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$L^{\mathcal{B}}$-valued general fuzzy automata and minimal determinization

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Abstract

Although a variety of methods have already been developed to convert and adapt a fuzzy automaton to its related language equivalent fuzzy deterministic finite automaton, they can still be applied merely for fuzzy automata which have been characterized over particular underlying sets of truth values. Filling this gap, thus, this study attempts to focus on developing a method for computing a minimal deterministic $L^{\mathcal{B}}$-valued general fuzzy automaton for an $L^{\mathcal{B}}$-valued GFA defined over a locally finite and divisible residuated lattice. This proposed method uses the concept of a reduction graph that helps us achieve a minimal deterministic $L^{\mathcal{B}}$-valued GFA. Accordingly, the present investigation aimed at establishing the notions related to L-valued language identified by $L^{\mathcal{B}}$-valued general fuzzy automata ($L^{\mathcal{B}}$-valued GFA) and also crisp deterministic $L^{\mathcal{B}}$-valued GFA $\tilde{F}^c$ equivalent to $L^{\mathcal{B}}$-valued GFA $\tilde{F}$. It then indicated the properties of $\tilde{F}^c$. The method of determinization through factorization of L-valued states and also a method concerning state reduction were proposed and studied in details. In particular, the main focus and contribution of this study was the automaton $\mathcal{H}(\tilde{F}^c)$ which is recognized as a deterministic $L^{\mathcal{B}}$-valued GFA that assures the necessary conditions intended for minimality and that its size is always equal or lesser than a minimal crisp deterministic $L^{\mathcal{B}}$-valued GFA equivalent to that. The related concepts and the results obtained in this study have also been clarified and explicated through representative examples.

Keywords: $L^{\mathcal{B}}$-valued general fuzzy automaton, minimal determinization method, factorization of L-valued states, locally finite lattices

1 Introduction

The concept of non-deterministic finite automata (NfAs) is regarded as appropriate mathematical models used to design language recognizer though; however, they are more computationally effective. It has been recognized as the main motive for devising sufficient processes in order to transfer a NfA into its equivalent DfA [11, 14]. The worst-case time complexity of this type of conversion is distinguished to be exponential concerning the input NfA size. One of the most suitable ways to compensate for this cost is that the determinization procedure leads to outputting minimal DfA rather than any other DfA which is equivalent to the input automaton. As reported, double reversal type of determinization algorithm by Brzozowski is regarded as one of the most prominent procedures with respect to minimal determinization [9].

Fuzzy finite automata (FfAs) have efficiently generalized NfAs, demonstrating actual useful applications in circumstances where there exist natural uncertainties. As concrete examples, it includes decision making, clinical monitoring, fault diagnosis, artificial intelligence, and also model checking (e.g. [6, 20, 21, 25, 33]). In these practical conditions, it is highly appropriate to change a FfA to an equivalent deterministic type. With regard to an output related to the determinization method, the previous studies have employed the concept of the fuzzy deterministic finite automaton (FDfA) [12]. As it is in a FDfA, initial and final states as well as transitions have been labeled with any truth value. As a particular case, FDfAs therefore include regular DfAs which have been provided with fuzzy final states. Moreover,

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these fuzzy automata have been known as crisp deterministic fuzzy finite automata (cDFfAs) in the existing literature [7]. An FDfA is considered minimal in the case when it demonstrates the least size amongst all FDfAs which are equivalent to that. According to the definitions proposed for FDfAs and cDFfAs, it can be concluded that: 1) the size for a minimal FDfA is regarded to be always equal or lesser than for a minimal cDFfA which is equivalent to that because a cDFfA has been a special type of an FDfA; and 2) a minimal FDfA may not be regarded as unique, since there may be many equivalent FDfAs with the same size which may demonstrate diverse values in their own components, with diverse topology, or both.

In their study, Doostfatemeh and Kremer [10], have explicated the concept of fuzzy automata, through which they proposed the notion of general fuzzy automata. Regarding that, the key impetus was the inadequacy of the obtain- able literature to handle the applications which employed fuzzy automata in the form of a modeling tool which allocates membership values to active states of related fuzzy automaton. A zero-weight transition has meant no transition in all types of conventional automata. In this approach that we have employed for general fuzzy automata, however, a zero-weight transition have not necessarily required no transition. It has been the main reason that we apply $[0, 1]$ as the fuzzy interval. The concept known as L^B-valued general fuzzy automata (L^B-valued GFA) has also been established in the studies by Abolpour and Zahedi [5], in which B is considered as a set of propositions concerning the general fuzzy automata, and where its underlying structure is a complete infinitely distiributive lattice. In addition, in their works, Zahedi and Abolpour and also a number of other researchers in the field have studied the procedures of how fuzzy automata theory have been developed [1, 2, 3, 4, 5, 22, 23, 24, 25, 27, 26, 28, 29, 31, 32].

In this study, $L^{\mathcal{B}}$-valued GFA concerning membership values which is in a complete residuated lattice is investigated. It aims at designing a determinization procedure for $L^{\mathcal{B}}$-valued GFAs with respect to membership values in a divisible and also locally finite $\mathcal{L}$ that brings back a minimal deterministic $L^{\mathcal{B}}$-valued GFA which is equivalent to the input $L^{\mathcal{B}}$-valued GFA. The design for this kind of procedure will be demanding since there may not be a unique minimal $L^{\mathcal{B}}$-valued GFA.

FfAs which are over a locally finite $\mathcal{L}$ only distinguish fuzzy languages related to finite image [7, 19]; though these are said to be determinizable in all cases via the fuzzy accessible subset construction [15]. The output as cDFfA is regarded as the Nerode automa- ton of A [16], indicated $N(A)$. Much more enhancements of this type of construction have also been obtained in [17, 18]. FfAs which are over a non-locally $\mathcal{L}$ finite may distinguish fuzzy languages of finite image. As a result, the methods of determinization for this type of automata must essentially bring back equivalent FDfAs. Determinization which is on the basis of fuzzy states factorization has been proposed in [13]. It converts a FfA A which has been over a divisible $\mathcal{L}$ into an equivalent FDfA, indicated $\mathcal{D}(A)$.

In this study, $\tilde{F}^c$ is denoted as a crisp deterministic $L^{\mathcal{B}}$-valued GFA which is equivalent to an $L^{\mathcal{B}}$-valued GFA $\tilde{F}$. After indicating common preliminaries for lattices, factorization and $L^{\mathcal{B}}$-valued GFAs, the properties of $\tilde{F}^c$ are presented. Deterministic $L^{\mathcal{B}}$-valued GFAs and their conditions for minimality are introduced in Subsection 3.1. In Subsection 3.2, the determinization method through factorization of L-valued states as well as a method concerning state reduction are proposed and studied. Subsection 3.3 encompasses the first contribution which is recognized as the automaton $\mathcal{H}(\tilde{F}^c)$, and is considered as the determinization through factorization of a crisp $L^{\mathcal{B}}$-valued GFA. It is a deterministic $L^{\mathcal{B}}$-valued GFA that fulfills the related necessary conditions for achieving minimality and that its size is said to be always equal or lesser than a minimal crisp deterministic $L^{\mathcal{B}}$-valued GFA which is equivalent to that. $\mathcal{H}(\tilde{F}^c)$ may not be a minimal deterministic $L^{\mathcal{B}}$-valued GFA; however, we demonstrate that, in this case, its L-valued states fulfill a strict order relation $\preceq$ that defines a graph, denoted $\zeta(\mathcal{H}(\tilde{F}^c))$, called the reduction graph of $\mathcal{H}(\tilde{F}^c)$. Using this graph, we can build all possible minimal deterministic $L^{\mathcal{B}}$-valued GFAs equivalent to $\tilde{F}$. This new contribution is provided in Section 4.

2 Preliminaries

In this part, some basic definitions and concepts which are associated with $L^{\mathcal{B}}$-valued general fuzzy automaton and complete residuated lattice are introduced and explicated.

Definition 2.1. [5] *A residuated lattice is an algebra* $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ *such that*
(i) $(L, \wedge, \vee, 0, 1)$ *is a lattice with the least element* 0 *and the greatest element 1,*
(ii) $(L, \otimes, 1)$ *is a commutative monoid with unit 1, and*
(iii) $\otimes$ *and* $\rightarrow$ *form an adjoint pair, i.e., for all* $a, b, c \in L, a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c$.
In addition, if $(L, \wedge, \vee, 0, 1)$ *is a complete lattice, then* $\mathcal{L}$ *is called a complete residuated lattice.*
The precomplement on L is the mapping $\neg : L \rightarrow L$ *such that* $\neg a = a \rightarrow 0, \forall a \in L$. *Some of the basic properties of complete residuated lattices, which we use, are as follows:*
(i) $a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c$;

(ii) $1 \to a = a$;
(iii) $(a \to b) \otimes (b \to c) \leq a \to c$;
(iv) $(a \otimes b) \to c = a \to (b \to c)$;
(v) $a \otimes (\,^{\circ} \vee_{i\in J} b_i) = \,^{\circ} \vee_{i\in J} (a \otimes b_i)$.
Much more properties of complete residuated lattices have been in [8]. *In the present study, we assume that* $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ *fulfills the conditions as follows:*
(i) *L is a totally ordered set w.r.t $\leq$.*
(ii) *$\mathcal{L}$ is divisible, i.e., for every $a, b \in L$ with $a \geq b$, there is $c \in L$ such that $a \otimes c = b$.*
(iii) *$(L, \otimes, 1)$ is zero-divisor free: for every $a, b \in L$, $a \otimes b = 0$ if and only if $a = 0$ or $b = 0$.*
(iv) *The algebra $\mathcal{L}^* = (L, \vee, \otimes, 0, 1)$, attained from $\mathcal{L}$, is locally finite. This is equivalent to assume that $(L, \otimes, 1)$ is locally finite* [19].

Let $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \tilde{\delta}, w, F_1, F_2)$ be a general fuzzy automaton. If we fix an input $a_k \in \Sigma$ at time t_i, the proposition $\alpha|_{a_k}$ can be calculated by $\mu^{t_i}(q_i)$ if the general fuzzy automaton $\tilde{F}$ is in the state q_i at time t_i otherwise $\alpha|_{a_k}$ is 0 if $\tilde{F}$ is not in the active state q_i. Accordingly, for each state $q_i \in Q$ we can examine the truth value of $\alpha|_{a_k}$, it is designated by $\alpha|_{a_k}(q_i)$. As mentioned above, $\alpha|_{a_k}(q_i) \in [0, 1]$. The aim of this section is therefore to drive the logic B which is a set of propositions about the general fuzzy automaton which is devised $\tilde{F}$ formulated by the observer and to construct a complete infinitely distributive lattice $\mathcal{B} = (B, \leq, \wedge, \vee, 0, 1)$. The order on B can be presented as follows:
For $\alpha, \beta \in B, \alpha \leq \beta$ if and only if $\alpha(q_i) \leq \beta(q_i)$ for all $q_i \in Q$. Concerning this, it can be examined instantly that the contradiction, i.e., the proposition with constant truth value 0, is the least component and the tautology, i.e., the proposition with constant truth value 1 is considered as the greatest component of the $\mathcal{B}$. Note that any component ith of 1 is the maximum membership values of active states at time t_i, for any $i \geq 0$.
Let $L = ([0, 1], \leq, \wedge, \vee, \otimes, \to)$ be a residuated lattice and $\mathcal{B} = (B, \leq, \wedge, \vee, 0, 1)$ be a complete infinitely distributive lattice of propositions about the general fuzzy automaton $\tilde{F}$.
The most explored and used structures of truth values, which have been defined on the real unit interval [0,1] with $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$, are related to the Łukasiewicz structure

$$(x \otimes y = \max(x + y - 1, 0), \qquad x \to y = \min(1 - x + y, 1)),$$

the product structure

$$(x \otimes y = x.y, x \to y = 1 \text{ if } x \leq y \text{ and } = \frac{y}{x} \text{ otherwise}),$$

and the Gödel structure

$$(x \otimes y = \min(x, y), x \to y = 1 \text{ if } x \leq y \text{ and } = y \text{ otherwise}).$$

We define $L^{\mathcal{B}}$-valued subset of $Q \times \Sigma \times Q$, i.e., a map $\delta : Q \times \Sigma \times Q \to L^{\mathcal{B}}$. The range set $L^{\mathcal{B}}$ allows us to consider $L^{\mathcal{B}}$ as a map assigning each (q, a_k, p) to $\delta_{a_k}(q, p) : B \to L$. This interpretation of transition map δ allows us to describe it as the family $\{\delta^{\alpha} : \alpha \in B\}$ of L-valued sets $\delta^{\alpha} \in L^{Q\times\Sigma\times Q}$ of $Q \times \Sigma \times Q$ ordered by the elements of B, where the L-valued sets δ^{α} are identified by

$$\delta^{\alpha}_{a_k}(q, p) = \delta_{a_k}(q, p)(\alpha) = \begin{cases} 1, & \text{if } q = p \\ \alpha|_{a_k}(q) \vee \alpha|_{a_k}(p), & \text{otherwise.} \end{cases}$$

Definition 2.2. [5] *An $L^{\mathcal{B}}$-valued general fuzzy automaton has been known as an 8- tuple $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$, where $\tilde{\delta}$ is an $L^{\mathcal{B}}$-valued subset of $(Q \times L) \times \Sigma \times Q$, i.e., a map $\tilde{\delta} : (Q \times L) \times \Sigma \times Q \to L^{\mathcal{B}}$ such that:*

$$\tilde{\delta}((q_i, \mu^t(q_i)), a_k, q_j)(\alpha) = F_1(\mu^t(q_i), \delta(q_i, a_k, q_j)(\alpha)).$$

Let Σ^* be a monoid which has been created by a nonempty set Σ. Define a map $\tilde{\delta}^* : (Q \times L) \times \Sigma^* \times Q \to L^{\mathcal{B}}$

$$\tilde{\delta}^*((q, \mu^t(q)), \wedge, p)(\alpha) = \tilde{\delta}((q, \mu^t(q)), \wedge, p)(\alpha) = \begin{cases} 1, & \text{if } \; q = p \\ 0, & \text{otherwise} \end{cases}$$

such that, $\forall q, p \in Q, \forall u \in \Sigma^*, \forall x \in \Sigma$ and $\forall \alpha \in \mathcal{B}$

$$\tilde{\delta^*}((q, \mu^{t_i}(q)), ux, p)(\alpha) = \vee\{\tilde{\delta}^*((q, \mu^{t_i}(q)), u, q')(\alpha) \otimes \tilde{\delta}((q', \mu^{t_j}(q')), x, p)(\alpha) | q' \in Q_{pred}(p, x)\}.$$

3 Minimal deterministic $L^{\mathcal{B}}$-valued general fuzzy automata

This section is subdivided into Subsections 3.1, 3.2 and 3.3. Deterministic $L^{\mathcal{B}}$-valued GFAs and their conditions for minimality are introduced in Subsection 3.1. In Subsection 3.2, the method of determinization through factorization of L-valued states as well as a method concerning state reduction have been proposed and studied in details. Subsection 3.3 encompasses the first contribution which is automaton $\mathcal{H}(\tilde{F}^c)$, and is regarded as the determinization through factorization of a crisp $L^{\mathcal{B}}$-valued GFA. It is a deterministic $L^{\mathcal{B}}$-valued GFA that fulfills the required conditions concerning minimality and its size is said to be always equal or less than a minimal crisp deterministic $L^{\mathcal{B}}$-valued GFA which is equivalent to that.

3.1 Deterministic $L^{\mathcal{B}}$-valued general fuzzy automata

Let us consider Q as a finite non-empty set of states. Any mapping $S : Q \to L^{\mathcal{B}}$, in brief $S \in (L^{\mathcal{B}})^Q$, is named $L^{\mathcal{B}}$-valued subset of Q. The range set $L^{\mathcal{B}}$ permits us to consider $L^{\mathcal{B}}$ as a map assigning each q to $S(q) : \mathcal{B} \to L$. This interpretation of map S permits us to show it as the family $\{S^\alpha : \alpha \in \mathcal{B}\}$ of L-valued sets $S^\alpha \in L^Q$ of Q ordered by the elements of $\mathcal{B}$, in which the L-valued sets S^α have been defined by $S^\alpha(q) = \alpha(q)$. The support of an L-valued state $S^\alpha \in L^Q$ is the subset of states defined by $Supp(S^\alpha) = \{q \in Q | S^\alpha(q) \neq 0\}$. In the present work, 0 denotes the L-valued state in which all states have value 0, i.e., $Supp(0) = \emptyset$. Correspondingly, the family $\{\delta_{a_k} : Q \times Q \to L^{\mathcal{B}} | a_k \in \Sigma\}$, in brief $\delta_{a_k} \in (L^{\mathcal{B}})^{Q\times Q}$, is known as $L^{\mathcal{B}}$-valued relation on Q for all $a_k \in \Sigma$. The degree of a transition from a state p to q under an $L^{\mathcal{B}}$-valued relation δ_{a_k} is demonstrated by the membership value $\delta^\alpha_{a_k}(p, q)$. The (crisp) equality relation on Q is indicated by $E^{\mathcal{B}}$, in which $E^\alpha_{a_k}(p, p) = 1$ for any $\alpha \in \mathcal{B}, a_k \in \Sigma$ and $p \in Q$ and 0 otherwise.

Let us consider $\delta^\alpha_{a_k}, \delta^\beta_{b_k} \in L^{Q\times Q}$; $S^\alpha, S^\beta \in L^Q$; and $\gamma \in B$. Subsequently, the products (a) $\delta^\alpha_{a_k} o \delta^\beta_{b_k} \in L^{Q\times Q}$, (b) $S^\alpha o \delta^\alpha_{a_k} \in L^Q$, (c) $\delta^\alpha_{a_k} o S^\beta \in L^Q$, (d) $S^\alpha o S^\beta \in L$, and (e) $\gamma \otimes S^\alpha \in L^Q$, are defined by:

(a) $(\delta^\alpha_{a_k} o \delta^\beta_{b_k})(p, q) = \vee_{p' \in Q} \delta^\alpha_{a_k}(p, p') \otimes \delta^\beta_{b_k}(p', q)$,

(b) $(S^\alpha o \delta^\alpha_{a_k})(q) = \vee_{p' \in Q} S^\alpha(p') \otimes \delta^\alpha_{a_k}(p', q)$,

(c) $(\delta^\alpha_{a_k} o S^\beta)(q) = \vee_{p' \in Q} \delta^\alpha_{a_k}(p, p') \otimes S^\beta(p')$,

(d) $S^\alpha o S^\beta = \vee_{p' \in Q} S^\alpha(p') \otimes S^\beta(p')$

(e) $(\gamma \otimes S^\alpha)(q) = \gamma(q) \otimes S^\alpha(q)$.

As $\otimes$ is commutative, product (e) can therefore be written $S^\alpha \otimes \gamma$. Trivially, this product is applied for $L^{\mathcal{B}}$-valued relations. As $\otimes$ distributes over $\vee$, product (a) is associative and its unity is $E^{\mathcal{B}}$.

Definition 3.1. *Given $\mathcal{L} = (L, \vee, \wedge, \otimes, \to, 0.1)$ and a set of states Q (non-empty). The function $k : (L^{\mathcal{B}})^Q \to L$ can be defined as follows:*
$k(S^\alpha) = \vee_{q \in Q} S^\alpha(q)$ for any $S^\alpha \in L^Q, S^\alpha \neq 0$; and $k(0) = 1$.
Through the use of the previous function k, a new function $h : (L^{\mathcal{B}})^Q \to (L^{\mathcal{B}})^Q$ can be defined as follows:

$$h(S^\alpha)(q) = k(S^\alpha) \to S^\alpha(q),$$

for any $S^\alpha \in L^Q$ and $q \in Q$. Let us assume that k and h have been well-defined functions. The pair of functions (h, k) fulfill the following properties for any $S^\alpha \in L^Q$, $q \in Q$ and $\gamma \in \mathcal{B}$:

(E_1) *$k(S^\alpha) \geq S^\alpha(q)$;*

(E_2) *$k(S^\alpha) > 0$;*

(E_3) *$h(S^\alpha) \geq S^\alpha$;*

(E_4) *$h(S^\alpha)(q) = 0 \Leftrightarrow S^\alpha(q) = 0$;*

(E_5) *$k(S^\alpha) \otimes h(S^\alpha) = S^\alpha$;*

(E_6) *$k(\gamma \otimes S^{\alpha}) = \gamma \otimes k(S^{\alpha})$ with $\gamma \neq 0$;*

(E_7) *$h(S^{\alpha}) \leq h(\gamma \otimes S^{\alpha})$ with $\gamma \neq 0$;*

(E_8) *$h(S^{\alpha}) = h(h(S^{\alpha}))$;*

(E_9) *$S^{\alpha} \neq 0 \Rightarrow \exists q : h(S^{\alpha})(q) = 1$.*

Through giving $\mathcal{L}$, an L-valued language is any mapping from Σ^* into L. L-valued languages can therefore be managed as formal power series over Σ^* and $\mathcal{L}$. The set of all possible L-valued languages on Σ^* and $\mathcal{L}$ is signified by L^{Σ^*}.
Let ℓ an L-valued language. The derivative of ℓ by a word x has been the L-valued language $x^{-1}\ell$ indicated by $(x^{-1})(\ell)(z) = \ell(xz)$ for any word z. Additionally, $y^{-1}(x^{-1}\ell) = (xy)^{-1}\ell$ for any words x and y.

Definition 3.2. *An $L^{\mathcal{B}}$-valued general fuzzy automaton ($L^{\mathcal{B}}-valued$) GFA is an 8-tuple $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ where Q is a finite nonempty set of states, Σ is an alphabet, $\tilde{R} \in L^Q$ is the initial L-valued state, $F \in L^Q$ is the final L-valued state, and $\tilde{\delta}_{a_k} : (Q \times L) \times Q \to L^{\mathcal{B}}$ defines an L-valued transition on Q for each symbol of the alphabet $a_k \in \Sigma$. Each state in Supp($\tilde{R}$) is called an initial state and each state in Supp(F) is called a final state. The extension of $\tilde{\delta} : \Sigma \to (L^{\mathcal{B}})^{(Q\times L)\times Q}$ to $\tilde{\delta}^* : \Sigma^* \to (L^{\mathcal{B}})^{(Q\times L)\times Q}$ is defined as follows:*
(i) *$\tilde{\delta}^{*\alpha}(\varepsilon) = E^{\mathcal{B}}$, and*
(ii) *$\tilde{\delta}^{*\alpha}(u\sigma) = \tilde{\delta}_u^{*\alpha} o \tilde{\delta}_{\sigma}^{\alpha}$ for any $u \in \Sigma^*, \sigma \in \Sigma$.*
To simplify notation, $\tilde{\delta}^*$ is also denoted by $\tilde{\delta}$. By associativity, $\tilde{\delta}^{\alpha}(xy) = \tilde{\delta}_x^{\alpha} o \tilde{\delta}_y^{\alpha}$ for any two words x and y. The L-valued language recognized by $\tilde{F}$, denoted by $[\tilde{F}]$, is defined as follows:

$$[\tilde{F}](x) = \tilde{R} o \tilde{\delta}_x^{\alpha} o F = \bigvee_{q\in Q, q_0 \in Q_{act}(t_0)} \tilde{R}(q_0) \otimes \tilde{\delta}_x^{\alpha}((q_0, \mu^{t_0}(q_0)), q) \otimes F(q) \quad \text{for any} \quad x \in \Sigma^*.$$

Lemma 3.3. *Consider that $\tilde{F}$ be an $L^{\mathcal{B}}$-valued GFA over $\mathcal{L}$. $[\tilde{F}]$ is an L-valued language of finite image.*

Proof. Since Q is finite and Σ^* is locally finite, the set $\{\tilde{R} o \tilde{\delta}_x^{\alpha} | x \in \Sigma^*\}$ is finite for the reason that the subsemiring induced by the finite set $\{\tilde{R}(q_0), \tilde{\delta}_{\sigma}^{\alpha}((q_0, \mu^{t_0}(q_0)), p) | p \in Q, q_0 \in Q_{act}(t_0), \sigma \in \Sigma\}$ is finite. The cardinality of the former set is $K^{|Q|}$ in which K is the number of different values in the subsemiring induced by the second set. Therefore, the set of values $\{\tilde{R} o \tilde{\delta}_x^{\alpha} o F | x \in \Sigma^*\}$ is finite and according to the previous definition, $[\tilde{F}]$ is an L-valued language of finite image. □

The size of an $L^{\mathcal{B}}$-valued GFA $\tilde{F}$, signified by $|\tilde{F}|$, is the cardinality of Q. Additionally, two $L^{\mathcal{B}}$-valued GFA, $\tilde{F}$ and $\tilde{F}'$, are (language) equivalent if and only if $[\tilde{F}] = [\tilde{F}']$.

Definition 3.4. *An $L^{\mathcal{B}}$-valued GFA $\tilde{F}$ over $\mathcal{L}$ is a crisp if it fulfills the conditions as follows:*
- (uniqe final state) $F = \{e/1\}$, i.e., it shows a distinctive final state with membership value 1.
- (crisp transition) $\tilde{\delta}_{a_k}^{\alpha}((p, \mu^t(p)), q) \in \{0, 1\}$ for any $p, q \in Q$ and $a_k \in \Sigma$.
- (complete bacward) For any $q \in Q$ and $a_k \in \Sigma$, there exists a state p such that $\tilde{\delta}_{a_k}^{\alpha}((p, \mu^t(p)), q) = 1$.
- (co-deterministic) For any $a_k \in \Sigma$ and $p, p', q \in Q$, if $\tilde{\delta}_{a_k}^{\alpha}((p, \mu^t(p)), q) = 1$ and $\tilde{\delta}_{a_k}^{\alpha}((p', \mu^t(p')), q) = 1$ then $p = p'$.
- (co-accessible) For every state $p \in Q$, there exists a word x such that $\tilde{\delta}_x^{\alpha}((p, \mu^t(p)), e) = 1$.

Let $\tilde{F}$ be a crisp $L^{\mathcal{B}}$-valued GFA. If two state p and p' assure that $\tilde{\delta}_x^{\alpha}((p, \mu^t(p)), e) = 1$ and $\tilde{\delta}_x^{\alpha}((p', \mu^t(p')), e) = 1$ for the same word x then $p = p'$. It happens since $\tilde{F}$ is co-deterministic. Since $\tilde{F}$ is complete and co-deterministic, the composition $\tilde{\delta}_x^{\alpha} o F = \tilde{\delta}_x^{\alpha} o \{e/1\}$ demonstrates a singleton crisp state $\{p/1\}$ in which p is the unique state such that $\tilde{\delta}_x^{\alpha}((p, \mu^t(p)), e) = 1$. This state is indicated by $e_{\bar{x}}$ (when $\tilde{F}$ is clear in the context). Actually, the set of states Q is the finite set $\{e_{\bar{x}} | x \in \Sigma^*\}$ since $\tilde{F}$ is crisp. This explanation leads us to compute the L-valued language distinguished by a crisp $L^{\mathcal{B}}$-valued GFA $\tilde{F}$.

$$S^{\alpha} o \tilde{\delta}_x^{\alpha} o \{e/1\} = S^{\alpha}(e_{\bar{x}}),$$

for any L-valued state $S^{\alpha} \in L^Q$ and word x. Thus,

$$[\tilde{F}](x) = \tilde{R} o \tilde{\delta}_x^{\alpha} o \{e/1\} = \tilde{R}(e_{\bar{x}}),$$

for any word x.

Definition 3.5. *Assume that $\tilde{F}$ be an $L^{\mathcal{B}}$-valued GFA. Then $\tilde{F}$ is considered to be a deterministic $L^{\mathcal{B}}$-valued general fuzzy automaton if the following properties are satisfied:*
(i) *$\tilde{F}$ has a unique initial state, which means the initial L-valued state can be shown as $\tilde{R} = \{q_0/\tilde{R}(q_0)\}$.*
(ii) *$\tilde{F}$ is complete, which means for any $a_k \in \Sigma$ and $p \in Q_{act}(t)$, there is $q \in Q$ such that $\tilde{\delta}^{\alpha}_{a_k}((p, \mu^t(p)), q) > 0$.*
(iii) *$\tilde{F}$ is deterministic, which means for any $a_k \in \Sigma$ and $p \in Q_{act}(t)$, $q, q' \in Q$, if $\tilde{\delta}^{\alpha}_{a_k}((p, \mu^t(p)), q) > 0$ and $\tilde{\delta}^{\alpha}_{a_k}((p, \mu^t(p)), q') > 0$ then $q = q'$.*
For a deterministic $L^{\mathcal{B}}$-valued general fuzzy automaton, $\tilde{F} = (Q, \Sigma, \{q_0/\tilde{R}(q_0)\}, Z, \omega, \tilde{\delta}, F_1, F_2)$, the $L^{\mathcal{B}}$-valued transition relation $\tilde{\delta}^{\alpha}$ induces $\tau_{\tilde{\delta}^{\alpha}} : Q \times \Sigma \to Q$ as follows:

$$\tau_{\tilde{\delta}^{\alpha}}(p, a_k) = q \Leftrightarrow \tilde{\delta}^{\alpha}_{a_k}((p, \mu^t(p)), q) > 0,$$

for any $a_k \in \Sigma$ and $p \in Q_{act}(t)$, $q \in Q$. Additionally, we have defined the extension $\tau^*_{\tilde{\delta}^{\alpha}} : Q \times \Sigma^* \to Q$ by $\tau^*_{\tilde{\delta}^{\alpha}}(p, \varepsilon) = p$ and $\tau^*_{\tilde{\delta}^{\alpha}}(p, xy) = \tau(\tau^*_{\tilde{\delta}^{\alpha}}(p, x), y)$ for any $y \in \Sigma, x \in \Sigma^*$ and $p \in Q$. The state $\tau^*_{\tilde{\delta}^{\alpha}}(p, x)$ characterizes the distinctive reachable state from p through the word x, which is simply denoted by p_x. Let us scrutinize that every state p is reachable by the empty word ε, which means $p = p_{\varepsilon}$. Through the use of previous notation, values of transitions between states are computed as follows:
$\tilde{\delta}^{\alpha}_{\varepsilon}((p, \mu^t(p)), p_{\varepsilon}) = 1$,
$\tilde{\delta}^{\alpha}_{x\sigma}((p, \mu^t(p)), p_{x\sigma}) = \tilde{\delta}^{\alpha}_{x}((p, \mu^t(p)), p_x) \otimes \tilde{\delta}^{\alpha}_{\sigma}((p_x, \mu^t(p_x)), p_{x\sigma})$, *and* $\tilde{\delta}^{\alpha}_{x}((p, \mu^t(p)), q) = 0$ *otherwise with $x \in \Sigma^*$ and $\sigma \in \Sigma$. By associativity,*

$$\tilde{\delta}^{\alpha}_{xy}((p, \mu^t(p)), p_{xy}) = \tilde{\delta}^{\alpha}_{x}((p, \mu^t(p)), p_x) \otimes \tilde{\delta}^{\alpha}_{y}((p_x, \mu^t(p_x)), p_{xy}),$$

for any $p \in Q_{act}(t)$ and words x and y.
Consider that for the zero-divisor free condition in $\mathcal{L}$, for any $x \in \Sigma^*$,

$$\tilde{\delta}^{\alpha}_{x}((p, \mu^t(p)), q) > 0 \Leftrightarrow q = p_x.$$

Consequently, a state q is regarded to be accessible if there exists a word x such that $q = u_x$, i.e., q is reachable from the initial state u by the word x. Then the L-valued language which has been identified by a deterministic $L^{\mathcal{B}}$- valued general fuzzy automaton $\tilde{F}$ is

$$[\tilde{F}](x) = \tilde{R}(u) \otimes \tilde{\delta}^{\alpha}_{x}((u, \tilde{R}(u)), u_x) \otimes F(u_x),$$

for any $x \in \Sigma^*$. By specifying a word x and an accessible state u_x, $[\tilde{F}_{u_x}]$ is the L-valued language shown by

$$[\tilde{F}_{u_x}](y) = \tilde{\delta}^{\alpha}_{y}((u_x, \mu^t(u_x)), u_{xy}) \otimes F(u_{xy}),$$

for any word y. As a result, it will not be difficult to prove that

$$x^{-1}[\tilde{F}] = (\tilde{R}(u) \otimes \tilde{\delta}^{\alpha}_{x}((u, \mu^t(u)), u_x)) \otimes [\tilde{F}(u_x)],$$

for any word x.
Here we declare that a deterministic $L^{\mathcal{B}}$-value GFA is accessible if any state $q \in Q$ is accessible, that is, the set of states Q of $\tilde{F}$ is the finite set $\{u_x | x \in \Sigma^*\}$.
Consider that, when a deterministic $L^{\mathcal{B}}$-valued GFA $\tilde{F}$ signifies $\tilde{R}(u) = 1$ and $\tilde{\delta}^{\alpha}_{a_k}$ is a crisp relation on Q, for every $a_k \in \Sigma$, then $[\tilde{F}](x) = F(u_x)$. In such a condition, $\tilde{F}$ is named crisp-deterministic $L^{\mathcal{B}}$-valued general fuzzy automaton. On the other hand, a crisp-deterministic $L^{\mathcal{B}}$-valued GFA is merely a specific case of a deterministic $L^{\mathcal{B}}$-valued GFA.

3.2 Conditions for a minimal deterministic $L^{\mathcal{B}}$-valued GFA

In the present study, we treat with the problem of finding a minimal deterministic $L^{\mathcal{B}}$-valued GFA which has been equivalent to a given $L^{\mathcal{B}}$-valued GFA. Distinct from ordinary deterministic finite automata, there may exist several topological different minimal deterministic $L^{\mathcal{B}}$-valued GFAs that are equivalent to the same $L^{\mathcal{B}}$-valued GFA. This is why we propose the minimality conditions for deterministic $L^{\mathcal{B}}$-valued GFAs in this subsection.

Definition 3.6. *A deterministic $L^{\mathcal{B}}$-valued GFA $\tilde{F}$ is regarded as a minimal deterministic $L^{\mathcal{B}}$-valued GFA if $|\tilde{F}| \leq |\tilde{F}'|$ for any deterministic $L^{\mathcal{B}}$-valued GFA $\tilde{F}'$ which is equivalent $\tilde{F}$.*

Property 3.7. *Consider that $\tilde{F} = (Q, \Sigma, \{u/\tilde{R}(u)\}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be a deterministic $L^{\mathcal{B}}$-valued GFA over $\mathcal{L}$. If $\tilde{F}$ is minimal then $\tilde{F}$ fulfills the next necessary conditions:*
1) *$\tilde{F}$ is an accessible deterministic $L^{\mathcal{B}}$-valued GFA.*
2) *$\tilde{F}$ is observable, i.e., for any $p, q \in Q$, $[\tilde{F}_p] = [\tilde{F}_q]$ demonstrates $p = q$.*

Some other mentioned necessary conditions are also proved in the same way to the given ordinary DFAs [15]. However, these properties are not regarded adequate for recognizing a minimal deterministic $L^{\mathcal{B}}$-valued GFA.

Property 3.8. *Let $\tilde{F} = (Q, \Sigma, \{u/\tilde{R}(u)\}, Z, \omega, \tilde{\delta}, F_1, F_2)$ be an accessible deterministic $L^{\mathcal{B}}$-valued GFA over $\mathcal{L}$, then $\neg(\exists x, y \in \Sigma^*, \ell \in L^{\Sigma^*}, \alpha, \beta \in \mathcal{B} : u_x \neq u_y$ and $x^{-1}[\tilde{F}] = \alpha \otimes \ell$ and $y^{-1}[\tilde{F}] = \beta \otimes \ell)$ implies $\tilde{F}$ is a minimal deterministic $L^{\mathcal{B}}$-valued GFA.*

Proof. As $\tilde{F}$ is accessible, $Q = \{u_x | x \in \Sigma^*\}$. Suppose that $||Q|| = n$. Since $\tilde{F}$ is a deterministic, there will be n different words, $x_1, \ldots, x_n$, such that $Q = \{u_{x_i} | i = 1, \ldots, n\}$. Consider that $\tilde{F}$ is not minimal. Then there will be a minimal automaton $\tilde{F'} = (Q', \Sigma, \{u'/\tilde{R}'(u)\}, Z, \omega', \tilde{\delta}', F_1, F_2)$ equivalent to $\tilde{F}$ with $||Q'|| < ||Q||$. As $\tilde{F}'$ is complete $u'_{x_i} \in Q'$ for each $i = 1, \ldots, n$. By the Pigeonhole Principle, at least there is two different words x_k and $x_{k'}$, with $1 \leq k \leq k' \leq n$, such that $u_{x_k} \neq u_{x_{k'}}$, and $u'_{x_k} \neq u'_{x_{k'}}$. Call x_k and $x_{k'}$ by x and y, respectively. In Consequently, $x^{-1}[\tilde{F}] = x^{-1}[\tilde{F}']$ and $y^{-1}[\tilde{F}] = y^{-1}[\tilde{F}']$ because $[\tilde{F}] = [\tilde{F}']$, they are equivalent. We have, $x^{-1}[\tilde{F}] = (\tilde{R}'(u') \otimes \tilde{\delta}_x'^{\alpha}((u', \mu^t(u')), u'_x)) \otimes [\tilde{F}'_{u'_x}]$ and $y^{-1}[\tilde{F}] = (\tilde{R}'(u') \otimes \tilde{\delta}_y'^{\alpha}((u', \mu^t(u')), u'_y)) \otimes [\tilde{F}'_{u'_y}]$. As $u'_x = u'_y$ then $[\tilde{F}'_{u'_x}] = [\tilde{F}'_{u'_y}] = \ell$. In brief, $u_x \neq u_y$ and $x^{-1}[\tilde{F}] = \alpha \otimes \ell$ and $y^{-1}[\tilde{F}] = \beta \otimes \ell$. Therefore, a contradiction takes place and the property attains. □

In general it means that these may be deterministic $L^{\mathcal{B}}$-valued GFAs that are accessible and observable, but they are not minimal deterministic GFAs.

3.3 Determinization of $L^{\mathcal{B}}$-valued GFA

As reported, the construction which is related to Nerode automaton of $\tilde{F}$, $N(\tilde{F})$ is regarded as a widespread method for determinization of $L^{\mathcal{B}}$-valued GFA $\tilde{F}$ over $\mathcal{L}$. This is a prominent method of determinization which is introduced by Ignjatovic et al. [15]. $N(\tilde{F})$ is equivalent to $\tilde{F}$ whose finite set of states is considered as $Q^{N(\tilde{F})} = \{u_x | x \in \Sigma^*\}$ where $u_x = \tilde{R} o \tilde{\delta}_x^{\alpha}$ for any word x. A generalization of this construction, which is on the concept of factorization of fuzzy states, was reported in [13]. This construction is in the following definition.

Definition 3.9. *For an $L^{\mathcal{B}}$-valued general fuzzy automaton $\tilde{F} = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ over $\mathcal{L}$, the determinization of $\tilde{F}$ through factorization (h, k) is the $L^{\mathcal{B}}$-valued GFA $\mathcal{H}(\tilde{F}) = (Q^{\mathcal{H}(\tilde{F})}, \Sigma, \tilde{R}^{\mathcal{H}(\tilde{F})}, Z, \omega^{\mathcal{H}(\tilde{F})}, \tilde{\delta}^{\mathcal{H}(\tilde{F})}, F_1, F_2)$ which is defined as:*
1) $Q^{\mathcal{H}(\tilde{F})} = \{u_x | x \in \Sigma^*\}$ *is the set of states whose elements are:*
(i) $u_\varepsilon = u = h(\tilde{R})$,
(ii) $u_{xy} = h(u_x o \tilde{\delta}_y^{\alpha}); x \in \Sigma^*, y \in \Sigma, \alpha \in \mathcal{B}$,
2) $\tilde{R}^{\mathcal{H}(\tilde{F})} = \{u/k(\tilde{R})\}$ *is the initial $L^{\mathcal{B}}$-valued state which contains the unique initial state u in the degree $k(\tilde{R})$.*
3) $\tilde{\delta}^{\mathcal{H}(\tilde{F})} : \Sigma \to (L^{\mathcal{B}})^{(Q^{\mathcal{H}(\tilde{F})} \times L) \times Q^{\mathcal{H}(\tilde{F})}}$ *is the $L^{\mathcal{B}}$-valued transition function fulfilling:*
For any $P, T \in Q^{\mathcal{H}(\tilde{F})}$ and $a_k \in \Sigma$:

$$\begin{cases} \tilde{\delta}_{a_k}^{\mathcal{H}(\tilde{F})\alpha}((P, \mu^t(P)), T) = k(P o \tilde{\delta}_{a_k}^{\alpha}) \Leftrightarrow P = u_x \text{ and } T = u_{xa_k}, \\ \tilde{\delta}_{a_k}^{\mathcal{H}(\tilde{F})\alpha}((P, \mu^t(P)), T) = 0 \quad \text{otherwise.} \end{cases}$$

4) $F^{\mathcal{H}(\tilde{F})}$ *is the final $L^{\mathcal{B}}$-valued state defined as $F^{\mathcal{H}(\tilde{F})}(u_x) = u_x o F$, for every $u_x \in Q^{\mathcal{H}(\tilde{F})}$.*

The following lemma has provided the major properties of $\mathcal{H}(\tilde{F})$.

Lemma 3.10. *Let $\tilde{F}$ be an $L^{\mathcal{B}}$-valued GFA over $\mathcal{L}$. Then $\mathcal{H}(\tilde{F})$ satisfies the following properties for any $x, y \in \Sigma^*$ and $a_k \in \Sigma$:*
1) $u_x o \tilde{\delta}_{a_k}^{\alpha} = \tilde{\delta}_{a_k}^{\mathcal{H}(\tilde{F})\alpha}((u_x, \mu^t(u_x)), u_{xa_k}) \otimes u_{xa_k}$,
2) $\tilde{R} o \tilde{\delta}_x^{\alpha} = \tilde{R}^{\mathcal{H}(\tilde{F})}(u) \otimes \tilde{\delta}_x^{\mathcal{H}(\tilde{F})\alpha}((u, \mu^t(u)), u_x) \otimes u_x$,
3) $[\tilde{F}] = [\mathcal{H}(\tilde{F})]$,
4) $[\mathcal{H}(\tilde{F})_{u_x}](y) = u_x o \tilde{\delta}_y^{\alpha} o F$.

Proof. The proof of the part 1, 2 and 3 are the same as for 1, 2 and 3 in [13]. Here, we show part 4 in which we have:

$$[\mathcal{H}(\tilde{F})_{u_x}](y) = \tilde{\delta}_y^{\mathcal{H}(\tilde{F})\alpha}((u_x, \mu^t(u_x)), u_{xy}) \otimes F^{\mathcal{H}(\tilde{F})}(u_{xy}).$$

Concerning this, when $x = \varepsilon$, we have $[\mathcal{H}(\tilde{F})_{u_x}](\varepsilon) = 1 \otimes F^{\mathcal{H}(\tilde{F})}(u_x) = u_x o F$. In case when $y = zp$,

$$[\mathcal{H}(\tilde{F})_{u_x}](y) = \tilde{\delta}_z^{\mathcal{H}(\tilde{F})\alpha}((u_x, \mu^t(u_x)), u_{xz}) \otimes \tilde{\delta}_p^{\mathcal{H}(\tilde{F})\alpha}((u_x z, \mu^t(u_x z)), u_{xy})) \otimes F^{\mathcal{H}(\tilde{F})}(u_{xy}).$$

By Definition 3.9, $F^{\mathcal{H}(\tilde{F})}(u_{xy}) = u_{xy}oF$. By Property 3.7 above, then,

$$[\mathcal{H}(\tilde{F})_{u_x}](y) = \tilde{\delta}_z^{\mathcal{H}(\tilde{F})\alpha}((u_x, \mu^t(u_x)), u_{xz}) \otimes (u_{xz}o\tilde{\delta}_p^{\alpha}oF).$$

Through similar reasoning backward unit $z = \varepsilon$, it has been followed that $[\mathcal{H}(\tilde{F})_{u_x}](y) = (u_x o\tilde{\delta}_y^{\alpha}oF)$. □

Let $\tilde{F}$ be an $L^{\mathcal{B}}$-valued GFA over $\mathcal{L}$. The structure of $\mathcal{H}(\tilde{F})$ gives us the opportunity to examine a reduction of its states because the states in $Q^{\mathcal{H}(\tilde{F})}$ are L-valued states of Q. Further, consider that $\mathcal{H}(\tilde{F})$ is a finite $L^{\mathcal{B}}$-valued GFA. To simplify the demonstration of this reduction, the truth value of the path is denoted from the initial state u to the state u_x which has been reached x by:

$$\mathcal{P}^{\mathcal{H}(\tilde{F})}(x) = \tilde{R}^{\mathcal{H}(\tilde{F})}(u) \otimes \tilde{\delta}_x^{\mathcal{H}(\tilde{F})\alpha}((u, \mu^t(u)), u_x),$$

for every $x \in \Sigma^*$. Furthermore, for each (accessible) state $S^\alpha \in Q^{\mathcal{H}(\tilde{F})}$ (S^α is an L-valued state, $S^\alpha \in L^Q$), we have defined the supremum value of all words that reach S^α from the initial state u,

$$\mathcal{P}_{sup,S^\alpha}^{\mathcal{H}(\tilde{F})} = \bigvee_{x\in\Sigma^*:S^\alpha=u_x} \mathcal{P}^{\mathcal{H}(\tilde{F})}(x).$$

Definition 3.11. *Given two state P and $T \in Q^{\mathcal{H}(\tilde{F})}$, we have indicated that P reduces T if and only if*

$$\mathcal{P}_{sup,T}^{\mathcal{H}(\tilde{F})} \otimes T = \mathcal{P}_{sup,T}^{\mathcal{H}(\tilde{F})} \otimes P.$$

The proposed intuition behind this kind of definition has been in the following examination. Assume that P reduces T. Every word x_k that reaches T from the initial state u, satisfies $u_{x_k} = T$ and $\mathcal{P}_{sup,T}^{\mathcal{H}(\tilde{F})} \geq \mathcal{P}^{\mathcal{H}(\tilde{F})}(x_k)$. As $\mathcal{L}$ is divisible, then

$$\mathcal{P}^{\mathcal{H}(\tilde{F})}(x_k) \otimes T = \mathcal{P}^{\mathcal{H}(\tilde{F})}(x_k) \otimes P.$$

By Lemma 3.10, and the fact $u_{x_k} = T$:

$$[\mathcal{H}(\tilde{F})](x_k\gamma) = (\mathcal{P}^{\mathcal{H}(\tilde{F})}(x_k) \otimes T)o\tilde{\delta}_\gamma^{\alpha}oF,$$

for every $\gamma \in \Sigma^*$, then

$$(\mathcal{P}^{\mathcal{H}(\tilde{F})}(x_k) \otimes T)o\tilde{\delta}_\gamma^{\alpha}oF = (\mathcal{P}^{\mathcal{H}(\tilde{F})}(x_k) \otimes P)o\tilde{\delta}_\gamma^{\alpha}oF.$$

The conclusion remark which is related to this relation is that every word that reaches T can alternatively reaches P without modifying the language which has been distinguished by $\mathcal{H}(\tilde{F})$. It is the main reason to indicate that P reduces T. Consequently, if P reduces T in $\mathcal{H}(\tilde{F})$, it will be possible to construct a new accessible deterministic $L^{\mathcal{B}}$-valued GFA $\mathcal{H}'$ equivalent to $\tilde{F}$ with the size less than $\mathcal{H}(\tilde{F})$. The construction of $\mathcal{H}' = (Q', \Sigma, \tilde{R}', Z, \omega', \tilde{\delta}', F_1, F_2)$ will be quite simple:

(1) construct $\mathcal{H}$ to be exactly equal to $\mathcal{H}(\tilde{F})$ and consider that for each $\tilde{\delta}_{a_k}^{\mathcal{H}(\tilde{F})\alpha}((S', \mu^t(S')), S) > 0$ in $\mathcal{H}(\tilde{F})$,

$$\tilde{\delta}_{a_k}^{\mathcal{H}\alpha}((S', \mu^t(S')), P) = \tilde{\delta}_{a_k}^{\mathcal{H}(\tilde{F})\alpha}((S', \mu^t(S')), S),$$

and $\tilde{\delta}_{a_k}^{\mathcal{H}\alpha}((S', \mu^t(S')), S) = 0$, that is, every one which is ending in S in $\mathcal{H}(\tilde{F})$ ends in P in $\mathcal{H}$. This kind of transformation leads to making $[\mathcal{H}] = [\mathcal{H}(\tilde{F})]$.

(2) constructing $\mathcal{H}'$ is in the following. Because in $\mathcal{H}$, S is not an accessible state, eliminate it and every non-accessible state reachable from S to get $\mathcal{H}'$. Again, $[\mathcal{H}'] = [\mathcal{H}]$ by this well-known process. Let $\tilde{F}^c = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ with final state $\{e/1\}$ be equivalent to the $L^{\mathcal{B}}$-valued GFA $\tilde{F}$. We have proved that $\mathcal{H}(\tilde{F}^c)$ is a deterministic $L^{\mathcal{B}}$-valued GFA equivalent to $\tilde{F}$ that has satisfied the minimality necessary conditions. In other words, $\mathcal{H}(\tilde{F}^c)$ is an accessible and observable deterministic $L^{\mathcal{B}}$-valued general fuzzy automaton.

Lemma 3.12. *Consider $\tilde{F}^c = (Q, \Sigma, \tilde{R}, Z, \omega, \tilde{\delta}, F_1, F_2)$ with final state $\{e/1\}$ over $\mathcal{L}$. For any $S^\alpha \in L^Q$ and $a_k \in \Sigma$,*

$$h(h(S^\alpha)o\tilde{\delta}_{a_k}^{\alpha}) = h(S^\alpha o\tilde{\delta}_{a_k}^{\alpha}),$$

where h is defined in Definition 3.1.

Proof. For any $a_k \in \Sigma$, the L-valued transition relation $\tilde{\delta}^{\alpha}_{a_k}$ is a crisp relations, and also for each $q \in Q$ there is a unique state $p_q \in Q$ such that $\tilde{\delta}^{\alpha}_{a_k}((p_q, \mu^t(p_q)), q) = 1$. Through this definition, $(h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k})(q) = h(S^{\alpha})(p_q)$ and $(S^{\alpha}o\tilde{\delta}^{\alpha}_{a_k})(q) = S^{\alpha}(p_q)$ for any $S^{\alpha} \in L^Q$ and $q \in Q$. Assume that by $(E_5), k(S^{\alpha}) \otimes h(S^{\alpha}) = S^{\alpha}$. Therefore, the next relation attained from (E_6) holds because by $(E_2), k(S^{\alpha}) > 0$ for any $S^{\alpha} \in L^Q$:

$$\begin{aligned} k(S^{\alpha}) \otimes k(h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k}) &= k(k(S^{\alpha}) \otimes (h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k})) \\ &= k((k(S^{\alpha}) \otimes h(S^{\alpha}))o\tilde{\delta}^{\alpha}_{a_k}) \\ &= k(S^{\alpha}o\tilde{\delta}^{\alpha}_{a_k}). \end{aligned}$$

For any $q \in Q$, there exists $h(h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k})(q) = 0$ if $(h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k})(q) = h(S^{\alpha})(p_q) = 0$. By (E_4), $S^{\alpha}(p_q) = 0$ and $(S^{\alpha}o\tilde{\delta}^{\alpha}_{a_k})(q) = 0$. Accordingly, by (E_4), $h(S^{\alpha}o\tilde{\delta}^{\alpha}_{a_k})(q)$= 0. Now consider that $h(h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k})(q) \neq 0$, then, $h(S^{\alpha})(p_q) \neq 0$. By $(E_4), S(p_q) \neq 0$. There exists

$$\begin{aligned} h(h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k})(q) &= k(h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k}) \to h(h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k})(q) \\ &= k(h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k}) \to h(S^{\alpha})(p_q) \\ &= k(h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k}) \to (k(S^{\alpha}) \to S^{\alpha}(p_q)) \\ &= k(S^{\alpha}) \otimes k(h(S^{\alpha})o\tilde{\delta}^{\alpha}_{a_k}) \to S^{\alpha}(p_q) \\ &= k(S^{\alpha}o\tilde{\delta}^{\alpha}_{a_k}) \to (S^{\alpha}o\tilde{\delta}^{\alpha}_{a_k})(q) \\ &= h(S^{\alpha}o\tilde{\delta}^{\alpha}_{a_k})(q). \end{aligned}$$

As a result, Lemma has been held. □

Lemma 3.13. *Consider $\tilde{F}^c$ with final state $\{e/1\}$ over $\mathcal{L}$. Then $\mathcal{H}(\tilde{F}^c)$ fulfills that, for each word $x \in \Sigma^*$:*

$$h(\tilde{R}_x) = u_x,$$

where $\tilde{R}_x = \tilde{R}o\tilde{\delta}^{\alpha}_x$; and

$$h(\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}(x) \otimes u_x) = u_x.$$

Proof. The set of states of $\mathcal{H}(\tilde{F}^c)$ is $Q^{\mathcal{H}(\tilde{F}^c)} = \{u_x | x \in \Sigma^*\}$. Each state in $Q^{\mathcal{H}(\tilde{F}^c)}$ is an L-valued state of L^Q. We prove $h(\tilde{R}_x) = u_x$ through induction on the length of x.
(i) Basis. Let $x = \varepsilon$. By Definition 3.9, $u_{\varepsilon} = h(\tilde{R}_{\varepsilon})$, since $\tilde{\delta}^{\alpha}_{\varepsilon} = E_Q$. The property holds.
(ii) Induction Hypothesis. Presume that $h(\tilde{R}_x) = u_x$ has been held for some word x.
(iii) Induction step. Let $y = xa_k$ in which $x \in \Sigma^*$ and $a_k \in \Sigma$. We have

$$\begin{aligned} u_{xa_k} &= h(u_x o\tilde{\delta}^{\alpha}_{a_k}) \\ &= h(h(\tilde{R}_x)o\tilde{\delta}^{\alpha}_{a_k}) \quad \text{(By induction hypothesis)} \\ &= h(\tilde{R}_x o\tilde{\delta}^{\alpha}_{a_k}) \quad \text{(By Lemma 3.12)} \\ &= h(\tilde{R}_{xa_k}). \quad (\text{since } \tilde{R}_x o\tilde{\delta}^{\alpha}_x o\tilde{\delta}^{\alpha}_{a_k} = \tilde{R}_x o\tilde{\delta}^{\alpha}_{xa_k}) \end{aligned}$$

Therefore, $h(\tilde{R}_x) = u_x$ for any word x. In addition, $h(\mathcal{P}^{\mathcal{H}(\tilde{F})} \otimes u_x) = u_x$ is derived directly by Lemma 3.10. □

Theorem 3.14. *Consider $\tilde{F}$ be an $L^{\mathcal{B}}$-valued GFA over $\mathcal{L}$. Let $\tilde{F}^c$ with final state $\{e/1\}$ be the $L^{\mathcal{B}}$-valued GFA equivalent to $\tilde{F}$. Then $\mathcal{H}(\tilde{F}^c)$ has been an accessible and observable deterministic $L^{\mathcal{B}}$-valued GFA equivalent to $\tilde{F}$. Furthermore, $|\mathcal{H}(\tilde{F}^c)| \leq |\mathcal{N}(\tilde{F}^c)|$.*

Proof. It has been evident that the automaton $\mathcal{H}(\tilde{F}^c)$ is an accessible deterministic $L^{\mathcal{B}}$-valued GFA equivalent to $\tilde{F}$. As $\tilde{F}^c$ is equivalent to $\tilde{F}$, then $\mathcal{H}(\tilde{F}^c)$ is therefore equivalent to $\tilde{F}$. We have proved that $\mathcal{H}(\tilde{F}^c)$ is a finite automaton. Let $Q^{\mathcal{N}(\tilde{F}^c)} = \{\tilde{R}_x | x \in \Sigma^*\}$ where $\tilde{R}_x = \tilde{R}o\tilde{\delta}^{\alpha}_x \in L^Q$ for any word x. We recall that $Q^{\mathcal{N}(\tilde{F}^c)}$ is the set of states of the Nerode automaton of $\tilde{F}^c$. Through the similar proof proposed in Lemma 3.3, as $\mathcal{L}$ is locally finite, then $Q^{\mathcal{N}(\tilde{F}^c)}$ is a finite set. We will define the function $\varphi : Q^{\mathcal{N}(\tilde{F}^c)} \to Q^{\mathcal{H}(\tilde{F}^c)}$ by $\varphi(\tilde{R}_x) = h(\tilde{R}_x)$ for each word x. We have $\varphi(\tilde{R}_x) = u_x$. This function φ has been well defined: for two words x and y, if $\tilde{R}_x = \tilde{R}_y$, then $\varphi(\tilde{R}_x) = h(\tilde{R}_x) = h(\tilde{R}_y) = \varphi(\tilde{R}_y)$, because h has been a well defined function. Moreover, for each word x, $u_x \in Q^{\mathcal{H}(\tilde{F}^c)}$ and $\tilde{R}_x \in Q^{\mathcal{H}(\tilde{F}^c)}$ with $h(\tilde{R}_x) = u_x$.

Therefore, for each $u' \in Q^{\mathcal{H}(\tilde{F}^c)}$ there exists $\tilde{R}' \in Q^{\mathcal{N}(\tilde{F}^c)}$ such that $u' = \varphi(\tilde{R}')$. In brief, the related properties of φ entail that $|\mathcal{H}(\tilde{F})| \leq |\mathcal{N}(\tilde{F}^c)|$ has been true and $\mathcal{H}(\tilde{F}^c)$ has been a finite automaton. As a result, $\mathcal{H}(\tilde{F}^c)$ is an accessible deterministic $L^{\mathcal{B}}$-valued GFA equivalent to $\tilde{F}$. At this time, we prove that $\mathcal{H}(\tilde{F}^c)$ has been observable. $\mathcal{H}(\tilde{F}^c)$ has been an accessible deterministic $L^{\mathcal{B}}$-valued GFA. Let us assume two different words x and y and the accessible state u_x and u_y. Consider that $u_x \neq u_y$ but $[\mathcal{H}(\tilde{F}^c)_{u_x}] = [\mathcal{H}(\tilde{F}^c)_{u_y}]$. For each word z, we have

$$[\mathcal{H}(\tilde{F}^c)_{u_x}](z) = u_x o \tilde{\delta}_z^{\alpha} o\{e/1\} = u_y o \tilde{\delta}_z^{\alpha} o\{e/1\} = [\mathcal{H}(\tilde{F}^c)_{u_y}](z).$$

Then $u_x(e_{\bar{z}}) = u_y(e_{\bar{z}})$. Since $\tilde{F}^c$ is a crisp $L^{\mathcal{B}}$-valued GFA, $Q = \{e_{\bar{z}} | z \in \Sigma^*\}$; in addition, $u_x, u_y \in L^Q$. Therefore, $u_x = u_y$. This has been a contradiction with the initial assumption. Therefore, $\mathcal{H}(\tilde{F}^c)$ is observable. The proof is the Theorem conclusion. □

The proof proposed in property 3.8, which characterizes the sufficient condition for minimality of a deterministic $L^{\mathcal{B}}$-valued GFA, has been applied here as a tool for investigating some other conditions over $\mathcal{H}(\tilde{F}^c)$ to guarantee that it is a minimal deterministic $L^{\mathcal{B}}$-valued GFA equivalent to $\tilde{F}$. Additionally, it has been significant to examine if such conditions have been useful enough to get a minimal determinization procedure. In the present section, we have formally found out such conditions. The following theorem characterizes the necessary conditions for the postulation that a $\mathcal{H}(\tilde{F}^c)$ has not been a minimal deterministic $L^{\mathcal{B}}$-valued GFA. Recall that $\tilde{F}^c$ is a crisp $L^{\mathcal{B}}$-valued GFA equivalent to a given $L^{\mathcal{B}}$-valued GFA $\tilde{F}$.

Theorem 3.15. *Let $\tilde{F}$ be an $L^{\mathcal{B}}$-valued GFA over $\mathcal{L}$. If $\mathcal{H}(\tilde{F}^c)$ is not a minimal deterministic $L^{\mathcal{B}}$-valued GFA, then there will be two words $x, y \in \Sigma^*$ such that actually one of the following two conditions is attained:*
(a) $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}(x) > \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}(y)$ *and u_x reduces u_y,*
(b) $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}(y) > \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}(x)$ *and u_y reduces u_x,*
in which, as usual, values $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}(x)$ and $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}(y)$ are defined by the definition, while $u_x, u_y \in Q^{\mathcal{H}(\tilde{F}^c)}$ are two states from the accessible deterministic $L^{\mathcal{B}}$-valued GFA $\mathcal{H}(\tilde{F}^c)$ accessible by x and y, respectively.

Proof. Note that there exists a minimal deterministic $L^{\mathcal{B}}$-valued GFA $\tilde{F}' = (Q', \Sigma, \{u'/\tilde{R}'(u')\}, Z, \omega', \tilde{\delta}', F_1, F_2)$ equivalent to $\tilde{F}$, and $|\mathcal{H}(\tilde{F}^c)| < |\tilde{F}|$. As we follow the proof of Property 3.8, there will be two different words x and y; two different accessible states (via x and y respectively) $u_x, u_y \in Q^{\mathcal{H}(\tilde{F}^c)}$; and a state $q' \in Q'$ that satisfies:

$$x^{-1}[\mathcal{H}(\tilde{F}^c)] = \mathcal{P}^{\tilde{F}'}(x) \otimes [\tilde{F}'_{q'}],$$

$$y^{-1}[\mathcal{H}(\tilde{F}^c)] = \mathcal{P}^{\tilde{F}'}(y) \otimes [\tilde{F}'_{q'}].$$

Consider that we have chosen the word x in such a way that x is the word that goes through the path to the state u_x with the supremum value, or in such a way that for every other word x_k with $u_x = u_{x_k}$, there exists $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}(x) \geq \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}(x_k)$. We have choosen y completely analogously. For the sake of simplicity, indicate the values $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}(x), \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}(y), \mathcal{P}^{(\tilde{F}')}(x)$ and $\mathcal{P}^{(\tilde{F}')}(y)$ with $\lambda(x), \lambda(y)$, $\gamma(x)$ and $\gamma(y)$, respectively. With respect to the proposed issue, consider that $\lambda(x) = \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,u_x}$ and $\lambda(y) = \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,u_y}$. Furthermore, we have

$$\lambda(x) \otimes [\mathcal{H}(\tilde{F}^c)_{u_x}] = \gamma(x) \otimes [\tilde{F}'_{q'}],$$

$$\lambda(y) \otimes [\mathcal{H}(\tilde{F}^c)_{u_y}] = \gamma(y) \otimes [\tilde{F}'_{q'}].$$

As $\mathcal{L}$ has been zero-divisor free, all the values $\lambda(x), \lambda(y), \gamma(x)$ and $\gamma(y)$ have been greater than 0. Through the use of Lemma 3.10 and that $\tilde{F}^c$ is a crisp $L^{\mathcal{B}}$-valued GFA, we have

$$\lambda(x) \otimes u_x(e_{\bar{z}}) = \gamma(x) \otimes [\tilde{F}'_{q'}](z),$$

$$\lambda(y) \otimes u_y(e_{\bar{z}}) = \gamma(y) \otimes [\tilde{F}'_{q'}](z),$$

for every $z \in \Sigma^*$. We mention that $Q = \{e_{\bar{z}} | z \in \Sigma^*\}$ because $\tilde{F}$ is finite. Moreover, u_x and u_y are L-valued states in L^Q. We have divided the rest of the proof into the subsequent related three claims through which it helps us attain the statement of the theorem. □

Claim 3.16. *$u_x \neq 0$ and $u_y \neq 0$.*

To this end, assume $u_x = 0$. We have $[\tilde{F}'_{q'}] = 0$. Then $u_y = 0$, a contradiction.

Claim 3.17. $\gamma(x) \neq \gamma(y)$.

To this end, assume that $\gamma(x) = \gamma(y)$. For every $z \in \Sigma^*$, $\lambda(x) \otimes u_x(e_{\bar{z}}) = \lambda(y) \otimes u_y(e_{\bar{z}})$, i.e, $\lambda(x) \otimes u_x = \lambda(y) \otimes u_y$. Then $u_x = h(\lambda(x) \otimes u_x) = h(\lambda(y) \otimes u_y) = u_y$, a contradiction.

Claim 3.18.
(a) $\lambda(x) > \lambda(y)$ *and* $\lambda(y) \otimes u_y = \lambda(y) \otimes u_x$*; or*
(b) $\lambda(y) > \lambda(x)$ *and* $\lambda(x) \otimes u_y = \lambda(x) \otimes u_x$.

To this end, by Claim 3.17 and that L is an ordered set, then $\gamma(x) > \gamma(y)$ or $\gamma(y) > \gamma(x)$. We examine the first possibility. As $\gamma(x) > \gamma(y)$ and $\mathcal{L}$ is divisible: $\exists z \in L : \gamma(x) \otimes z = \gamma(y)$. Consequently,

$$\lambda(y) \otimes u_y = (\lambda(x) \otimes z) \otimes u_x.$$

Through the use of Claim 3.16 and (E_9), we come to the point that $\exists q_1 \in Q : u_x(q_1) = 1$ and $\exists q_2 \in Q : u_y(q_2) = 1$. The existence was mentioned before and for monotonicity of $\otimes$ see the following:

$$\lambda(y) \geq \lambda(y) \otimes u_y(q_1) = \lambda(x) \otimes z \otimes 1,$$

and

$$\lambda(y) \otimes 1 = \lambda(x) \otimes z \otimes u_x(q_2) \leq \lambda(x) \otimes z.$$

It is concluded that, $\lambda(y) = \lambda(x) \otimes z$. Then $\lambda(y) \otimes u_y = \lambda(y) \otimes u_x$; and, by monotonicity, $\lambda(x) \geq \lambda(y)$. Therefore, $\gamma(x) > \gamma(y)$ implies
(a) $\lambda(x) \geq \lambda(y)$; and
(b) $\lambda(y) \otimes u_y = \lambda(y) \otimes u_x$;
and similarly, $\gamma(y) > \gamma(x)$ signifies
(a) $\lambda(y) \geq \lambda(x)$; and
(b) $\lambda(x) \otimes u_y = \lambda(x) \otimes u_x$.
Concerning both cases, $\lambda(x) \neq \lambda(y)$, because on contrary case, $\lambda(y) \otimes u_y = \lambda(x) \otimes u_x$ and this indicates $u_x = u_y$ (see also Claim 3.17). By the use of 3.17 $\gamma(x) \neq \gamma(y)$. It can be concluded that Claim 3.18 is correct. At this time, the proof of the theorem directly follows while using the concept of L-valued state reduction which has been presented in Definition 3.11. Consider that the conditions pointed out from Claim 3.18 can alternatively be rewritten. With regard to the fact that $\tilde{F^c}$ is a crisp $L^{\mathcal{B}}$-valued GFA and Lemma 3.13, we can restate it as follows:
$(a)\lambda(x) > \lambda(y)$ and $u_y = \lambda(y) \to (\lambda(y) \otimes u_x)$; or
$(b)\lambda(y) > \lambda(x)$ and $u_x = \lambda(x) \to (\lambda(x) \otimes u_y)$.
In fact, $h(\lambda(x) \otimes u_y) = h(\lambda(x) \otimes u_x)$. Then, $u_y = h(\lambda(y) \otimes u_x)$. Through using the definition k, (2) and the fact that $\exists q_1 \in Q$: $u_x(q_1) = 1$, then, $k(\lambda(y) \otimes u_x) = \lambda(y)$. Using the definition of h and (3), for any $q \in Q$,

$$h(\lambda(y) \otimes u_x)(q) = \lambda(y) \to \lambda(y) \otimes u_x(q).$$

In the same way, equation (b) mentioned above is derived.

4 Characterization of minimality via $\mathcal{H}(\tilde{F^c})$

The study carried out in this section allows us to obtain the following characterization of minimality of the automaton $\mathcal{H}(\tilde{F^c})$.

Definition 4.1. *Let $\tilde{F}$ be an $L^{\mathcal{B}}$-valued GFA over $\mathcal{L}$. Let $\tilde{F^c}$ be the equivalent to $\tilde{F}$. Given the automaton $\mathcal{H}(\tilde{F^c})$ whose set of states in $Q^{\mathcal{H}(\tilde{F^c})}$; then two states $P, T \in Q^{\mathcal{H}(\tilde{F^c})}$, satisfy the relation $P \preceq T$ if and only if*
(i) $P \neq T$ *and* $\mathcal{P}^{\mathcal{H}(\tilde{F^c})}_{sup,P} > \mathcal{P}^{\mathcal{H}(\tilde{F^c})}_{sup,T}$,
(ii) $T = \mathcal{P}^{\mathcal{H}(\tilde{F^c})}_{sup,T} \to (\mathcal{P}^{\mathcal{H}(\tilde{F^c})}_{sup,T} \otimes P), (P$ *reduces* $T)$.

The obtained main result is provided in the following Theorem whose proof is straightforward after the results and definitions are presented in this section.

Theorem 4.2. *Let $\tilde{F}$ be an $L^{\mathcal{B}}$-valued GFA over $\mathcal{L}$. Let $\tilde{F^c}$ be the equivalent to $\tilde{F}$. The construction $\mathcal{H}(\tilde{F^c})$ is a minimal deterministic $L^{\mathcal{B}}$-valued GFA equivalent to $\tilde{F}$ if and only if there are not two states P and T of $\mathcal{H}(\tilde{F^c})$ such that $P \preceq T$.*

Proof. By contradiction. Suppose that there are two states P and T in $Q^{\mathcal{H}(\tilde{F}^c)}$ such that $P \preceq T$ and $\mathcal{H}(\tilde{F}^c)$ is a minimal deterministic $L^{\mathcal{B}}$-valued GFA. By Definition 4.1 (ii) P reduces T. Then, by the result given in subsection 3.3, $\mathcal{H}(\tilde{F}^c)$ is not a minimal deterministic $L^{\mathcal{B}}$-valued GFA. Now, consider that there are not two states P and T in $Q^{\mathcal{H}(\tilde{F}^c)}$ such that $P \preceq T$ and $\mathcal{H}(\tilde{F}^c)$ is not minimal. Then, by Claim 3, there are two different states P and T in $Q^{\mathcal{H}(\tilde{F}^c)}$ such that $P \preceq T$. In both directions, a contradiction happens and Theorem is correct. □

A very simple corollary to this Theorem is stated when a crisp deterministic $L^{\mathcal{B}}$-valued GFA, that it is a minimal deterministic $L^{\mathcal{B}}$-valued GFA.

Corollary 4.3. *Let $\tilde{F}$ be an $L^{\mathcal{B}}$-valued GFA over $\mathcal{L}$. Let $\tilde{F}^c$ with final state $\{e/1\}$ be the equivalent to $\tilde{F}$. If the construction $\mathcal{H}(\tilde{F}^c)$ is a crisp determinstic $L^{\mathcal{B}}$-valued GFA, then it is a minimal deterministic $L^{\mathcal{B}}$-valued GFA equivalent to $\tilde{F}$.*

Proof. If $\mathcal{H}(\tilde{F}^c)$ is a crisp deterministic $L^{\mathcal{B}}$-valued GFA, then each transition is a crisp transition (in $\{0,1\}$). Since every state $T \in Q^{\mathcal{H}(\tilde{F}^c)}$ is accessible, then $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,T} = 1$ for every $T \in Q^{\mathcal{H}(\tilde{F}^c)}$. By Definition 4.1 (i) there is no states P and T, such that $P \preceq T$. By Theorem 4.2, $\mathcal{H}(\tilde{F}^c)$ is a minimal deterministic $L^{\mathcal{B}}$-valued GFA equivalent to $\tilde{F}$. Let us observe that by Definition 4.1, for two states $P, T \in Q^{\mathcal{H}(\tilde{F}^c)}$ such that $P \preceq T$, as P reduces T (Definition 4.1 (ii)) then $T \geq P$. In addition, $Supp(P) = Supp(T)$. Therefore, the reducer state P is not a crisp state of $Q^{\mathcal{H}(\tilde{F}^c)}$. □

Lemma 4.4. *Let $\tilde{F}$ be an $L^{\mathcal{B}}$-valved GFA over $\mathcal{L}$. Let $\tilde{F}^c$ with final state $\{e/1\}$ be the equivalent to $\tilde{F}$. Given the automaton $\mathcal{H}(\tilde{F}^c)$, then the pair $(Q^{\mathcal{H}(\tilde{F}^c)}, \preceq)$ is a strict partial ordered set.*

Proof. By Definition 4.1 (i) it is trivial that $\preceq$ is reflexive and symetric. Let us consider that $P \preceq T$ and $T \preceq M$ for $P, T, M \in Q^{\mathcal{H}(\tilde{F}^c)}$. By Definition 4.1 (i), $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,P} > \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,T}$ and $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,T} > \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,M}$. Thus, $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,P} > \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,M}$. By the paragraph above, we have that $T \geq P$, $T \neq P$, $M \geq T$, and $M \neq T$. Thus, for some $q \in Q^{\tilde{F}^c}$, $M(q) > T(q) \geq P(q)$, i.e., $M \neq P$. By Definition 4.1 (ii) P reduces T and T reduces M, that is, $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,M} \otimes M = \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,M} \otimes T$. Since $\mathcal{L}$ is divisible, $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,M} = \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,T} \otimes a$ for some $a \in L$. Then $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,M} \otimes M = \mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,M} \otimes P$ holds. Therefore, $P \preceq M$. This concludes that $\preceq$ is transitive. The strict partial order $\preceq$ can be trivially converted in a non-strict partial order $\preceq$. In this way, the graph representing the poset $(Q^{\mathcal{H}(\tilde{F}^c)}, \preceq)$ is called the reduction graph of $\mathcal{H}(\tilde{F}^c)$, dentoted $\zeta(\mathcal{H}(\tilde{F}^c))$. In this graph, the nodes are states of $Q^{\mathcal{H}(\tilde{F}^c)}$; and the arcs represent the relation $\preceq$ between two states. Let us observe that if the construction $\mathcal{H}(\tilde{F}^c)$ is not a minimal deterministic $L^{\mathcal{B}}$-valued GFA, the structure of the graph $\zeta(\mathcal{H}(\tilde{F}^c))$ represents all the ways that minimal deterministic $L^{\mathcal{B}}$-valued GFAs equivalent to $\tilde{F}$ may be constructed from the automaton $\mathcal{H}(\tilde{F}^c)$. We recall that $\preceq$ is obtained via the Pigeonhole Principle with respect to the states of a minimal deterministic GFA equivalent to $\tilde{F}$. Thus, each isolated node in $\zeta(\mathcal{H}(\tilde{F}^c))$ is naturally associated to a unique state of this minimal deterministic $L^{\mathcal{B}}$-valued GFA. In addition, each node in $\zeta(\mathcal{H}(\tilde{F}^c))$, which is the left extreme of a path in the graph, is also associated to a unique state of this minimal determinstic $L^{\mathcal{B}}$-valued GFA. A node that is a left extreme of a path is a reducer state of all the rest of nodes (states) for its path. Therefore, the cardinal of the set of isolated nodes plus the cardinal of left extremes is the cardinal of any minimal deterministic $L^{\mathcal{B}}$-valued GFA equivalent to $\tilde{F}$. □

Example 4.5. *In this example, we show how to build all the possible minimal deterministic $L^{\mathcal{B}}$-valued GFAs for a given fuzzy language via the reduction graph of $\mathcal{H}(\tilde{F}^c)$. Let $\ell = \{\varepsilon/1, a^+/0.9, ba^*/0.4, ca*/0.7, a^+d/0.4, ba^*d/0.3, ca^*d/0.5\}$ be a fuzzy language over the alphabet $\Sigma = \{a, b, c, d\}$ and $\mathcal{L}$, where $\mathcal{L} = ([0,1], \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is based on a Continuous t-norm defined by*

$$x \otimes y = \begin{cases} \max(0.3, x + y - 1), & \text{if } (x, y) = [0.3, 1]^2 \\ \min(x, y), & \text{otherwise} \end{cases}$$

and whose residum is

$$x \rightarrow y = \begin{cases} 1, & \text{if } x \leq y \\ 1 - x + y, & \text{if } 0.3 < y < x \leq 0.7 \\ y, & \text{otherwise.} \end{cases}$$

By using the method to construct an $L^{\mathcal{B}}$-valued GFA from a fuzzy regular expression, and applying the construction $\mathcal{H}(\tilde{F}^c)$ to the $L^{\mathcal{B}}$-valued GFA $\tilde{F}$ that recognizes ℓ, we obtain the deterministic $L^{\mathcal{B}}$-valued GFA as in Figure 1. The Crisp $L^{\mathcal{B}}$-valued GFA $\tilde{F}^c$ that recognizes ℓ has 8 states. The set of states of $\mathcal{H}(\tilde{F}^c)$ are indicated in Table I. Each state u_x in Table I is an L-valued state of $Q^{\tilde{F}^c}$ and each row show the membership values $u_x(q_i) \in [0,1]$ with $q_i \in Q^{F^c}$. The *last Column in Table I indicates the values $\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{sup,u_x}$ for each state u_x. In this example, the maximum value of the paths reach each state from the initial state. Let us observe the operation to compute such values in the t-norm as indicated above.*

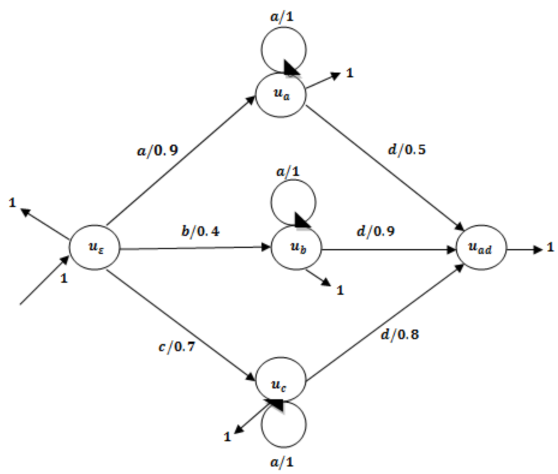

Figure 1: The deterministic $L^{\mathcal{B}}$-valued GFA that recognizes the fuzzy language ℓ obtaind via $\mathcal{H}(\tilde{F}^c)$.

Table I

P	$(q_1, \ldots, q_8)$	$\mathcal{P}^{\mathcal{H}(\tilde{F}^c)}_{max,P}$
u_ε	$(1, 0.9, 0.4, 0.7, 0, 0.4, 0.3, 0.5)$	1
u_a	$(1, 1, 0, 0, 0.5, 0.5, 0, 0)$	0.9
u_b	$(1, 1, 0, 0, 0.9, 0.9, 0, 0)$	0.4
u_c	$(1, 1, 0, 0, 0.8, 0.8, 0, 0)$	0.7
u_{ad}	$(1, 0, 0, 0, 0, 0, 0, 0)$	0.5

The deterministic $L^{\mathcal{B}}$-valued GFA in Figure 1 is accessible and observable but it is not a minimal deterministic $L^{\mathcal{B}}$-valued GFA as we see in the following. By applying Definition 4.1 to each pair of states $(P, T) \in Q^{\mathcal{H}(\tilde{F}^c)} \times Q^{\mathcal{H}(\tilde{F}^c)}$, we can define the adjacency matrix of reduction graph of $\mathcal{H}(\tilde{F}^c)$. This matrix is illustrated in Table II.

Table II

	u_ε	u_a	u_b	u_c	u_{ad}
u_ε	$-$	$-$	$-$	$-$	$-$
u_a			$\preceq$	$-$	$-$
u_b					
u_c			$\preceq$		$-$
u_{ad}			$-$		

In this simple example, the graph $\zeta(\mathcal{H}(\tilde{F}^c))$ contains two areas: $u_a \preceq u_b$ and $u_c \preceq u_b$. A minimal deterministic $L^{\mathcal{B}}$-valued GFA equivalent to $\mathcal{H}(\tilde{F}^c)$ has only 4 states and there are two minimal deterministic $L^{\mathcal{B}}$-valued GFAs equivalent. The former is obtained by reducing u_b to u_a and the second one, by reducing u_b to u_c. The method of reduction was outlined in subsection 3.3. In both cases, the two minimal deterministic $L^{\mathcal{B}}$-valued GFAs are obtained moving ending in u_b in Figure 1 to u_a or, alternatively, to u_c. In both cases, u_b is not accessible and it is removed. The two solutions are shown in Figure 2 and Figure 3, respectively. The algorithms to obtain the minimum automata ((i) the calculation of the maximum values of the paths, (ii) the construction of the adjacency matrix of the reduction graph, and (iii) the subsequent reduction of states from (the said matrix) are polynomial algorithms in the size $\mathcal{H}(\tilde{F}^c)$.

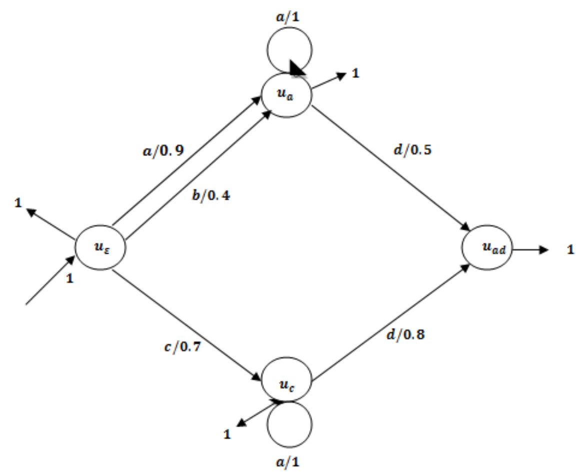

Figure 2: First option to obtain a minimal deterministic $L^{\mathcal{B}}$-valued GFA.

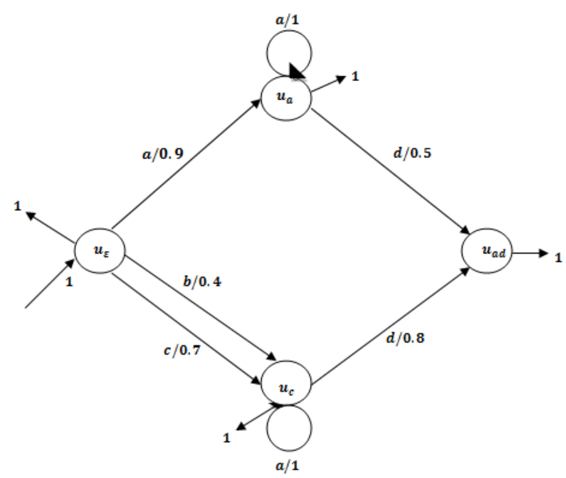

Figure 3: Second option to obtain a minimal deterministic $L^{\mathcal{B}}$-valued GFA.

5 Conclusions

The determinization of fuzzy automata is regarded as a closely examined and a well-investigated issue in the realm of theoretical computer science which has been distinguished for its highly practical useful applications. Even though a variety of methods have already been established to convert and adapt a fuzzy automaton to its related language equivalent fuzzy deterministic finite automaton (FDfA), they can still be employed merely for fuzzy automata which have been characterized over particular underlying sets of truth values. Filling this gap, therefore, the present investigation aimed to establish the concepts related to L-valued language identified by $L^{\mathcal{B}}$-valued general fuzzy automata as well as crisp deterministic $L^{\mathcal{B}}$-valued GFA $\tilde{F}^c$ equivalent to $L^{\mathcal{B}}$-valued GFA $\tilde{F}$, concentrating on the properties of $\tilde{F}^c$. Further, determinization method through factorization of L-valued states together with a method concerning state reduction were introduced and explicated. The main contribution of this study was then the automaton $\mathcal{H}(\tilde{F}^c)$, a deterministic $L^{\mathcal{B}}$-valued GFA that was expected to assure the related necessary conditions intended to achieve minimality and that its size is always equal or lesser than a minimal crisp deterministic $L^{\mathcal{B}}$-valued GFA equivalent to that. In our future research, we will study the formation of other reduction graphs more complicated than the one developed in Example 1, whose objective has been simply to illustrate the proposed minimal determinization procedure.

References

[1] Kh. Abolpour, *On L^B-valued GFA: An L^B-valued operator oriented view with t-norm/t-conorm and implicators*, New Mathematics and Natural Computation, **18**(3) (2022), 573-592. https://doi.org/10.1142/S1793005722500296

[2] Kh. Abolpour, *A new characterization of congruence and the discrete Sugeno integral on LB-valued general fuzzy automata*, Fuzzy Sets and Systems, **460** (2023), 186-199. https://doi.org/10.1016/j.fss.2023.01.012

[3] Kh. Abolpour, A. Broumand Saeid, *Fundamental group of L^B-valued general fuzzy automata*, New Mathematics and Natural Computation, **18**(2) (2022), 545-558. https://doi.org/10.1142/S1793005722500272

[4] Kh. Abolpour, M. M. Zahedi, *New directions in L^B-valued general fuzzy automata: A topological view*, Filomat, **35**(1) (2021), 251-270. https://doi.org/10.2298/FIL2101251A

[5] Kh. Abolpour, M. M. Zahedi, *L^B-valued general fuzzy automata*, Fuzzy Sets and Systems, **442** (2022), 288-308. https://doi.org/10.1016/j.fss.2021.08.017

[6] G. Bailador, G. Trivino, *Pattern recognition using temporal fuzzy automata*, Fuzzy Sets and Systems, **161** (2010), 37-55. https://doi.org/10.1016/j.fss.2009.08.005

[7] R. Belohlavek, *Determinism and fuzzy automata*, Information Sciences, **143** (2002), 205-209. https://doi.org/10.1016/S0020-0255(02)00192-5

[8] R. Belohlavek, V. Vychodil, *Fuzzy equational logic, studies in fuzziness and soft computing*, Springer, Berlin-Heidelberg, 2005. https://doi.org/10.1007/b105121

[9] J. A. Brzozowski, *Canonical regular expressions and minimal state graphs for definite events*, In Proc. Sympos. Math. Theory of Automata (New York, 1962), Polytechnic Press of Polytechnic Inst. of Brooklyn, Brooklyn, N.Y., (1963), 529-561.

[10] M. Doostfatemeh, S. C. Kremer, *New directions in fuzzy automata*, International Journal of Approximate Reasoning, **38** (2005), 175-214. https://doi.org/10.1016/j.ijar.2004.08.001

[11] R. van Glabbeek, B. Ploeger, *Five determinization algorithms*, Conference: Implementation and Applications of Automata, 13th International Conference, CIAA 2008, San Francisco, California, USA, July 21-24, (2008). Proceedings. https://doi.org/10.1007/978-3-540-70844-5_17

[12] J. R. Gonzalez de Mendivil, *A generalization of Myhill-Nerode theorem*, Fuzzy Sets and Systems, **301** (2016), 103-115. https://doi.org/10.1016/j.fss.2015.12.011

[13] J. R. Gonzalez de Mendivil, J. R. Garitagoitia, *Determinization of fuzzy automata via factorization of fuzzy states*, Information Sciences, **283** (2014), 165-179. https://doi.org/10.1016/j.ins.2018.08.033

[14] J. E. Hopcroft, R. Motwani, J. D. Ullman, *Introduction to automata theory*, 3rd Edition. Addison-Wesley, 2007.

[15] J. Ignjatovic, M. Ciric, S. Bogdanovi, *Determinization of fuzzy automata with membership values in complete residuated lattices*, Information Sciences, **178** (2008), 164-180. https://doi.org/10.1016/j.ins.2007.08.003

[16] J. Ignjatovic, M. Ciric, S. Bogdanovic, T. Petkovic, *Myhill-Nerode type theory for fuzzy languages and automata*, Fuzzy Sets and Systems, **161** (2010), 1288-1324. https://doi.org/10.1016/j.fss.2009.06.007

[17] Z. Jančić, J. Ignjatović, M. Ćirić, *An improved algorithm for determinization of weighted and fuzzy automata*, Information Sciences, **181** (2011), 1358-1368. https://doi.org/10.1016/j.ins.2010.12.008

[18] Z. Jančić, I. Micic, J. Ignjatović, M. Ćirić, *Further improvement of determinization methods for fuzzy finite automata*, Fuzzy Sets and Systems, **301** (2015), 79-102. https://doi.org/10.1016/j.fss.2015.11.019

[19] Y. M. Li, W. Pedrycz, *Fuzzy finite automata and fuzzy regular expressions with membership values in lattice ordered monoids*, Fuzzy Sets and Systems, **156** (2005), 68-92. https://doi.org/10.1016/j.fss.2005.04.004

[20] Y. Li, J. Wei, *Possibilistic fuzzy linear temporal logic and its model checking*, IEEE Transactions on Fuzzy Systems, **29** (2021), 1899-1913. https://doi.org/10.1109/TFUZZ.2020.2988848

[21] J. Mordeson, D. Malik, *Fuzzy automata and languages: Theory and applications*, Chapman and Hall, CRC Press, London, Boca Raton, FL., 2002.

[22] D. Qiu, *Automata theory based on quantum logic: Some characterizations*, Information and Computation, **190** (2004), 179-195. https://doi.org/10.1016/j.ic.2003.11.003

[23] D. Qiu, *Characterizations of fuzzy finite automata*, Fuzzy Sets and Systems, **141** (2004), 391-414. https://doi.org/10.1016/S0165-0114(03)00202-1

[24] D. Qiu, *Automata theory based on quantum logic: Reversibilities and pushdown automata*, Theoretical Computer Science, **386** (2007), 38-56. https://doi.org/10.1016/j.tcs.2007.05.026

[25] G. G. Rigatos, *Fault detection and isolation based on fuzzy automata*, Information Sciences, **179**(12) (2009), 1893-1902. https://doi.org/10.1016/j.ins.2009.01.015

[26] M. Shamsizadeh, M. M. Zahedi, Kh. Abolpour, *Bisimulation for BL-general fuzzy automata*, Iranian Journal of Fuzzy Systems, **13** (2016), 35-50. https://doi.org/10.22111/IJFS.2016.2594

[27] M. Shamsizadeh, M. M. Zahedi, Kh. Abolpour, *Admissible partition for Bl-general fuzzy automaton*, Iranian Journal of Fuzzy Systems, **15** (2018), 79-90. https://doi.org/10.22111/IJFS.2018.4283

[28] A. K. Srivastava, S. P. Tiwari, *A topology for fuzzy automata*, Lecture Notes in Artificial Intelligence, **2275** (2002), 485-490. https://doi.org/10.1007/3-540-45631-7_66

[29] A. K. Srivastava, S. P. Tiwari, *On relationships among fuzzy approximation operators, fuzzy topology, and fuzzy automata*, Fuzzy Sets and Systems, **138** (2003), 197-204. https://doi.org/10.1016/S0165-0114(02)00442-6

[30] S. Stanimirovic, M. Ciric, J. Ignjatovic, *Determinization of fuzzy automata by factorizations of fuzzy states and right invariant fuzzy quasiorders*, Information Sciences, **469**(12) (2018), 79-100. https://doi.org/10.1016/j.ins.2018.08.033

[31] S. P. Tiwari, S. Sharan, *Fuzzy automata based on lattice-ordered monoids with algebraic and topological aspects*, Fuzzy Information and Engineering, **4** (2012), 155-164. https://doi.org/10.1007/s12543-012-0108-y

[32] S. P. Tiwari, S. Sharan, *ℓ-valued automata and associated topologies*, International Journal of Granular Computing, Rough Sets and Intelligent Systems, **3** (2013), 85-94. https://doi.org/10.1504/IJGCRSIS.2013.054128

[33] Q. Wu, Z. Han, Q. E. Wu, *Application of fuzzy automata decisionmaking system in target control*, Journal of Computer and Communications, **5**(10) (2017), 16-25. https://doi.org/10.4236/jcc.2017.510003.