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Original paper

On modularity property for uninorms with continuous underlying functions

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Abstract

In literature, for the four usual classes of uninorms, the modularity equation has been solved except for the kind of ones having continuous underlying functions. This paper is devoted to solving the modularity equation involving two uninorms with continuous underlying functions. We discuss this modularity equation in detail by dividing the main section into two parts. The structure characterization of the two uninorms is almost completely obtained and it is found that they are equal in the unit square except in a subdomain.

Keywords: Aggregation functions, uninorms, modularity, functional equations.

1 Introduction

As mixed aggregation functions, the term uninorm was first introduced in [26] and then be further studied [1, 5, 6, 7, 11, 15, 17]. It has been proved in literature that such functions are useful in expert systems [2], fuzzy logic [9], neural networks [16], fuzzy systems [25] and so on. Because of the strong applications, uninorms have also been extensively studied theoretically. Many scholars have theoretically characterized the structure of uninorms with certain properties. There are many examples of studies on this research line, such as migrativity [14, 24], conditional distributivity or distributivity [10, 12, 19, 21, 22], and modularity [4, 13, 20].

Modularity has been attracted great attention from researchers since it can be seen as a generalized associativity equation with a restricted condition and it is also closely related to distributivity equation. In the earlier literature, for the modularity between two uninorms, researchers studied it when both of the two uninorms in the modularity equation belong to one of the three common kinds of uninorms: $\mathcal{U}_{\min}$ or $\mathcal{U}_{\max}$ [13], idempotent uninorms [18], or uninorms continuous in $(0,1)^2$ [4]. Later, the authors in [20] restudied it when one of the uninorms lies in the four well-known classes of uninorms (i.e., $\mathcal{U}_{\min}$ or $\mathcal{U}_{\max}$, representable uninorms, idempotent uninorms and uninorms continuous in $(0,1)^2$), and the other uninorm is arbitrary. The results in [20] show that when one uninorm is in one of the four common classes of uninorms, then the other uninorm must belong to the same class. Modularity equations also have been discussed between other operators, see [13, 27, 28, 29, 30].

As we all know, uninorms having continuous underlying functions are a new class of uninorms and become one of research hotspots. We attempt in this work to solve the modularity equation involving uninorms U_1 and U_2 with continuous underlying functions. Based on $e_1 > e_2$ or $e_2 > e_1$, we discuss the modularity equation in detail, respectively. The structure of the two uninorms are almost completely characterized and we find that they are equal in the unit square except in a subdomain. We think our work is a good complement to the previous ones in literature for this topic because uninorms having continuous underlying functions are an important class of uninorms.

The structural layout of the paper is as follows. The second section is used to recall some basic concepts and facts. The main results of this work are put in the third section. A simple conclusion is in the fourth part of the text.

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2 Preliminaries

In this section, we recall some concepts and results used in the text.

Definition 2.1. [8] *A decreasing function $N : [0,1] \to [0,1]$ is called a strong negation if it is involutive, i.e., $N(N(x)) = x$, for any $x \in [0,1]$.*

Definition 2.2. [8] *Let $F : [0,1]^2 \to [0,1]$ be a binary operator and N be a strong negation. A function $G : [0,1]^2 \to [0,1]$ is called the N-dual of F if $G(x,y) = N(F(N(x),N(y)))$, for any $x, y \in [0,1]$.*

Definition 2.3. [8] *A function $T : [0,1]^2 \to [0,1]$ is called a triangular norm (t-norm for short) if T satisfies the associativity, the commutativity and the monotonicity, and has 1 as its neutral element, i.e., $T(x,1) = x$, for any $x \in [0,1]$.*

A t-norm T is called Archimedean if for any $x, y \in (0,1)$, there exists some $n \in \mathbf{N}$ such that $x_T^{(n)} < y$, where $x_T^{(n)} = T(\underbrace{x, x, \cdots, x}_{n \ times})$; it is called strict if it is continuous and strictly monotonic, i.e., $T(x,y) < T(x,z)$ when $x > 0$ and $y < z$; it is called nilpotent if for each $x \in (0,1)$, there exists some $n \in \mathbf{N}^+$ such that $x_T^{(n)} = 0$.

Remark 2.4. *For a continuous t-norm T, it is Archimedean if and only if $T(x,x) < x$ for any $x \in (0,1)$. Any continuous Archimedean t-norm is either strict or nilpotent.*

Theorem 2.5. [8] *Let T be a t-norm. Then T is continuous if and only if T is uniquely representable as an ordinal sum of continuous Archimedean t-norms, i.e., there exists a uniquely determined (finite or countable infinite) index set I, a family of uniquely determined pairwise disjoint open subintervals $(a_i, b_i)_{i\in I}$ and a family of uniquely determined continuous Archimedean t-norms $(T_i)_{i\in I}$ such that $T = (\langle a_i, b_i, T_i\rangle)_{i\in I}$, or,*

$$T(x,y) = \begin{cases} a_i + (b_i - a_i)T_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}) & (x,y) \in [a_i,b_i]^2, \\ \min(x,y) & otherwise. \end{cases}$$

For any t-conorm S, since it is N-dual to a t-norm T, where N is the standard strong negation $N(x) = 1 - x$, then all facts about t-conorms can be obtained similarly to t-norms.

Definition 2.6. [26] *A function $U : [0,1]^2 \to [0,1]$ is called a uninorm if U satisfies the associativity, the commutativity and the monotonicity, and has $e \in [0,1]$ as its neutral element, i.e., $U(e,x) = x$, for all $x \in [0,1]$.*

It has been proved that $U(0,1) \in \{0,1\}$ [6]. U is called conjunctive if $U(0,1) = 0$ and disjunctive if $U(0,1) = 1$. A uninorm becomes a t-conorm if its neutral element is 0, and is a t-norm if its neutral element is 1. A uninorm U is called a *proper* uninorm if its neutral element e is in $(0,1)$.

Proposition 2.7. [6] *Let U be a uninorm and e be the neutral element of U. Then U is given as*

$$U(x,y) = \begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & (x,y) \in [e,1]^2, \\ C(x,y) & otherwise, \end{cases}$$

where T_U is a t-norm, S_U is a t-conorm and C fulfills $\min(x,y) \le C(x,y) \le \max(x,y)$ for all $(x,y) \in [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

Here, we call T_U and S_U the underlying t-norm and the underlying t-conorm of U, respectively. A uninorm has continuous underlying functions if its underlying t-norm and underlying t-conorm are both continuous. The class of uninorms with continuous underlying functions is denoted as **COU**.

Lemma 2.8. [15] *If $a \in [0,1]$ is an idempotent element of $U \in \mathbf{COU}$, then $U(a,x) \in \{a,x\}$ for any $x \in [0,1]$.*

Lemma 2.9. [17] *Let U be a proper uninorm in **COU** with neutral element e. If $U(x_0,y_0) = x_0$ for some $(x_0,y_0) \in [0,e) \times (e,1]$, then $U(x,y) = x$ for any $[0,x_0] \times [e,y_0]$. If $U(x_0,y_0) = y_0$, for $(x_0,y_0) \in [0,e) \times (e,1]$, then $U(x,y) = y$, for any $[x_0,e] \times [y_0,1]$.*

Definition 2.10. [6] *Let $e \in (0,1)$. An operator $U : [0,1]^2 \to [0,1]$ is called a* representable *uninorm if and only if there exists a continuous strictly increasing function $h : [0,1] \to [-\infty, +\infty]$ with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that $U(x,y) = h^{-1}(h(x) + h(y))$, for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$.*

Here, h is called an additive generator of U.

Theorem 2.11. [5] *Let U be a proper uninorm with neutral element e such that both T_U and S_U are strict. Then U is representable if and only if there exists some $(x_0, y_0) \in A(e)$ such that $\min(x_0, y_0) < U(x_0, y_0) < \max(x_0, y_0)$.*

Proposition 2.12. [23] *Let U be a proper uninorm with continuous Archimedean underlying functions and neutral element e. Then the values of U on the domain $(0, e) \times (e, 1) \cup (e, 1) \times (0, e)$ must be one of the following cases:*

(i) *$U(x, y) = \min(x, y)$, for any $(x, y) \in (0, e) \times (e, 1) \cup (e, 1) \times (0, e)$.*

(ii) *$U(x, y) = \max(x, y)$, for any $(x, y) \in (0, e) \times (e, 1) \cup (e, 1) \times (0, e)$.*

(iii) *$\min(x, y) < U(x, y) < \max(x, y)$, for any $(x, y) \in (0, e) \times (e, 1) \cup (e, 1) \times (0, e)$. In this case, the underlying functions T_U and S_U are both strict.*

Definition 2.13. [13] *Let F and G be two binary operators on $[0, 1]$. We say F is modular over G if*

$$F(x, G(y, z)) = G(F(x, y), z) \text{ for any } x, y, z \in [0, 1] \text{ and } z \le x.$$

Proposition 2.14. [20] (i) *A t-norm T_1 is modular over a t-norm T_2 if and only if $T_1 = T_2$.*

(ii) *A t-conorm S_1 is modular over a t-conorm S_2 if and only if $S_1 = S_2$.*

(iii) *A t-norm T is modular over a t-conorm S if and only if $T = T_M$ and $S = S_M$.*

(iv) *A t-conorm S is never modular over a t-norm T.*

Proposition 2.15. [20] *Let F and G be two binary operators on $[0, 1]$, N be a strong negation and F_1 and G_1 be the N-duals of F and G, respectively. If F and G are commutative, then F is modular over G if and only if G_1 is modular over F_1.*

Theorem 2.16. [20] *Let U_1 and U_2 be two proper uninorms with neutral elements e_1 and e_2, respectively.*

(i) *Suppose U_1 is a representable uninorm with an additive generator h_1. Then U_1 is modular over U_2 if and only if U_2 is a representable uninorm with additive generator $h_2(x) = c(h_1(x) - h_1(e_2))$, and U_1 and U_2 both are conjunctive or both are disjunctive, where $c > 0$ is a constant.*

(ii) *Suppose U_2 is a representable uninorm with an additive generator h_2. Then U_1 is modular over U_2 if and only if U_1 is a representable uninorm with additive generator $h_1(x) = c(h_2(x) - h_2(e_1))$, and U_1 and U_2 both are conjunctive or both are disjunctive, where $c > 0$ is a constant.*

3 Main results

In this section, we first show that the modularity property is not satisfied between proper uninorms and continuous Archimedean t-norms (t-conorms). Then we discuss the modularity equation involving two proper uninorms with continuous underlying functions.

Proposition 3.1. *Let U be a proper uninorm and S a continuous Archimedean t-conorm. Then U is not modular over S.*

Proof. Let $e \in (0, 1)$ be the neutral element of U. Assume that U is modular over S. Due to the Archimedean property of S, we have $S(e, e) > e$. Again by the continuity of S, there exists some $x_0 \in (0, e)$ such that $S(x_0, x_0) = e$. According to the modularity equation, we get that $x_0 = U(x_0, e) = U(x_0, S(x_0, x_0)) = S(U(x_0, x_0), x_0)$, i.e., $x_0 = S(U(x_0, x_0), x_0)$. Let $t_0 = U(x_0, x_0)$. Clearly, $t_0 \le x_0 < e$ and $x_0 = S(t_0, x_0)$.

If $t_0 > 0$, then $x_0 = S(t_0, x_0) = S(t_0, S(t_0, x_0)) = S(S(t_0, t_0), x_0)$. By iteration, we further have that $x_0 = S(t_0, x_0) = \cdots = S((t_0)_S^{(n)}, x_0)$ for any $n \in \mathbf{N}^+$, where $(t_0)_S^{(n)} = S(\underbrace{x_0, x_0, \cdots, x_0}_{n \text{ times}})$. By the continuity and Archimedean property of S, it holds that $x_0 = S(\lim\limits_{n\to\infty} (t_0)_S^{(n)}, x_0) = S(1, x_0) = 1$, a contradiction.

If $t_0 = 0$, then $U(x_0, S(0, x_0)) = U(x_0, x_0) = t_0 = 0$. But $S(U(x_0, 0), x_0) = S(0, x_0) = x_0$. It is a contradiction as well.

Summing up, U is not modular over S. □

Remark 3.2. (1) *If T is a continuous Archimedean t-norm and U is a proper uninorm, then T is not modular over U by Proposition 2.15.*

(2) *In [20], the authors have proved that a uninorm U is modular over a t-norm T if and only if $U = T$* (cf. [20, Theorem 7]), *and a t-conorm S is modular over a uninorm U if and only if $U = S$* (cf. [20, Theorem 5]). *So proper uninorms are not modular over t-norms, and t-conorms are not modular over proper uninorms.*

Proposition 3.3. *Let U_1 and U_2 be two proper uninorms with respective neutral elements e_1 and e_2. Suppose both U_1 and U_2 have continuous Archimedean underlying functions and $e_2 < e_1$. If U_1 is modular over U_2, then both U_1 and U_2 are representable uninorms.*

Proof. Clearly, $U_2(e_1, e_1) > e_1$ since U_2 has continuous Archimedean underlying functions. We first prove U_1 is representable. In fact, according to Proposition 2.12, the values of U_1 on $(0, e_1) \times (e_1, 1)$ must fulfill one of the following cases:

i) $U_1(x, y) = \min(x, y)$ for any $(x, y) \in (0, e_1) \times (e_1, 1)$.

ii) $U_1(x, y) = \max(x, y)$ for any $(x, y) \in (0, e_1) \times (e_1, 1)$.

iii) $\min(x, y) < U_1(x, y) < \max(x, y)$ for any $(x, y) \in (0, e_1) \times (e_1, 1)$. In this case, the underlying functions T_{U_1} and S_{U_1} must be strict.

Assume i) holds. Then for any $x \in (e_1, 1)$, it holds that

$$x = U_1(x, e_1) = U_1(x, U_2(e_2, e_1)) = U_2(U_1(x, e_2), e_1) = U_2(\min(x, e_2), e_1) = U_2(e_2, e_1) = e_1,$$

i.e, $x = e_1$. This is a contradiction.

Assume ii) holds. Then for any $y \in (e_1, 1)$, it holds that

$$y = U_1(y, e_1) = U_1(y, U_2(e_2, e_1)) = U_2(U_1(y, e_2), e_1) = U_2(\max(y, e_2), e_1) = U_2(y, e_1),$$

i.e., $U_2(e_1, y) = y$ for any $y \in (e_1, 1)$. It follows from the continuity of S_{U_2} that

$$U_2(e_1, e_1) = \lim_{y \to e_1^+} U_2(e_1, y) = \lim_{y \to e_1^+} y = e_1,$$

i.e., $U_2(e_1, e_1) = e_1$, which contradicts $U_2(e_1, e_1) > e_1$.

Then iii) must hold. Hence U_1 is a representable uninorm by Theorem 2.11. Again in line with Theorem 2.16, we have that U_2 is a representable uninorm, too. □

Proposition 3.4. *Let U_1 and U_2 be two proper uninorms with respective neutral elements e_1 and e_2. Suppose both U_1 and U_2 have continuous Archimedean underlying functions and $e_1 < e_2$. If U_1 is modular over U_2, then both U_1 and U_2 are representable uninorms.*

Proof. Obviously, $U_2(e_1, e_1) < e_1$ since U_2 has continuous Archimedean underlying functions. First, we prove that U_2 is representable.

Just as the above proposition, if i) holds for U_2, then for any $x \in (e_2, 1)$, we obtain that

$$x = U_1(x, e_1) = U_1(x, U_2(e_1, x)) = U_2(U_1(x, e_1), x) = U_2(x, x),$$

i.e., $U_2(x, x) = x$ for any $x \in (e_2, 1)$, which contradicts the Archimedean property of S_{U_2}.

If ii) holds for U_2, then for any $x \in (e_2, 1)$, it is true that

$$U_1(x, x) = U_1(x, U_2(e_1, x)) = U_2(U_1(x, e_1), x) = U_2(x, x),$$

i.e., $U_1(x, x) = U_2(x, x)$. Hence

$$U_1(e_2, e_2) = \lim_{x \to e_2^+} U_1(x, x) = \lim_{x \to e_2^+} U_2(x, x) = e_2,$$

by the continuity of S_{U_1} and S_{U_2}, i.e., $U_1(e_2, e_2) = e_2$. This contradicts the Archimedean property of S_{U_1}.

Hence, only iii) holds and then U_2 is a representable uninorm by Theorem 2.11. Again by Theorem 2.16, we know that U_1 is representable. □

In the above two propositions, we discuss the cases $e_2 < e_1$ and $e_1 < e_2$ for two uninorms with continuous Archimedean underlying functions, respectively. Since for any uninorm U_1 and U_2, if $e_1 = e_2$ and U_1 is modular over U_2, then it must hold $U_1 = U_2$ [13]. So there is no need to discuss the case $e_1 = e_2$ in the above for two uninorms with continuous Archimedean underlying functions.

In the below section, we will divide it into two subsections, one is for the case $e_1 > e_2$, the other is for the case $e_2 > e_1$ (the case $e_1 = e_2$ means that $U_1 = U_2$ when U_1 is modular over U_2 [13]).

3.1 Case $e_1 > e_2$

Condition $e_1 > e_2$ implies that $U_2(e_1, e_1) \geq e_1$. We will consider the modularity equation for the case $U_2(e_1, e_1) > e_1$, since the results for the case $U_2(e_1, e_1) = e_1$ can be found in Theorem 9 of [20].

Lemma 3.5. *Let U_1 and U_2 be two proper uninorms in* **COU** *with respective neutral elements e_1 and e_2 ($e_1 > e_2$). If $U_2(e_1, e_1) > e_1$ and U_1 is modular over U_2, then $U_1(e_2, e_2) < e_2$ and there exists an idempotent element d of U_2 ($d > e_1$) such that $U_1(d, d) = d$ and $U_2|_{[e_2,d]^2}$ is isomorphic to a continuous Archimedean t-conorm.*

Proof. Suppose $S_{U_2} = (\langle \frac{c_m - e_2}{1-e_2}, \frac{d_m - e_2}{1-e_2}, S_m \rangle)_{m\in I}$, where $1 \geq d_m > c_m \geq e_2$ and S_m is a continuous Archimedean t-conorm. Note that e_1 is not an idempotent element of U_2, then exists some $m_0 \in I$ such that $e_2 \leq c_{m_0} < e_1 < d_{m_0} \leq 1$.

From the modularity equation, we have that

$$U_1(d_{m_0}, d_{m_0}) = U_1(d_{m_0}, U_2(e_1, d_{m_0})) = U_2(U_1(d_{m_0}, e_1), d_{m_0}) = U_2(d_{m_0}, d_{m_0}) = d_{m_0}.$$

So $U_1(d_{m_0}, d_{m_0}) = d_{m_0}$.

In the following, we will prove that $c_{m_0} = e_2$ and $U_1(e_2, e_2) < e_2$. In fact, it is obvious that $U_1(c_{m_0}, c_{m_0}) \leq c_{m_0}$.

If $U_1(c_{m_0}, c_{m_0}) \in [e_2, c_{m_0}]$, then

$$U_1(c_{m_0}, c_{m_0}) = U_1(c_{m_0}, U_2(c_{m_0}, c_{m_0})) = U_2(U_1(c_{m_0}, c_{m_0}), c_{m_0}) = \max(U_1(c_{m_0}, c_{m_0}), c_{m_0}) = c_{m_0}.$$

So $U_1(c_{m_0}, c_{m_0}) = c_{m_0}$. Consequently, $U_1|_{[c_{m_0}, d_{m_0}]}$ is isomorphic to a uninorm U_1', which is clearly modular over the continuous Archimedean t-conorm S_{m_0}. But it is impossible by Proposition 3.1. So $U_1(c_{m_0}, c_{m_0}) < e_2$. Because of the continuity of T_{U_1} and $U_1(c_{m_0}, c_{m_0}) < e_2 \leq U_1(c_{m_0}, e_1) = c_{m_0}$, there exists some $x_0 \in (c_{m_0}, e_1]$ such that $U_1(c_{m_0}, x_0) = e_2$. Then

$$e_2 = U_1(x_0, c_{m_0}) = U_1(x_0, U_2(c_{m_0}, c_{m_0})) = U_2(U_1(x_0, c_{m_0}), c_{m_0}) = U_2(e_2, c_{m_0}) = c_{m_0}.$$

That is to say, $e_2 = c_{m_0}$ and hence $U_1(e_2, e_2) = U_1(c_{m_0}, c_{m_0}) < e_2$.

Replacing d_{m_0} with d, we end the proof. □

Lemma 3.6. *Let U_1 and U_2 be two proper uninorms in* **COU** *with respective neutral elements e_1 and e_2 ($e_1 > e_2$). If $U_1(e_2, e_2) < e_2$ and U_1 is modular over U_2, then $U_2(e_1, e_1) > e_1$ and there exists some idempotent element a of U_1 ($a < e_2$) such that $U_2(a, a) = a$ and $U_1|_{[a,e_1]^2}$ is isomorphic to a continuous Archimedean t-norm.*

Proof. Suppose $T_{U_1} = (\langle \frac{a_n}{e_1}, \frac{b_n}{e_1}, T_n \rangle)_{n\in J}$, where $a_n < b_n \leq e_1$ and T_n is a continuous Archimedean t-norm. Since $U_1(e_2, e_2) < e_2$, then there exists some $n_0 \in J$ such that $a_{n_0} < e_2 < b_{n_0} \leq e_1$.

$$a_{n_0} = U_1(a_{n_0}, a_{n_0}) = U_1(a_{n_0}, U_2(e_2, a_{n_0})) = U_2(U_1(a_{n_0}, e_2), a_{n_0}) = U_2(a_{n_0}, a_{n_0}).$$

So $U_2(a_{n_0}, a_{n_0}) = a_{n_0}$.

In the following, we prove $e_1 = b_{n_0}$ and $U_2(e_1, e_1) > e_1$. It is obvious that $U_2(b_{n_0}, b_{n_0}) \geq b_{n_0}$.

If $U_2(b_{n_0}, b_{n_0}) \in [b_{n_0}, e_1]$, then

$$b_{n_0} = U_1(b_{n_0}, U_2(b_{n_0}, b_{n_0})) = U_2(U_1(b_{n_0}, b_{n_0}), b_{n_0}) = U_2(b_{n_0}, b_{n_0}),$$

i.e., $U_2(b_{n_0}, b_{n_0}) = b_{n_0}$, which together $U_2(a_{n_0}, a_{n_0}) = a_{n_0}$ shows that $U_2|_{[a_{n_0}, b_{n_0}]^2}$ is isomorphic to a uninorm U_2'. Consequently, T_{n_0} is modular over U_2', which is impossible by Remark 3.2.

So $U_2(b_{n_0}, b_{n_0}) > e_1$. Notice that $U_2(b_{n_0}, b_{n_0}) > e_1 \geq U_2(b_{n_0}, e_2) = b_{n_0}$ and the continuity of S_{U_2}, then there exists some $x_1 \in [e_2, b_{n_0})$ such that $U_2(b_{n_0}, x_1) = e_1$. Hence,

$$b_{n_0} = U_1(b_{n_0}, e_1) = U_1(b_{n_0}, U_2(b_{n_0}, x_1)) = U_2(U_1(b_{n_0}, b_{n_0}), x_1) = U_2(b_{n_0}, x_1) = e_1,$$

i.e., $b_{n_0} = e_1$. So $U_2(e_1, e_1) > e_1$.

To end the proof, Replace a_{n_0} with a. □

According to the above two lemmas, we get the following corollary.

Corollary 3.7. *Let U_1 and U_2 be two proper uninorms with respective neutral elements e_1 and e_2. Suppose $e_1 > e_2$. If U_1 is modular over U_2, then $U_1(e_2, e_2) < e_2$ if and only if $U_2(e_1, e_1) > e_1$.*

Lemma 3.8. *Let U_1' and U_2' be two proper uninorms in* **COU** *with respective neutral elements e_1' and e_2' ($e_1' > e_2'$). Suppose that $T_{U_1'}$ and $S_{U_2'}$ are continuous Archimedean. If U_1' is modular over U_2', then both $S_{U_1'}$ and $T_{U_2'}$ are continuous Archimedean.*

Proof. We first prove $S_{U_1'}$ is continuous Archimedean. Otherwise, there exists an $x_0 \in (e_1', 1)$ such that $U_1'(x_0, x_0) = x_0$. Then $U_1'(x_0, e_2') = x_0$ or e_2' by Lemma 2.8.

If $U_1'(x_0, e_2') = x_0$, then

$$x_0 = U_1'(x_0, x_0) = U_1'(x_0, U_2'(e_2', x_0)) = U_2'(U_1'(x_0, e_2'), x_0) = U_2'(x_0, x_0).$$

So $U_2'(x_0, x_0) = x_0$, which contradicts the Archimedean property of $S_{U_2'}$.

If $U_1'(x_0, e_2') = e_2'$, then for any $y \in (e_1', x_0)$, it holds that

$$U_1'(x_0, y) = U_1'(x_0, U_2'(e_2', y)) = U_2'(U_1'(x_0, e_2'), y) = U_2'(e_2', y) = y,$$

i.e., $U_1'(x_0, y) = y = \min(x_0, y)$. However, $U_1'(x_0, y) \geq \max(x_0, y) = x_0$ since $x_0 > e_1'$ and $y > e_1'$. It leads to a contradiction.

As a consequence, $S_{U_1'}$ is a continuous Archimedean t-conorm.

Now we prove that $T_{U_2'}$ is continuous Archimedean. Otherwise, there exists some $x_1 \in (0, e_2')$ such that $U_2'(x_1, x_1) = x_1$. Then $U_2'(x_1, e_1') = x_1$ or e_1'.

If $U_2'(e_1', x_1) = x_1$, then

$$U_1'(x_1, x_1) = U_1'(x_1, U_2'(e_1', x_1)) = U_2'(U_1'(x_1, e_1'), x_1) = U_2'(x_1, x_1) = x_1.$$

So $U_1'(x_1, x_1) = x_1$, which contradicts the Archimedean property of $T_{U_1'}$.

If $U_2'(e_1', x_1) = e_1'$, then for any $y \in (x_1, e_2')$,

$$y = U_1'(y, e_1') = U_1'(y, U_2'(e_1', x_1)) = U_2'(U_1'(y, e_1'), x_1) = U_2'(y, x_1),$$

i.e., $U_2'(y, x_1) = y = \max(y, x_1)$. However, $U_2'(y, x_1) \leq \min(y, x_1) = x_1$ since $y < e_1'$ and $x_1 < e_1'$. It is a contradiction.

So, $T_{U_2'}$ also is a continuous Archimedean t-norm. □

Remark 3.9. *If $U_2'(e_1', e_1') > e_1'$ and the conditions in Lemma 3.8 are satisfied, then from Lemma 3.8 and Proposition 3.3, we can obtain that both U_1' and U_2' are representable uninorms.*

Theorem 3.10. *Let U_1 and U_2 be two proper uninorms in* **COU** *with respective neutral elements e_1 and e_2 ($e_1 > e_2$). Suppose $U_2(e_1, e_1) > e_1$. Then U_1 is modular over U_2 if and only if there exist two elements a and d, a continuous t-norm T, a continuous t-conorm S, two representable uninorms R_1 and R_2 with respective neutral elements $e_1' = \frac{e_1 - a}{d-a}$ and $e_2' = \frac{e_2 - a}{d-a}$ such that $0 \leq a < e_2 < e_1 < d \leq 1$, U_1 and U_2 are given by, respectively,*

$$U_1(x,y) = \begin{cases} aT(\frac{x}{a}, \frac{y}{a}) & (x,y) \in [0,a]^2, \\ a + (d-a)R_1(\frac{x-a}{d-a}, \frac{y-a}{d-a}) & (x,y) \in [a,d]^2, \\ d + (1-d)S(\frac{x-d}{1-d}, \frac{y-d}{1-d}) & (x,y) \in [d,1]^2, \\ \min(x,y) & (x,y) \in [0,a] \times [a,d) \cup [a,d) \times [0,a], \\ \max(x,y) & (x,y) \in (a,d] \times [d,1] \cup [d,1] \times (a,d], \\ U_1(x,y) & \textit{otherwise}, \end{cases} \tag{1}$$

$$U_2(x,y) = \begin{cases} aT(\frac{x}{a}, \frac{y}{a}) & (x,y) \in [0,a]^2, \\ a + (d-a)R_2(\frac{x-a}{d-a}, \frac{y-a}{d-a}) & (x,y) \in [a,d]^2, \\ d + (1-d)S(\frac{x-d}{1-d}, \frac{y-d}{1-d}) & (x,y) \in [d,1]^2, \\ \min(x,y) & (x,y) \in [0,a] \times [a,d) \cup [a,d) \times [0,a], \\ \max(x,y) & (x,y) \in (a,d] \times [d,1] \cup [d,1] \times (a,d], \\ U_1(x,y) & \textit{otherwise}, \end{cases} \tag{2}$$

where R_1 is modular over R_2.

Proof. (Necessity) Corollary 3.7 shows $U_1(e_2, e_2) < e_2$. According to Lemmas 3.5 and 3.6, there exist an idempotent element a of U_1 and an idempotent element d of U_2 such that $a < e_2 < e_1 < d$ and $U_1(d,d) = d$, $U_2(a,a) = a$. Since $U_1(a,a) = a$, $U_1(d,d) = d$ and $e_1 \in (a,d)$, then there exists a uninorm U_1' such that $U_1(x,y) = a + (d-a)U_1'(\frac{x-a}{d-a}, \frac{y-a}{d-a})$ for $x, y \in [a,d]$. Similarly, there exists a uninorm U_2' such that $U_2(x,y) = a + (d-a)U_2'(\frac{x-a}{d-a}, \frac{y-a}{d-a})$ for $x, y \in [a,d]$. It is clear that U_1' is modular over U_2'. Hence, $T_{U_1'}$ $S_{U_1'}$, $T_{U_2'}$ and $S_{U_2'}$ are continuous Archimedean by Lemmas 3.5, 3.6 and 3.8.

According to Remark 3.9, it holds that U_1' and U_2' are representable uninorms with neutral elements $e_1' = \frac{e_1-a}{d-a}$ and $e_2' = \frac{e_2-a}{d-a}$, respectively. Here, we use R_1 and R_2 to replace U_1' and U_2', respectively.

Clearly, $U_1|_{[0,a]^2}$ is isomorphic to a continuous t-norm T_1 and $U_2|_{[0,a]^2}$ is isomorphic to a continuous t-norm T_2. Then $T_1 = T_2$ since $U_1|_{[0,a]^2}$ is modular over $U_2|_{[0,a]^2}$. We denote $T_1 = T_2 = T$. Similarly, both $U_1|_{[d,1]^2}$ and $U_2|_{[d,1]^2}$ are isomorphic to a continuous t-conorm S.

It is not surprising that $U_1(x,y) = \min(x,y)$ for $(x,y) \in [0,a] \times [a,e_1] \cup [a,e_1] \times [0,a]$, and $U_1(x,y) = \max(x,y)$ for $(x,y) \in [e_1,d] \times [d,1] \cup [d,1] \times [e_1,d]$.

In addition, $U_1(x,y) = \min(x,y)$ for $(x,y) \in [0,a] \times [e_1,d) \cup [e_1,d) \times [0,a]$ and $U_1(x,y) = \max(x,y)$ for $(x,y) \in (a,e_1] \times [d,1] \cup [d,1] \times (a,e_1]$ by $U_1(a,y) = a$ for any $y \in [e_1,d)$ and $U_1(x,d) = d$ for any $x \in (a,e_1]$ and Lemma 2.9.

Similarly, we can prove that $U_2(x,y) = \min(x,y)$ for $(x,y) \in [0,a] \times [e_1,d) \cup [e_1,d) \times [0,a]$ and $U_2(x,y) = \max(x,y)$ for $(x,y) \in (a,e_1] \times [d,1] \cup [d,1] \times (a,e_1]$.

For any $x \le a$ and $y \ge d$, it holds that $U_1(y,x) = U_1(y, U_2(e_1,x)) = U_2(U_1(y,e_1),x) = U_2(y,x)$, i.e., $U_1(x,y) = U_2(x,y)$ for $(x,y) \in [0,a] \times [d,1] \cup [d,1] \times [0,a]$.

(Sufficiency) The first thing we need to know is that $U_1(x,y) \le a$ or $U_1(x,y) \ge d$ for $(x,y) \in [0,a] \times [d,1] \cup [d,1] \times [0,a]$ by Theorem 5.1 in [3]. Then we prove that $U_1(x, U_2(y,z)) = U_2(U_1(x,y),z)$ for any $x,y,z \in [0,1]$ with $z \le x$.

1. $z \le x \le a$

1.1. $y \le a$. Then $U_1(x, U_2(y,z)) = U_2(U_1(x,y),z)$ because of the associativity of T.

1.2. $y \in (a,d)$. Then $U_1(x, U_2(y,z)) = U_1(x,z) = U_2(U_1(x,y),z)$.

1.3. $y \ge d$. Then

$$\begin{aligned} U_1(x, U_2(y,z)) &= U_1(x, U_1(y,z)) \\ &= U_1(U_1(x,y),z) = \begin{cases} T(T(x,y),z) & U_1(x,y) \le a, \\ U_1(U_1(x,y),z) & U_1(x,y) \ge d, \end{cases} \end{aligned}$$

and

$$U_2(U_1(x,y),z) = \begin{cases} T(T(x,y),z) & U_1(x,y) \le a, \\ U_1(U_1(x,y),z) & U_1(x,y) \ge d. \end{cases}$$

2. $z \le a < x < d$.

2.1. $y \le a$. Then $U_1(x, U_2(y,z)) = U_2(y,z) = U_2(U_1(x,y),z)$ since $U_2(y,z) \le a$.

2.2. $y \in (a,d)$. Then $U_1(x, U_2(y,z)) = U_1(x,z) = z = U_2(U_1(x,y),z)$ since $U_1(x,y) \in (a,d)$.

2.3. $y \in [d,1]$. Then $U_1(x, U_2(y,z)) = U_2(y,z)$ whatever $U_2(y,z) \le a$ or $U_2(y,z) \ge d$, and $U_2(U_1(x,y),z) = U_2(y,z)$.

3. $z \le a < d \le x$.

3.1. $y \le a$. Then $U_1(x, U_2(y,z)) = U_1(x, U_1(y,z)) = U_1(U_1(x,y),z) = U_2(U_1(x,y),z)$ whatever $U_1(x,y) \le a$ or $U_1(x,y) \ge d$.

3.2. $y \in (a,d)$. Then $U_1(x, U_2(y,z)) = U_1(x,z) = U_2(x,z) = U_2(U_1(x,y),z)$.

3.3. $y \in [d,1]$. Then $U_1(x, U_2(y,z)) = U_1(x, U_1(y,z)) = U_1(U_1(x,y),z) = U_2(U_1(x,y),z)$ since $U_1(x,y) \ge d$.

4. $a < z \le x < d$.

4.1. $y \le a$. Then $U_1(x, U_2(y,z)) = U_1(x,y) = y = U_2(y,z) = U_2(U_1(x,y),z)$.

4.2. $y \in (a,d)$. Then $U_1(x, U_2(y,z)) = U_2(U_1(x,y),z)$ since R_1 is modular over R_2.

4.3. $y \in [d,1]$. Then $U_1(x, U_2(y,z)) = U_1(x,y) = y = U_2(y,z) = U_2(U_1(x,y),z)$.

5. $a < z < d \le x$.

5.1. $y \le a$. Then $U_1(x, U_2(y,z)) = U_1(x,y) = U_2(U_1(x,y),z)$ since $U_1(x,y) \le a$ or $U_1(x,y) \ge d$.

5.2. $y \in (a,d)$. Then $U_1(x, U_2(y,z)) = x = U_2(x,z) = U_2(U_1(x,y),z)$ since $U_2(y,z) \in (a,d)$.

5.3. $y \in [d,1]$. Then $U_1(x, U_2(y,z)) = U_1(x,y) = U_2(U_1(x,y),z)$ since $U_1(x,y) \ge d$.

6. $d \le z \le x$.

6.1. $y \le a$. Then $U_1(x, U_2(y,z)) = U_1(x, U_1(y,z)) = U_1(U_1(x,y),z) = U_2(U_1(x,y),z)$.

6.2. $y \in (a,d)$. Then $U_1(x, U_2(y,z)) = U_1(x,z) = U_2(x,z) = U_2(U_1(x,y),z)$.

6.3. $y \in [d,1]$. Then $U_1(x, U_2(y,z)) = U_2(U_1(x,y),z)$ since S is associative. □

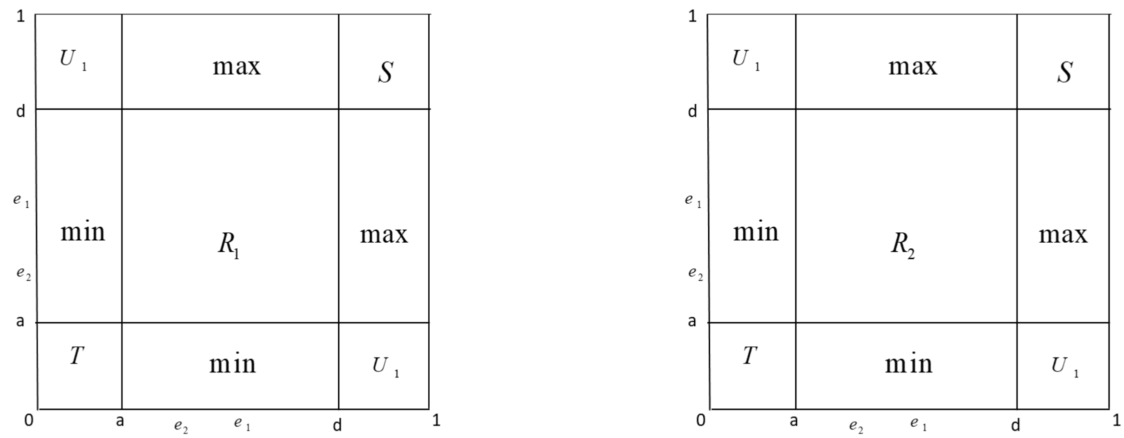

Figure 1: U_1 in Eq. (1) (left) and U_2 in Eq. (2) (right)

Remark 3.11. (i) *Since R_1 is modular over R_2 in Theorem 3.10, if h_1 is an additive generator of R_1, then we from Theorem 2.16 obtain that h_2 given by $h_2(x) = c(h_1(x) - h_1(\frac{e_2-a}{d-a}))$ is an additive generator of R_2, where $c > 0$ is a constant.*

(ii) *Although the value of U_1 or U_2 on $[0,a] \times [d,1] \cup [d,1] \times [0,a]$ in Theorem 3.10 can not be obtained directly, it does not affect the pair (U_1, U_2) satisfying the modularity equation.*

(iii) *In particular, if $a = 0$ or $d = 1$ in the above theorem, then U_1 and U_2 become uninorms continuous on $(0,1)^2$.*

(iv) *If $a > 0$ and $d < 1$ in the above theorem, then such U_1 and U_2 are uninorms belonging to* **COU**, *but not in the four well-known classes of uninorms, namely, $\mathcal{U}_{\min}$ or $\mathcal{U}_{\min}$, idempotent uninorms, representable uninorms and uninorms continuous on $(0,1)^2$.*

(v) *Since 1 is an idempotent point of U_1, then $U_1(1,a) = a$ or 1. Especially if $U_1(1,a) = a$, then $U_1(x,y) = \min(y,x)$ on $[0,a] \times [d,1] \cup [d,1] \times [0,a]$ by Lemma 2.9 and then*

$$U_1(x,y) = \begin{cases} d + (1-d)S(\frac{x-d}{1-d}, \frac{y-d}{1-d}) & (y,x) \in [d,1]^2, \\ a + (d-a)R_1(\frac{x-a}{d-a}, \frac{y-a}{d-a}) & (y,x) \in [a,d]^2, \\ aT(\frac{x}{a}, \frac{y}{a}) & (y,x) \in [0,a]^2, \\ \max(y,x) & (x,y) \in (a,d] \times [d,1] \cup [d,1] \times (a,d], \\ \min(y,x) & \textit{otherwise}, \end{cases}$$

$$U_2(x,y) = \begin{cases} d + (1-d)S(\frac{x-d}{1-d}, \frac{y-d}{1-d}) & (y,x) \in [d,1]^2, \\ a + (d-a)R_2(\frac{x-a}{d-a}, \frac{y-a}{d-a}) & (y,x) \in [a,d]^2, \\ aT(\frac{x}{a}, \frac{y}{a}) & (y,x) \in [0,a]^2, \\ \max(y,x) & (x,y) \in (a,d] \times [d,1] \cup [d,1] \times (a,d], \\ \min(y,x) & \textit{otherwise}. \end{cases}$$

(vi) *Similarly, if $U_1(0,d) = d$, then we get*

$$U_1(x,y) = \begin{cases} d + (1-d)S(\frac{x-d}{1-d}, \frac{y-d}{1-d}) & (y,x) \in [d,1]^2, \\ a + (d-a)R_1(\frac{x-a}{d-a}, \frac{y-a}{d-a}) & (y,x) \in [a,d]^2, \\ aT(\frac{x}{a}, \frac{y}{a}) & (y,x) \in [0,a]^2, \\ \max(y,x) & (x,y) \in [0,d] \times [d,1] \cup [d,1] \times [0,d], \\ \min(y,x) & \textit{otherwise}, \end{cases}$$

$$U_2(x,y) = \begin{cases} d + (1-d)S(\frac{x-d}{1-d}, \frac{y-d}{1-d}) & (y,x) \in [d,1]^2, \\ a + (d-a)R_2(\frac{x-a}{d-a}, \frac{y-a}{d-a}) & (y,x) \in [a,d]^2, \\ aT(\frac{x}{a}, \frac{y}{a}) & (y,x) \in [0,a]^2, \\ \max(y,x) & (y,x) \in [0,d] \times [d,1] \cup [d,1] \times [0,d], \\ \min(y,x) & \textit{otherwise}. \end{cases}$$

3.2 Case $e_2 > e_1$

Condition $e_2 > e_1$ implies that $U_2(e_1, e_1) \leq e_1$. Su et al. [20] have proved that if $0 < e_1 < e_2 < 1$ and U_1 is modular over U_2, then $U_2(e_1, e_1) = e_1$ is impossible. So we only need to discuss the case $U_2(e_1, e_1) < e_1$ in this subsection. In the following, we only give the proof of Lemma 3.15 since others can be similarly obtained as in subsection 3.1.

Lemma 3.12. *Let U_1 and U_2 be two proper uninorms in* **COU** *with respective neutral elements e_1 and e_2 ($e_2 > e_1$). If $U_2(e_1, e_1) < e_1$ and U_1 is modular over U_2, then $U_1(e_2, e_2) > e_2$ and there exists some idempotent element c of U_2 ($c < e_1$) such that $U_1(c, c) = c$ and $U_2|_{[c,e_2]^2}$ is isomorphic to a continuous Archimedean t-norm.*

Proof. The proof is similar to Lemma 3.5. □

Lemma 3.13. *Let U_1 and U_2 be two proper uninorms in* **COU** *with respective neutral elements e_1 and e_2 ($e_2 > e_1$). If $U_1(e_2, e_2) > e_2$ and U_1 is modular over U_2, then $U_2(e_1, e_1) < e_1$ and there exists some idempotent element b of U_1 ($b > e_2$) such that $U_2(b, b) = b$ and $U_1|_{[e_1,b]^2}$ is isomorphic to a continuous Archimedean t-conorm.*

Proof. The proof is similar to Lemma 3.6. □

Corollary 3.14. *Let U_1 and U_2 be two proper uninorms in* **COU** *with respective neutral elements e_1 and e_2 ($e_2 > e_1$). If U_1 is modular over U_2, then $U_1(e_2, e_2) > e_2$ if and only if $U_2(e_1, e_1) < e_1$.*

Lemma 3.15. *Let U'_1 and U'_2 be two proper uninorms in* **COU** *with respective neutral elements e'_1 and e'_2 ($e'_2 > e'_1$). Suppose that $S_{U'_1}$ and $T_{U'_2}$ are continuous Archimedean. If U'_1 is modular over U'_2, then both $T_{U'_1}$ and $S_{U'_2}$ are continuous Archimedean.*

Proof. We first prove that $T_{U'_1}$ is a continuous Archimedean t-norm. Otherwise, there exists some $x_0 \in (0, e'_1)$ such that $U'_1(x_0, x_0) = x_0$ and hence $U'_1(x_0, e'_2) = x_0$ or $U'_1(x_0, e'_2) = e'_2$ by Lemma 2.8.

If $U'_1(x_0, e'_2) = x_0$, then $x_0 = U'_1(x_0, x_0) = U'_1(x_0, U'_2(e'_2, x_0)) = U'_2(U'_1(x_0, e'_2), x_0) = U'_2(x_0, x_0)$, i.e., $U'_2(x_0, x_0) = x_0$. This is a contradiction with the Archimedean property of $T_{U'_2}$.

If $U'_1(x_0, e'_2) = e'_2$, then $U'_1(x_0, U'_2(e'_1, x_0)) = U'_2(U'_1(x_0, e'_1), x_0) = U'_2(x_0, x_0)$. Since $U'_2(e'_1, x_0) \leq x_0$, then we obtain $U'_1(x_0, U'_2(e'_1, x_0)) = \min(x_0, U'_2(e'_1, x_0)) = U'_2(e'_1, x_0)$. So $U'_2(e'_1, x_0) = U'_2(x_0, x_0)$. Therefore, it holds that $U'_1(e'_2, U'_2(x_0, e'_1)) = U'_2(U'_1(e'_2, x_0), e'_1) = U'_2(e'_2, e'_1) = e'_1$, but $U'_1(e'_2, U'_2(x_0, x_0)) = U'_2(U'_1(e'_2, x_0), x_0) = U'_2(e'_2, x_0) = x_0$. From the above steps, this leads to a contradiction.

Consequently, $T_{U'_1}$ is a continuous Archimedean t-norm.

In the following, we will prove that $S_{U'_2}$ is a continuous Archimedean t-conorm. Otherwise, there exists some $x_1 \in (e'_2, 1)$ such that $U'_2(x_1, x_1) = x_1$. Then $U'_2(x_1, e'_1) = x_1$ or e'_1 by Lemma 2.8.

If $U'_2(x_1, e'_1) = x_1$, then $U'_1(x_1, x_1) = U'_1(x_1, U'_2(e'_1, x_1)) = U'_2(U'_1(x_1, e'_1), x_1) = U'_2(x_1, x_1) = x_1$, i.e., $U'_1(x_1, x_1) = x_1$, which contradicts the Archimedean property of $S_{U'_1}$.

If $U'_2(x_1, e'_1) = e'_1$, then it holds that

$$U'_1(x_1, x_1) = U'_1(x_1, U'_2(e'_2, x_1)) = U'_2(U'_1(x_1, e'_2), x_1) = \max(U'_1(x_1, e'_2), x_1) = U'_1(x_1, e'_2),$$

since $U'_1(x_1, e'_2) \geq x_1$. That is to say $U'_1(x_1, x_1) = U'_1(x_1, e'_2)$. On the other hand, $x_1 = U'_1(x_1, e'_1) = U'_1(x_1, U'_2(x_1, e'_1)) = U'_2(U'_1(x_1, x_1), e'_1)$, and $e'_2 = U'_1(e'_2, e'_1) = U'_1(e'_2, U'_2(x_1, e'_1)) = U'_2(U'_1(e'_2, x_1), e'_1)$. From the above, we obtain that $x_1 = e'_2$. A contradiction. □

Using Lemmas 3.12-3.15 and Proposition 3.4, we get the following theorem.

Theorem 3.16. *Let U_1 and U_2 be two proper uninorms in* **COU** *with respective neutral elements e_1 and e_2 ($e_2 > e_1$). Suppose $U_2(e_1, e_1) < e_1$. Then U_1 is modular over U_2 if and only if there exist two elements b and c, a continuous t-norm T, a continuous t-conorm S, two representable uninorms R_1 and R_2 with respective neutral elements $e'_1 = \frac{e_1 - c}{b - c}$ and $e'_2 = \frac{e_2 - c}{b - c}$ such that $0 \leq c < e_1 < e_2 < b \leq 1$, U_1 and U_2 are given by, respectively,*

$$U_1(x, y) = \begin{cases} cT(\frac{x}{c}, \frac{y}{c}) & (x, y) \in [0, c]^2, \\ c + (b - c)R_1(\frac{x-c}{b-c}, \frac{y-c}{b-c}) & (x, y) \in [c, b]^2, \\ b + (1 - b)S(\frac{x-b}{1-b}, \frac{y-b}{1-b}) & (x, y) \in [b, 1]^2, \\ \min(x, y) & (x, y) \in [0, c] \times [c, b) \cup [c, b) \times [0, c], \\ \max(x, y) & (x, y) \in (c, b] \times [b, 1] \cup [b, 1] \times (c, b], \\ U_1(x, y) & \textit{otherwise}, \end{cases} \quad (3)$$

$$U_2(x,y)=\begin{cases} cT(\frac{x}{c},\frac{y}{c}) & (x,y)\in[0,c]^2,\\ c+(b-c)R_2(\frac{x-c}{b-c},\frac{y-c}{b-c}) & (x,y)\in[c,b]^2,\\ b+(1-b)S(\frac{x-b}{1-b},\frac{y-b}{1-b}) & (x,y)\in[b,1]^2,\\ \min(x,y) & (x,y)\in[0,c]\times[c,b)\cup[c,b)\times[0,c],\\ \max(x,y) & (x,y)\in(c,b]\times[b,1]\cup[b,1]\times(c,b],\\ U_1(x,y) & \textit{otherwise},\end{cases} \tag{4}$$

where R_1 is modular over R_2.

Remark 3.17. (i) *According to Theorem 3.10 and Theorem 3.16, one can notices that U_1 always equals U_2 in the unit square except for $(a,d)^2$ or $(c,b)^2$.*

(ii) *If $U_1(c,1)=c$, then*

$$U_1(x,y)=\begin{cases} b+(1-b)S(\frac{x-b}{1-b},\frac{y-b}{1-b}) & (y,x)\in[b,1]^2,\\ c+(b-c)R_1(\frac{x-c}{b-c},\frac{y-c}{b-c}) & (y,x)\in[c,b]^2,\\ cT(\frac{x}{c},\frac{y}{c}) & (y,x)\in[0,c]^2,\\ \min(y,x) & (y,x)\in[0,c]\times[c,1]\cup[c,1]\times[0,c],\\ \max(y,x) & (y,x)\in(c,b]\times[b,1]\cup[b,1]\times(c,b],\end{cases}$$

$$U_2(x,y)=\begin{cases} b+(1-b)S(\frac{x-b}{1-b},\frac{y-b}{1-b}) & (y,x)\in[b,1]^2,\\ c+(b-c)R_2(\frac{x-c}{b-c},\frac{y-c}{b-c}) & (y,x)\in[c,b]^2,\\ cT(\frac{x}{c},\frac{y}{c}) & (y,x)\in[0,c]^2,\\ \min(y,x) & (y,x)\in[0,c]\times[c,1]\cup[c,1]\times[0,c],\\ \max(y,x) & (y,x)\in(c,b]\times[b,1]\cup[b,1]\times(c,b].\end{cases}$$

(iii) *If $U_1(0,b)=b$, then*

$$U_1(x,y)=\begin{cases} b+(1-b)S(\frac{x-b}{1-b},\frac{y-b}{1-b}) & (y,x)\in[b,1]^2,\\ c+(b-c)R_1(\frac{x-c}{b-c},\frac{y-c}{b-c}) & (y,x)\in[c,b]^2,\\ cT(\frac{x}{c},\frac{y}{c}) & (y,x)\in[0,c]^2,\\ \min(y,x) & (y,x)\in[0,c]\times[c,b)\cup[c,b)\times[0,c],\\ \max(y,x) & (y,x)\in[0,b]\times[b,1]\cup[b,1]\times[0,b],\end{cases}$$

$$U_2(x,y)=\begin{cases} b+(1-b)S(\frac{x-b}{1-b},\frac{y-b}{1-b}) & (y,x)\in[b,1]^2,\\ c+(b-c)R_2(\frac{x-c}{b-c},\frac{y-c}{b-c}) & (y,x)\in[c,b]^2,\\ cT(\frac{x}{c},\frac{y}{c}) & (y,x)\in[0,c]^2,\\ \min(y,x) & (y,x)\in[0,c]\times[c,b)\cup[c,b)\times[0,c],\\ \max(y,x) & (y,x)\in[0,b]\times[b,1]\cup[b,1]\times[0,b].\end{cases}$$

4 Conclusion

We discussed in the work the modularity equation between two uninorms having continuous underlying functions. We mainly studied it for the case $e_1 > e_2$ and almost obtained the complete structure characterization of the two uninorms in the modularity equation. As for the case $e_2 > e_1$, similar results were gotten. Since uninorms having continuous underlying functions are a new kind of uninorms, we think that our work would fill the gap for this topic.

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