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Characterizations for the alpha-cross-migrativity of continuous t-conorms over generated implications

F. M. He [1] and B. W. Fang [2]

[1,2] *School of Science, Wuhan University of Technology, Wuhan 430070, PR China*

hefumei@whut.edu.cn, bowenfang.math@whut.edu.cn

Abstract

The α-cross-migrativity can be regarded as weaker form of the commuting equation. It has been extensively investigated between some aggregation functions including t-norms, overlap functions, uninorms, and semi-t-operators. Recently, Fang [10] has proposed the α-cross-migrativity of t-conorms over fuzzy implications. This paper continues to consider this research topic and mainly focuses on the fuzzy implications generated by additive (resp. multiplicative) generators of continuous Archimedean t-norms and t-conorms. Full characterizations for the α-cross-migrativity of continuous t-conorms over (f, g)-, k-, h- and (θ, t)-generated implications are obtained. Moreover, some supporting examples for solutions are given.

Keywords: α-cross-migrative, continuous Archimedean t-norm, continuous t-conorm, generated implication.

1 Introduction

1.1 Brief review on fuzzy implications

Fuzzy implication is an extension of classical implication which generalizes the classical truth values from $\{0, 1\}$ to $[0, 1]$. It has been widely investigated both in theory and applications. There are so many applications regarding fuzzy implications, such as fuzzy control and approximate reasoning [13, 22], image processing [16], fuzzy morphological operations [4], computing with words [34], and data mining [33]. From the theoretic perspective, it is important to construct and characterize different classes of fuzzy implications.

There exist many ways to construct fuzzy implications [1, 2, 5, 6, 8, 9, 11, 14, 21, 25, 37]. The method to construct fuzzy implications generated by unary functions and the usual addition or multiplication is firstly proposed by Yager [32]. In his method, generating functions are additive generators of continuous Archimedean t-norms (resp. t-conorms). Considering the multiplicative generators of continuous Archimedean t-conorm as the generating function, Balasubramaniam [15] introduces the h-generated implication. Along with the same research line, Zhou [39] defines the k-generated implications, where the function k is a multiplicative generator of continuous Archimedean t-norm. Motivated by generalizing Yager's implications, Xie and Liu [31] propose the (f, g)-generated implication, where $g : [0, 1] \to [0, 1]$ is an increasing function satisfying $g(0) = 0$ and $g(1) = 1$. Consider the general modus ponens, Zhou [40] constructs (θ, t)-generated implications induced by additive generators of t-norm and multiplicative generators of t-norm. In particular, it satisfies the flexible ordering property and law of importation. Moreover, the generated implications induced by additive generators of representable uninorms have been extensively investigated [20, 23]. In this context, we mainly discuss the generated implications induced by generators of t-norms and t-conorms.

Corresponding Author: B. W. Fang

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1.2 Short introduction of α-cross-migrativity property

As a weaker form of the classical computing equation [27], the α-cross-migrativity equation is firstly considered by Fodor, Klement, and Mesiar [12]. They completely characterize the α-cross-migrativity for continuous t-norms over some strict or nilpotent t-norm. Depending on whether α is an idempotent element of the t-norm, Li, Qin, and Fodor [19] characterize the α-cross-migrativity for a continuous t-norm over a fixed and continuous t-norm. Then Su et al. [28] obtain the complete characterization of the class of continuous t-norms satisfying α-cross-migrativity. The α-cross-migrativity involving other aggregation functions have been extensively investigated, such as uninorms [18, 35, 36], overlap functions [26], and semi-t-operators [38].

In [10], Fang defines the α-cross-migrativity of t-conorms over fuzzy implications. To understand it clearly, we restate the derivation process. Consider (T, T_P) is α-cross-migrative. That is, for all $x, y \in [0,1]$

$$T(\alpha x, y) = xT(\alpha, y). \tag{1}$$

Inspired by [3, 24], consider $N(x) = 1 - x$ and $S(x, y) = 1 - T(1 - x, 1 - y)$. Eq. (1) can be rewritten as

$$1 - S(1 - \alpha x, 1 - y) = x(1 - S(1 - \alpha, 1 - y)).$$

Then, we have

$$S(1 - x + (1 - \alpha)x, 1 - y) = 1 - x + xS(1 - \alpha, 1 - y)).$$

Notice the analytic formula $1 - x + (1 - \alpha)x$ can be expressed by the Reichenbach implication $I_{RC}(x, 1 - \alpha) = 1 - x + (1 - \alpha)x$. That is

$$S(I_{RC}(x, 1 - \alpha), 1 - y) = I_{RC}(x, S(1 - \alpha, 1 - y)).$$

Let $\beta = 1 - \alpha$ and $z = 1 - y$, then for all $x, z \in [0,1]$ and $\beta \in [0,1]$ is fixed

$$S(I_{RC}(x, \beta), z) = I_{RC}(x, S(\beta, z)). \tag{2}$$

It is easy to see that Eq. (2) follows Eq. (1). It is natural to consider whether the other pair (S, I) satisfies property for (S, I_{RC}) in Eq. (2), i.e., the α-cross-migrativity of t-conorms over fuzzy implications.

1.3 Motivation of our research

In [10], Fang obtains the characterizations of α-cross-migrativity for continuous t-conorms over fuzzy implications satisfying the order property, (S, N)-implications, and Yager's implications. Defining fuzzy implications by using additive or multiplicative generators is one of the main construction methods. Members in this class are mainly Yager's implications and their generalizations from different perspectives. However, the characterizations involving other types of generated implications remain unknown. The main aim of this paper is to fill this gap.

We continue to consider the α-cross-migratity property and mainly focus on the fuzzy implications generated by generators of continuous, Archimedean t-norms and t-conorms. Though one can obtain the multiplicative generator (resp. the additive generator) of the N-dual t-conorm from the additive generator (resp. the multiplicative generator) of the t-norm and vice versa (see [17], pp.80-81), where $N(x) = 1 - x$, the generated implications are different. From the theoretical point of view, we need to study all generated implications. Then we can obtain some new constructions of t-conorms and fuzzy implications satisfying the α-cross-migrativity property.

1.4 Outline of the content

The main results of this paper are full characterizations of α-cross-migrativity for continuous t-conorms over (f, g)-, k-, h- and (θ, t)-generated implications, which are summarized in Tabel 1 at the end of the paper. The remainder of this paper is organized as follows: Section 2 reviews briefly related concepts on t-norms, t-conorms, fuzzy implications, and their related properties. Section 3 focuses on characterizations of α-cross-migrativity for continuous t-conorms over fuzzy implications generated by generators of continuous, Archimedean t-norms and t-conorms. Section 4 concludes our research and further work.

2 Preliminaries

In this section, we recall some fundamental concepts and definitions which shall be needed in the sequel. For more information regarding t-norms and implications, see the monographs [1, 17].

Definition 2.1. [29] *Let $f: [a,b] \longrightarrow [c,d]$ be an increasing or decreasing function. The function $f^{(-1)}: [c,d] \longrightarrow [a,b]$ defined by*

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a,b] \mid f(x) < y\}, & \text{if } f(a) < f(b), \\ \sup\{x \in [a,b] \mid f(x) > y\}, & \text{if } f(a) > f(b), \\ a, & \text{if } f(a) = f(b) \end{cases}$$

is called the pseudo-inverse of f.

Remark 2.2. [30] *If a function $f: [a,b] \longrightarrow [c,d]$ is increasing (resp. decreasing), then $f^{(-1)}$ is also increasing (resp. decreasing). If f is strict monotone, then $f^{(-1)}$ is continuous, $f^{(-1)} \circ f = Id_{[a,b]}$ and $f \circ f^{(-1)}(x) = x$ if and only if $x \in Ran(f)$, where $Ran(f)$ is denoted as the range or image of the function f.*

A triangular norm (resp. conorm) [17], or t-norm (resp. t-conorm) in short, is a nondecreasing, associative and commutative function $T : [0,1] \times [0,1] \to [0,1]$ (resp. $S : [0,1] \times [0,1] \to [0,1]$) that satisfies the boundary condition: for all $x \in [0,1]$, $T(x,1) = x$ (resp. for all $x \in [0,1]$, $S(x,0) = x$).

Example 2.3. [17] *The following are some basic t-norms and t-conorms, for $x, y \in [0,1]$.*

(i) *Minimum and maximum pair: $T_M(x,y) = \min\{x,y\}$, $S_M(x,y) = \max\{x,y\}$;*

(ii) *Product and probabilistic sum pair: $T_P(x,y) = xy$, $S_P(x,y) = x + y - xy$;*

(iii) *Łukasiewicz pair: $T_{LK}(x,y) = \max\{x+y-1, 0\}$, $S_{LK}(x,y) = \min\{x+y, 1\}$.*

Remark 2.4. [17]

(i) *A t-conorm S is strict (i.e., both continuous and strictly monotone) if and only if there exists a strictly decreasing bijection $\varphi : [0,1] \to [0,1]$ such that $S(x,y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$.*

(ii) *A t-conorm S is nilpotent (i.e., S is continuous and for each $x \in]0,1[$, $x_S^{(n)} = 1$ for some $n \in \mathbb{N}$) if and only if there exists a strictly increasing bijection $\varphi : [0,1] \to [0,1]$ such that $S(x,y) = \varphi^{-1}(\min\{\varphi(x) + \varphi(y), 1\})$.*

Proposition 2.5. [17] *Let $(S_\alpha)_{\alpha \in A}$ be a family of t-conorms and $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ be a family of non-empty pairwise disjoint open subintervals of $[0,1]$, where A is an index set. The cardinality of A is either finite or countably infinite. Then the following function S defined by*

$$S(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) S_\alpha\left(\frac{x-a_\alpha}{e_\alpha - a_\alpha}, \frac{y-a_\alpha}{e_\alpha - a_\alpha}\right), & (x,y) \in [a_\alpha, e_\alpha]^2, \\ \max\{x,y\}, & \text{otherwise} \end{cases}$$

is a t-conorm which is called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, S_\alpha \rangle$, $\alpha \in A$, and we shall write

$$S = (\langle a_\alpha, e_\alpha, S_\alpha \rangle)_{\alpha \in A}.$$

Remark 2.6. [17]

(i) *A continuous t-conorm S (resp. t-norm T) is Archimedean if and only if $S(x,x) > x$ (resp. $T(x,x) < x$) for all $x \in]0,1[$.*

(ii) *A continuous Archimedean t-conorm is either strict or nilpotent.*

(iii) *A continuous t-conorm belongs to one of the three classes of t-conorms, the idempotent t-conorms, continuous Archimedean t-conorms, or ordinal sum t-conorms.*

Definition 2.7. [17]

(i) *An additive generator of a continuous Archimedean t-norm T is a continuous, strictly decreasing unary function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$, such that $T(x,y) = t^{(-1)}(t(x) + t(y))$, for all $(x,y) \in [0,1]^2$.*

(ii) *A multiplicative generator of a continuous Archimedean t-norm T is a continuous, strictly increasing unary function $\theta : [0,1] \to [0,1]$ with $\theta(1) = 1$, such that $T(x,y) = \theta^{(-1)}(\theta(x) \cdot \theta(y))$, for all $(x,y) \in [0,1]^2$.*

(iii) *An additive generator of a continuous Archimedean t-conorm S is a continuous, strictly increasing unary function $s : [0,1] \to [0,\infty]$ with $s(0) = 0$, such that $S(x,y) = s^{(-1)}(s(x) + s(y))$, for all $(x,y) \in [0,1]^2$.*

(iv) *A multiplicative generator of a continuous Archimedean t-conorm S is a continuous, strictly decreasing unary function $\sigma : [0,1] \to [0,1]$ with $\sigma(0) = 1$, such that $S(x,y) = \sigma^{(-1)}(\sigma(x) \cdot \sigma(y))$, for all $(x,y) \in [0,1]^2$.*

A fuzzy implication [1] is a function $I : [0,1] \times [0,1] \to [0,1]$ satisfying $I(1,0) = 0$ and $I(1,1) = I(0,1) = I(0,0) = 1$, for all $x \in [0,1]$, $I(\cdot, x)$ is decreasing and $I(x,\cdot)$ is increasing. The set of all fuzzy implications is denoted as $\mathcal{FI}$.

Note that, from the definition, it follows that $I(0,x) = I(x,1) = 1$ for all $x \in [0,1]$.

Example 2.8. [1] *The following are some basic fuzzy implications, for all $x,y \in [0,1]$,*

(i) *Lukasiewicz implication: $I_{LK}(x,y) = \min(1, 1 - x + y)$.*

(ii) *Gödel implication:*

$$I_{GD}(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ y, & \text{if } x > y. \end{cases}$$

(iii) *Goguen implication:*

$$I_{GG}(x,y) = \begin{cases} 1, & \text{if } x \le y, \\ \frac{y}{x}, & \text{if } x > y. \end{cases}$$

Definition 2.9. [10] *Consider $\alpha \in [0,1]$. A t-conorm $S : [0,1]^2 \to [0,1]$ is said to be α-cross-migrativity over a fuzzy implication $I : [0,1]^2 \to [0,1]$ (or, say that (S,I) is α-cross-migrative) if*

$$I(x, S(\alpha, y)) = S(y, I(x,\alpha)), \tag{3}$$

for all $x,y \in [0,1]$. Moreover, we call (S,I) cross-migrative if (S,I) is α-cross-migrative, for all $\alpha \in [0,1]$.

Remark 2.10. *Notice that, for $\alpha = 1$, Eq. (3) is automatically true. Moreover, it is easy to verify when (S,I) is 0-cross-migrative, the fuzzy implication I must be an (S,N)-implication. Thus, in the sequel, we only consider the case $\alpha \in]0,1[$.*

Definition 2.11. [32] *Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing and continuous function satisfying $f(1) = 0$, and $g : [0,1] \to [0,\infty]$ is a strictly increasing and continuous function satisfying $g(0) = 0$. Then*

(i) *The function $I_f : [0,1]^2 \to [0,1]$ defined by $I_f(x,y) = f^{(-1)}(xf(y))$, for all $x,y \in [0,1]$ with the understanding $0 \cdot \infty = 0$ is called an f-generated implication.*

(ii) *The function $I_g : [0,1]^2 \to [0,1]$ defined by $I_g(x,y) = g^{(-1)}(\frac{1}{x} g(y))$, for all $x,y \in [0,1]$ with the understanding $\frac{1}{0} = \infty$ and $\infty \cdot 0 = \infty$ is called a g-generated implication.*

Theorem 2.12. [10] *Take $\alpha \in]0,1[$. Let S be a t-conorm and I_f be an f-generated implication. Then (S, I_f) is α-cross-migrative, if and only if*

$$S(x,y) = f^{-1}\left(\frac{f(y)}{f(\alpha)} \cdot f(S(\alpha,x))\right), \text{ for all } (x,y) \in [0,1] \times [\alpha,1]. \tag{4}$$

Theorem 2.13. [10] *Take $\alpha \in]0,1[$. Let S be a continuous t-conorm and I_f be an f-generated implication. Then (S, I_f) is α-cross-migrative if and only if one of the following statements holds:*

1) *S is strict. There exists a strictly decreasing bijection $\varphi : [0,1] \to [0,1]$ such that $S(x,y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$, and*

$$h(\varphi(\alpha)) \cdot h(\varphi(x) \cdot \varphi(y)) = h(\varphi(y)) \cdot h(\varphi(\alpha) \cdot \varphi(x)), \text{ for all } (x,y) \in [0,1] \times [\alpha,1],$$

where $h : [0, \varphi(\alpha)] \to [0, f(\alpha)]$ is a strictly increasing and continuous function satisfying $h(x) = f \circ \varphi^{-1}(x)$.

2) *S is an ordinal sum t-conorm.* $S = (\langle 0, \beta, S_a \rangle, \langle \beta, 1, S_b \rangle)$, *where* $\beta < \alpha$ *and* S_b *is strict. There exists a strictly decreasing bijection* $\varphi : [0,1] \to [0,1]$ *such that* $S_b(x,y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$, *and*

$$h(\phi(\alpha)) \cdot h(\phi(x) \cdot \phi(y)) = h(\phi(y)) \cdot h(\phi(\alpha) \cdot \phi(x)), \text{ for all } (x,y) \in [\beta,1] \times [\alpha,1],$$

where $\phi : [\beta, 1] \to [0,1]$ *is a strictly decreasing and continuous function satisfying* $\phi(x) = \varphi\left(\frac{x-\beta}{1-\beta}\right)$, *and* $h : [0, \phi(\alpha)] \to [0, f(\alpha)]$ *is a strictly increasing and continuous function satisfying* $h(x) = f \circ \phi^{-1}(x)$.

3) *S is an ordinal sum t-conorm.* $S = (\langle 0, \beta, S_a \rangle, \langle \beta, 1, S_b \rangle)$, *where* $\beta = \alpha$ *and* S_b *is strict and S is in the form as follows*

$$S(x,y) = \begin{cases} \alpha S_a(\frac{x}{\alpha}, \frac{y}{\alpha}), & \text{if } (x,y) \in [0,\alpha]^2, \\ f^{-1}(\frac{f(x) \cdot f(y)}{f(\alpha)}), & \text{if } (x,y) \in [\alpha,1]^2, \\ \max(x,y), & \text{otherwise.} \end{cases} \tag{5}$$

Theorem 2.14. [10] *Take* $\alpha \in]0,1[$. *Let S be a t-conorm and* I_g *be a g-generated implication with* $g(1) = \infty$. *Then* (S, I_g) *is* α*-cross-migrative if and only if S and* I_g *are constructed as the form in an arbitrary item of Theorem 2.13, with* $f(x) = \frac{1}{g(x)}$.

Theorem 2.15. [10] *Take* $\alpha \in]0,1[$. *Let S be a t-conorm and* I_g *be a g-generated implication with* $g(1) < \infty$. *Then* (S, I_g) *is* α*-cross-migrative if and only if one of the following statements holds.*

1) *S is nilpotent. There exists a strictly increasing bijection* $\varphi : [0,1] \to [0,1]$ *such that*

$$\eta(\varphi(x) + \varphi(y)) \cdot \eta(1 - \varphi(x)) = \eta(\varphi(y)) \cdot \eta(1), \text{ for all } y \in [\alpha, 1],\ x \le \varphi^{-1}(1 - \varphi(y)),$$

where $\eta : [\varphi(\alpha), 1] \to [g(\alpha), g(1)]$ *is given by* $\eta(x) = g \circ \varphi^{-1}(x)$.

2) *S is an ordinal sum t-conorm given by* $S = (\langle 0, \beta, S_a \rangle, \langle \beta, 1, S_b \rangle)$, *where* $\beta < \alpha$ *and* S_b *is nilpotent. There exists a strictly increasing bijection* $\varphi : [0,1] \to [0,1]$ *such that*

$$\eta(\phi(x) + \phi(y)) \cdot \eta(1 - \phi(x)) = \eta(\phi(y)) \cdot \eta(1), \text{ for all } y \in [\alpha, 1],\ \beta \le x \le \varphi^{-1}(1 - \varphi(y)),$$

where $\phi(x) = \varphi\left(\frac{x-\beta}{1-\beta}\right)$ *and* $\eta : [\phi(\alpha), 1] \to [g(\alpha), g(1)]$ *is given by* $\eta(x) = g \circ \phi^{-1}(x)$.

3) *S is an ordinal sum t-conorm given by* $S = (\langle 0, \alpha, S_a \rangle, \langle \alpha, 1, S_b \rangle)$, *where* S_b *is nilpotent. Moreover, S is in the form as follows.*

$$S(x,y) = \begin{cases} \alpha S_a(\frac{x}{\alpha}, \frac{y}{\alpha}), & \text{if } (x,y) \in [0,\alpha]^2, \\ g^{(-1)}\left(\frac{g(x) \cdot g(y)}{g(\alpha)}\right), & \text{if } (x,y) \in [\alpha,1]^2, \\ \max(x,y), & \text{otherwise.} \end{cases} \tag{6}$$

3 Main results

In this section, we characterize the α-cross-migrativity of continuous t-conorms over fuzzy implications generated by generators of t-norms and t-conorms, i.e., (f,g)-generated implications (see Section 3.1), k-generated implications (see Section 3.2), h-generated implications (see Section 3.3), and (θ, t)-generated implications (see Section 3.4). Some supporting examples for solutions are given.

3.1 Case of (f,g)-generated implications

Definition 3.1. [31] *A function* $I_{f,g} : [0,1]^2 \to [0,1]$ *defined by*

$$I_{f,g}(x,y) = f^{(-1)}(g(x) \cdot f(y)),$$

with the understanding $0 \cdot \infty = 0$, *is called an* (f,g)*-generated implication, where f is an f-generator,* $g : [0,1] \to [0,1]$ *is an increasing function satisfying* $g(0) = 0$ *and* $g(1) = 1$.

Theorem 3.2. *Take $\alpha \in]0,1[$. Let S be a t-conorm and $I_{f,g}$ be an (f,g)-generated implication. Then $(S, I_{f,g})$ is α-cross-migrative, if and only if*

$$S(x,y) = f^{-1}\left(\frac{f(y)}{f(\alpha)} \cdot f(S(\alpha,x))\right), \text{ for all } (x,y) \in [0,1] \times [\alpha,1].$$

Proof. ($\Rightarrow$). Since $(S, I_{f,g})$ is α-cross-migrative, we have

$$f^{(-1)}(g(x) \cdot f(S(\alpha,y))) = S(y, f^{(-1)}(g(x) \cdot f(\alpha))).$$

Denote $f(z) = g(x) \cdot f(\alpha)$. Notice that

$$\alpha = f^{(-1)}(g(1) \cdot f(\alpha)) \leq f^{(-1)}(g(x) \cdot f(\alpha)) = f^{(-1)}(f(z)) = z \leq f^{(-1)}(g(0) \cdot f(\alpha)) = 1.$$

Hence, $z \in [\alpha,1]$. One obtains

$$S(y,z) = f^{(-1)}\left(\frac{f(z)}{f(\alpha)} \cdot f(S(\alpha,y))\right), \text{ for all } (y,z) \in [0,1] \times [\alpha,1].$$

Notice that $\frac{f(z)}{f(\alpha)} \cdot f(S(\alpha,y)) \subseteq [0, f(S(\alpha,y))] \subseteq Ran(f)$. Hence we have

$$S(x,y) = f^{-1}\left(\frac{f(y)}{f(\alpha)} \cdot f(S(\alpha,x))\right), \text{ for all } (x,y) \in [0,1] \times [\alpha,1].$$

($\Leftarrow$). The converse is immediate. □

According to Theorems 2.12 and 3.2, we can see the necessary and sufficient conditions for the pair (S, I_f) and $(S, I_{f,g})$ satisfying the α-cross-migrative are the same. Hence, one can easily obtain the characterization for the α-cross-migrative of continuous t-conorms over $I_{f,g}$. The theorem is given as follows. The proof is easily obtained according to Theorem 2.13.

Theorem 3.3. *Take $\alpha \in]0,1[$. Let S be a continuous t-conorm and $I_{f,g}$ be an (f,g)-generated implication. Then $(S, I_{f,g})$ is α-cross-migrative if and only if one of the following statements holds:*

1) *S is strict. There exists a strictly decreasing bijection $\varphi : [0,1] \to [0,1]$ such that $S(x,y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$, and*

$$h(\varphi(\alpha)) \cdot h(\varphi(x) \cdot \varphi(y)) = h(\varphi(y)) \cdot h(\varphi(\alpha) \cdot \varphi(x)), \text{ for all } (x,y) \in [0,1] \times [\alpha,1],$$

where $h : [0, \varphi(\alpha)] \to [0, f(\alpha)]$ is a strictly increasing and continuous function satisfying $h(x) = f \circ \varphi^{-1}(x)$.

2) *S is an ordinal sum t-conorm. $S = (\langle 0,\beta,S_a\rangle, \langle \beta,1,S_b\rangle)$, where $\beta < \alpha$ and S_b is strict. There exists a strictly decreasing bijection $\varphi : [0,1] \to [0,1]$ such that $S_b(x,y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$, and*

$$h(\phi(\alpha)) \cdot h(\phi(x) \cdot \phi(y)) = h(\phi(y)) \cdot h(\phi(\alpha) \cdot \phi(x)), \text{ for all } (x,y) \in [\beta,1] \times [\alpha,1],$$

where $\phi : [\beta,1] \to [0,1]$ is a strictly decreasing and continuous function satisfying $\phi(x) = \varphi\left(\frac{x-\beta}{1-\beta}\right)$, and $h : [0, \phi(\alpha)] \to [0, f(\alpha)]$ is a strictly increasing and continuous function satisfying $h(x) = f \circ \phi^{-1}(x)$.

3) *S is an ordinal sum t-conorm. $S = (\langle 0,\beta,S_a\rangle, \langle \beta,1,S_b\rangle)$, where $\beta = \alpha$ and S_b is strict and S is in the form as follows*

$$S(x,y) = \begin{cases} \alpha S_a(\frac{x}{\alpha}, \frac{y}{\alpha}), & \text{if } (x,y) \in [0,\alpha]^2, \\ f^{-1}(\frac{f(x)\cdot f(y)}{f(\alpha)}), & \text{if } (x,y) \in [\alpha,1]^2, \\ \max(x,y), & \text{otherwise.} \end{cases} \tag{7}$$

Proposition 3.4. *Take $\alpha \in]0,1[$. Let $I_{f,g}$ be an (f,g)-generated implication, and S be an ordinal sum t-conrom given by $S = \langle (0,\alpha,S_a), (\alpha,1,S_b)\rangle$, where S_b is strict. If $(S, I_{f,g})$ is α-cross-migrative, then $(S, I_{f,g})$ is γ-cross-migrative, for all $\gamma \in [\alpha,1[$.*

Remark 3.5. *Consider the results in Theorems 2.13 and 3.3 from another viewpoint: Construct a generated implication which is α-cross-migrativity over a continuous t-conorm. Since g is no need to be continuous, we can obtain some fuzzy implications which are non-continuous.*

Example 3.6. *One can easily verify the following pairs of (S, I) satisfying the α-cross-migrativity.*

(i) $S_\varphi(x,y) = \sqrt[n]{x^n + y^n - x^n \cdot y^n}$ *and* $I_{f,g}(x,y) = \sqrt[n]{1 - x^2 + x^2 y^n}$, *where* $f(x) = 1 - x^n$ *and* $g(x) = x^2$. *For* $\alpha \in]0,1[$, $(S_\varphi, I_{f,g})$ *is α-cross-migrative.*

(ii)

$$S_f(x,y) = \begin{cases} x + y - exy, & \text{if } (x,y) \in [0, \frac{1}{e}]^2 \\ e^{-\ln x \cdot \ln y}, & \text{if } (x,y) \in [\frac{1}{e}, 1]^2 \\ \max\{x,y\}, & \text{otherwise,} \end{cases} \qquad I_{f,g}(x,y) = \begin{cases} 1, & \text{if } x \in [0, \frac{1}{4}[, \\ y^{\frac{1}{2}}, & \text{if } x \in [\frac{1}{4}, \frac{1}{2}[, \\ y, & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

where $f(x) = -\ln x$ *and* $g(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{4}[, \\ \frac{1}{2}, & \text{if } x \in [\frac{1}{4}, \frac{1}{2}[\\ 1, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$ *For* $\alpha \in [\frac{1}{e}, 1[$, $(S_f, I_{f,g})$ *is α-cross-migrative.*

3.2 Case of k-generated implication

Definition 3.7. [39] *Let $k : [0,1] \to [0,1]$ be a strictly increasing and continuous function with $k(1) = 1$. Define a binary function $I_k : [0,1]^2 \to [0,1]$ by*

$$I_k(x,y) = k^{(-1)}\left(\frac{1}{x} \cdot k(y)\right), \text{ for all } x, y \in [0,1],$$

with the understanding $\frac{0}{0} = 1$ and $\frac{1}{0} = \infty$. Then I_k is a fuzzy implication called k-generated implication.

Theorem 3.8. *Take $\alpha \in]0,1[$. Let S be a t-conorm and I_k be a k-generated implication. Then (S, I_k) is α-cross-migrative if and only if*

$$S(x,y) = k^{(-1)}\left(\frac{k(y)}{k(\alpha)} \cdot k(S(\alpha,x))\right), \text{ for all } (x,y) \in [0,1] \times [\alpha, 1]. \tag{8}$$

Proof. $(\Rightarrow)$. Because (S, I_k) is α-cross-migrative, then the following equation holds.

$$k^{(-1)}\left(\frac{1}{x} \cdot k(S(\alpha,y))\right) = S\left(y, k^{(-1)}\left(\frac{1}{x} \cdot k(\alpha)\right)\right).$$

When $x \in [k(\alpha), 1]$, it is easy to verify that $\frac{1}{x} \cdot k(\alpha) \in Ran(k)$. Then, denote $z = k^{(-1)}\left(\frac{1}{x} \cdot k(\alpha)\right) = k^{-1}\left(\frac{1}{x} \cdot k(\alpha)\right)$, we can infer that $z \in [\alpha, 1]$. One obtains Eq. (8).

$(\Leftarrow)$. The converse is immediate. □

Proposition 3.9. *Take $\alpha \in]0,1[$. Let S be a t-conorm and I_k be a k-generated implication. If S has no 1-divisor, then (S, I_k) is not α-cross-migrative.*

Proof. Suppose on the contrary that (S, I_k) is α-cross-migrative. Since S has no 1-divisor, we know $S(x,y) = 1$ if and only if $x = 1$ or $y = 1$. Take $y_0 \in]\alpha, 1[$, and $x_0 \in]k(\alpha), k(S(\alpha, y_0))[$. One has $\frac{1}{x_0} \cdot k(\alpha) < 1 < \frac{1}{x_0} \cdot k(S(\alpha, y_0))$. That is, $\frac{1}{x_0} \cdot k(\alpha) \in Ran(k)$ and $\frac{1}{x_0} \cdot k(S(\alpha, y_0)) \in]k(1), \infty[$. Hence, we have

$$1 = k^{(-1)}\left(\frac{1}{x_0} \cdot k(S(\alpha, y_0))\right) = I_k(x_0, S(\alpha, y_0)) = S(y_0, I_k(x_0, \alpha)) = S\left(y_0, k^{(-1)}\left(\frac{1}{x} \cdot k(\alpha)\right)\right) \neq 1,$$

which leads to a contradiction. □

Proposition 3.10. *Take $\alpha \in]0,1[$. Let S be a t-conorm and I_k be a k-generated implication. If (S, I_k) is α-cross-migrative, then $S(x,x) > x$ for all $x \in]\alpha, 1[$.*

Proof. Suppose on the contrary that there exists some $x_0 \in]\alpha, 1[$ such that $S(x_0, x_0) = x_0$. Then take $x = x_0$ and $y \in]\alpha, x_0[$ in Eq. (8). One has

$$x_0 = S(x_0, y) = k^{(-1)}\left(\frac{k(y)}{k(\alpha)} \cdot k(S(\alpha, x_0))\right) = k^{(-1)}\left(\frac{k(y) \cdot k(x_0)}{k(\alpha)}\right) = \sup\left\{x \in [0,1] \mid k(x) < \frac{k(y) \cdot k(x_0)}{k(\alpha)}\right\},$$

which implies $k(y) = k(\alpha)$ leading to a contradiction. □

Theorem 3.11. *Take $\alpha \in]0, 1[$. Let S be a t-conorm and I_k be a k-generated implication. Then (S, I_k) is α-cross-migrative if and only if one of the following statements holds:*

1) *S is nilpotent. There exists a strictly increasing bijection $\varphi : [0,1] \to [0,1]$ such that*

$$\eta(\varphi(x) + \varphi(y)) \cdot \eta(1 - \varphi(x)) = \eta(\varphi(y)), \text{ for all } y \in [\alpha, 1],\ x \le \varphi^{-1}(1 - \varphi(y)),$$

where $\eta : [\varphi(\alpha), 1] \to [k(\alpha), 1]$ is given by $\eta(x) = k \circ \varphi^{-1}(x)$.

2) *S is an ordinal sum t-conorm given by $S = (\langle 0, \beta, S_a\rangle, \langle \beta, 1, S_b\rangle)$, where $\beta < \alpha$ and S_b is nilpotent. There exists a strictly increasing bijection $\varphi : [0,1] \to [0,1]$ such that*

$$\eta(\phi(x) + \phi(y)) \cdot \eta(1 - \phi(x)) = \eta(\phi(y)), \text{ for all } y \in [\alpha, 1],\ \beta \le x \le \phi^{-1}(1 - \phi(y)),$$

where $\phi(x) = \varphi\left(\frac{x-\beta}{1-\beta}\right)$ and $\eta : [\varphi(\alpha), 1] \to [k(\alpha), 1]$ is given by $\eta(x) = k \circ \varphi^{-1}(x)$.

3) *S is an ordinal sum t-conorm given by $S = (\langle 0, \alpha, S_a\rangle, \langle \alpha, 1, S_b\rangle)$, where S_b is nilpotent. Moreover, S is in the form as follows.*

$$S(x, y) = \begin{cases} \alpha S_a(\frac{x}{\alpha}, \frac{y}{\alpha}), & \text{if } (x, y) \in [0, \alpha]^2, \\ k^{(-1)}\left(\frac{k(x) \cdot k(y)}{k(\alpha)}\right), & \text{if } (x, y) \in [\alpha, 1]^2, \\ \max\{x, y\}, & \text{otherwise.} \end{cases} \tag{9}$$

Proof. It can be proved similarly as shown in Theorem 2.15. □

Due to Theorem 3.11, the following proposition is easily verified.

Proposition 3.12. *Take $\alpha \in]0, 1[$. Let I_k be a k-generated implication, and S be an ordinal sum t-conrom given by $S = \langle (0, \alpha, S_a), (\alpha, 1, S_b)\rangle$, where S_b is nilpotent. If (S, I_k) is α-cross-migrative, then (S, I_k) is γ-cross-migrative, for all $\gamma \in [\alpha, 1[$.*

To illustrate the existence of cases in Theorem 3.11, we give some examples in the following.

Example 3.13. (i) *Consider $\alpha \in]0, 1[$, $S_{LK} = \min\{x + y, 1\}$ and*

$$I_k(x, y) = \begin{cases} y - \ln x, & \text{if } x > e^{y-1}, \\ 1, & \text{if } x \le e^{y-1}. \end{cases}$$

One can easily verify that I_k is a k-generated implication with $k(x) = e^{x-1}$ (see Fig. 1(a)). Moreover, one knows $\varphi(x) = x$ and $h(x) = k(x) = e^{x-1}$. For any $\alpha \in]0, 1[$, we have

$$I_k(x, S_{LK}(\alpha, y)) = \begin{cases} I_k(x, y + \alpha), & \text{if } y < 1 - \alpha, \\ 1, & \text{if } y \ge 1 - \alpha, \end{cases} = \begin{cases} y + \alpha - \ln x, & \text{if } x > e^{y+\alpha-1}, \\ 1, & \text{otherwise }, \end{cases}$$

$$S_{LK}(y, I_k(x, \alpha)) = \begin{cases} S_{LK}(y, \alpha - \ln x), & \text{if } x > e^{\alpha-1}, \\ 1, & \text{if } x \le e^{\alpha-1}, \end{cases} = \begin{cases} y + \alpha - \ln x, & \text{if } x > e^{y+\alpha-1}, \\ 1, & \text{otherwise }. \end{cases}$$

Therefore, (S_{LK}, I_k) is α-cross-migrative for all $\alpha \in]0, 1[$.

(ii) *Consider* $\alpha \in [0.5, 1[$,

$$S_k(x) = \begin{cases} x + y - 2xy, & \text{if } (x, y) \in [0, 0.5]^2, \\ \min\{x + y - 0.5, 1\}, & \text{if } (x, y) \in [0.5, 1]^2, \\ \max\{x, y\}, & \text{otherwise.} \end{cases}$$

It is easy to see S *is an ordinal sum t-conorm (see Fig. 1(b)). For* $\alpha \in [0.5, 1[$, *we have*

$$I_k(x, S_k(\alpha, y)) = \begin{cases} I_k(x, \alpha + y - 0.5), & \text{if } y \in [0.5, 1.5 - \alpha], \\ I_k(x, \alpha), & \text{if } y \in [0, 0.5], \\ 1, & \text{if } y \in [1.5 - \alpha, 1], \end{cases}$$

$$= \begin{cases} \alpha + y - 0.5 - \ln x, & \text{if } x > e^{\alpha + y - 1.5}, y \in [0.5, 1.5 - \alpha], \\ \alpha - \ln x, & \text{if } x > e^{\alpha - 1}, y \in [0, 0.5], \\ 1, & \text{otherwise,} \end{cases}$$

$$S_k(y, I_k(x, \alpha)) = \begin{cases} S_k(y, \alpha - \ln x), & \text{if } x > e^{\alpha - 1}, \\ 1, , & \text{if } x \leq e^{\alpha - 1}, \end{cases}$$

$$= \begin{cases} \alpha + y - 0.5 - \ln x, & \text{if } x > e^{\alpha + y - 1.5}, y \in [0.5, 1.5 - \alpha], \\ \alpha - \ln x, & \text{if } x > e^{\alpha - 1}, y \in [0, 0.5], \\ 1, & \text{otherwise.} \end{cases}$$

Therefore, (S_k, I_k) *is* α*-cross-migrative for all* $\alpha \in [0.5, 1[$.

(a) (b)

Figure 1: Plot of (a). I_k and (b). S_k presented in Example 3.13

3.3 Case of h-generated implications

Definition 3.14. [15] *Let* $h : [0, 1] \to [0, 1]$ *be a strictly decreasing and continuous function with* $h(0) = 1$. *Define a binary function* $I_h : [0, 1]^2 \to [0, 1]$ *by*

$$I_h(x, y) = h^{(-1)}(x \cdot h(y)), \text{ for all } x, y \in [0, 1].$$

Then I_h *is a fuzzy implication called* h*-generated implication.*

Theorem 3.15. *Take* $\alpha \in]0, 1[$. *Let* S *be a t-conorm and* I_h *be an* h*-generated implication. Then* (S, I_h) *is* α*-cross-migrative if and only if*

$$S(x, y) = h^{(-1)} \left(\frac{h(y)}{h(\alpha)} \cdot h(S(\alpha, x)) \right), \text{ for all } (x, y) \in [0, 1] \times [\alpha, 1]. \tag{10}$$

Proof. (⇒). Since (S, I_h) is α-cross-migrative, then the following equation holds

$$h^{(-1)}(x \cdot h(S(\alpha, y))) = S(y, h^{(-1)}(x \cdot h(\alpha))).$$

Consider whether $h(1) = 0$ or not. We divide the proof into two parts as follows.

Case 1. $h(1) = 0$. One knows h is a special f-generator. Hence, according to Theorem 2.12, Eq. (10) holds.

Case 2. $h(1) > 0$. Denote $x_\alpha = \frac{h(1)}{h(\alpha)}$. Then denote $z = h^{-1}(x \cdot h(\alpha))$. It is clear that $z \in [\alpha, 1]$. Then we obtain $x = \frac{h(z)}{h(\alpha)}$ and

$$S(y, z) = h^{(-1)}\left(\frac{h(z)}{h(\alpha)} \cdot h(S(\alpha, y))\right), \text{ for all } (y, z) \in [0, 1] \times [\alpha, 1].$$

(⇐). The converse is immediate. □

Proposition 3.16. *Take $\alpha \in]0, 1[$. Let S be a t-conorm and I_h be an h-generated implication. Consider one of the following condition holds:*

1) *$h(1) = 0$ and S has a 1-divisor;*

2) *$h(1) > 0$ and S has no 1-divisor.*

Then (S, I_h) is not α-cross-migrative.

Proof. (1) When $h(1) = 0$, one knows h is a special f-generator. Hence, according to Proposition 4.3 in [10], the conclusion is immediate.

(2) When $h(1) > 0$, suppose on the contrary that (S, I_h) is α-cross-migrative. Since S has no 1-divisor, we know $S(x, y) = 1$ if and only if $x = 1$ or $y = 1$. Take $y_0 \in]\alpha, 1[$, and $x_0 \in \left[\frac{h(1)}{h(\alpha)}, \frac{h(1)}{h(S(\alpha,y_0))}\right]$. One has $x_0 \cdot h(S(\alpha, y_0)) < h(1) < x_0 \cdot h(\alpha)$. That is, $x_0 \cdot h(\alpha) \in Ran(h)$ and $x_0 \cdot h(S(\alpha, y_0)) \in [0, h(1)[$. Hence, we have

$$1 = h^{(-1)}(x_0 \cdot h(S(\alpha, y_0))) = I_h(x_0, S(\alpha, y_0)) = S(y_0, I(x_0, \alpha)) = S(y_0, h^{(-1)}(x_0 \cdot h(\alpha))) \neq 1,$$

which leads to a contradiction.

□

Proposition 3.17. *Take $\alpha \in]0, 1[$. Let S be a t-conorm and I_h be an h-generated implication. If (S, I_h) is α-cross-migrative, then $S(x, x) > x$ for all $x \in]\alpha, 1[$.*

Proof. Suppose on the contrary that there exists some $x_0 \in]\alpha, 1[$ such that $S(x_0, x_0) = x_0$. Then let $x = x_0$ and $y \in]\alpha, x_0[$ in Eq. (10). One has

$$x_0 = S(x_0, y) = h^{(-1)}\left(\frac{h(y)}{h(\alpha)} \cdot h(S(\alpha, x_0))\right) = h^{(-1)}\left(\frac{h(x_0) \cdot h(y)}{h(\alpha)}\right),$$

which implies $h(y) = h(\alpha)$ leading to a contradiction. □

When $h(1) = 0$, one knows h is a special f-generator. Hence the characterization for the α-cross-migrative of continuous t-conorms over I_h with $h(1) = 0$ is immediate.

Theorem 3.18. *Take $\alpha \in]0, 1[$. Let S be a t-conorm and I_h be an h-generated implication with $h(1) = 0$. Then (S, I_h) is α-cross-migrative if and only if S is constructed as the form in Theorem 2.13 with $h(x) = f(x)$.*

In the next theorem, we consider the characterization for the case that $h(1) > 0$.

Theorem 3.19. *Take $\alpha \in]0, 1[$. Let S be a t-conorm and I_h be an h-generated implication with $h(1) > 0$. Then (S, I_h) is α-cross-migrative if and only if one of the following statements holds:*

1) *S is nilpotent. There exists a strictly increasing bijection $\varphi : [0, 1] \to [0, 1]$ such that*

$$\eta(\varphi(x) + \varphi(y)) \cdot \eta(1 - \varphi(x)) = \eta(\varphi(y)) \cdot \eta(1), \text{ for all } y \in [\alpha, 1],\ x \leq \varphi^{-1}(1 - \varphi(y)),$$

where $\eta : [\varphi(\alpha), 1] \to [h(\alpha), h(1)]$ is given by $\eta(x) = h \circ \varphi^{-1}(x)$.

2) *S is an ordinal sum t-conorm given by $S = (\langle 0, \beta, S_a\rangle, \langle \beta, 1, S_b\rangle)$, where $\beta < \alpha$ and S_b is nilpotent. There exists a strictly increasing bijection $\varphi : [0, 1] \to [0, 1]$ such that*

$$\eta(\phi(x) + \phi(y)) \cdot \eta(1 - \phi(x)) = \eta(\phi(y)) \cdot \eta(1), \text{ for all } y \in [\alpha, 1], \ \beta \le x \le \varphi^{-1}(1 - \varphi(y)),$$

where $\phi(x) = \varphi\left(\frac{x-\beta}{1-\beta}\right)$ and $\eta : [\phi(\alpha), 1] \to [h(\alpha), h(1)]$ is given by $\eta(x) = h \circ \phi^{-1}(x)$.

3) *S is an ordinal sum t-conorm given by $S = (\langle 0, \alpha, S_a\rangle, \langle \alpha, 1, S_b\rangle)$, where S_b is nilpotent. Moreover, S is in the form as follows:*

$$S(x, y) = \begin{cases} \alpha S_a(\frac{x}{\alpha}, \frac{y}{\alpha}), & \text{if } (x, y) \in [0, \alpha]^2, \\ h^{(-1)}\left(\frac{h(x)\cdot h(y)}{h(\alpha)}\right), & \text{if } (x, y) \in [\alpha, 1]^2, \\ \max\{x, y\}, & \text{otherwise.} \end{cases} \tag{11}$$

Proof. It can be proved similarly as shown in Theorem 2.15. □

Due to Theorems 3.18 and 3.19, the following proposition is easily verified.

Proposition 3.20. *Take $\alpha \in]0, 1[$. Let I_h be an h-generated implication, and S be an ordinal sum t-conrom given by $S = (\langle 0, \alpha, S_a\rangle, \langle \alpha, 1, S_b\rangle)$. If (S, I_h) is α-cross-migrative, then (S, I_h) is γ-cross-migrative, for all $\gamma \in [\alpha, 1[$.*

Remark 3.21. *Continue to consider Example 3.13. Let $h(x) = e^{-x}$. It is easy to see I_k can be regarded as an h-generated implication. It is similar to the ordinal sum t-conorm S_k. Therefore, the cases in Theorem 3.19 exist.*

3.4 Case of (θ, t)-generated implications

Definition 3.22. [40] *Let $\theta : [0, 1] \to [0, 1]$ be a strictly increasing and continuous function satisfying $\theta(1) = 1$ and $t : [0, 1] \to [0, \infty]$ be a strictly decreasing and continuous function satisfying $t(1) = 0$. Then the pair (θ, t) will be called a (θ, t)-generator if $t(0) \ge \theta(1^-) - \theta(0)$. A binary function $I_{\theta,t} : [0, 1]^2 \to [0, 1]$ defined by*

$$I_{\theta,t}(x, y) = \theta^{(-1)}(\min\{t(x) + \theta(y), 1\}), \text{ for all } x, y \in [0, 1],$$

is a fuzzy implication, called (θ, t)-generated implication.

Theorem 3.23. *Take $\alpha \in]0, 1[$. Let S be a t-conorm and $I_{\theta,t}$ be a (θ, t)-generated implication. Then $(S, I_{\theta,t})$ is α-cross-migrative if and only if*

$$S(x, y) = \theta^{(-1)}(\min\{\theta(y) - \theta(\alpha) + \theta(S(\alpha, x)), 1\}), \text{ for all } (x, y) \in [0, 1] \times [\alpha, 1]. \tag{12}$$

Proof. $(\Rightarrow)$. Since $(S, I_{\theta,t})$ is α-cross-migrative, the following equation holds:

$$\theta^{(-1)}(\min\{t(x) + \theta(S(\alpha, y)), 1\}) = S(y, \theta^{(-1)}(\min\{t(x) + \theta(\alpha), 1\})).$$

Denote $x_\alpha = t^{(-1)}(1 - \theta(\alpha))$. Then, consider $z = \theta^{(-1)}(t(x) + \theta(\alpha))$ for all $x \in [x_\alpha, 1]$. We know $z \in [\alpha, 1]$ and $t(x) + \theta(\alpha) \in [\theta(\alpha), 1] \subseteq Ran(\theta)$. Hence, we have $\theta(z) = t(x) + \theta(\alpha)$ and

$$S(y, z) = \theta^{(-1)}(\min\{\theta(z) - \theta(\alpha) + \theta(S(\alpha, y)), 1\}), \text{ for all } (y, z) \in [0, 1] \times [\alpha, 1].$$

$(\Leftarrow)$. The converse is immediate. □

Proposition 3.24. *Take $\alpha \in]0, 1[$. Let S be a t-conorm and $I_{\theta,t}$ be a (θ, t)-generated implication. If S has no 1-divisor, then $(S, I_{\theta,t})$ is not α-cross-migrative.*

Proof. Suppose on the contrary that $(S, I_{\theta,t})$ is α-cross-migrative. Since S has no 1-divisor, we know $S(x, y) = 1$ if and only if $x = 1$ or $y = 1$. Take $y_0 \in]\alpha, 1[$, and $x_0 \in]t^{-1}(1 - \theta(\alpha)), t^{-1}(1 - \theta(S(\alpha, y_0)))[$. One has $t(x_0) + \theta(\alpha) < 1 < t(x_0) + \theta(S(\alpha, y_0))$. Hence, we have

$$1 = \theta^{(-1)}(\min\{t(x_0) + \theta(S(\alpha, y_0)), 1\}) = S(y_0, \theta^{(-1)}(\min\{t(x_0) + \theta(\alpha), 1\})) = S(y_0, \theta^{(-1)}(t(x_0) + \theta(\alpha))) \neq 1,$$

which leads to a contradiction. □

Proposition 3.25. *Take $\alpha\in]0,1[$. Let S be a t-conorm and $I_{\theta,t}$ be a (θ,t)-generated implication. If $(S,I_{\theta,t})$ is α-cross-migrative, then $S(x,x)>x$ for all $x\in]\alpha,1[$.*

Proof. Suppose on the contrary that there exists some $x_0\in]\alpha,1[$ such that $S(x_0,x_0)=x_0$. Then, take $x=x_0$ and $y\in]\alpha,x_0[$ in Eq. (12). One has

$$x_0=S(x,y)=\theta^{(-1)}(\min\{\theta(y)-\theta(\alpha)+\theta(S(\alpha,x_0)),1\})=\theta^{(-1)}(\min\{\theta(y)-\theta(\alpha)+\theta(x_0),1\})>\theta^{(-1)}(\theta(x_0))=x_0,$$

which leads to a contradiction. □

Theorem 3.26. *Take $\alpha\in]0,1[$. Let S be a t-conorm and $I_{\theta,t}$ be a (θ,t)-generated implication. Then $(S,I_{\theta,t})$ is α-cross-migrative if and only if one of the following statements holds:*

1) *S is nilpotent. There exists a strictly increasing bijection $\varphi:[0,1]\to[0,1]$ such that*

$$\eta(\varphi(x)+\varphi(y))+\eta(1-\varphi(y))=1+\eta(\varphi(y)),\ \text{for all } y\in[\alpha,1],\ x\le\varphi^{-1}(1-\varphi(y)),$$

where $\eta:[\varphi(\alpha),1]\to[\theta(\alpha),1]$ is given by $\eta(x)=\theta\circ\varphi^{-1}(x)$.

2) *S is an ordinal sum t-conorm given by $S=(\langle 0,\beta,S_a\rangle,\langle\beta,1,S_b\rangle)$, where $\beta<\alpha$ and S_b is nilpotent. There exists a strictly increasing bijection $\varphi:[0,1]\to[0,1]$ such that*

$$\eta(\varphi(x)+\varphi(y))+\eta(1-\varphi(y))=1+\eta(\varphi(y)),\ \text{for all } y\in[\alpha,1],\ \beta\le x\le\varphi^{-1}(1-\varphi(y)),$$

where $\phi(x)=\varphi\left(\frac{x-\beta}{1-\beta}\right)$ and $\eta:[\varphi(\alpha),1]\to[\theta(\alpha),1]$ is given by $\eta(x)=\theta\circ\varphi^{-1}(x)$.

3) *S is an ordinal sum t-conorm given by $S=(\langle 0,\alpha,S_a\rangle,\langle\alpha,1,S_b\rangle)$, where S_b is nilpotent. Moreover, S is in the form as follows:*

$$S(x,y)=\begin{cases}\alpha S_a(\frac{x}{\alpha},\frac{y}{\alpha}), & \text{if } (x,y)\in[0,\alpha]^2,\\ \theta^{(-1)}(\min\{\theta(x)+\theta(y)-\theta(\alpha),1\}), & \text{if } (x,y)\in[\alpha,1]^2,\\ \max\{x,y\}, & \text{otherwise.}\end{cases}\tag{13}$$

Proof. (⇒). Consider that S is a continuous t-conorm and $I_{\theta,t}$ is a (θ,t)-generated implication. According to Propositions 3.24 and 3.25, we know S is neither strict nor S_M. There are two cases to be discussed: nilpotent and ordinal sum.

Case 1. S is nilpotent. Due to Remark 2.4, there exists a strictly increasing bijection $\varphi:[0,1]\to[0,1]$ such that $S(x,y)=\varphi^{-1}(\min\{\varphi(x)+\varphi(y),1\})$. Denote $x_\alpha\in[0,1]$ such that $S(\alpha,x_\alpha)=\varphi^{-1}(\varphi(\alpha)+\varphi(x_\alpha))=1$. Take $x_0\in[0,x_\alpha]$ and $y_0=\varphi^{-1}(1-\varphi(x_0))$. We know $\varphi(x_0)+\varphi(y_0)=1$ and $y_0\in[\alpha,1]$. By Eq. (12), we have

$$1=S(x_0,y_0)=S\left(x_0,\varphi^{-1}(1-\varphi(x_0))\right)=\theta^{(-1)}(\min\left\{\theta\circ\varphi^{-1}(1-\varphi(x_0))-\theta(\alpha)+\theta(S(\alpha,x_0)),1\right\}).$$

Notice that for any $\epsilon>0$, $S(x_0,y_0-\epsilon)<S(x_0,y_0)=1=S(x_0,y_0+\epsilon)$. It implies that

$$\theta\circ\varphi^{-1}(1-\varphi(x_0))-\theta(\alpha)+\theta(S(\alpha,x_0))=1,\ \text{for all } x_0\in[0,x_\alpha].$$

Notice that $\varphi(x)+\varphi(y)>1$ if and only if $\theta(y)-\theta\circ\varphi^{-1}(1-\varphi(x))>0$. Consider Eq. (12) again. We have

$$\varphi^{-1}(\varphi(x)+\varphi(y))=\theta^{-1}\left(1+\theta(y)-\theta\circ\varphi^{-1}(1-\varphi(x))\right),\ \text{for all } y\in[\alpha,1],\ x\le\varphi^{-1}(1-\varphi(y)).$$

Therefore, one obtains

$$\eta(\varphi(x)+\varphi(y))+\eta(1-\varphi(y))=1+\eta(\varphi(y)),\ \text{for all } y\in[\alpha,1],\ x\le\varphi^{-1}(1-\varphi(y)).$$

Case 2. S is an ordinal sum t-conorm given by $S=(\langle 0,\beta,S_a\rangle,\langle\beta,1,S_b\rangle)$. Due to Proposition 3.25, we know $\beta\le\alpha$. According to Proposition 3.24, one knows S_b must be nilpotent. Due to Remark 2.4, there exists a strictly increasing bijection $\varphi:[0,1]\to[0,1]$ such that $S_b(x,y)=\varphi^{-1}(\min\{\varphi(x)+\varphi(y),1\})$. Moreover, according to Proposition 2.5, we know

$$S(x,y)=\begin{cases}\beta S_a(\frac{x}{\beta},\frac{y}{\beta}), & \text{if } (x,y)\in[0,\beta]^2,\\ \phi^{-1}(\min\{\phi(x)+\phi(y),1\}), & \text{if } (x,y)\in[\beta,1]^2,\\ \max\{x,y\}, & \text{otherwise,}\end{cases}$$

where $\phi(x)=\varphi(\frac{x-\beta}{1-\beta})$. In the next step, we divide the proof into two parts as follows.

(i) $\beta < \alpha$. Consider $(x,y) \in [0,\beta] \times [\alpha,1]$ in Eq. (12). One obtains

$$S(x,y) = y = \theta^{(-1)}(\min\{\theta(y) - \theta(\alpha) + \theta(\alpha), 1\}) = \theta^{(-1)}(\min\{\theta(y) - \theta(\alpha) + \theta(S(\alpha,x)), 1\}).$$

It is automatically true. Then denote $x_\alpha = \phi^{-1}(1-\phi(\alpha))$. It is easy to see that $x_\alpha > \beta$ for $\alpha = S(\alpha,\beta) < S(\alpha,x_\alpha) = 1$. When $x \in [x_\alpha, 1]$, we have

$$S(x,\alpha) = \phi^{-1}(\min\{\phi(x) + \phi(\alpha), 1\}) \geq \phi^{-1}(\min\{\phi(x_\alpha) + \phi(\alpha), 1\}) = 1.$$

Notice that S is non-decreasing in both variables. Hence we know $S(x,y) = 1$ for all $(x,y) \in [x_\alpha,1] \times [\alpha,1]$. It is in line with Eq. (12). Then for all $(x,y) \in [\beta,x_\alpha] \times [\alpha,1]$, we have

$$S(x,y) = \phi^{-1}(\min\{\phi(x) + \phi(y), 1\}) = \theta^{-1}(\min\{\theta(y) - \theta(\alpha) + \theta(S(\alpha,x)), 1\}).$$

For all $x_0 \in [\beta,x_\alpha]$, denote $y_0 = \phi^{-1}(1-\phi(x_0))$. Then for any $\epsilon > 0$, we know $S(x_0, y_0-\epsilon) < S(x_0,y_0) = 1 = S(x_0, y_0+\epsilon)$. It implies that

$$\theta(y_0) - \theta(\alpha) + \theta(S(\alpha,x_0)) = 1.$$

Hence

$$\theta(S(\alpha,x_0)) = 1 - \theta \circ \phi^{-1}(1-\phi(x_0)) + \theta(\alpha), \text{ for all } x_0 \in [\beta, x_\alpha].$$

Notice that $\phi(x) + \phi(y) > 1$ if and only if $\theta(y) - \theta \circ \phi^{-1}(1-\phi(x)) > 0$. Consider Eq. (12) again. We have

$$\phi^{-1}(\phi(x) + \phi(y)) = \theta^{-1}(1 + \theta(y) - \theta \circ \phi^{-1}(1-\phi(x))), \text{ for all } y \in [\alpha,1],\ \beta \leq x \leq \phi^{-1}(1-\varphi(y)).$$

Therefore, one obtains

$$\eta(\phi(x) + \phi(y)) + \eta(1-\phi(y)) = 1 + \eta(\phi(y)), \text{ for all } y \in [\alpha,1],\ x \leq \phi^{-1}(1-\phi(y)).$$

(ii) $\beta = \alpha$. For all $(x,y) \in [0,\alpha] \times [\alpha,1]$. One obtains

$$S(x,y) = y = \theta^{(-1)}(\min\{\theta(y) - \theta(\alpha) + \theta(\alpha), 1\}) = \theta^{(-1)}(\min\{\theta(y) - \theta(\alpha) + \theta(S(\alpha,x)), 1\}).$$

It is in line with Eq. (12). Then considering $(x,y) \in [\alpha,1]^2$, we know $S(\alpha,x) = x$. One has

$$S(x,y) = \theta^{(-1)}(\min\{\theta(x) + \theta(y) - \theta(\alpha), 1\}), \text{ for all } (x,y) \in [\alpha,1]^2. \tag{14}$$

In the next step, we show $S_1 = S\mid_{[\alpha,1]^2}$ is isomorphic to a nilpotent t-conorm. According to Eq. (14), it is easy to know S_1 is commutative and strictly increasing with respect to each variable. Moreover, S_1 is continuous, because g is continuous. Take $x = \alpha$ in Eq. (14). One has

$$S(\alpha,y) = \theta^{(-1)}(\min\{\theta(\alpha) + \theta(y) - \theta(\alpha), 1\}) = \theta^{(-1)}(\theta(y)) = y,$$

which implies S_1 has α as its neutral element. Considering the associativity, we want to verify that $S_1(x, S_1(y,z)) = S_1(S_1(x,y),z)$ for all $x,y,z \in [\alpha,1]$. There exist three cases.

(a) $\theta(x)+\theta(y) \geq 1+\theta(\alpha)$. Then $S_1(S_1(x,y),z) = S_1(1,z) = 1$. Since $S_1(y,z) \geq y$, we know $S_1(x,S_1(y,z)) \geq S_1(x,y) = 1$. Hence the equation $S_1(x,S_1(y,z)) = S_1(S_1(x,y),z)$ holds.

(b) $\theta(y) + \theta(z) \geq 1 + \theta(\alpha)$. It can be proved in a similar way.

(c) $\theta(x) + \theta(y) < 1 + \theta(\alpha)$ and $\theta(y) + \theta(z) < 1 + \theta(\alpha)$. One obtains

$$S_1(x, S_1(y,z)) = \theta^{-1}(\theta(x) + \theta(y) + \theta(z) - 2\theta(\alpha)) = S_1(S_1(x,y),z).$$

Therefore, S_1 is associative. Suppose that S_1 has an idempotent element $x_0 \in]\alpha,1[$, then

$$x_0 = S(x_0,x_0) = \theta^{(-1)}(\min\{\theta(x_0) + \theta(x_0) - \theta(\alpha), 1\}) = \theta^{-1}(2\theta(x_0) - \theta(\alpha)),$$

which implies $\theta(x_0) = \theta(\alpha)$. It is a contradiction to θ being strictly increasing. So S_1 is isomorphic to a continuous Archimedean t-conorm. Moreover, for each $x_0 \in]\alpha,1[$, we have $\theta(x_0) > \theta(\alpha)$. Then

$$(n+1)\theta(x_0) - n\theta(\alpha) > n\theta(x_0) - (n-1)\theta(\alpha) > \cdots > 2\theta(x_0) - \theta(\alpha) > \theta(x_0).$$

It implies for each $x_0 \in]\alpha,1[$, there exists some $n \in \mathbb{N}$ such that $n\theta(x_0) - (n-1)\theta(\alpha)$. That is, $x_{S_1}^{(n)} = \theta^{(-1)}(\min\{n\theta(x_0) - (n-1)\theta(\alpha), 1\}) = 1$, for some $n \in \mathbb{N}$. Therefore, S_1 is isomorphic to a nilpotent t-conrom. Then it is nilpotent itself.

(⇐). The converse is immediate. □

Due to Theorem 3.26, the following proposition is easily verified.

Proposition 3.27. *Take $\alpha \in]0,1[$. Let $I_{\theta,t}$ be a (θ,t)-generated implication, and S be an ordinal sum t-conrom given by $S = (\langle 0,\alpha,S_a\rangle, \langle \alpha,1,S_b\rangle)$. If $(S, I_{\theta,t})$ is α-cross-migrative, then $(S, I_{\theta,t})$ is γ-cross-migrative, for all $\gamma \in [\alpha,1[$.*

To illustrate the existence of cases in Theorem 3.26, we give some examples in the following.

Example 3.28. (i) *Consider $\alpha \in]0,1[$, $S_{LK} = \min\{x+y,1\}$, and $I_{\theta,t}(x,y) = \min\{2-2x+y,1\}$. It is easy to see $I_{\theta,t}$ is a (θ,t)-generated implication with $\theta(x) = 0.6+0.4x$ and $t(x) = 0.8-0.8x$ (see Fig. 2). Then for any $\alpha \in]0,1[$, we have*

$$I_{\theta,t}(x, S_{LK}(\alpha,y)) = \begin{cases} I_{\theta,t}(x,\alpha+y), & \text{if } y < 1-\alpha, \\ 1, & \text{if } y \geq 1-\alpha, \end{cases}$$

$$= \begin{cases} 2-2x+\alpha+y, & \text{if } x > 0.5(\alpha+y+1), \\ 1, & \text{otherwise}, \end{cases}$$

$$S_{LK}(y, I_{\theta,t}(x,\alpha)) = \begin{cases} S_{LK}(y, 2-2x+\alpha), & \text{if } x > 2\alpha-1, \\ 1, & \text{otherwise}, \end{cases}$$

$$= \begin{cases} 2-2x+\alpha+y, & \text{if } x > 0.5(\alpha+y+1), \\ 1, & \text{otherwise}. \end{cases}$$

Therefore, $(S_{LK}, I_{\theta,t})$ is α-cross–migrative for all $\alpha \in]0,1[$.

(ii) *Consider $\alpha \in [0.5,1[$, and*

$$S_{\theta,t}(x) = \begin{cases} x+y-2xy, & \text{if } (x,y) \in [0,0.5]^2, \\ \min\{x+y-0.5,1\}, & \text{if } (x,y) \in [0.5,1]^2, \\ \max\{x,y\}, & \text{otherwise}. \end{cases}$$

We know $S_{\theta,t} = S_k$ in Example 3.13. It is an ordinal sum t-conorm. For any $\alpha \in [0.5,1[$, we have

$$I_{\theta,t}(x, S_{\theta,t}(\alpha,y)) = \begin{cases} I_{\theta,t}(x,\alpha+y-0.5), & \text{if } 0.5 \leq y \leq 1.5-\alpha, \\ I_{\theta,t}(x,\alpha), & \text{if } y \leq 0.5, \\ 1, & \text{if } y \geq 1.5-\alpha, \end{cases}$$

$$= \begin{cases} 1.5-2x+\alpha+y, & \text{if } x > 0.25+0.5\alpha+0.5y, y \geq 0.5, \\ 2-2x+\alpha, & \text{if } x > 0.5+0.5\alpha, y \leq 0.5, \\ 1, & \text{otherwise}, \end{cases}$$

$$S_{\theta,t}(y, I_{\theta,t}(x,\alpha)) = \begin{cases} S_{\theta,t}(y, 2-2x+\alpha), & \text{if } x > 0.5+0.5\alpha, \\ 1, & \text{if } x \geq 0.5+0.5\alpha, \end{cases}$$

$$= \begin{cases} 1.5-2x+\alpha+y, & \text{if } x > 0.25+0.5\alpha+0.5y, y \geq 0.5, \\ 2-2x+\alpha, & \text{if } x > 0.5+0.5\alpha, y \leq 0.5, \\ 1, & \text{otherwise}. \end{cases}$$

Therefore, $(S_{\theta,t}, I_{\theta,t})$ is α-cross-migrative for all $\alpha \in [0.5,1[$.

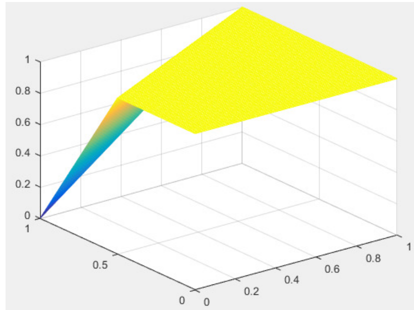

Figure 2: Plot of $I_{\theta,t}$ in presented in Example 3.28

In Table 1, we list the key results, for their detailed requirements and characterizations, please see the corresponding theorems.

Table 1: Characterizations for the α-cross-migrativity of continuous t-conorms over generated implications

Generated implications	Continuous t-conorms
I_f	iff, S is strict or ordinal sum $(\langle 0,\beta,S_a\rangle,\langle \beta,1,S_b\rangle)$, where $\beta\le\alpha$, S_b is strict, satisfying the requirements in Theorem 2.13.
I_g $(g(1)=\infty)$	iff, S is strict or ordinal sum $(\langle 0,\beta,S_a\rangle,\langle \beta,1,S_b\rangle)$, where $\beta\le\alpha$, S_b is strict, satisfying the requirements in Theorem 2.14.
I_g $(g(1)<\infty)$	iff, S is nilpotent or ordinal sum $(\langle 0,\beta,S_a\rangle,\langle \beta,1,S_b\rangle)$, where $\beta\le\alpha$, S_b is nilpotent, satisfying the requirements in Theorem 2.15.
$I_{f,g}$	iff, S is strict or ordinal sum $(\langle 0,\beta,S_a\rangle,\langle \beta,1,S_b\rangle)$, where $\beta\le\alpha$, S_b is strict, satisfying the requirements in Theorem 3.3.
I_k	iff, S is nilpotent or ordinal sum $(\langle 0,\beta,S_a\rangle,\langle \beta,1,S_b\rangle)$, where $\beta\le\alpha$, S_b is nilpotent, satisfying the requirements in Theorem 3.11.
$I_h(h(1)=0)$	iff, S is strict or ordinal sum $(\langle 0,\beta,S_a\rangle,\langle \beta,1,S_b\rangle)$, where $\beta\le\alpha$, S_b is strict, satisfying the requirements in Theorem 3.18.
$I_h(h(1)>0)$	iff, S is nilpotent or ordinal sum $(\langle 0,\beta,S_a\rangle,\langle \beta,1,S_b\rangle)$, where $\beta\le\alpha$, S_b is nilpotent, satisfying the requirements in Theorem 3.19.
$I_{\theta,t}$	iff, S is nilpotent or ordinal sum $(\langle 0,\beta,S_a\rangle,\langle \beta,1,S_b\rangle)$, where $\beta\le\alpha$, S_b is nilpotent, satisfying the requirements in Theorem 3.26.

Consider these results from another viewpoint: α is the idempotent element of the continuous t-conorm S or not. We can rewrite the theorem of the full characterizations for α-cross-migrativity of $S=(\langle 0,\alpha,S_a\rangle,\langle \alpha,1,S_b\rangle)$ over the generated implications as follows.

Theorem 3.29. *Take $\alpha\in]0,1[$. Let S be an ordinal sum t-conorm which has the idempotent element α, i.e., $S=(\langle 0,\alpha,S_a\rangle,\langle \alpha,1,S_b\rangle)$. Then the following statements hold:*

1) *Let I be an f- or (f,g)-generated implication. Then (S,I) is α-cross-migrativity if and only if S has the form as follows.*

$$S(x,y)=\begin{cases}\alpha S_a\left(\frac{x}{\alpha},\frac{y}{\alpha}\right), & \text{if } (x,y)\in[0,\alpha]^2,\\ f^{-1}\left(\frac{f(x)\cdot f(y)}{f(\alpha)}\right), & \text{if } (x,y)\in[\alpha,1]^2,\\ \max\{x,y\}, & \text{otherwise.}\end{cases}$$

2) *Let I_g be an g-generated implication. Then (S,I_g) is α-cross-migrativity if and only if S has the form as follows.*

$$S(x,y)=\begin{cases}\alpha S_a\left(\frac{x}{\alpha},\frac{y}{\alpha}\right), & \text{if } (x,y)\in[0,\alpha]^2,\\ g^{(-1)}\left(\frac{g(x)\cdot g(y)}{g(\alpha)}\right), & \text{if } (x,y)\in[\alpha,1]^2,\\ \max\{x,y\}, & \text{otherwise.}\end{cases}$$

3) *Let I_k be an k-generated implication. Then (S, I_k) is α-cross-migrativity if and only if S has the form as follows.*

$$S(x,y)=\begin{cases}\alpha S_a\left(\frac{x}{\alpha},\frac{y}{\alpha}\right), & \text{if } (x,y)\in[0,\alpha]^2,\\ k^{(-1)}\left(\frac{k(x)\cdot k(y)}{k(\alpha)}\right), & \text{if } (x,y)\in[\alpha,1]^2,\\ \max\{x,y\}, & \text{otherwise.}\end{cases}$$

4) *Let I_h be an h-generated implication. Then (S, I_h) is α-cross-migrativity if and only if S has the form as follows.*

$$S(x,y)=\begin{cases}\alpha S_a\left(\frac{x}{\alpha},\frac{y}{\alpha}\right), & \text{if } (x,y)\in[0,\alpha]^2,\\ h^{(-1)}\left(\frac{h(x)\cdot h(y)}{h(\alpha)}\right), & \text{if } (x,y)\in[\alpha,1]^2,\\ \max\{x,y\}, & \text{otherwise.}\end{cases}$$

5) *Let $I_{\theta,t}$ be a (θ,t)-generated implication. Then $(S, I_{\theta,t})$ is α-cross-migrativity if and only if S has the form as follows.*

$$S(x,y)=\begin{cases}\alpha S_a\left(\frac{x}{\alpha},\frac{y}{\alpha}\right), & \text{if } (x,y)\in[0,\alpha]^2,\\ \theta^{(-1)}(\min\{\theta(x)+\theta(y)-\theta(\alpha),1\}), & \text{if } (x,y)\in[\alpha,1]^2,\\ \max\{x,y\}, & \text{otherwise.}\end{cases}$$

Remark 3.30. (i) *From Theorem 3.29, it is easy to see the solutions for S is completely determined by the restriction on a portion of domain of S, i.e., $[\alpha,1]^2$, and have nothing to do with the remaining of $[0,1]^2$.*

(ii) *The characterization for the α-cross-migrativity of ordinal sum t-conorm $S = (\langle 0,\alpha,S_a\rangle, \langle \alpha,1,S_b\rangle)$ over $I_{f,g}$ is completely determined by the function f and independent of the function g.*

(iii) *Notice the g-generated implication I_g. When $g(1)=\infty$, $S\mid_{[\alpha,1]^2}(x,y) = g^{-1}\left(\frac{g(x)\cdot g(y)}{g(\alpha)}\right)$ is isomorphism to a strict t-conorm; When $g(1)<\infty$, $S\mid_{[\alpha,1]^2}(x,y) = g^{(-1)}\left(\frac{g(x)\cdot g(y)}{g(\alpha)}\right)$ is isomorphism to a nilpotent t-conorm. The function h has a similar principle.*

(iv) *For the case that α is not the idempotent element of the continuous t-conorm, we have already obtained the necessary and sufficient conditions. However, they are not in concise forms.*

4 Conclusions and future work

In this paper, we mainly focus on the α-cross-migrativity involving fuzzy implications generated by generators of continuous Archimedean t-norms and t-conorms. Full characterizations for the α-cross-migrativity of continuous t-conorms over (f,g)-, k-, h-, and (θ,t)-generated implications are obtained. When the t-conorm S is a continuous Archimedean t-conorm, the solutions for S are completely determined by the restriction on a portion of the domain of S, i.e., $[0,1]\times[\alpha,1]$; On the other hand, when the t-conorm S is an ordinal sum t-conorm, the solutions for S are completely determined by the restriction on a portion of domain of S, i.e., $[\alpha,1]^2$. The key results are summarized in Table 1.

Our work has contributed to the study of the α-cross-migrativity of t-conorms over fuzzy implications. As further theoretical work, we are concerned with the following issues.

- To find a concise form of the characterizations for the α-cross-migrativity of continuous t-conorm, which does not have α as its idempotent element, over generated implications.
- To obtain the characterizations for the continuous disjunctive uninorms over fuzzy implication generated by additive generators of representable uninorms [20, 23].

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