in I_b^a$*.*

Proof. Let P be an idempotent semi-t-operator on L with $P(0, 1) = a$ and $P(1, 0) = b$.

(1) Assume $x, y \in I_a^b$. If $x = y$, it is obviously $P(x, y) \in I_a^b$. If $P(x, y) \in [0, a]$, then by Proposition 3.1 (1), we get $P(0, P(x, y)) = P(x, y)$. On the other hand, $P(P(0, x), y) = P(x \wedge a, y) = (x \wedge a) \vee (y \wedge a)$ from Proposition 3.1 (2) and (5). Therefore, $P(x, y) = (x \wedge a) \vee (y \wedge a)$ from associativity of P. Similarly, if $P(x, y) \in [b, 1]$, then $P(x, y) = P(1, P(x, y)) = P(P(1, x), y) = P(b, y) = b$ from Proposition 2.8 I (9). If $P(x, y) \in [a, b]$, then $a = P(0, P(x, y)) = P(P(0, x), y) = (x \wedge a) \vee (y \wedge a)$ from Proposition 2.7 (i), Proposition 3.1 (2) and (5).

(2) If $P(x,y) \in [a,b]$ for all $x,y \in I_a^b$, then by Proposition 2.7 (i), we have $P(0,P(x,y)) = a$. Further, $P(P(0,x),y) = P(x \wedge a, y) = (x \wedge a) \vee (y \wedge a)$ from Proposition 3.1 (2) and (5). Hence $(x \wedge a) \vee (y \wedge a) = a$ by associativity of P. On the contrary, if $(x \wedge a) \vee (y \wedge a) = a$, we obtain $P(x,y) \in [a,b]$ since $(x \wedge a) \vee (y \wedge a) \leq P(x,y) \leq x \vee a$ for all $x,y \in I_a^b$ from Proposition 2.8 I (6).

(3) Suppose $x,y \in I_b^a$. If $x = y$, it is clearly $P(x,y) \in I_b^a$. If $P(x,y) \in [0,a]$, then by Proposition 3.1 (1), we get $P(0,P(x,y)) = P(x,y)$. On the other hand, $P(P(0,x),y) = P(a,y) = a$ from Proposition 2.8 I (1). Therefore, $P(x,y) = a$ by associativity of P. Similarly, if $P(x,y) \in [b,1]$, then $P(x,y) = P(1,P(x,y)) = P(P(1,x),y) = P(x \vee b, y) = (x \vee b) \wedge (y \vee b)$ by Proposition 3.1 (6). If $P(x,y) \in [a,b]$, then $b = P(1,P(x,y)) = P(P(1,x),y) = (x \vee b) \wedge (y \vee b)$.

(4) If $P(x,y) \in [a,b]$ for all $x,y \in I_b^a$, then by Proposition 2.7 (ii), we have $P(1,P(x,y)) = b$. Furthermore, $P(P(1,x),y) = P(x \vee b, y) = (x \vee b) \wedge (y \vee b)$ from Proposition 3.1 (3) and (6). Hence $(x \vee b) \wedge (y \vee b) = b$ by associativity of P. On the contrary, if $(x \vee b) \wedge (y \vee b) = b$, we obtain $P(x,y) \in [a,b]$ since $x \wedge b \leq P(x,y) \leq (x \vee b) \wedge (y \vee b)$ for all $x,y \in I_b^a$ from Proposition 2.8 I (5). □

4 Construction methods of idempotent semi-t-operators

In this section, we introduce some construction methods of idempotent semi-t-operators on bounded lattices. We only list two methods of idempotent semi-t-operators with $a < b$ such that $a,b \in L \setminus \{0,1\}$. By using the same ideas and properties introduced in section 3, we can obtain other two similar methods of idempotent semi-t-operators with $b \leq a$. Hence, we omit them.

In the following Theorem 4.1, we consider the bounded lattices possessing only two elements $\{u,v\} \in I_a^b$ and $u \parallel v$.

Theorem 4.1. *Let $(L,\wedge,\vee,0,1)$ be a bounded lattice, $a,b \in L \setminus \{0,1\}$ and $a < b$. If $I_a^b = \{u,v\}$, $I_b^a = \emptyset$ and $I_{a,b} = \emptyset$, then the function $P_1 : L^2 \to L$ defined by*

$$P_1(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0,a]^2, \\ x \wedge y & \text{if } (x,y) \in [b,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,b] \cup [0,a] \times [b,1], \\ x \vee (y \wedge a) & \text{if } (x,y) \in [0,a] \times I_a^b, \\ (x \wedge a) \vee y & \text{if } (x,y) \in I_a^b \times [0,a], \\ x & \text{if } (x,y) \in [a,b] \times L \cup \{(u,u),(v,v)\}, \\ (x \vee a) \wedge (y \vee a) & \text{if } (x,y) \in I_a^b \times (a,b] \cup I_a^b \times [b,1] \cup \{(u,v),(v,u)\}, \\ b & \text{otherwise}, \end{cases} \tag{1}$$

is an idempotent semi-t-operator on L if and only if $(u \wedge a) \vee (v \wedge a) = (u \vee a) \wedge (v \vee a) = a$ for all $u \neq v$.

Proof. It is clear that P_1 is idempotent, $P_1(0,0) = 0$ and $P_1(1,1) = 1$. We can get that $(P_1)_0 \in \mathcal{F}(L,[0,a] \setminus I_b^a)$, $(P_1)^0 \in \mathcal{F}(L,[0,b] \setminus I_a^b)$, $(P_1)_1 \in \mathcal{F}(L,[b,1] \setminus I_a^b)$, $(P_1)^1 \in \mathcal{F}(L,[a,1] \setminus I_b^a)$.

Necessity. Let P_1 be an idempotent semi-t-operator on L with $P_1(0,1) = a$ and $P_1(1,0) = b$. Suppose that $I_a^b = \{u,v\}$ with $u \neq v$ and $(u \vee a) \wedge (v \vee a) \neq a$. That is, $(u \vee a) \wedge (v \vee a) > a$. By Proposition 2.6 (1) (i), we get $u \vee a, v \vee a \in (a,b]$. Let $(x,y) = (u,v)$ and $z \in (0,a)$, then $P_1(x,P_1(y,z)) = P_1(u,(v \wedge a) \vee z) = (u \wedge a) \vee (v \wedge a) \vee z \leq a$, $P_1(P_1(x,y),z) = P_1((u \vee a) \wedge (v \vee a), z) = (u \vee a) \wedge (v \vee a) > a$. Therefore, $P_1(x,P_1(y,z)) \neq P_1(P_1(x,y),z)$. This is a contradiction.

Assume that $I_a^b = \{u,v\}$ with $u \neq v$ and $(u \wedge a) \vee (v \wedge a) \neq a$. That is, $(u \wedge a) \vee (v \wedge a) < a$. By Proposition 2.6 (1) (ii), we have $u \wedge a, v \wedge a \in [0,a)$. Let $(y,z) = (u,v)$ and $x = 0$, then $P_1(x,P_1(y,z)) = P_1(0,(u \vee a) \wedge (v \vee a)) = a$, $P_1(P_1(x,y),z) = P_1(0 \vee (u \wedge a), v) = (u \wedge a) \vee (v \wedge a) < a$. Therefore, $P_1(x,P_1(y,z)) \neq P_1(P_1(x,y),z)$. This is a contradiction.

Sufficiency. First of all, we have to check that P_1 satisfies monotonicity, that is, if $x \leq y$, then $P_1(x,z) \leq P_1(y,z)$ for all $z \in L$. In the view of Theorem 2.9 and Theorem 4.3 [28], we only consider the cases as follows.

1. Let $x \in [0,a]$.

 1.1 $y \in [a,b]$ and $z \in I_a^b$.
 $P_1(x,z) = x \vee (z \wedge a) \leq a \leq y = P_1(y,z)$.

 1.2 $y \in [b,1]$ and $z \in I_a^b$.
 $P_1(x,z) = x \vee (z \wedge a) \leq a \leq b = P_1(y,z)$.

 1.3 $y \in I_a^b$.
 1.3.1 $z \in [0,a]$. $P_1(x,z) = x \vee z \leq (y \wedge a) \vee z = P_1(y,z)$.

1.3.2 $z \in (a, b] \cup [b, 1]$. $P_1(x, z) = a \le (y \vee a) \wedge (z \vee a) = P_1(y, z)$.
1.3.3 $z \in I_a^b$.
1.3.3.1 $y = z$, $P_1(x, z) = x \vee (z \wedge a) = x \vee (y \wedge a) = y \wedge a \le y = P_1(y, z)$.
1.3.3.2 $(y, z) \in \{(u, v), (v, u)\}$,
$P_1(x, z) = x \vee (z \wedge a) \le a = (y \vee a) \wedge (z \vee a) = P_1(y, z)$.
2. Let $x \in [a, b]$, $y \in [b, 1]$ and $z \in I_a^b$.
$P_1(x, z) = x \le b = P_1(y, z)$.
3. Let $x \in I_a^b$.
3.1 $y \in [a, b]$.
3.1.1 $z \in [a, b] \cup [b, 1]$,
$P_1(x, z) = (x \vee a) \wedge (z \vee a) \le x \vee a \le y \vee a = y = P_1(y, z)$.
3.1.2 $z \in I_a^b$,
3.1.2.1 $x = z$, $P_1(x, z) = x \le y = P_1(y, z)$.
3.1.2.2 $(x, z) \in \{(u, v), (v, u)\}$,
$P_1(x, z) = (x \vee a) \wedge (z \vee a) = a \le x \vee a \le y \vee a = y = P_1(y, z)$.
3.2 $y \in [b, 1]$.
3.2.1 $z \in [a, b]$, $P_1(x, z) = (x \vee a) \wedge (z \vee a) \le x \vee a \le b = P_1(y, z)$.
3.2.2 $z \in [b, 1]$, $P_1(x, z) = (x \vee a) \wedge z \le y \wedge z = P_1(y, z)$.
3.2.3 $z \in I_a^b$,
3.2.3.1 $x = z$, $P_1(x, z) = x < b = P_1(y, z)$.
3.2.3.2 $(x, z) \in \{(u, v), (v, u)\}$,
$P_1(x, z) = (x \vee a) \wedge (z \vee a) = a \le x \vee a \le b = P_1(y, z)$.
3.3 $y \in I_a^b$. Then $x = y$. $P_1(x, z) = P_1(y, z)$ for all $z \in L$.

Next, we prove $P_1(x, y) \le P_1(x, z)$ for $x, y, z \in L$ and $y \le z$.
1. Let $y \in [0, a]$.
1.1 $z \in [a, b] \cup [b, 1]$ and $x \in I_a^b$.
$P_1(x, y) = (x \wedge a) \vee y \le a \le (x \vee a) \wedge (z \vee a) = P_1(x, z)$.
1.2 $z \in I_a^b$.
1.2.1 $x \in [0, a]$. $P_1(x, y) = x \vee y = x \vee (y \wedge a) \le x \vee (z \wedge a) = P_1(x, z)$.
1.2.2 $x \in [a, b]$. $P_1(x, y) = x = P_1(x, z)$.
1.2.3 $x \in [b, 1]$. $P_1(x, y) = b = P_1(x, z)$.
1.2.4 $x \in I_a^b$.
1.2.4.1 $x = z$,
$P_1(x, y) = (x \wedge a) \vee y = (x \wedge a) \vee (y \wedge a) \le (x \wedge a) \vee (z \wedge a) = x \wedge a \le x = P_1(x, z)$.
1.2.4.2 $(x, z) \in \{(u, v), (v, u)\}$,
$P_1(x, y) = (x \wedge a) \vee y \le a = (x \vee a) \wedge (z \vee a) = P_1(x, z)$.
2. Let $y \in [a, b] \cup [b, 1]$. Then we have $z \in [a, b] \cup [b, 1]$.
When $x \in I_a^b$, $P_1(x, y) = (x \vee a) \wedge (y \vee a) \le (x \vee a) \wedge (z \vee a) = P_1(x, z)$.
3. Let $y \in I_a^b$.
3.1 $z \in [a, b] \cup [b, 1]$.
3.1.1 $x \in [0, a]$. $P_1(x, y) = x \vee (y \wedge a) \le a = P_1(x, z)$.
3.1.2 $x \in [a, b]$. $P_1(x, y) = x = P_1(x, z)$.
3.1.3 $x \in [b, 1]$. $P_1(x, y) = b \le P_1(x, z)$.
3.1.4 $x \in I_a^b$.
3.1.4.1 $x = y$,
$P_1(x, y) = x \le (x \vee a) \wedge (z \vee a) = P_1(x, z)$.
3.1.4.2 $(x, y) \in \{(u, v), (v, u)\}$,
$P_1(x, y) = (x \vee a) \wedge (y \vee a) = a \le (x \vee a) \wedge (z \vee a) = P_1(x, z)$.

Furthermore, we prove the associativity of P_1, that is, $P_1(x, P_1(y, z)) = P_1(P_1(x, y), z)$ for all $x, y, z \in L$. In the light of Theorem 2.9 and Theorem 4.3 [28], we only need to consider the cases as follows.

Case 1 : Let $x \in [0, a]$.
1.1 $y \in [a, b] \cup [b, 1]$ and $z \in I_a^b$.
$P_1(x, P_1(y, z)) = a = a \vee (z \wedge a) = P_1(a, z) = P_1(P_1(x, y), z)$.
1.2 $y \in I_a^b$.

1.2.1 $z \in [a,b] \cup [b,1]$.

$$\begin{aligned} P_1(x, P_1(y,z)) &= P_1(x, (y \vee a) \wedge (z \vee a)) = a \\ &= P_1(x \vee (y \wedge a), z) = P_1(P_1(x,y), z) \end{aligned}$$

1.2.2 $z \in I_a^b$.
1.2.2.1 $y = z$,

$$\begin{aligned} P_1(x, P_1(y,z)) &= P_1(x,y) = x \vee (y \wedge a) = x \vee (y \wedge a) \vee (z \wedge a) \\ &= P_1(x \vee (y \wedge a), z) = P_1(P_1(x,y), z) \end{aligned}$$

1.2.2.2 $(y,z) \in \{(u,v),(v,u)\}$, since $(y \wedge a) \vee (z \wedge a) = a$, then

$$\begin{aligned} P_1(x, P_1(y,z)) &= P_1(x, (y \vee a) \wedge (z \vee a)) = a = x \vee a \\ &= x \vee (y \wedge a) \vee (z \wedge a) \\ &= P_1(x \vee (y \wedge a), z) = P_1(P_1(x,y), z). \end{aligned}$$

Case 2 : Let $x \in [b,1]$.
2.1 $y \in [b,1]$ and $z \in I_a^b$.
$P_1(x, P_1(y,z)) = P_1(x,b) = b = P_1(x \wedge y, z) = P_1(P_1(x,y), z)$.
2.2 $y \in I_a^b$ and $\forall z \in L$.
$P_1(x, P_1(y,z)) = b = P_1(b,z) = P_1(P_1(x,y), z)$.
Case 3 : Let $x \in I_a^b$.
3.1 $y \in [0,a]$.
3.1.1 $z \in [a,b] \cup [b,1]$.

$$\begin{aligned} P_1(x, P_1(y,z)) &= P_1(x,a) = (x \wedge a) \vee a = a \\ &= P_1((x \wedge a) \vee y, z) = P_1(P_1(x,y), z). \end{aligned}$$

3.1.2 $z \in I_a^b$.

$$\begin{aligned} P_1(x, P_1(y,z)) &= P_1(x, y \vee (z \wedge a)) = (x \wedge a) \vee y \vee (z \wedge a) \\ &= P_1((x \wedge a) \vee y, z) = P_1(P_1(x,y), z). \end{aligned}$$

3.2 $y \in [a,b]$ and $\forall z \in L$. Since $x \vee a \in (a,b]$, then

$$\begin{aligned} P_1(x, P_1(y,z)) &= P_1(x,y) = (x \vee a) \wedge (y \vee a) = (x \vee a) \wedge y \\ &= P_1((x \vee a) \wedge y, z) = P_1(P_1(x,y), z). \end{aligned}$$

3.3 $y \in [b,1]$
3.3.1 $z \in [0,a] \cup [a,b] \cup I_a^b$.

$$\begin{aligned} P_1(x, P_1(y,z)) &= P_1(x,b) = (x \vee a) \wedge (b \vee a) = x \vee a = P_1(x \vee a, z) \\ &= P_1((x \vee a) \wedge y, z) = P_1(P_1(x,y), z). \end{aligned}$$

3.3.2 $z \in [b,1]$.

$$\begin{aligned} P_1(x, P_1(y,z)) &= P_1(x, y \wedge z) = (x \vee a) \wedge ((y \wedge z) \vee a) \\ &= (x \vee a) \wedge (y \wedge z) = x \vee a \\ &= P_1((x \vee a) \wedge y, z) = P_1(P_1(x,y), z) \end{aligned}$$

3.4 $y \in I_a^b$.
3.4.1 $z \in [0,a]$.
3.4.1.1 $x = y$.

$$\begin{aligned} P_1(x, P_1(y,z)) &= P_1(x, (y \wedge a) \vee z) = (x \wedge a) \vee (y \wedge a) \vee z \\ &= (x \wedge a) \vee z = P_1(x,z) = P_1(P_1(x,y), z) \end{aligned}$$

3.4.1.2 $(x, y) \in \{(u, v), (v, u)\}$.
Since $(x \wedge a) \vee (y \wedge a) = (x \vee a) \wedge (y \vee a) = a$, then

$$\begin{aligned} P_1(x, P_1(y, z)) &= P_1(x, (y \wedge a) \vee z) \\ &= (x \wedge a) \vee (y \wedge a) \vee z = a \vee z = a = P_1(a, z) \\ &= P_1((x \vee a) \wedge (y \vee a), z) = P_1(P_1(x, y), z). \end{aligned}$$

3.4.2 $z \in [a, b] \cup [b, 1]$.
3.4.2.1 $x = y$.

$$\begin{aligned} P_1(x, P_1(y, z)) &= P_1(x, (y \vee a) \wedge (z \vee a)) \\ &= (x \vee a) \wedge (((y \vee a) \wedge (z \vee a)) \vee a) = (x \vee a) \wedge z \\ &= (x \vee a) \wedge (z \vee a) = P_1(x, z) = P_1(P_1(x, y), z). \end{aligned}$$

3.4.4.2 $(x, y) \in \{(u, v), (v, u)\}$ and since $(x \vee a) \wedge (y \vee a) = a$.

$$\begin{aligned} P_1(x, P_1(y, z)) &= P_1(x, (y \vee a) \wedge (z \vee a)) = (x \vee a) \wedge (y \vee a) \wedge z \\ &= a \wedge z = a = P_1(a, z) = P_1((x \vee a) \wedge (y \vee a), z) \\ &= P_1(x, P_1(y, z)) = P_1(P_1(x, y), z). \end{aligned}$$

3.4.3 $z \in I_a^b$. Since there are only two elements in I_a^b, then $x = y = z = u$ or $x = y = z = v$ or two elements of x, y, z are same.
3.4.3.1 $x = y = z$. Obviously, $P_1(x, P_1(y, z)) = P_1(P_1(x, y), z)$.
3.4.3.2 When two elements of x, y, z are same. Without loss of generality, let $x = z = u, y = v$.

$$\begin{aligned} P_1(x, P_1(y, z)) &= P_1(u, (v \vee a) \wedge (u \vee a)) = (u \vee a) \wedge (v \vee a) \\ &= P_1((u \vee a) \wedge (v \vee a), v) = P_1(P_1(x, y), z). \end{aligned}$$

□

In Theorem 4.1, the condition $(u \wedge a) \vee (v \wedge a) = (u \vee a) \wedge (v \vee a) = a$ for all $I_a^b = \{u, v\}$ such that $u \neq v$ cannot be neglected. We give an example to illustrate it.

Example 4.2. (1) *Consider the lattice* $(L_1 = \{0, a, b, c, m, n, u, v, 1\}, \leq, 0, 1)$ *described by the Hasse diagram in Fig.1, it is obvious that* $I_{a,b} = \emptyset$, $I_b^a = \emptyset$ *and* $u, v \in I_a^b$. *We can verify that* $(u \vee a) \wedge (v \vee a) = a$, *but* $(u \wedge a) \vee (v \wedge a) = c \neq a$. *Next, we show that the function* P_1 *doesn't satisfy associativity. Indeed,*

$P_1(u, P_1(v, c)) = P_1(u, (v \wedge a) \vee c) = P_1(u, c) = (u \wedge a) \vee c = c \neq a = P_1(a, c) = P_1(P_1(u, v), c)$.

(2) *Consider the lattice* $(L_2 = \{0, a, b, c, d, p, q, u, v, 1\}, \leq, 0, 1)$ *described by the Hasse diagram in Fig.2, it is clear that* $I_{a,b} = \emptyset$, $I_b^a = \emptyset$ *and* $u, v \in I_a^b$. *We can check that* $(u \wedge a) \vee (v \wedge a) = a$, *but* $(u \vee a) \wedge (v \vee a) = d \neq a$. *Next, we show that the function* P_1 *doesn't satisfy associativity. Indeed,*

$P_1(u, P_1(v, c)) = P_1(u, (v \wedge a) \vee c) = P_1(u, q) = (u \wedge a) \vee q = a$, *however,* $P_1(P_1(u, v), c) = P_1(d, v) = d$. *Obviously,* $P_1(u, P_1(v, c)) \neq P_1(P_1(u, v), c)$.

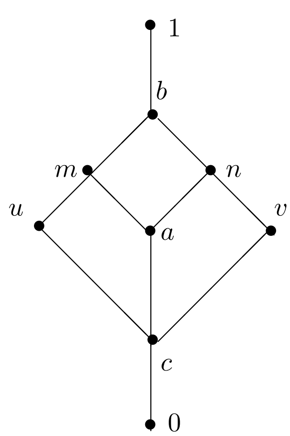

Fig.1 The lattice L_1

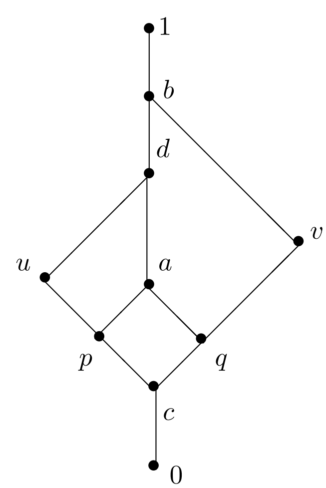

Fig.2 The lattice L_2

Remark 4.3. (i) *In Theorem 4.1, if the only two elements in* I_a^b *are always equal, then the equation (1) coincides with equation (15) in* [28].

(ii) *If a bounded lattice is distributive, then the equation (1) holds without the condition* $(u \wedge a) \vee (v \wedge a) = (u \vee a) \wedge (v \vee a) = a$ *for all* $I_a^b = \{u, v\}$.

Example 4.4. *Let* $(L_3 = \{0, c, d, a, e, f, b, u, v, 1\}, \leq, 0, 1)$ *and* $(L_4 = \{0, h, a, m, b, 1\}, \leq, 0, 1)$ *be two bounded lattices whose Hasse diagram can be depicted by Fig.3 and Fig.4, respectively. It follows from Theorem 4.1 that* P_1 *on* L_3 *in Table 1 is an idempotent semi-t-operator, from Theorem 4.3 [28], we obtain an idempotent semi-t-operator* P *displayed in Table 2.*

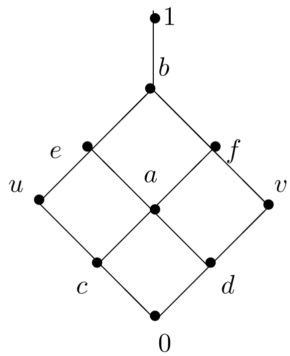

Fig.3 The lattice L_3

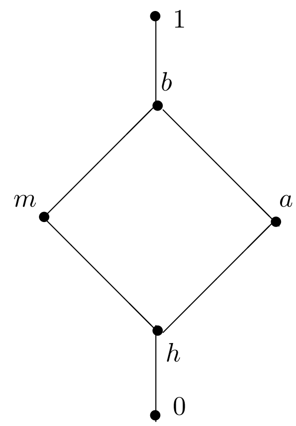

Fig.4 The lattice L_4

Table 1: Idempotent semi-t-operator P_1

$P_1(x,y)$	0	c	d	a	e	f	b	u	v	1
0	0	c	d	a	a	a	a	c	d	a
c	c	c	a	a	a	a	a	c	a	a
d	d	a	d	a	a	a	a	a	d	a
a	a	a	a	a	a	a	a	a	a	a
e	e	e	e	e	e	e	e	e	e	e
f	f	f	f	f	f	f	f	f	f	f
b	b	b	b	b	b	b	b	b	b	b
u	c	c	a	a	e	a	e	u	a	e
v	d	a	d	a	a	f	f	a	v	f
1	b	b	b	b	b	b	b	b	b	1

Table 2: Idempotent semi-t-operator P

$P(x,y)$	0	h	m	a	b	1
0	0	h	h	a	a	a
h	h	h	h	a	a	a
m	h	h	m	a	b	b
a	a	a	a	a	a	a
b	b	b	b	b	b	b
1	b	b	b	b	b	1

In the following theorem, we consider the bounded lattices possessing only two elements $\{s, t\} \in I_b^a$ and $s \parallel t$.

Theorem 4.5. *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $a, b \in L \setminus \{0,1\}$ and $a < b$. If $I_a^b = \emptyset$ and $I_{a,b} = \emptyset$, then the function $P_2 : L^2 \to L$ defined by*

$$P_2(x,y) = \begin{cases} x \vee y & \text{if } (x,y) \in [0,a]^2, \\ x \wedge y & \text{if } (x,y) \in [b,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times \{L \setminus [0,a]\}, \\ x \wedge (y \vee b) & \text{if } (x,y) \in [b,1] \times I_b^a, \\ (x \vee b) \wedge y & \text{if } (x,y) \in I_b^a \times [b,1], \\ x & \text{if } (x,y) \in [a,b] \times L \cup \{I_b^a \times I_b^a, x = y\}, \\ (x \wedge b) \vee (y \wedge b) & \text{if } (x,y) \in I_b^a \times [a,b) \cup I_b^a \times [0,a] \cup \{(s,t),(t,s)\}, \\ b & \text{otherwise}, \end{cases} \tag{2}$$

is an idempotent semi-t-operator on L if and only if $(s \wedge b) \vee (t \wedge b) = (s \vee b) \wedge (t \vee b) = b$ for all $I_b^a = \{s,t\}$ such that $s \neq t$.

Proof. We can get the proof similarly to Theorem 4.1. □

In Theorem 4.5, the condition $(s \wedge b) \vee (t \wedge b) = (s \vee b) \wedge (t \vee b) = b$ for all $I_b^a = \{s,t\}$ such that $s \neq t$ cannot be omitted. We illustrate this fact with an example.

Example 4.6. (1) *Consider the lattice $L_5 = \{0, a, b, c, d, x_1, x_2, s, t, 1\}$ described by the Hasse diagram in Fig.5, it is obviously that $I_{a,b} = \emptyset$, $I_a^b = \emptyset$ and $s, t \in I_b^a$. We can verify that $(s \wedge b) \vee (t \wedge b) = b$, while $(s \vee b) \wedge (t \vee b) = d$. Afterwards, we explain that the function P_2 doesn't fulfill associativity. Indeed,*

$$\begin{aligned} P_2(d, P_2(s,t)) &= P_2(d, (s \wedge b) \vee (t \wedge b)) = P_2(d, b) = d \wedge b \\ &= b \neq d = d \wedge (t \vee b) = P_2(d, t) = P_2(d \wedge (s \vee b), t) \\ &= P_2(P_2(d, s), t). \end{aligned}$$

(2) *Consider the lattice $L_6 = \{0, a, b, e, g, h, s, t, 1\}$ described by the Hasse diagram in Fig.6, it is clearly that $I_{a,b} = \emptyset$, $I_a^b = \emptyset$ and $s, t \in I_b^a$. We can check that $(s \vee b) \wedge (t \vee b) = b$, but $(s \wedge b) \vee (t \wedge b) = e \neq b$. Then, we illustrate that the function P_2 doesn't satisfy associativity due to*

$P_2(s, P_2(t, g)) = P_2(s, (t \vee b) \wedge g) = P_2(s, b) = (s \vee b) \wedge b = b$, but $P_2(P_2(s,t), g) = P_2(e, g) = e$. Obviously, $P_2(s, P_2(t, g)) \neq P_2(P_2(s,t), g)$.

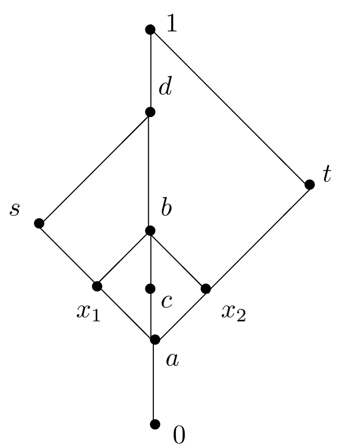

Fig.5 The lattice L_5

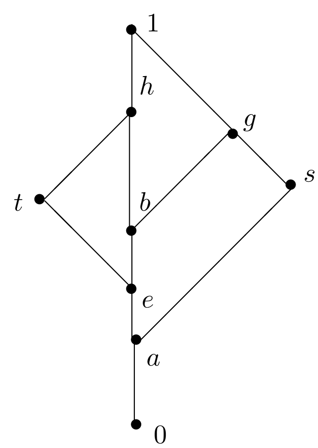

Fig.6 The lattice L_6

Corollary 4.7. *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $a, b \in L \setminus \{0,1\}$ and $a < b$. If there are only two elements $u, v \in I_a^b$ in a sublattice which is contained in L and is isomorphic to one of sublattices characterized by Hasse diagram in Fig.7-Fig.10, then there is no idempotent semi-t-operator on L.*

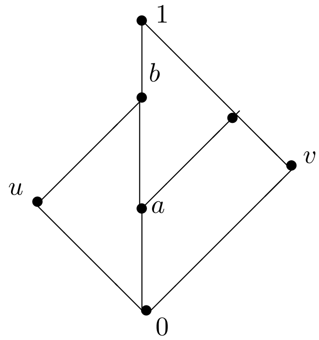

Fig.7.

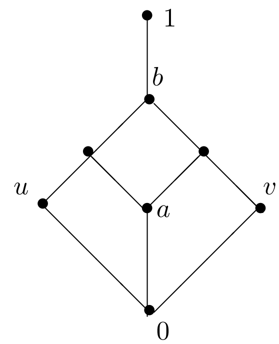

Fig.8.

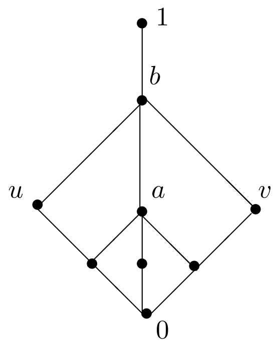

Fig.9.

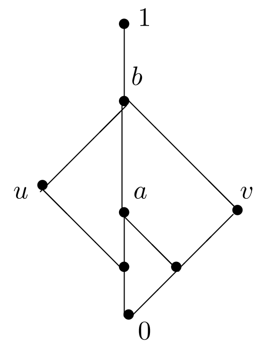

Fig.10.

Corollary 4.8. *Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice, $a, b \in L \setminus \{0, 1\}$ and $a < b$. If there are only two elements $s, t \in I_b^a$ in a sublattice which is contained in L and is isomorphic to one of sublattices characterized by Hasse diagram in Fig.11-Fig.14, then there is no idempotent semi-t-operator on L.*

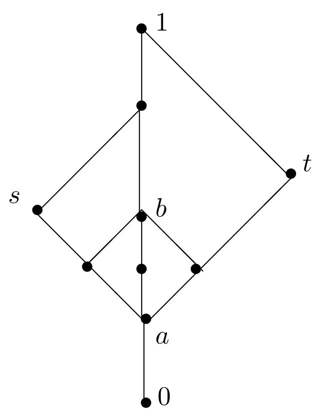

Fig.11.

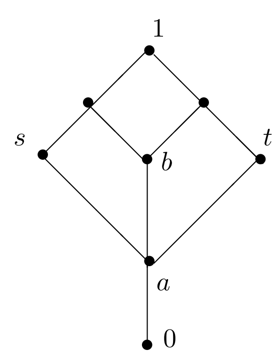

Fig.12.

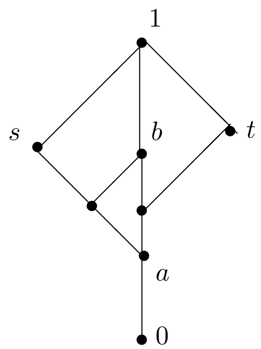

Fig.13.

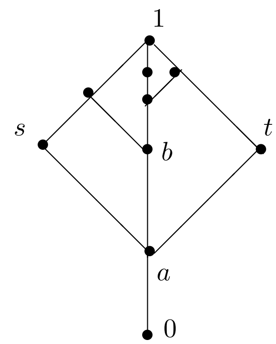

Fig.14.

Remark 4.9. (i) *In Theorem 4.5, if the only two elements in I_b^a are always equal, then the equation (2) coincides with equation (16) in* [28].

(ii) *If a bounded lattice is distributive, then Theorem 4.5 holds without the condition* $(s\wedge b)\vee(t\wedge b) = (s\vee b)\wedge(t\vee b) = b$ *for all* $I_b^a = \{s,t\}$.

5 Conclusions

Semi-t-operator on a bounded lattice is a new research hot spot. Because it does not satisfy commutative property, so compared with nullnorm, uninorm, etc., its properties and construction methods are somewhat complicated. Therefore, it is more challenging and meaningful to study it. In this paper, we have investigated some basic properties of idempotent semi-t-operators on bounded lattices. Then, we have presented some methods of constructing idempotent semi-t-operators on several types of special bounded lattices.

Two challenging questions can arise:

(1) Can idempotent semi-t-operators be characterized without any constraints on bounded lattices?

(2) If there are more than two elements in I_a^b (I_b^a) which are incomparable, can we obtain methods of constructing idempotent semi-t-operators on bounded lattices?

Acknowledgement

This research is supported by a grant of National Natural Science Foundation of China (No.12101492), Natural Science Foundation of Shaanxi Province (2022JM-017).

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