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Approximate reasoning based on similarity of Z-numbers

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Abstract

The concept of Z-number was introduced by Zadeh in order to deal with partial reliability of information. This concept describes a fusion of fuzzy and probabilistic types of uncertainty. In turn, one of the main fields of dealing with imperfect information is approximate reasoning. For the case of pure fuzzy information this field is well-developed. In contrast, existing studies on reasoning with Z-valued "if-then" rules are scarce. One of the main reasons is high analytical and computational complexity. In this work, we develop an approach to reasoning with such kind of rules. The original approach proposed here allows to deal with sparse rule base and is characterized by relatively low computational complexity. The new concept of similarity of Z-numbers based on Jaccard similarity index and measure of divergence of probability distributions is introduced. Based on similarity degrees of current input Z-numbers and Z-numbers located in rule antecedents, weights of linear combination of Z-numbers in rule consequents are determined. The linear combination is based on operations with Z-numbers proposed by authors. Applications of the proposed approach to evaluation of economic development level for a country and control problem are considered.

Keywords: Approximate reasoning, If-Then rules, fuzzy number, probability density function, reliability, Z-number.

1 Introduction

Approximate reasoning is a well-developed field of study originated in the work of Zadeh [38]. The recent studies and their applications can be found in [8, 14, 16, 17, 18, 21, 22, 28, 29, 32, 33, 34, 37, 41]. Mainly, two types of approaches to reasoning exist: compositional rule of inference-based approach [8, 36] and interpolative reasoning [19, 39]. In the first approach, the reasoning is implemented by using a composition of a current fuzzy input with a fuzzy relation matrix [30]. The main problem is that it can be used only when a current fuzzy input intersects with antecedents of at least one rule in a rule base. However, it is not often possible when fuzzy rule base is sparse. The second approach was developed in order to deal with sparse rule base. For a current fuzzy input, the corresponding fuzzy output is computed as an interpolation of rule consequents. It is not required for current input to intersect antecedents of any rule. This technique is characterized by relatively low computational complexity. The first works of interpolative reasoning were proposed in [19, 39]. In these works, the resulting output of reasoning is computed as a weighted combination of rule consequents. The weights are determined based on the distances between a current input and rule antecedents.

Let us overview some recent works in the field of approximate reasoning. In [10], a new interpolative reasoning method is proposed. The method is based on involving similarity measure for polygonal fuzzy sets. A novel technique

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is used to resolve contradictory results of fuzzy interpolative reasoning. An application to diarrheal disease prediction problem is considered.

To cope with uncertainty and high complexity of information, recent works are devoted to reasoning with extensions of fuzzy sets and/or combination of fuzzy information and probabilistic information. A reasoning method based on a new similarity measure for interval-valued fuzzy sets is proposed in [23]. The method is developed for Fuzzy Modus Ponens and Fuzzy Modus Tollens rules. The authors also discuss robustness of the method and consider its generalization.

Novel similarity measures of spherical fuzzy sets are introduced in [11]. The proposed measures are applied to a diagnosis problem.

In [12] the authors propose four kinds of intuitionistic similarities based on implication operator. These similarities are analyzed by using topological method. A method of intuitionistic reasoning is analyzed by using the considered similarities.

A series of works is devoted to studying connections between fuzzy interpolative and compositional inference methods. An approach to similarity-based inference that allows to describe relation between similarity and existing fuzzy inference approaches is proposed in [31]. An axiomatic foundation for the formation of a suitable similarity measure is given. The proposed method is applied to fuzzy control of inverted pendulum. An innovative fuzzy inference approach is introduced in [24]. This approach allows to select nearest weighted rules for firing in compositional inference and interpolative reasoning. A systematic experimental analysis of the proposed approach is conducted. A new method of similarity-based inference for interval-valued fuzzy information and its combination with conventional compositional approach is proposed in [13]. In [27] they propose similarity index to modify a fuzzy relation in fuzzy inference system. A comparative analysis of different existing similarity indexes is conducted. One of the indexes is used for rule selection and fuzzy relation modification. Finally, an integration of Zadeh's compositional rule of inference and similarity-based inference approaches is proposed. An application to modeling DC shunt motor is considered.

A series of works is devoted to reasoning under fuzziness and probabilistic uncertainty. In [25], fuzzy sets with randomized membership functions based on a probability density function are considered. However, in real-world problems, information needed to obtain an actual probability density function is incomplete. Thus, the use of probabilistic fuzzy If-Then rules may not be adequate. In [15, 35] fuzzy belief-rule based systems are considered where each consequent part is composed of a fuzzy value and an associated precise probability. However, assigning a precise accurate probability of a fuzzy value is counterintuitive and difficult due to absence of relevant information.

As a new granular formalization of fuzziness and probabilistic information, Zadeh introduced the concept of a Z-number [40]. A systematic review of the works devoted to this concept may be found in [9]. A Z-number is a pair of fuzzy numbers used to describe a value of a random variable X. A is a fuzzy restriction on values of X. B is considered as a fuzzy reliability of A, and is described as a value of probability measure of A. Imprecision of B stems from imprecise information about actual probability distribution related to X. In other words, fuzziness of B implies existence of a fuzzy set G of probability distributions of X. Existing studies on reasoning with Z-valued if-then rules are scarce. In [7] they consider a generalization of fuzzy interpolative reasoning for Z-valued if-then rules. Similarity of current input Z-number with Z-number in rule antecedent relies on the Jaccard similarity index for A and B parts of Z-numbers. However, latent probability distributions are not considered. Based on the obtained similarity values for all antecedent variables, similarity levels of current input vector with each rule are computed. These levels are then used to compute interpolation weights for rule consequents. The resulting output is found as weighted sum of Z-number-valued consequents. Applications in educational performance and job satisfaction evaluation problems are considered. Alternative approaches are proposed in [1, 2]. However, these works also lack from consideration of internal structure of Z-number.

It is of interest to develop a systematic approach to interpolative reasoning due to two main reasons. On the one hand, Z-rules are naturally scarce because Z-number encompasses a fusion of fuzzy and probabilistic uncertainties. It is difficult to obtain Z-numbers covering all the universe of discourse due to lack of information and knowledge. On the other hand, compositional rule of inference-based approach for Z-rules would be quite complex. Based on these two aspects, we propose an approach to reasoning with Z-valued rules based on Z-numbers similarity. A definition of the similarity measure is introduced relying on similarity degrees of A, G, and B parts of Z-numbers. As a way to quantify similarity of probability distributions, well-known Jensen-Shannon divergence measure is used. Based on similarity degrees of current input Z-numbers and Z-numbers in rule antecedents, weights of linear combination of Z-numbers in rule consequents are found. Applications of the proposed approach to evaluation of economic development level for hypothetic country and a control problem are considered.

The paper is structured as follows. Necessary definitions, including definitions of a Z-number, divergence of probability distributions, similarity of Z-numbers etc. are given in Section 2. In Section 3 we describe the proposed approach to reasoning with Z-rules. In Section 4, we consider real-world applications of the proposed approach in a problem of evaluation of economic development level for a country and a control problem. Section 5 offers some conclusions.

2 Preliminaries

Definition 2.1. [42] *Jaccard similarity S of two fuzzy sets A_1, A_2 defined over the same space is defined as*

$$S(A_1, A_2) = \frac{|A_1 \cap A_2|}{|A_1 \cup A_2|} \tag{1}$$

where $||$ *denotes cardinality of a fuzzy set.*

Definition 2.2. [5, 6, 40] *A continuous Z-number is an ordered pair $Z = (A, B)$. A is a continuous fuzzy number with membership function $\mu_A(x)$ playing a role of a fuzzy restriction on a value that a random variable X may take:*
Value of X is A.

In other words, A is used to describe imprecise information about a value of X. A degree of reliability of A is described as a value of probability measure $P(A) = \int_R \mu_A(x)p(x)dx$, where p is probability density function (pdf) of X. If p is precisely known, P(A) is a number. However, in real-world problems an actual p may not be precisely known, and one has to consider a set of pdfs. This requires dealing with a fuzzy restriction on a value of $P(A)$. Thus, to describe uncertainty related to p, a continuous fuzzy number B with a membership functionμ_B: $[0, 1] \to [0, 1]$ is used as a fuzzy restriction:
The value of $P(A)$ is B

This implies that fuzzy uncertainty related to p induces fuzziness of a value of $P(A)$. Thus, due to fuzziness and probabilistic uncertainty, a value of random variable X can be described as a Z-number $Z = (A, B)$: A is a fuzzy estimation of a value and B is a fuzzy reliability of this estimation.

A concept of a continuous Z^+-number is closely related to the concept of a continuous Z-number. Given a continuous Z-number$Z = (A, B)$, Z^+-number Z^+ is a pair consisting of a continuous fuzzy number,A , and pdf p_R [31]:

$$Z^+ = (A,\ p_R),$$

where A plays the same role as it does in a continuous Z-number $Z = (A, B)$ and $P(A) = \int_{\mathcal{R}} \mu_A(x)p_R(x)\text{dx}$, $P(A) \in supp(B)$, $\mathcal{R}$ is the set of real numbers.

A discrete Z-number is defined analogously:

Definition 2.3. [4, 6] *A discrete Z-number is an ordered pair $Z = (A, B)$ where A is a discrete fuzzy number playing a role of a fuzzy constraint on values that a random variable X may take:*

$$X \quad is \quad A,$$

and B is a discrete fuzzy number with a membership function μ_B:$\{b_1, ..., b_m\} \to [0, 1]$, $\{b_1, ..., b_m\} \subset [0, 1]$, playing a role of a fuzzy constraint on a value of probability measure $P(A) = \sum_{x \in \mathrm{X}} \mu_A(x)p(x)$ of A:

$$P(A) \quad is \quad B.$$

Both in continuous and discrete settings, the concept of Z-number is related to a set of probability distributions. Let us explain this issue more clearly.

A fuzzy set of probability density functions. A Z-number $Z = (A, B)$ is characterized by fuzzy number A, fuzzy number B and underlying fuzzy set G of pdfs p_R (or discrete probability distributions). Indeed, a set of pdfs is characterized by a possibilistic restriction π_G [40]:

$$\pi_G(p_R) = \mu_B\left(\int_{\mathcal{R}} \mu_A(x)p_R(x)dx\right). \tag{2}$$

For discrete case:

$$\pi_G(p_R) = \mu_B\left(\sum_{x \in \mathrm{X}} \mu_A(x)p_R(x)\right). \tag{3}$$

This implies that a degree of feasibility $\pi_G(p_R)$ of pdf p_R is equal to membership degree of $P(A) = \int_{\mathcal{R}} \mu_A(x)p_R(x)dx$ in fuzzy set B. That is, fuzzy set G of pdfs p_R is defined as:

$$G = \left\{(\pi_G(p_R), p_R) \middle| \pi_G(p_R) = \mu_B\left(\int_{\mathcal{R}} \mu_A(x)p_R(x)dx\right)\right\}. \tag{4}$$

For convenience of notation, we will use p instead of p_R.
For discrete case, one has:

$$G=\left\{(\pi_G(p_k),p_k)\,\middle|\,\pi_G(p_k)=\mu_B\left(\sum_{x\in \mathrm{X}}\mu_A(x)p_k(x)\right)\right\},k=1,\cdots,K, \quad (5)$$

where K is the cardinality of the support of discrete fuzzy number B (number of points), because each probability distribution p_k is induced by some $b_k \in B$, that is, p_k satisfies:

$$b_k=\sum_{x\in \mathrm{X}}\mu_A(x)p_k(x). \quad (6)$$

According to the structure of $Z=(A,B)$, we will propose a technique for measuring similarity between two Z-numbers. Consider the following concepts.

Definition 2.4. [20] *Given two pdfs p and q of a continuous random variable X, KLD is defined as*

$$KLD(p,q)=\int p(x)\log\left(\frac{p(x)}{q(x)}\right)dx. \quad (7)$$

For a discrete random variable X, KLD of its probability distributions p and q is defined as

$$KLD(p,q)=\sum_{x\in X}\left(p(x)log\left(\frac{p(x)}{q(x)}\right)\right), \quad (8)$$

where X is the range of discrete random variable X.

KLD is a well-known measure used to evaluate difference of two pdfs. As this measure is not symmetric, the following concept is used:

Definition 2.5. [26] *Jensen-Shannon Divergence (JSD) is defined as*

$$JSD(p,q)=\frac{1}{2}\left(KLD\left(p,\frac{p+q}{2}\right)+KLD\left(q,\frac{p+q}{2}\right)\right). \quad (9)$$

Being the total divergence to the average distribution $\frac{p+q}{2}$, JSD is a famous symmetrization of the KLD measure. It satisfies the relationships $0\leq JSD(p,q)\leq\log 2$.

Definition 2.6. *Similarity of two fuzzy sets of distributions $G_j, j=1,2$ denoted $S(G_1,G_2)$ is defined as*

$$S\left(G_1,G_2\right)=1-D_{set}(G_1,G_2), \quad (10)$$

where $D_{set}(G_1,G_2)=\frac{\sum_{i=1}^{n}\alpha_i D_{set}(G_1^{\alpha_i},G_2^{\alpha_i})}{\sum_{i=1}^{n}\alpha_i}$. $G_1^{\alpha_i},G_2^{\alpha_i}$ are classical (non-fuzzy) sets of distributions as α-cuts of fuzzy sets G_1,G_2:

$$G_j^{\alpha_i}=\{p{:}\pi_G\left(p\right)\geq\alpha_i\},\ \alpha_i\in\left(0,1\right],\ j=1,2. \quad (11)$$

To define $D_{set}(G_1^{\alpha_i},G_2^{\alpha_i})$, we adopt the concept of Hausdorff distance. $D_{set}(G_1^{\alpha_i},G_2^{\alpha_i})$ is the divergence between $G_1^{\alpha_i},G_2^{\alpha_i}$ defined as (on the basis of JSD measure):

$$D_{set}\left(G_1^{\alpha_i},G_2^{\alpha_i}\right)=\max\{max_{p_1\in G_1^{\alpha_i}}(min_{p_2\in G_2^{\alpha_i}}\ JSD(p_1,p_2)),max_{p_2\in G_2^{\alpha_i}}(min_{p_1\in G_1^{\alpha_i}}\ JSD(p_1,p_2))\}. \quad (12)$$

Similarity of fuzzy sets of discrete probability distributions is defined analogously.

Definition 2.7. *Similarity of Z-numbers Z_1,Z_2 is defined as*

$$S(Z_1,Z_2)=v_1S(A_1,A_2)+v_2S(B_1,B_2)+v_3S(G_1,G_2), \quad (13)$$

where $S(A_1,A_2)$ and $S(B_1,B_2)$ are values of Jaccard similarity of fuzzy numbers, v_l is a weight describing importance of $v_l\epsilon\left[0,1\right],\sum_{l=1}^{3}v_l=1$.

Note that B and G are closely related (B is induced by G) and an emphasis can be put only to one of them, e.g. $v_1 = v_3 \gg v_2$. In a special case, when one is not certain about values of mean, variance or other characteristics of distributions (when extracting latent distributions, to only rely on value of $P(A)$) a simple average, $v_1 = v_2 = v_3 = \frac{1}{3}$, can be used. That is, one can use $v_3 = v_2$ to account for B parts, when the sets of distributions G are considered as some approximation. In the paper we will use (13) as simple average.

Similarity of Z+-numbers $Z^+{}_1, Z^+{}_2$ is defined as

$$S(Z^+{}_1, Z^+{}_2) = \frac{1}{2}\left(S(A_1, A_2) + S(G_1, G_2)\right), \tag{14}$$

where $S(G_1, G_2) = S(p_1, p_2) = 1 - JSD(p, q)$.
Consider some illustrative examples.

Example 2.8. *(Similarity of continuous Z-numbers) Compute $S(Z_1, Z_2)$ for Z-numbers with components described as triangular fuzzy numbers (TFNs): $Z_1 = ((1,2,3),(0.7,0.8,0.9))$, $Z_2 = ((2,3,4),(0.6,0.7,0.8))$. The obtained results: $S(A_1, A_2) = 0.144$, $S(B_1, B_2) = 0.14$, $S(G_1, G_2) = 0.85$. According to Definition 2.7, $S(Z_1, Z_2) = 0.38$.*

Assume now that $B_1 = B_2 = (0.6, 0.7, 0.8)$. Naturally, $S(A_1, A_2) = 0.144$, $S(B_1, B_2) = 1$. In turn, $S(G_1, G_2) = 0.87$. Thus, $S(Z_1, Z_2) = 0.67$. It is intuitive that overall similarity increases once B parts become equal.

Example 2.9. *(Similarity of discrete Z-numbers) Compute $S(Z_1, Z_2)$ for $Z_1 = ((1, 1.5, 4),(0.8, 0.85, 0.9))$, $Z_2 = ((1, 2, 4),(0.8, 0.85, 0.9))$. At first, we computed similarity values of A and B parts: $S(A_1, A_2) = 0.75$, $S(B_1, B_2) = 1$. Next, we compute $S(G_1, G_2)$. For simplicity, we consider fuzzy sets with 3 distributions in each G_i, $i = 1, 2; k = 1, \ldots, 3$:*

$$G_i = \{(\pi_{G_i}(p_{i,1}) = 0.1, p_{i,1}),\ (\pi_{G_i}(p_{i,2}) = 0.1, p_{i,2}), (\pi_{G_3}(p_{i,3}) = 0.1, p_{i,3})\}$$

The discrete probability distributions $p_{1,k}$ are given in Tables 1 and 2.

Table 1. Fuzzy set G_1 of probability distributions

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	...	4.0
$p_{1,1}$(x)	0	0	0.8	0	0.1	0.1	0	0	0	...	0
$p_{1,2}$(x)	0	0	0.5	0.2	0.2	0.1	0	0	0	...	0
$p_{1,3}$(x)	0	0	0.2	0.5	0.2	0.1	0	0	0	...	0

Table 2. Fuzzy set G_2 of probability distributions

(x)	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	...	4.0
$p_{1,1}$(x)	0	0	0	0	0.29	04	0.26	0.21	0.13	...	0
$p_{1,2}$(x)	0	0	0	0	0.2	0.3	0.2	0.2	0.1	...	0
$p_{1,3}$(x)	0	0	0	0	0.1	0.5	0.2	0.1	0.1	...	0

Let us compute $S(G_1, G_2)$ (Definition 2.6). First, we obtained $JSD(p_1, p_2)$ $p_1 \epsilon G_1^{\alpha_i}, p_2 \epsilon G_2^{\alpha_i}$, $\alpha_1 = 0.1$, $\alpha_2 = 1$, as shown below.

For $\alpha_1 = 0.1$:

$$JSD(p_1, p_2) = JSD(p_{1,1}(x), p_{2,1}(x)) = 0.75, JSD(p_1, p_2) = JSD(p_{1,1}(x), p_{2,2}(x)) = 0.69,$$

$$JSD(p_1, p_2) = JSD(p_{1,1}(x), p_{2,3}(x)) = 0.69;$$

$$JSD(p_1, p_2) = JSD(p_{1,2}(x), p_{2,1}(x)) = 0.68, JSD(p_1, p_2) = JSD(p_{1,2}(x), p_{2,2}(x)) = 0.63,$$

$$JSD(p_1, p_2) = JSD(p_{1,2}(x), p_{2,1}(x)) = 0.64;$$

$$JSD(p_1, p_2) = JSD(p_{1,3}(x), p_{2,1}(x)) = 0.64, JSD(p_1, p_2) = JSD(p_{1,3}(x), p_{2,2}(x)) = 0.59,$$

$$JSD(p_1, p_2) = JSD(p_{1,3}(x), p_{2,3}(x)) = 0.62.$$

For $\alpha_1 = 1$: $JSD(p_1, p_2) = JSD(p_{1,2}(x), p_{2,2}(x)) = 0.63$.

According to (12), we have:

$$D_{set}(G_1^{\alpha_1}, G_2^{\alpha_1}) = 0.69, D_{set}(G_1^{\alpha_2}, G_2^{\alpha_2}) = 0.63.$$

and $D_{set}(G_1, G_2) = \frac{\sum_{i=1}^{2} \alpha_i D_{set}(G_1^{\alpha_i}, G_2^{\alpha_i})}{\sum_{i=1}^{2} \alpha_i} = 0.634$. By using (10), we obtain $S(G_1, G_2) = 0.37$. Then, $S(Z_1, Z_2) = 0.7$.

Example 2.10. *(Similarity of discrete Z+-numbers) Compute* $S(Z^+{}_1, Z^+{}_2)$ *for* $Z^+{}_1 = ((1, 1.5, 4), p_1)$, $Z^+{}_2 = ((1, 2, 4), p_2)$, *where* $p_1(x)$ *and* $p_2(x)$ *are shown in Table 3.*

Table 3. Discrete probability distributions

x	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	...	4.0
$p_1(x)$	0.0	0.0	0.5	0.2	0.2	0.1	0.0	0.0	0.0	...	0.0
$p_2(x)$	0.0	0.0	0.0	0.0	0.2	0.3	0.2	0.2	0.1	...	0.0

Let us compute similarity $S(Z^+{}_1, Z^+{}_2)$ *. According to Definition 2.1,* $S(A_1, A_2) = 0.75$*. Based on Definition 2.5,* $JSD\,(p_1, p_2) = 0.63$*. Then, according to Definition 2.7,* $S\,(G_1, G_2) = S\,(p_1, p_2) = 1 - JSD\,(p, q) = 0.37$ *and* $S\,(Z^+{}_1, Z^+{}_2) = 0.56$*.*

3 Similarity-based reasoning with Z-valued rules

We propose an approach to reasoning that allows to consider latent probability distributions when measuring similarity of current input and rule antecedents. The resulting output is computed as weighted sum of rule consequents.

Let us consider a problem of approximate reasoning with Z-valued rules [7] described below.

Given the following Z-rules:

Rule 1: If X_1*is* $Z_{X_1,1} = (A_{X_1,1}, B_{X_1,1})$ *and,..., and* X_m*is* $Z_{X_m,1} = (A_{X_m,1}, B_{X_m,1})$*then* Y*is* $Z_Y = (A_{Y,1}, B_{Y,1})$,

Rule 2: If X_1*is* $Z_{X_1,2} = (A_{X_1,2}, B_{X_1,2})$ *and,..., and* X_m*is* $Z_{X_m,2} = (A_{X_m,2}, B_{X_m,2})$*then* Y*is* $Z_Y = (A_{Y,2}, B_{Y,2})$,

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.

.

Rule n: If X_1*is* $Z_{X_1,n} = (A_{X_1,n}, B_{X_1,n})$ *and,..., and* X_m*is* $Z_{X_m,n} = (A_{X_m,n}, B_{X_m,n})$*then* Y*is* $Z_Y = (A_{Y,n}, B_{Y,n})$,

and a current observation

X_1 *is*$Z'_{X_1} = (A'_{X_1}, B'_{X_1})$ *and ,..., and* X_m*is*$Z'_{X_m} = (A'_{X_m}, B'_{X_m})$,

find the Z-value of Y, Z'_Y*.*

The resulting output is determined as

$$Z'_Y = \sum_{j=1}^{n} w_j Z_{Y,j}, \tag{15}$$

where $Z_{Y,j}$ is the Z-valued consequent of the j-th rule, w_j, $j = 1, ..., n$, are coefficients of linear interpolation, n is the number of Z-rules. The operations of scalar multiplication and addition are defined as in [4, 5, 6].

The solution algorithm is as follows.

Algorithm Reasoning with Z-valued rules

Step 0. Assign threshold to similarity $S_v(Z', Z_j)$: $c\epsilon[0, 1]$.

Step 1. Get current observation vector $Z' = (Z'_{X_1}, ...Z'_{X_m})$.

Step 2. For rule j=1,...,n, compute similarity of current observation vector $Z' = (Z'_{X_1}, ...Z'_{X_m})$ and rule antecedent vector $Z_j = (Z_{X_1,j}, ..., Z_{X_m,j})$, as a minimum similarity of the vectors components:

$$S_v(\mathbf{Z}', \mathbf{Z}_j) = \min_{i=1,...,m} S(Z'_{X_i} Z_{X_i,j}), \tag{16}$$

where similarity of Z-numbers $Z'_{X_i} = \left(A'_{X_i}, B'_{X_i}\right)$, $Z_{X_i,j} = (A_{X_i,j}, B_{X_i,j})$ is computed by using Definition 2.7, formula (13).

Step 3. Determine the resulting output:

- Compute interpolation weights:

$$w_j = \frac{S_v(\mathbf{Z}', \mathbf{Z}_j)}{\sum S_v(\mathbf{Z}', \mathbf{Z}_k)}, \tag{17}$$

where threshold condition is satisfied: $S_v\left(\mathbf{Z}', \mathbf{Z}_j\right), S_v(\mathbf{Z}', \mathbf{Z}_k) \geq c$

- Resulting output is computed by using (15).

  Note that '*min*' operation is adopted in (16) as one of the most often used in approximate reasoning (e.g. to measure similarity of input vector with vector of rule antecedents in Mamdani and Sugeno systems). Alternatively, other t-norm operations as 'product' can also be used (Larsen model), according to existing tradition.

A flowchart of the algorithm is shown in Fig. 1.

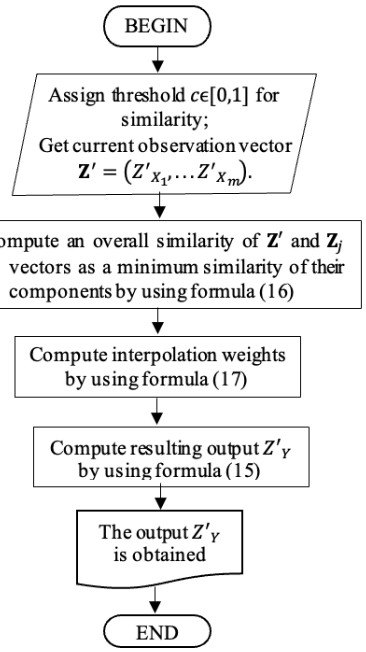

Figure 1: A flowchart of the algorithm.

In [7] the authors proposed a generalization of fuzzy interpolative reasoning for Z-valued if-then rules. A similarity of current input vector of Z-numbers with rule antecedents is computed based on distance between Z-numbers, or by using Jaccard similarity index. However, in both cases, latent probability distributions are not considered. This leads to loss of information and to departure from the concept of Z-number as a fusion of fuzzy and probabilistic information. In the approach proposed in this paper, we determine similarity between Z-number taking into account fuzzy numbers and related probability distributions.

In general, two types of the methods of approximate reasoning exist: the methods based on composition rule of inference and interpolation-based methods. The motivation of development and approach for reasoning with Z-number-based rules relying on the interpolation-based method include [its advantages [19]: 1) the capability of dealing with sparse rule base 2) reduced computational complexity. Indeed, for case of Z-valued rule base, development and application of composition method would be quite complex. This would require to propose a technique for reasoning with fuzzy set of distributions G. This, in turn, may require employing copula methods as a fundamental basis of probabilistic reasoning. As a result, a fuzzy restriction over copula functions would be considered to describe the relation between sets of distributions. That is, a complex approach to reasoning with fusion of fuzziness and probabilistic information should be developed.

In the proposed approach, complexity stems from a number of activated rules (may be reduced by using a threshold as considered above). Another main source of complexity is the structure of a set of probability distributions within a Z-number. One problem is to find a tradeoff between a number of considered probability distributions and accuracy of results (reducing number of considered distributions until similarity between two Z-numbers satisfies a predefined threshold). Analogously, an acceptable size of support of a distribution can be found. On the other hand, only probability distributions which satisfy reasonable restrictions on a type of distribution, values of parameters (mean, variance, entropy measure) can be considered. When such information is available, the complexity level may be reduced.

4 Applications

4.1 An evaluation of a level of economic development of a country

In [32], it is considered a problem of fuzzy If-Then rules-based evaluation of a level of economic development of a country given the influential factors as GDP, GDP per capita and unemployment rate. The author used three different rule bases, each base for countries with low, medium and high population. Let us consider a generalization of the rule base for countries with small population to the case of Z-number-valued rules. This implies we assume that information on these factors and the development level is characterized by fuzziness and probabilistic uncertainty. In view of this, consider the following Z-valued If-Then rules:

Rule 1: If GDP is (low, very sure) and GDP per capita is (low, very sure) and unemployment rate is (medium, very sure), then the level of economic development is (low, sure)

Rule 2: If GDP is (medium, very sure) and GDP per capita is (medium, very sure) and unemployment rate is (low, very sure), then the level of economic development is (medium, sure).

Rule 3: If GDP is (high, very sure) and GDP per capita is (medium, very sure) and unemployment rate is (medium, very sure), then the level of economic development is (medium, sure).

Rule 4: If GDP is (medium, very sure) and GDP per capita is (medium, very sure) and unemployment rate is (high, very sure), then the level of economic development is (low, sure).

Rule 5: If GDP is (high, very sure) and GDP per capita is (high, very sure) and unemployment rate is (low, very sure), then the level of economic development is (high, sure).

Rule 6. If GDP is (medium, very sure) and GDP per capita is (high, very sure) and unemployment rate is (medium, very sure), then the level of economic development is (high, sure).

Rule 7. If GDP is (low, very sure) and GDP per capita is (high, very sure) and unemployment rate is (high, very sure), then the level of economic development is (medium, sure).

Assume that random variables in the considered rules are described by normal pdfs. Codebooks of linguistic terms used in the rules are shown in Figs. 2-5 (A parts of Z-numbers) and Fig. 6 (B parts of Z-numbers).

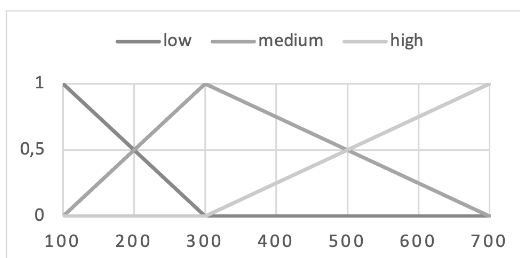

Figure 2: Membership functions for GDP (Million USD)

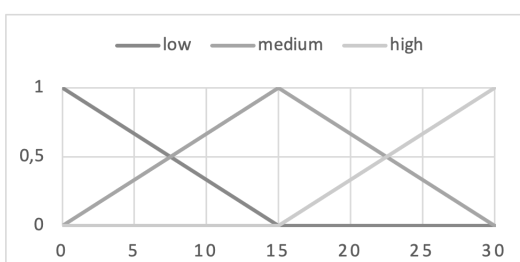

Figure 3: Membership functions for GDP per capita.

For example, the components of Z-number (medium, very sure) are described in Fig. 7. Let us apply the proposed approach to reasoning with these rules. Assume that input and output random variables in the rules are Gaussian. Consider the following current input values:

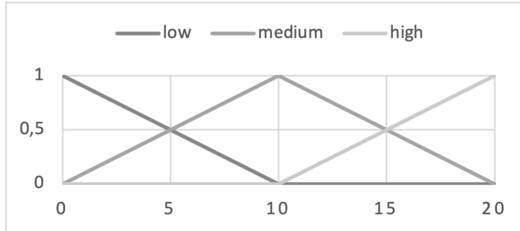

Figure 4: Membership functions for Unemployment.

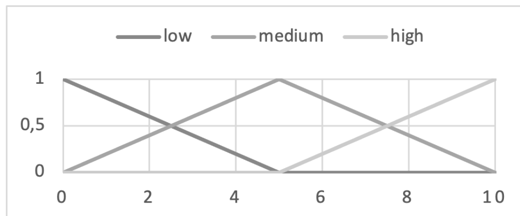

Figure 5: Membership functions for the level of economic development.

GDP is ((100,200,300),(0.8,0.85,0.9)),
GDP per capita is ((15,20,30),(0.8,0.85,0.9)),
unemployment rate is ((0,5,10),(0.75,0.8,0.85)).

According to Section 3, at step 0, we assign threshold value. At first, consider c=0. At step 1, we compute similarity of current observation $\mathbf{Z}' = ({Z'}_{X_1}, ...{Z'}_{X_3})$ and vector of rule antecedents $\mathbf{Z}_j = (Z_{X_1,j}, ..., Z_{X_3,j})$, for rule j=1,. . . ,7 (see (16)):

$$S_v(\mathbf{Z'}, \mathbf{Z}_j) = \min_{i=1,...,3} S({Z'}_{X_i} Z_{X_i,j}).$$

At first, we need to compute $S\left({Z'}_{X_i} Z_{X_i,j}\right)$, $i=1,\ldots,3$ (see Definition 2.7) to evaluate $S_v(\mathbf{Z'}, \mathbf{Z}_j)$ for each rule. For example, for rule 1, we compute:

$$S\left(Z'_{X_1} Z_{X_1,1}\right) = S\left(\left(Z'_{X_1} = \left((100, 200, 300), (0.8, 0.85, 0.9)\right), Z_{X_1,1} = (\text{low}, \ \text{very sure})\right)\right) = 0.53,$$

$$S\left(Z'_{X_2} Z_{X_2,1}\right) = S\left(\left(Z'_{X_2} = \left((15, 20, 30), (0.8, 0.85, 0.9)\right), Z_{X_2,1} = (\text{low}, \ \text{very sure})\right)\right) = 0.33,$$

$$S\left(Z'_{X_3} Z_{X_3,1}\right) = S\left(\left(Z'_{X_3} = \left((0, 5, 10), (0.75, 0.8, 0.85)\right), Z_{X_3,1} = (\text{medium}, \ \text{very sure})\right)\right) = 0.3.$$

Thus, similarity of the current input with the rule 1 antecedents is found as:

$$S_v(\mathbf{Z'}, \mathbf{Z}_1) = \min\left(S\left(Z'_{X_1} Z_{X_1,1}\right), S\left(Z'_{X_2} Z_{X_2,1}\right), S\left(Z'_{X_3} Z_{X_3,1}\right)\right) = \min\left(0.53, 0.33, 0.3\right) = 0.3.$$

Analogously, we obtained the results for all the remaining rules. The similarity values are shown in Table 4.

Table 4. Title Similarity of current observation with all the rules antecedents

Rule, j	1	2	3	4	5	6	7
$S_v(\mathbf{Z'}, \mathbf{Z}_j)$	0.3	0.269	0.3	0.067	0.269	0.3	0.067

At *step* 2 we infer the resulting output. At first, we obtain the coefficients of linear interpolation of rule consequents, $w_j = \frac{S_v(\mathbf{Z'},\mathbf{Z}_j)}{\sum_{k=1}^{n} S_v(\mathbf{Z'},\mathbf{Z}_k)}$. The obtained results are: $w_1 = 0.19$, $w_2 = 0.17$, $w_3 = 0.19$, $w_4 = 0.04$, $w_5 = 0.17$, $w_6 = 0.19$, $w_7 = 0.04$.

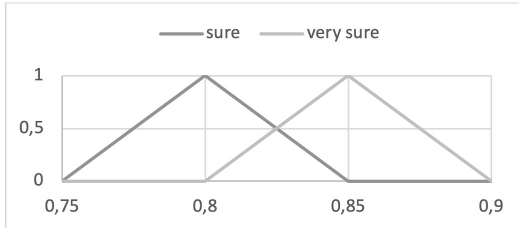

Figure 6: Membership functions for reliability of Z-numbers

Figure 7: The components of Z-number (medium, very sure): a) A part b) B part c) G set (some of distributions).

Finally, we computed the resulting output:

$$Z'_Y = \sum_{j=1}^{7} w_j Z_{Y,j} = (\,(1.8, 5.6, 8.8)\,, (0.81, 0.85, 0.89)\,).$$

According to the codebook (Figs. 2-6), $Z'_Y = (medium, very\ sure)$.

Let us analyze sensitivity of the obtained result. For this purpose, we changed a little only the Z-number of the 1^{st} input: *unemployment rate* is ((0.1,5.5,11),(0.8,0.85,0.9)). The obtained result is:

$$Z'_Y = \sum_{j=1}^{7} w_j Z_{Y,j} = (\,(1.7, 5.6, 8.9)\,, (0.81, 0.85, 0.88)\,).$$

Thus, the resulting input changes slightly at the slight change of input value.

Let us now analyze how the value of threshold c impact the results. In the considered example, assign threshold value c=0.27. According to Table 4, only 3 rules will be activated: rules# 1, 3 and 6. The resulting output is obtained as:

$$Z'_Y = w_1 Z_{Y,1} + w_3 Z_{Y,3} + w_6 Z_{Y,6} = (\,(1.7, 5, 8.3)\,, (0.83, 0.86, 0.9)\,).$$

This result is close to that obtained for *c=0*, the similarity is 0.77. However, three Z-numbers instead of six need to be used in obtaining the resulting output. Thus, the use of threshold value *c*=0.27 allows to reduce computational complexity.

Let us also conduct sensitivity analysis in terms of the coefficients of linear interpolation (as influencing factors of the technique). In the previous case, we have $w_1 = w_3 = w_6 = \frac{1}{3}$. For slightly changed values $w_1 = 0.37,\ w_3 = 0.33, w_6 = 0.3$, the following result is obtained:

$$Z'_Y = ((1.5, 4.67, 8.16)\,, (0.88, 0.9, 0.93)).$$

This results slightly differs with the previous one, and the linguistic approximation is the same $Z'_Y = (medium, very\ sure)$. Assume now that the coefficients are changed significantly: $w_1 = 0.67,\ w_3 = 0.33, w_6 = 0$. In this case we obtained the result which deviates much from the previous one:

$$Z'_Y = ((0, 1.65, 6.65)\,, (0.86, 0.89, 0.91)).$$

The linguistic approximation is also different to the previous one: $Z'_Y = (\text{low}, \text{very sure})$. Thus, coefficients of linear interpolation (induced by values of similarity of current inputs with rule antecedents) influence the obtained output significantly.

Let us consider influence of reliability parts in rule base on resulting output. We decreased the values of reliability (mainly in the rules with the highest similarity degrees $S_v(\mathbf{Z'}, \mathbf{Z}_j)$ to the current input). For new values of reliability, TFN-based description is used):

Rule 1: If GDP is (low, (0.5, 0.55, 0.6)) and GDP per capita is (low, (0.75, 0.80, 0.85)) and unemployment rate is (medium, (0.65, 0.7, 0.75)), then the level of economic development is (low, sure)

Rule 3: If GDP is (high, very sure) and GDP per capita is (medium, very sure) and unemployment rate is (medium, (0.65, 0.7, 0.75)), then the level of economic development is (medium, sure).

Rule 6: If GDP is (medium, very sure) and GDP per capita is (high, very sure) and unemployment rate is (medium, (0.65, 0.7, 0.75)), then the level of economic development is (high, sure).

Rule 7. If GDP is (low, (0.5, 0.55, 0.6)) and GDP per capita is (high, very sure) and unemployment rate is (high, very sure), then the level of economic development is (medium, sure).

The obtained result for the current input considered above is $Z'_Y = ((2.12, 6.6, 9.5), (0.8, 0.84, 0.88))$. The similarity of this result to that produced by using the initial rule base, $Z'_Y = ((1.8, 5.6, 8.8), (0.81, 0.85, 0.89))$, is 0.5. Thus, when reliability values in rule base change significantly, the result also changes significantly.

4.2 Reasoning with single-input-single-output Z-rules of controller

Consider Z-valued rules as a generalization of single-input-single-output fuzzy rules used in fuzzy controller [3]. The input is control error, E, and the output is control action, U. The rules are as follows:

If E *is (negative big, sure)* Then U *is (negative big, very sure)*
If E *is (negative medium, sure)* Then U *is (negative medium, very sure)*
If E *is (negative small, sure)* Then U *is (negative small, very sure)*
If E *is (zero, sure)* Then U *is (zero, very sure)*
If E *is (positive small, sure)* Then U *is (positive small, very sure)*
If E *is (positive medium, sure)* Then U *is (positive medium, very sure)*
If E *is (positive big, sure)* Then U *is (positive big, very sure)*

The codebooks for components of Z-numbers are given in Tables 5 and 6:

Table 5. Codebooks for A parts of input and output variables

Linguistic Term	TFN of A part of E	TFN of A part of U
negative big	(-10,-10,-7)	(-1,-1,-0.7)
negative medium	(-10,-7,-3)	(-1,-0.7,-0.3)
negative small	(-7,-3,0)	(-0.7,-0.3,0)
zero	(-3,0,3)	(-0.3,0,0.3)
positive small	(0,3,7)	(0,0.3,0.7)
positive medium	(3,7,10)	(0.3,0.7,1)
positive big	(7,10,10)	(0.7,1,1)

Table 6. Codebook for B parts

Linguistic Term	TFN
sure	(0.75,0.8,0.85)
very sure	(0.8,0.85,0.9)

Let us apply the proposed approach to reasoning with these rules. Assume that input and output random variables in the rules are normal. Let the current input is $Z_E = ((-10, -8, -6), (0.8, 0.9, 0.95))$. The obtained resulting output:

$$Z'_U = ((-0.74 - 0.5, -0.16), (0.82, 0.86, 0.9))$$

According to the codebooks (Tables 5,6), $Z'_U = (\textit{negative small}, \textit{very sure})$.

For different input (positive value of Error) $Z_E = ((4, 5, 6), (0.8, 0.9, 0.95))$, the obtained output is $Z'_U = ((0, 0.26, 0.57), (0.8, 0.86, 0.9))$. According to the codebooks, $Z'_U = (\textit{positive medium}, \textit{very sure})$. The rule base describes control law of chemical process and mainly imitates smart operator behavior. Results of huge experiments were estimated by human knowledge and fit his/her results. Here we show only two results of experiments.

5 Conclusion

The studies on reasoning with Z-valued If-Then rules are scarce. In existing works, information on probability distributions is missed in evaluation of similarity of current information with rule antecedents. A and B parts as fuzzy numbers (solely) are only considered. In the proposed work, latent probability distributions of Z-numbers are considered in all the stages of reasoning. For fuzzy sets of latent probability distributions, an aggregation of Jensen-Shannon divergence measure is adopted to compute similarity level. A motivation of using interpolation technique in development of the approach (as compared to composition rule of inference-based one) is discussed. Ways to decrease complexity of the approach stemming from consideration of Z-numbers instead of fuzzy counterparts are proposed which are related to the structure of a set of probability distributions.

Two applications are used to illustrate the approach: evaluation of economic development of a country and a control problem. The obtained results illustrate validity of the used approach. Particularly, sensitivity of inference results to influential factors are considered.

Future works may be related to application of the approach to more complex practical problems with imprecision and probabilistic information.

References

[1] R. H. Abiyev, *Dynamic system control using Z-number based fuzzy neural network.* In: C. Kahraman, S. Cevik Onar, B. Oztaysi, I. Sari, S. Cebi, A. Tolga, (eds) Intelligent and Fuzzy Techniques: Smart and Innovative Solutions. (Proceedings of International Conference on Intelligent and Fuzzy Systems, INFUS 2020, July 2020, Istanbul. Advances in Intelligent Systems and Computing, **1197**, 964-72. https://doi.org/10.1007/978-3-030-51156-2_112

[2] R. H. Abiyev, I. S. Günsel, N. Akkaya, E. Aytac, A. Çağman, S. Abizada, *Fuzzy control of omnidirectional robot*, Procedia Computer Science, **120** (2017), 608-616. https://doi.org/10.1016/j.procs.2017.11.286

[3] R. A. Aliev, F. Aliev, M. Babaev, *Fuzzy process control and knowledge engineering in petrochemical and robotic manufacturing*, Verlag, 1991.

[4] R. A. Aliev, A. V. Alizadeh, O. H. Huseynov, *The arithmetic of discrete Z-numbers*, Information Sciences, **290** (2015), 134-155. https://doi.org/10.1016/j.ins.2014.08.024

[5] R. A. Aliev, O. H. Huseynov, L. M. Zeinalova, *The arithmetic of continuous Z-numbers*, Information Sciences, **373** (2016), 441-460. https://doi.org/10.1016/j.ins.2016.08.078

[6] R. A. Aliev, W. Pedrycz, O. H. Huseynov, R. R. Aliyev, *Eigensolutions of partially reliable decision preferences described by matrices of Z-numbers*, International Journal of Information Technology and Decision Making, **19**(06) (2020), 1429-1450. https://doi.org/10.1142/s0219622020500340

[7] R. A. Aliev, W. Pedrycz, O. H. Huseynov, S. Z. Eyupoglu, *Approximate reasoning on a basis of Z-number-valued "If-Then" rules*, IEEE Transactions on Fuzzy Systems, **25**(6) (2017), 1589-600. https://doi.org/10.1109/tfuzz.2016.2612303

[8] R. A. Aliev, A. Tserkovny, *Systemic approach to fuzzy logic formalization for approximate reasoning*, Information Sciences, **181**(6) (2011), 1045-1059.https://doi.org/10.1016/j.ins.2010.11.021

[9] R. Banerjee, S. K. Pal, J. K. Pal, *A decade of the Z-numbers*, IEEE Transactions on Fuzzy Systems, **30**(8) (2022), 2800-2812. https://doi.org/10.1109/tfuzz.2021.3094657

[10] S. M. Chen, D. Barman, *Adaptive fuzzy interpolative reasoning based on similarity measures of polygonal fuzzy sets and novel move and transformation techniques*, Information Sciences, **489** (2019), 303-315. https://doi.org/10.1016/j.ins.2019.03.034

[11] Y. Donyatalab, F. Kutlu Gündoğdu, F. Farid, S. A. Seyfi-Shishavan, E. Farrokhizadeh, C. Kahraman, *Novel spherical fuzzy distance and similarity measures and their applications to medical diagnosis*, Expert Systems with Applications, **191** (2022). https://doi.org/10.1016/j.eswa.2021.116330

[12] J. Duan, X. Li, *Similarity of intuitionistic fuzzy sets and its applications*, International Journal of Approximate Reasoning, **137** (2021), 166-180. https://doi.org/10.1016/j.ijar.2021.07.009

[13] Z. Feng, C. Liu, H. Huang, *Knowledge modeling based on interval-valued fuzzy rough set and similarity inference: Prediction of welding distortion*, Journal of Zhejiang University-Science C, **15** (2014), 636-650. https://doi.org/10.1631/jzus.c1300370

[14] E. Hanyu, Y. Cui, W. Pedrycz, A. Robinson Fayek, Z. Li, J. Li, *Design of fuzzy rule-based models with fuzzy relational factorization*, Expert Systems with Application, (2022). https://doi.org/10.1016/j.eswa.2022.117904

[15] L. Jiao, Q. Pan, T. Denœux, Y. Liang, X. Feng, *Belief rule-based classification system: Extension of FRBCS in belief functions framework*, Information Sciences, **309** (2015), 26-49.

[16] I. U. Karim, M. F. Akkash, S. Raha, *Approximate reasoning with fuzzy soft set*, Iranian Journal of Fuzzy Systems, **19**(4) (2022), 107-124. https://doi.org/10.22111/ijfs.2022.7090

[17] Y. W. Kerk, K. M. Tay, C. P. Lim, *Monotone fuzzy rule interpolation for practical modeling of the zero-order TSK fuzzy inference system*, IEEE Transactions on Fuzzy Systems, **30**(5) (2022), 1248-1259. https://doi.org/10.1109/tfuzz.2021.3057239

[18] J. Kerr-Wilson, W. Pedrycz, *Generating a hierarchical fuzzy rule-based model*, Fuzzy Sets and Systems, **381** (2020), 124-139. https://doi.org/10.1016/j.fss.2019.07.013

[19] L. T. Kóczy, K. Hirota, *Approximate reasoning by linear rule interpolation and general approximation*, International Journal of Approximate Reasoning, **9**(3) (1993), 197-225. https://doi.org/10.1016/0888-613x(93)90010-b

[20] S. Kullback, R. A. Leibler, *On information and sufficiency*, The Annals of Mathematical Statistics, **22**(1) (1951), 79-86. https://doi.org/10.1214/aoms/1177729694

[21] Y. Li, X. Hu , W. Pedrycz, F. Yang, Z. Liu, *Multivariable fuzzy rule-based models and their granular generalization: A visual interpretable framework*, Applied Soft Computing, **134** (2023). https://doi.org/10.1016/j.asoc.2022.109958

[22] J. Li, Y. Liu, *Property analysis of triple implication method for approximate reasoning on Atanassov's intuitionistic fuzzy sets*, Iranian Journal of Fuzzy Systems, **15**(5) (2018), 95-116. https://doi.org/10.22111/ijfs.2018.4161

[23] M. Luo, Y. Wang, R. Zhao, *Interval-valued fuzzy reasoning method based on similarity measure*, Journal of Logical and Algebraic Methods in Programming, **113** (2020). https://doi.org/10.1016/j.jlamp.2020.100541

[24] H. Lv, F. Li, C. Shang, Q. Shen, *W-Infer-polation: Approximate reasoning via integrating weighted fuzzy rule inference and interpolation*, Knowledge-Based Systems, **258** (2022). https://doi.org/10.1016/j.knosys.2022.109995

[25] A. H. Meghdadi, T. M. R. Akbarzadeh, *Probabilistic fuzzy logic and probabilistic fuzzy systems*, 10th IEEE International Conference on Fuzzy Systems (Cat No01CH37297), (2001). https://doi.org/10.1109/fuzz.2001.1008853

[26] F. Nielsen, *On the Jensen-Shannon symmetrization of distances relying on abstract means*, Entropy, **21**(5) (2019), 485. https://doi.org/10.3390/e21050485

[27] A. Pal, B. Mondal, N. Bhattacharyya, S. Raha, *Similarity in fuzzy systems*, Journal of Uncertainty Analysis and Applications, **2**(1) (2014). https://doi.org/10.1186/s40467-014-0018-0

[28] W. Pedrycz, *Granular computing: Analysis and design of intelligent systems (Industrial Electronics)*, London: CRC Press, 2017.

[29] W. Pedrycz, *Granular modeling for the realization of fundamentals of control, systems analysis, and decision making*, Studies in Systems, Decision and Control, (2022), 3-25. https://doi.org/10.1007/978-3-030-99776-2-1

[30] W. Pedrycz, F. Gomide, *Fuzzy systems engineering: Toward human-centric computing*, New Jersey: Wiley-IEEE Press; 2007. https://doi.org/10.1002/9780470168967

[31] S. Raha, A. Hossain, S. Ghosh, *Similarity based approximate reasoning: Fuzzy control*, Journal of Applied Logic, **6**(1) (2008), 47-71. https://doi.org/10.1016/j.jal.2007.01.001

[32] G. Stojic, *Using fuzzy logic for evaluating the level of countries' (Regions') economic development*, Panoeconomicus, **3** (2012), 293-310. https://doi.org/10.2298/PAN1203293S

[33] Y. Tang, F. Ren, *Universal triple I method for fuzzy reasoning and fuzzy controller*, Iranian Journal of Fuzzy Systems, **10**(5) (2013), 1-24. https://doi.org/10.22111/ijfs.2013.1204

[34] T. Vetterlein, A. Zamansky, *Reasoning with graded information: The case of diagnostic rating scales in healthcare*, Fuzzy Sets and Systems, **298** (2016), 207-221. https://doi.org/10.1016/j.fss.2015.11.002

[35] Q. Wang, Z. Su, B. Rezaee, P. Wang, *Constructing T-S fuzzy model from imprecise and uncertain knowledge represented as fuzzy belief functions*, Neurocomputing, **166** (2015), 319-336. https://doi.org/10.1016/j.neucom.2015.03.068

[36] R. R. Yager, *On theory of approximate reasoning*, SBA Controle and Automacao, **4** (1994), 116-125.

[37] L. Yang, Q. Shen, *Adaptive fuzzy interpolation*, IEEE Transactions on Fuzzy Systems, **19**(6) (2011), 1107-1126. https://doi.org/10.1109/tfuzz.2011.2161584

[38] L. A. Zadeh, *Fuzzy logic and approximate reasoning*, Synthese, **30**(3-4) (1975), 407-28. https://doi.org/10.1007/bf00485052

[39] L. A. Zadeh, *Interpolative reasoning in fuzzy logic and neural network theory*, [1992 Proceedings] IEEE International Conference on Fuzzy Systems, (1992). https://doi.org/10.1109/fuzzy.1992.258757

[40] L. A. Zadeh, *A note on Z-numbers*, Information Science, **181** (2011), 2923-2932. https://doi.org/10.1016/j.ins.2011.02.022

[41] P. Zhang, C. Shang, Q. Shen, *Fuzzy rule interpolation with K-neighbors for TSK models*, IEEE Transactions on Fuzzy Systems, **30**(10) (2022), 4031-4043. https://doi.org/10.1109/tfuzz.2021.3136359

[42] H. J. Zimmermann, *Fuzzy set theory and its applications*, Springer, 2001, 514 pages.https://doi.org/10.1007/978-94-010-0646-0