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A hybrid model for choosing the optimal stock portfolio under intuitionistic fuzzy sets

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Abstract

In the dynamic world of financial investment, crafting an optimal stock portfolio that judiciously balances risk, return, and efficiency emerges as a critical challenge. Despite the wealth of research on financial portfolio optimization, prevailing methodologies predominantly emphasize either risk minimization or return maximization, often overlooking the imperative for a holistic strategy that simultaneously boosts efficiency and effectiveness. Addressing this gap in the literature, this study introduces an innovative four-objective model that intricately blends risk, return, and efficiency considerations for the strategic selection of stock portfolios.
This model ingeniously integrates the foundational principles of Markowitz's mean-variance analysis with the sophisticated network data envelopment analysis (NDEA) techniques, significantly refining the portfolio selection methodology. It further distinguishes itself by incorporating returns represented as trapezoidal intuitionistic fuzzy numbers, adeptly capturing the inherent uncertainties in financial returns. Additionally, the model employs the network data envelopment analysis's cross-efficiency principle, providing a nuanced measure of company performance. To effectively navigate the complexities of this model, we deploy the Non-dominated Sorting Genetic Algorithm II (NSGA-II) and a multi-objective genetic algorithm, demonstrating the model's capability to unearth optimal solutions efficiently. The comparative analysis highlights that the proposed model significantly outperforms the efficiency and effectiveness of existing models, marking a substantial advancement in portfolio optimization strategies.

Keywords: Portfolio optimization, Markowitz mean-variance model, network data envelopment analysis, cross-efficiency, intuitionistic fuzzy sets.

1 Introduction

Investing in the stock market offers potential avenues for wealth generation, but it also comes with inherent uncertainties and challenges. One of the major challenges faced by investors is the task of identifying an optimal stock portfolio that can optimize returns while effectively managing risk [36]. Portfolio optimization has been extensively studied in the field of finance, with researchers and professionals striving to develop robust methodologies to guide investment decisions. Choosing the optimal stock portfolio offers several key advantages for investors [34]. Firstly, an optimal portfolio aims to maximize returns while effectively managing risk. By carefully selecting a combination of stocks with varying risk profiles and potential returns, investors can strike a balance that aligns with their risk tolerance and investment goals [14].

Considerable research has been conducted regarding the selection of an optimal stock portfolio. Ramen et al. [36] proposed a suggested method for choosing the best portfolio of stocks using variable length NSGA-II. Nourahmadi

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and Sadeqi [34] proposed a Machine Learning-Based Hierarchical Risk Parity Approach in a Case Study of a Portfolio Consisting of Stocks of the Top 30 Companies in Tehran Stock.

Fuzzy mean–semi variance and Data Envelopment Analysis (DEA) cross efficiency were incorporated in a fuzzy multi-objective portfolio selection model by Chen et al. [13]. Mehlawat and Gupta [31] incorporate short-term returns, long-term returns, and liquidity in the portfolio in the portfolio optimization problem. Mehlawat et al. [32] proposed a multi-objective portfolio selection problem where the objectives included mean, variance, skewness, kurtosis, and efficiency. Rezaei Pouya et al. [39] solve a multi-objective portfolio optimization problem using invasive weed optimization based on fifty top companies of the Tehran Stock Exchange. Wang et al. [46] use the Sharpe ratio and Value-at-Risk ratio in fuzzy environments for portfolio optimization.

But the previous articles did not provide a model to reduce variance and increase efficiency and effectiveness at the same time. For this reason, in the present study, in order to reduce the variance and increase the efficiency and effectiveness of the stock portfolio, we present a four-objective model at the same time.

The present study introduces a novel model based on risk, return, and efficiency for choosing an appropriate stock portfolio. The model integrates the principles of Markowitz mean-variance analysis and network data envelopment analysis to enhance the portfolio selection process. By considering multiple objectives, including maximizing average stock return and effectiveness while minimizing associated risks, the model aims to provide a comprehensive approach to portfolio optimization. To address the uncertainty associated with the results, the study employs the concept of trapezoidal intuitionistic fuzzy numbers to represent returns. This allows for the representation of uncertainty and provides a more comprehensive understanding of the potential outcomes of the recommended portfolio. In various intuitionistic fuzzy sets, this investigation suggested a novel ranking function taking into account the communication of the membership functions with non-membership types.

The effectiveness of each decision-making unit can be better understood with the help of network data envelopment analysis because it makes it possible to look inside each unit's internal workings. The NDEA cross-efficiency model is employed in this research to analyze the firm's effectiveness [19]. To illustrate the uncertainties in returns, we take into account intuitionistic fuzzy returns. We also present a new four-objective model that incorporates the NDEA cross-efficiency into the Markowitz model with intuitionistic fuzzy efficiencies. To solve the model, the recommended new approach employs the NSGAII.

Furthermore, the paper incorporates the principle of network data envelopment analysis cross-efficiency to effectively measure company efficiency. By considering the efficiency of individual companies within the portfolio selection process, the model aims to enhance the overall performance of the chosen portfolio. To solve the complex optimization problem, the Non-Dominated Sorting Genetic Algorithm (NSGA-II) and multi-objective genetic algorithm are employed. These algorithms offer efficient and effective solutions by exploring the trade-off between multiple objectives and providing a set of optimal solutions.

The most popular mathematical model for assessing the performance of businesses is the data envelopment analysis (DEA) model presented by Charnes and Cooper in 1978. One of the useful tools in performance measurement that can offer effective units for portfolio formation is the cross-efficiency model using Data Envelopment Analysis. Traditional DEA models, nevertheless, take a black-box approach and overlook the internal organization of the units. In order to assess systems with more than one stage, Network Data Envelopment Analysis (NDEA) was developed [45].

The Markowitz model is combined with the network data envelopment analysis cross-efficiency moded trapezoidal intuitionistic fuzzy returns in this article. The financial structure of this hybrid model is determined by taking into account the market values of companies. In order to reduce variance and increase the return and effectiveness of the stock portfolio, we concurrently present a four-objective model.

The rest of the paper is organized as follows: A Literature Review is presented in Section 2. Details about the proposed method are presented in Section 3 Numerical Experiments are presented in Section 4. Results and Model parameters and Input and output values are presented in Section 5. Section 6 concludes the paper.

2 Literature review

Transaction costs were included in the constraints introduced by Yoshimoto [49], who also used a multi-period model. The number of asset constraints in the Markowitz model portfolio was also counted by Chang et al. [9] in 2000. The upper and lower constraints of each portfolio asset were taken into account by Gómez and Sharma [22], who then introduced the Markowitz mean-variance model with limited components. Due to the fact that risk is described by variance in the Markowitz model, this criterion quantifies the excess or deficiency of return relative to investment risk. A new definition of risk has also been proposed by several investigators. Other than Markowitz's model, different definitions of risk have been employed, including half-variance and mean absolute deviation, by Chang et al. [11].

Consideration of a fixed return on assets can be less compliant with actual problems because there are constantly a variety of uncertain factors affecting the return on assets. Fuzzy returns can be taken into account as one of the best models for asset returns since the Markowitz model was developed. Other researchers have defined other types of return uncertainty and presented other models. Watada [48], and León et al. [28] investigated the problem of stock portfolio selection by employing the fuzzy decision theory. Interval planning models for supply chain design and portfolio selection were developed by Wang and Zhu [47], and Giove et al. [21]. Carlsson et al. [7] suggested a new method for selecting the most favorable portfolio based on the assumption that asset returns are trapezoidal fuzzy numbers. A fuzzy combined programming model was developed by Carlsson et al. [6] for selecting the best portfolio by taking prospective cash flows represented by trapezoidal fuzzy numbers into account.

To unravel the Markowitz mean-variance model and create a fuzzy portfolio selection model, Guo et al. [24] employed the fuzzy numbers concept. Edirisinghe and Zhang [18] suggested utilizing financial proportions to investigate company performance and the data envelopment analysis technique to determine a relative financial strength index (RFS), a criterion of a company's competitiveness relative to other corporates. Network models can address the problems because traditional data envelopment analysis models perform like black boxes. When introducing the network range-adjusted measure (NRAM) model in 2012, Avkiran and McCrystal [3] also compared the proposed model to the network slacks-based measure (NSBM) model and examined its sensitivity. The impact of the CNG2020 strategy on airline environmental performance was examined by Li [29] using the NRAM.

Chang et al. [10] assessed the portfolio using NDEA. The Markowitz and data envelopment analysis models for assessing the effectiveness of businesses do not take into account the concepts of risk and return. In other words, the effectiveness of companies in selecting the stock portfolio is not taken into account in the developed Markowitz models; this problem can be addressed by integrating Markowitz models with data envelopment analysis. Non-dominated sorting genetic algorithm II (NSGAII) was used by Omrani and Mashayekhi [35] in their presentation of a multi-objective model for portfolio selection. In addition to assessing the company's performance using financial ratio indicators, this model concurrently examines risk and return using the Markowitz model. Additionally, they measured company performance using the idea of cross-efficiency.

The 52 active companies from the Tehran Stock Exchange market make up the executed model. This model was compared to the NSGAII algorithm by Mashayekhi and Omrani [30], who viewed the return on assets as fuzzy trapezoidal numbers. The findings demonstrate that the returns, in this case, were substantially improved. Intuitionistic fuzzy data is one type of vague data that has been taken into account by some researchers and discussed in several investigations. Utilizing intuitionistic fuzzy input and output data, Puri and Yadav [40] conducted a groundbreaking analysis of optimistic and pessimistic efficiency. A triangular intuitionistic fuzzy environment was employed to introduce an extended DEA model by Edalatpanah [17]. Rasoulzadeh et al. [41] presented a four-objective model by combining Markowitz and DEA cross-efficiency models with intuitionistic fuzzy returns. A summary of the conducted research and research discussion is given in Table 1.

Table 1: Research literature review

Ref	Purpose	Intuitionistic Fuzzy Set	DEA	NDEA	Markowitz models	Method	The number of decision units	Case study	Uncertainty
García-Melón et al [20]	prioritize project portfolio	No	No	No	No	RAI	14	The electrical sector of Venezuela	No
Bjerring et al [5]	Mitigates the problem of excessive portfolio	No	No	No	Yes	Out-of-sample Back-tests	49	Kenneth French data library	Yes
Peralta and Zareei [38]	Improve the portfolio selection process	No	No	No	Yes	In-sample and out-of-sample analysis	200	NYSE	No
Huang and Di [25]	Optimal portfolio selection	No	No	No	Yes	Multi-Greedy heuristic	20	Shanghai Stock Exchange	No
Rather et al [42]	Finding similar time windows and predicting the market behavior	No	No	No	Yes	KMNLOG KMDLOG SPCLOG and HRCLOG	30	—	NO
Pedersen and Peskir [37]	Compare the optimal portfolio with background risk with that without background risk	No	No	No	No	Uncertainty theory	20	Shanghai Stock Exchange	Yes
Zhou and Xu [51]	Forming an optimal portfolio of stocks	Yes	No	No	Yes	AI	20	Stock Exchange	No
Zhou Xiaoyang et al [50]	Stock portfolio selection	Yes	Yes	No	Yes	DEA	33	Chinese stock market	NO
Cesarone et al [8]	Prediction of stocks	No	No	No	Yes	Monte Carlo method Generating Gaussian random noise	7	Shanghai stock	Yes
Salehpoor et al [43]	Reduce estimation error and highly concentrated optimal portfolios	No	No	No	Yes	Markov-Switching (M-S)	30	US stocks	No
Khedmati and Azin [26]	the nonlinear mean-variance optimal control	No	No	No	Yes	Lagrange multipliers	20	—	No
Shadabfar and Cheng [44]	Choosing the Optimal Stock Portfolio	Yes	No	Yes	Yes	Electromagnetism-like algorithm (EM), particle swarm optimization (PSO), genetic algorithm (GA), genetic network programming (GNP) and simulated annealing (SA)	50	Iranian stock exchange	No
Chen et al [13]	Portfolio selection based on qualitative information	Yes	No	No	No	HFS	3	Stock market in China	Yes
Amin and Hajjami [1]	The nonlinear mean-variance optimal control	Yes	No	No	No	VaR ϵ-constraint method	10	Chinese stock market	No
Oscar et al [16]	Solve the portfolio selection problem	No	No	No	No	multi-objective optimization Genetic algorithm	50	Shanghai Stock Exchange	Yes
Current	Choosing the Optimal Stock Portfolio	Yes	Yes	Yes	Yes	NSGAII and intuitionistic fuzzy efficiencies	50	Tehran Stock Exchange	Yes

2.1 Contribution of the present study

In various intuitionistic fuzzy sets, this investigation suggested a novel ranking function taking into account the communication of the membership functions with non-membership types. The effectiveness of each decision-making unit can be better understood with the help of network data envelopment analysis because it makes it possible to look inside each unit's internal workings. The NDEA cross-efficiency model [33] is employed in this research to analyze the firm's effectiveness. To illustrate the uncertainties in returns, we take into account intuitionistic fuzzy returns. We also present a new four-objective model that incorporates the NDEA cross-efficiency into the Markowitz model with intuitionistic fuzzy efficiencies. To solve the model, the recommended new approach employs the NSGAII.

3 Methodology

In this section, the research methodology is addressed.

3.1 Markowitz model

Many improvements and refinements have been made to the original Markowitz model since it was first introduced because of its widespread acceptance as a useful resource for investors seeking portfolio optimization. The following describes the two-objective Markowitz model, in which the objective functions are maximizing returns and reducing risk:

$$Max \sum_{i=1}^{N} W_i \bar{R}_i,$$

$$Min\ \sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} W_i W_j cov\left(R_i, R_j\right),$$

s.t.

$$\sum_{i=1}^{N} W_i = 1,$$

$$W_i \geqslant 0 (i = 1, 2, 3, \cdots, N).$$

Where $\bar{R}_i$ denotes the mean return on assets i, $\text{cov}(R_i, R_j)$ represents the return covariance on assets i and j, N represents the frequency of investable assets, and w_i represents the weight of asset i^{th} in the investment portfolio.

3.2 Intuitionistic fuzzy numbers

A trapezoidal intuitionistic fuzzy number $\widetilde{A}^I$ can be regarded as $A = (a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ in which membership functions $\mu_{\widetilde{A}^I}$ and non-membership $\nu_{\widetilde{A}^I}$ are expressed as follows [40]:

$$\mu_{\tilde{A}^I} = \begin{cases} f_A(x) & a_1 \leq x < a_2 \\ 1 & a_2 \leq x \leq a_3 \\ g_A(x) & a_3 < x \leq a_4 \\ 0 & otherwise \end{cases}$$

$$\nu_{\tilde{A}^I} = \begin{cases} h_A(x) & b_1 \leq x < b_2 \\ 0 & b_2 \leq x \leq b_3 \\ k_A(x) & b_3 < x \leq b_4 \\ 1 & otherwise \end{cases}$$

Where $0 \leq \mu_{\widetilde{A}^I}(x) + \nu_{\widetilde{A}^I}(x) \leq 1$ and $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{R}$ and $b_1 \leq a_1 \leq b_2 \leq a_2 \leq a_3 \leq b_3 \leq a_4 \leq b_4$. The functions f_A and k_A represent the non-descending continuous functions in the intervals $[a_1, a_2)$ and $(b_3, b_4]$ respectively, and the functions g_A and h_A are the non-ascending continuous functions in the intervals $(a_3, a_4]$ and $[b_1, b_2)$ respectively. The anticipated distance of an Intuitionistic phase number $\widetilde{A}^I$ defined as below [23]:

$$E_L\left(\tilde{A}^I\right) = \frac{b_1 + a_2}{2} + \frac{1}{2}\int_{b_1}^{b_2} h_A(x)\, dx - \frac{1}{2}\int_{a_1}^{a^2} f_A(x)\, dx,$$

$$E_U\left(\tilde{A}^I\right)=\frac{a_3+b_4}{2}+\frac{1}{2}\int_{a_3}^{a_4} g_A(x)\,dx-\frac{1}{2}\int_{b_3}^{b_4} k_A(x)\,dx.$$

Where E_L is the lower limit of the expected value of intuitionistic fuzzy, E_U is the upper limit of the expected value of intuitionistic fuzzy, and E is the global expected value.

As a result, the center of the expected interval of that intuitionistic fuzzy number is determined as follows:

$$EV\left(\tilde{A}^I\right)=\frac{E_L\left(\tilde{A}^I\right)+E_U\left(\tilde{A}^I\right)}{2},$$

by hypothesis

$$f_A(x)=\frac{x-a_1}{a_2-a_1},$$

$$g_A(x)=\frac{x-a_4}{a_3-a_4},$$

$$h_A(x)=\frac{x-b_2}{b_1-b_2},$$

$$k_A(x)=\frac{x-b_3}{b_4-b_3},$$

In this case, we have [23]:

$$EV(A)=\frac{1}{8}\left(\sum_{i=1}^{4} a_i+\sum_{i=1}^{4} b_i\right),$$

To calculate the variance, we have [23]:

$$VAR(X)=E\left(X^2\right)-(E(X))^2,$$

$$E_L\left(X^2\right)=\frac{b_1+a_2}{2}+\frac{1}{2}\int_{b_1}^{b_2} h_A\left(X^2\right)dx-\frac{1}{2}\int_{a_1}^{a_2} f_A\left(X^2\right)dx,$$

$$E_U\left(X^2\right)=\frac{a_3+b_4}{2}+\frac{1}{2}\int_{a_3}^{a_4} g_A\left(X^2\right)dx-\frac{1}{2}\int_{b_3}^{b_4} k_A\left(X^2\right)dx,$$

$$E_L\left(X^2\right)=-\frac{1}{3}\left(a_1^2+a_2^2+b_1^2+b_2^2+a_1a_2+b_1b_2\right)+a_1+a_2+b_1+b_2,$$

$$E_U\left(X^2\right)=-\frac{1}{3}\left(a_3^2+a_4^2+b_3^2+b_4^2+a_3a_4+b_3b_4\right)+a_3+a_4+b_3+b_4,$$

$$E\left(X^2\right)=\frac{E_L\left(X^2\right)+E_U\left(X^2\right)}{2}.$$

By placing relations (18) and (19) in relation (20), the value of $E(x^2)$ is obtained, and then with the help of relations (14) and (15), the variance of a trapezoidal intuitionistic fuzzy number $\widetilde{A}^I=(a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4)$ can be calculated as follows [23].

$$VAR\left(\tilde{A}^I\right)=\frac{1}{4}\left(\sum_{i=1}^{4} a_i+\sum_{i=1}^{4} b_i\right)-\frac{1}{12}\left(a_1a_2+a_3a_4+b_1b_2+b_3b_4+\sum_{i=1}^{4} a_i^2+\sum_{i=1}^{4} b_i^2\right)-\frac{1}{64}\left(\sum_{i=1}^{4} a_i+\sum_{i=1}^{4} b_i\right)^2.$$

3.3 Data envelopment analysis cross-efficiency model

The DEA model compares the effectiveness of many DMUs that make use of various inputs to generate various outputs. Many scholars have built upon the original DEA model given by Charnes, Cooper, and Rhodes [12]. We assume that there are n DMUs with m inputs and s outputs. The following model will find the set of weights that maximize the efficiency score of DMU o where x_{io} and y_{ro} represent the inputs and outputs of the decision unit o:

$$Max \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} \nu_i x_{io}},$$

s.t.

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} \nu_i x_{ij}} \leq 1,$$

$$u_r, \nu_j \geq 0 (r = 1, \cdots, s)(i = 1, \cdots, m)$$

Where x_{ij} and y_{rj} represent the inputs and outputs of the decision unit j; u_r and v_i represent the inputs and outputs' weights, respectively. Maximizing the ratio of the weighted sum of the outputs to the weighted sum of the inputs is entailed in the objective function as much as possible [3].

Other DMU units' cross-efficiency assessed by the unit K can be determined using the relationship shown below if u_r^* and v_i^* are the best answers from the aforementioned model for the unit K.

$$e_{kl} = \frac{\sum_{r=1}^{s} u_r^* y_{rl}}{\sum_{i=1}^{m} \nu_r^* x_{il}}.$$

If the above method is repeated for other DMUs, we can form an $n \times n$ matrix of e_{ij} where $i, j = 1, \ldots, n$. The crossover efficiency matrix, which is the resulting matrix, is indicated by the symbol E. When calculating the average for each column in the performance matrix above, such as column l, we refer to the result as a cross-efficiency score [27].

$$\bar{e}_l = \frac{1}{n} \sum_{k=1}^{n} e_{kl}.$$

3.4 Cross-performance network data envelopment analysis model

Network DEA (Data Envelopment Analysis) is an extension of the traditional DEA methodology that allows for the evaluation of the relative efficiency and performance of interconnected units or entities within a network or system. It is particularly useful in assessing the efficiency and effectiveness of entities that depend on each other and operate collectively as a network [4].

In a network DEA analysis, the entities or decision-making units (DMUs) are interconnected through input-output relationships. The performance of each DMU is evaluated by considering both its individual efficiency and its contribution to the overall network efficiency. This approach enables the assessment of the network as a whole and identifies the synergies and interdependencies among the DMUs. The network DEA model takes into account the inputs and outputs of each DMU, as well as the network structure and the relationships among the entities. It aims to identify the optimal resource allocation and operational strategies that can improve the efficiency and overall performance of the network [3, 10, 30].

The network DEA analysis provides insights into various aspects of the network, such as the efficiency of individual DMUs, the efficiency of the overall network, and the potential for improving the network efficiency through changes in resource allocation or operational practices. It helps in identifying the best-performing entities within the network and understanding the factors that contribute to their success. By applying network DEA, decision-makers can identify inefficiencies, allocate resources more effectively, and enhance the overall performance of interconnected entities within a network [10, 30].

Overall, network DEA is a powerful analytical tool for evaluating the efficiency and performance of interconnected entities in a network setting. It provides a comprehensive understanding of the network's dynamics and enables informed decision-making for optimizing resource allocation and improving the overall efficiency of the network. In the present study, we employed network DEA because our study involved analyzing interconnected entities within a network. We wanted to assess the efficiency and performance of these entities while considering their interdependencies and the overall network dynamics. By using network DEA, we were able to capture the synergies, optimize resource allocation, and obtain a more comprehensive evaluation of the network's performance, which would not have been possible with classical DEA [30].

Network DEA offers several advantages over classical data envelopment analysis (DEA) in the context of evaluating interconnected entities within a network. Here are some of the advantages and the reasons for using this approach in our research article:

1. Consideration of Interdependencies: Network DEA takes into account the interdependencies and relationships among entities within the network. This enables a more realistic and comprehensive evaluation of efficiency and performance, considering the collective impact of the entities on the overall network. In contrast, classical DEA analyzes entities in isolation and does not consider the interconnected nature of the network.

2. Synergy Analysis: Network DEA allows for the identification and quantification of synergy effects within the network. It captures the potential benefits and efficiencies that can be achieved by entities working together and leveraging their interdependencies. Classical DEA lacks the ability to evaluate and measure these synergy effects.

3. Resource Allocation Optimization: By assessing the efficiency and performance of interconnected entities, network DEA helps optimize resource allocation within the network. It provides insights into the most effective allocation of resources among entities to enhance the overall efficiency and performance of the network. This aspect is not addressed in classical DEA, which focuses on individual efficiency without considering resource allocation across the network.

4. Comprehensive Performance Evaluation: Network DEA provides a more holistic evaluation of the network by considering both individual entity efficiency and the overall network efficiency. This comprehensive approach allows for a deeper understanding of the network's performance, strengths, and areas for improvement. Classical DEA, on the other hand, only focuses on individual efficiency without capturing the broader network perspective.

NDEA model developed by Färe and Grosskopf [19] is employed to completely review the black box information. Classical Radial DEA (CRS) models should not be used because some inputs and outputs in financial models may have negative values. As a result, Classical Radial DEA models can be employed when some inputs or outputs are negative [15]. The Range Adjusted Measure (RAM) model with incompetence criteria will be as follows, given the preceding information:

$$
\begin{aligned}
& Min - \frac{1}{m+s}\left(\sum_{i=1}^{m}\frac{s_{io}^{-}}{R_i^{-}} + \sum_{r=1}^{s}\frac{s_{ro}^{+}}{R_r^{+}}\right), \\
& \text{s.t.} \\
& x_{io} = \sum_{j=1}^{n}\lambda_j x_{ij} + s_{io}^{-} (i = 1, \cdots, m), \\
& y_{ro} = \sum_{j=1}^{n}\lambda_j y_{rj} - s_{ro}^{+} (r = 1, \cdots, s), \\
& \sum_{j=1}^{n}\lambda_j = 1, \\
& \lambda_j, s_{io}^{-}, s_{ro}^{+} \geq 0 (j = 1, \cdots, n).
\end{aligned}
$$

$X = x_{ij} \in \mathbb{R}^{m\times n}$ and $Y = y_{rj} \in \mathbb{R}^{s\times n}$ indicate the data matrix of input and output in which all columns demonstrate one of the units, and all rows show the level of one of the factors of the relative DMU. s_{io}^{-} and s_{ro}^{+} stand for the slacks of the ith input and rth output.

Also, R^{-} and R^{+} are expressed as follows:

$$
R^{-} = \left(\frac{1}{R_1^{-}}, \frac{1}{R_2^{-}}, \frac{1}{R_3^{-}}, \cdots, \frac{1}{R_m^{-}}\right),
$$

$$
R^{+} = \left(\frac{1}{R_1^{+}}, \frac{1}{R_2^{+}}, \frac{1}{R_3^{+}}, \cdots, \frac{1}{R_s^{+}}\right),
$$

$$R_i^- = \max_{j=1,\cdots,n}\{x_{ij}\} - \min_{j=1,\cdots,n}\{x_{ij}\},$$

$$R_i^+ = \max_{j=1,\cdots,n}\{y_{rj}\} - \min_{j=1,\ldots,n}\{y_{rj}\}.$$

The Figure 1 provides an overview of a network model for DMU number l.

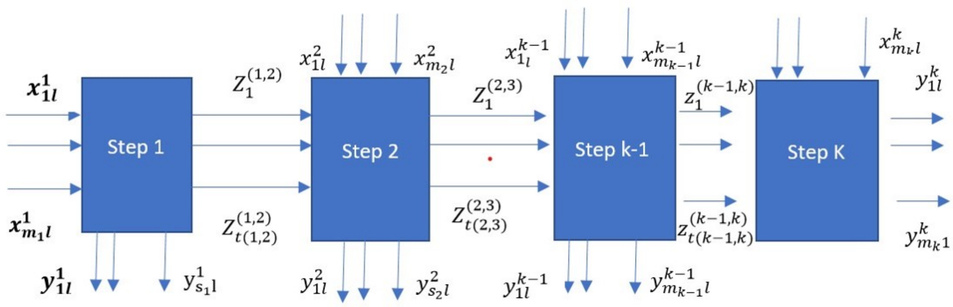

Figure 1: Overview of a network model for DMU number l

Where:
x_{ij}^k: is the ith input for the jth unit in the kth step, where $k = 1, \ldots, K$, $i = 1, \ldots, m_k$, and $j = 1, \ldots, n$.

y_{rj}^k: is the rth output for the jth unit in the kth step, where $k = 1, \ldots, K$, $r = 1, \ldots, s_k$, and $j = 1, \ldots, n$.

$Z_j^{(i-1,i)}$: jth output in the $i-1$th step and input of the ith step (internal link).

$t(i-1, i)$: output count of the $i-1$th stage and ith inputs (internal link).

The Min model with the performance criterion is as follows:

$$Min \sum_{k=1}^{K} - \frac{t_k}{m_k + s_k} \left(\sum_{i=1}^{m_k} \frac{s_i^{k-}}{R_i^-} + \sum_{r=1}^{m_k} \frac{s_r^{k+}}{R_r^+} \right).$$

s.t.

$$x_o^k = \sum_{j=1}^{n} \lambda_j^k x_{ij}^k + s_i^{k-} (i = 1, \cdots, m_k; k = 1, \cdots, K),$$

$$y_o^k = \sum_{j=1}^{n} \lambda_j^k y_{rj}^k - s_i^{k+} (r = 1, \cdots, s_k; k = 1, \cdots, K),$$

$$\sum_{j=1}^{n} \lambda_j^k Z_j^{(k.h)} = \sum_{j=1}^{n} \lambda_j^h Z_j^{(k.h)} (h, k = 1, \cdots, K),$$

$$\sum_{j=1}^{n} \lambda_j^k = 1 (k = 1, \cdots, K),$$

$$\lambda, s^-, s^+ \geq 0 (j = 1, \cdots, n),$$

where $\lambda^k \in \mathbb{R}^n$ is the intensity vector corresponding to Division k $(k = 1, \ldots, K)$. t_k is the weight of Division k, and $s_i^{(k-)}$ and $s_r^{(k+)}$ are the slacks of the ith input and rth output for Division k.

The following is the dual of the models mentioned above:

$$Max \sum_{k=1}^{K} \sum_{r=1}^{s_k} p_r^k y_o^k - \sum_{k=1}^{K} \sum_{i=1}^{m_k} q_i^k x_o^k + \sum_{k=1}^{K} \xi_k,$$

s.t.

$$\sum_{r=1}^{s_1} p_r^1 y_{rl}^1 - \sum_{i=1}^{m_1} q_i^1 x_{il}^1 + \sum_{j=1}^{t(1.2)} c_j^1 Z_j^{(1.2)} + \xi_1 \leq 0(l=1,\ldots,n),$$

$$\sum_{r=1}^{s_k} p_r^k y_{rl}^k - \sum_{i=1}^{m_k} q_i^k x_{il}^k - \sum_{j=1}^{t(k-1.k)} c_j^k Z_j^{(k-1.k)} + \xi_k \leq 0(l=1,\ldots,n),$$

$$\sum_{r=1}^{s_k} p_r^k y_{rl}^k - \sum_{i=1}^{m_k} q_i^k x_{il}^k - \sum_{j=1}^{t(k-1.k)} c_j^{k-1} Z_j^{(k-1.k)} + \sum_{j=1}^{t(k.k+1)} c_j^k Z_j^{(k.k+1)} + \xi_k \leq 0(l=1,\ldots,n; k=2,...,k-1),$$

$$p_i^j \geq \frac{w_j}{(m_j+s_j)\,R_i^+}(i=1,\ldots,s_j; j=1,\ldots,K),$$

$$q_i^j \geq \frac{w_j}{(m_j+s_j)\,R_i^-}(i=1,\ldots,m_j; j=1,\ldots,K).$$

Where **p**, **q**, and **c** vectors represent the price of output and the cost of input and units connection. ξ_k is a positive infinitesimal value. The model maximizes DMU's efficiency score and optimizes the weight for all DMUs simultaneously. Let * represent the optimal solution of the model. The efficiency score of other DMUs is obtained by using the weights that DMU k has chosen. The cross-efficiency of DMU l with the weights of DMU k (e_{kl}) can be expressed as follows [14]:

$$e_{kl}^* = p_k^* y_k - q_k^* x_k + \xi_k.$$

4 Markowitz model integrated with intuitionistic fuzzy efficiencies and network data envelopment analysis cross-efficiency

In order to concurrently evaluate risk, return, and efficiency, Omrani and Mashayekhi [35] presented a four-objective Markowitz model incorporated with the data envelopment analysis cross-efficiency model. Returns can be thought of in these models as either fuzzy or definite numbers. Despite the fact that fuzzy numbers are less precise and more uncertain than crisp ones, it may be assumed that a topic has a certain level of membership when there may be misgivings regarding that level. Take into account instances in which the variables in question are linguistic in nature, i.e., their values are not described numerically but rather through the use of words like "good," "bad," "short," "low," etc. There is no method available in fuzzy set theory to incorporate this ambiguity in membership levels.

The Intuitionistic Fuzzy Set (IFS) recommended by Atanassov [2] could be used to solve this problem. For a more realistic and precise assessment in this study, the returns are taken into consideration as intuitionistic fuzzy numbers. Additionally, the NDEA model has been employed to analyze performance because DEA models are comparable to the "black box" and have deficiencies in sub-process analysis. It means that these models are seen as complex and opaque. They take inputs and produce outputs, but the inner workings and mechanisms are not easily understandable or transparent.

This lack of transparency can make it difficult to gain insights into the specific sub-processes or components that contribute to overall performance. To overcome this limitation, the NDEA (Nested Data Envelopment Analysis) model has been employed. The NDEA model is a more advanced version of DEA that allows for a more detailed analysis of sub-processes. It provides a framework for breaking down the overall performance into smaller components and assessing the efficiency of each individual sub-process. The newly discovered model is as follows:

$$Max \qquad E\left(\sum_{i=1}^{N} \tilde{R}_i^I w_i\right) = \frac{1}{8}\left(\sum_{i=1}^{N} (a_{i1}+a_{i2}+a_{i3}+a_{i4}+b_{i1}+b_{i2}+b_{i3}+b_{i4})\, w_i\right),$$

$$Min \qquad VAR\left(\sum_{i=1}^{N} \tilde{R}_i^I w_i\right),$$

$$= \frac{1}{4}\sum_{i=1}^{N}\left(\left(\sum_{j=1}^{4}a_{ij}+\sum_{j=1}^{4}b_{ij}\right)-\frac{1}{12}\left(a_{i1}a_{i2}+a_{i3}a_{i4}+b_{i1}b_{i2}+b_{i3}b_{i4}+\sum_{j=1}^{4}a_{ij}^{2}+\sum_{j=1}^{4}b_{ij}^{2}\right)-\frac{1}{64}\left(\sum_{j=1}^{4}a_{ij}+\sum_{j=1}^{4}b_{ij}\right)^{2}w_i\right),$$

$$Max \qquad \sum_{i=1}^{N} w_i\tilde{e}_i,$$

$$Min \qquad \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j cov\left(e_i, e_j\right),$$

s.t.

$$l_i z_i \leq w_i \leq u_i z_i (i = 1, ..., N),$$

$$\sum_{i=1}^{N} w_i = 1,$$

$$w_i \geq 0 (i = 1, \ldots, N).$$

Where the z_i represents a zero-one variable. The value is 0 unless the asset ith is added to the portfolio, at which point it becomes 1. Maximum and minimum allocations of the portfolio's total investment budget to the ith variable are indicated by l_i and u_i, respectively. The efficiency score of the network intersection of ith DMU is demonstrated by $\bar{e}_i$, enhancing portfolio effectiveness. The covariance of the network intersecting effectiveness between $e_{(.i)}$ (DMUi) and $e_{(.j)}$ (DMUj) is indicated by cov(e_i *and* e_j).

5 Numerical experiments

To solve a numerical expression and compare the answers, we use information about Tehran Stock Exchange companies. Fifty firms listed on the Tehran Stock Exchange in 2020 have been chosen for this purpose. These companies have not been added to the list of desired businesses because banks, insurance, and investment firms have distinct financial evaluations than others. Moreover, we employ the documented data in the financial declarations up to 2020 to assess the companies' effectiveness.

Performance has been more precisely calculated using a two-stage network model. Table 2 shows the inputs and outputs for the initial phase of the above model. While also increasing the firm's productivity and improving its parameters, the results of the second phase demonstrate the management's capacity to make profitable utilization of the company's resources. The effectiveness and results of the company's parameters are shown by these outputs. The market value of the company at the end of 2020, therefore, is the output of the second phase. Thus, the efficiency of the company and its effect on shareholder wealth can be better explained by this network. For shareholders and investors, this is the most important indicator of the company's productivity.

5.1 Input and output values

The efficiency in the initial phase was assessed using seven input and seven output criteria. Table 2 shows a collection of input and output criteria. Data has been extracted from and evaluated for each piece of information found in the audited financial statements up to 2020.

Table 2: Research Criteria

Description	Criteria Type	input and output
Period income divided by accounts receivable	Turnover of accounts receivable	Input
Period income divided by inventories	Inventory turnover	Input
Revenue shared by wealth	Asset turnover	Input
Existing wealth over existing wealth	Current ratio	Input
Acidic assets on existing rabies debt	Quick ratio	Input
Overall debt shared by shareholders' justice	Debt to equity ratio	Input
The total debt on overall wealth	Debt ratio	Input
Net profit on equity	Return on equity (ROE)	Output
Net profit on assets	Return on assets (ROA)	Output
Net profit on sales	Net profit margin	Output
Net profit on the number of shares	Earnings per share (EPS)	Output
Existing period income divided by the earlier period income minus one	Income growth rate	Output
Current period net profit shared by previous period net profit minus one	Net profit growth rate	Output
Existing period EPS shared by earlier period EPS minus one	Earnings growth rate per share	Output

Additionally, the Tehran Stock Exchange's website was used to obtain the second stage's output, which is the market value of businesses in 2020 as the last business day for the companies under evaluation. Table 3 shows the returns for the fifty corporates as intuitionistic fuzzy numbers. The MATLAB version 2014 software and NSGA-II algorithm were used to solve the presented model.

5.2 Model parameters

In this model, the population size N_{pop} is considered one hundred, $p_c = 0.8$, $p_m = 0.1$, maximum number of repetitions $= 200$, $\mu = 2$; the minimum investment per share (l_i) is considered 10% and the maximum investment per share (u_i) is equal to 30%, and $t_1 = 0.7$ and $t_2 = 0.3$.

5.3 Results

The suggested model has been put into practice in this section, and we've got a list of Pareto-optimal answers as a result. These findings can be used to develop an investment portfolio for investors. The favored portfolio can be chosen by taking into account a variety of factors. The most efficient responses are taken into account in this model. Tables 4-7 demonstrate the acquired findings.

Table 3: Returns as trapezoidal intuitionistic fuzzy numbers

Asset No	*IF Returns*	*Asset No*	*IF Returns*
1	*(0.01,0.14,0.262,0.39,-0.29,0.1,0.29,1.24)*	*26*	*(0.1,0.14,0.181,0.22,-0.25,0.13,0.19,0.69)*
2	(-0.05,0.05,0.152,0.25,-0.29,0.02,0.18,0.56)	27	(0.04,0.1,0.166,0.23,-0.23,0.08,0.18,0.55)
3	(0.01,0.07,0.122,0.18,-0.2,0.05,0.14,0.83)	28	(-0.06,0,0.063,0.12,-0.27,-0.01,0.08,0.67)
4	(-0.08,0,0.075,0.15,-0.4,-0.02,0.09,0.59)	29	(-0.02,0.04,0.097,0.16,-0.39,0.03,0.11,0.52)
5	(0.07,0.12,0.173,0.22,-0.22,0.11,0.19,0.73)	30	(0.07,0.1,0.127,0.16,-0.08,0.09,0.13,0.45)
6	(0.03,0.09,0.142,0.2,-0.25,0.07,0.16,0.56)	31	(-0.03,0.01,0.043,0.08,-0.42,0,0.05,0.25)
7	(-0.01,0.07,0.148,0.23,-0.31,0.05,0.17,0.81)	32	(-0.03,0.12,0.272,0.42,-0.3,0.08,0.31,1.42)
8	(0.04,0.07,0.095,0.12,-0.16,0.06,0.1,0.4)	33	(0.02,0.11,0.207,0.3,-0.17,0.09,0.23,1.01)
9	(-0.04,0.02,0.071,0.12,-0.3,0,0.08,0.54)	34	(0,0.03,0.05,0.07,-0.12,0.02,0.06,0.58)
10	(-0.01,0.02,0.063,0.1,-0.3,0.01,0.07,0.59)	35	(-0.05,0.03,0.122,0.21,-0.33,0.01,0.14,0.74)
11	(-0.02,0.03,0.085,0.14,-0.28,0.02,0.1,0.45)	36	(-0.06,0.02,0.098,0.18,-0.47,0,0.12,0.56)
12	(0.09,0.15,0.205,0.26,-0.2,0.14,0.22,0.63)	37	(-0.25,0.09,0.419,0.75,-0.47,0,0.5,1.77)
13	(-0.04,0.01,0.065,0.12,-0.28,0,0.08,0.51)	38	(-0.05,0.06,0.171,0.28,-0.27,0.03,0.2,1.17)
14	(0.09,0.14,0.184,0.23,-0.32,0.12,0.2,0.56)	39	(0.01,0.05,0.081,0.12,-0.21,0.04,0.09,0.45)
15	(-0.05,0.05,0.156,0.26,-0.28,0.03,0.18,0.51)	40	(0.08,0.16,0.241,0.32,-0.23,0.14,0.26,0.84)
16	(0.05,0.1,0.155,0.21,-0.12,0.09,0.17,0.51)	41	(-0.02,0.04,0.093,0.15,-0.31,0.02,0.11,0.71)
17	(-0.02,0.02,0.062,0.1,-0.23,0.01,0.07,0.51)	42	(0.01,0.04,0.063,0.09,-0.21,0.03,0.07,0.42)
18	(0.07,0.13,0.193,0.25,-0.24,0.12,0.21,0.9)	43	(0,0.03,0.062,0.09,-0.19,0.03,0.07,0.47)
19	(0,0.08,0.151,0.23,-0.24,0.06,0.17,0.79)	44	(0.05,0.08,0.116,0.15,-0.17,0.07,0.12,0.65)
20	(0.06,0.15,0.238,0.33,-0.36,0.13,0.26,0.88)	45	(-0.16,0.1,0.361,0.62,-0.45,0.03,0.43,1.72)
21	(0,0.07,0.145,0.22,-0.16,0.06,0.16,0.61)	46	(0.09,0.11,0.124,0.14,-0.07,0.1,0.13,0.4)
22	(-0.07,-0.01,0.053,0.12,-0.27,-0.03,0.07,0.44)	47	(0.02,0.08,0.148,0.21,-0.27,0.06,0.16,0.47)
23	(0.05,0.13,0.202,0.28,-0.33,0.11,0.22,0.84)	48	(0.01,0.06,0.1,0.14,-0.26,0.04,0.11,0.6)
24	(0,0.09,0.176,0.27,-0.37,0.06,0.2,0.99)	49	(-0.04,0.07,0.193,0.31,-0.33,0.04,0.22,0.83)
25	(0.01,0.14,0.262,0.39,-0.29,0.1,0.29,1.24)	50	(0.15,0.18,0.207,0.23,-0.18,0.17,0.21,0.59)

Table 4: Assets' weights with the IFS Network DEA model

Asset No	1	31	36	44
Weights	0.21	0.23	0.24	0.32

Table 5: Assets' weights with the IFS DEA model

Asset No	31	36	44	50
Weights	0.21	0.35	0.25	0.19

Table 6: Assets' weights with fuzzy set Network DEA model

Asset No	22	36	37	42	49
Weights	0.20	0.17	0.19	0.25	0.19

Table 7: Assets' weights with fuzzy set DEA model

Asset No	1	31	32	36	39
Weights	0.20	0.14	0.21	0.21	0.24

Table 8 also shows the findings for the objective function of both models.

Table 8: Optimal values of the objective functions

Objective	**Objective Type**	**Target Function**	**Intuitionistic Fuzzy & Network**	**Fuzzy & Network**	**Intuitionistic Fuzzy**	**Fuzzy**
z1	Maximum	Expected Return Rate	0.33	0.31	0.30	0.28
z2	Minimum	Return Rate-Based Portfolio Risk	0.19	0.18	0.20	0.12
z3	Maximum	Portfolio Effectiveness	0.66	0.55	0.51	0.53
z4	Minimum	Effectiveness-Based Portfolio Risk	0.09	0.10	0.15	0.16

Based on the provided data in Table 8, here is a comparison and the corresponding result:
Objective: Maximum Expected Return Rate

- Intuitionistic Fuzzy & Network: 0.33
- Fuzzy & Network: 0.31
- Intuitionistic Fuzzy: 0.30
- Fuzzy: 0.28

The highest expected return rate is associated with the Intuitionistic Fuzzy objective, with a value of 0.33. The Fuzzy & Network objective follows closely with a value of 0.31, while the Intuitionistic Fuzzy and Fuzzy objectives have lower values of 0.30 and 0.28, respectively.

Objective: Minimum Return Rate-Based Portfolio Risk

- Intuitionistic Fuzzy & Network: 0.19
- Fuzzy & Network: 0.18
- Intuitionistic Fuzzy: 0.20
- Fuzzy: 0.12

The lowest return rate-based portfolio risk is observed with the Fuzzy objective, having a value of 0.12. The Fuzzy & Network and Intuitionistic Fuzzy & Network objectives have slightly higher values of 0.18 and 0.19, respectively. The Intuitionistic Fuzzy objective has the highest value of 0.20, indicating a relatively higher risk.

Objective: Maximum Portfolio Effectiveness

- Intuitionistic Fuzzy & Network: 0.66

- Fuzzy & Network: 0.55
- Intuitionistic Fuzzy: 0.51
- Fuzzy: 0.53

The maximum portfolio effectiveness is achieved with the Intuitionistic Fuzzy & Network objective, having a value of 0.66. The Fuzzy & Network and Fuzzy objectives have values of 0.55 and 0.53, respectively, indicating a slightly lower level of effectiveness. The Intuitionistic Fuzzy objective has the lowest value of 0.51, indicating a relatively lower level of effectiveness.

Objective: Minimum Effectiveness-Based Portfolio Risk

- Intuitionistic Fuzzy & Network: 0.09
- Fuzzy & Network: 0.10
- Intuitionistic Fuzzy: 0.15
- Fuzzy: 0.16

The lowest effectiveness-based portfolio risk is associated with the Intuitionistic Fuzzy & Network objective, having a value of 0.09. The Fuzzy & Network and Fuzzy objectives have slightly higher values of 0.10 and 0.16, respectively. The Intuitionistic Fuzzy objective has the highest value of 0.15, indicating a relatively higher risk.

The intuitionistic fuzzy model outperforms the fuzzy model in terms of rate of return and effectiveness, as demonstrated by the Table above. Additionally, the effectiveness risk has decreased notwithstanding the fact that the risk of portfolio returns has increased significantly. Furthermore, when compared to the non-network model, the findings have improved dramatically, notwithstanding the effectiveness enhancement and risk reduction. Eventually, we can conclude that more pertinent findings in terms of effectiveness and efficiency can be acquired by taking into account the intuitionistic fuzzy returns as well as the network structure. The Figure 2 determines the final result of the problem.

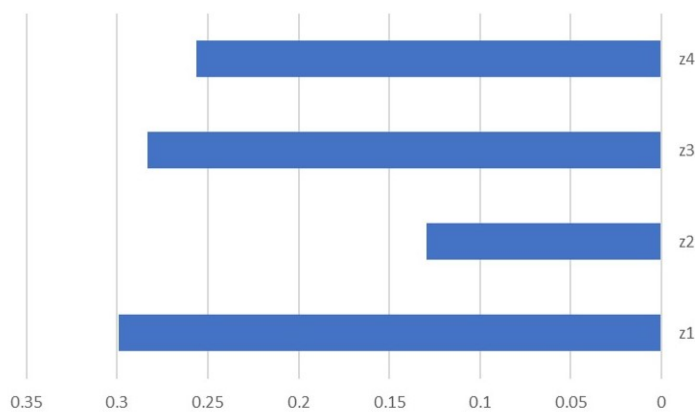

Figure 2: Final result

These weights represent the relative importance or priority assigned to each objective in the decision-making process. The target functions associated with each objective represent the specific metrics used to evaluate and measure the corresponding aspects. Based on the provided data, the objective with the highest weight is "Expected Return Rate" with a weight of 0.2992. This objective has the greatest relative importance in the decision-making process compared to the other objectives. On the other hand, the objective with the lowest weight is "Return Rate-Based Portfolio Risk" with a weight of 0.1297. This objective has the least relative importance among the given objectives. The weight assigned to each objective represents the relative importance or priority given to that objective in the decision-making process. The fact that the "Expected Return Rate" objective has a higher weight compared to the other objectives suggests that it is considered more important in the decision-making process.

There could be several reasons why the "Expected Return Rate" objective has a higher weight:

1. Investment goals: The decision-makers may have determined that maximizing the expected return rate is the primary goal of the investment strategy. This could be driven by factors such as the need for high profitability or meeting specific financial targets.

2. Risk appetite: The decision-makers may have a higher tolerance for risk and prioritize higher returns over other factors. They might believe that the potential for higher returns justifies taking on greater risk.

3. Strategic focus: The decision-makers may be placing a strategic emphasis on generating higher returns as a means of achieving competitive advantage or meeting specific organizational objectives.

4. Investor preferences: The weight assigned to the "Expected Return Rate" objective could also reflect the preferences and priorities of the investors or stakeholders involved in the decision-making process. They may have explicitly expressed a stronger preference for higher returns.

6 Conclusion

In this study, the intuitionistic fuzzy returns, network data envelopment analysis, and Markowitz models were integrated. Using financial statement data and company market value, the four-objective model takes into account the returns and risk, as well as the units' efficiency. The model was executed based on the price fluctuations over the course of three years from 2020 until 2023, as well as the financial data and market value of the most recent year for fifty publicly-traded companies. In the instances of fuzzy and intuitionistic fuzzy efficiencies, in addition to network and non-network DEA, the models were solved. The findings demonstrated that, when compared to the other models, the models' efficiency and effectiveness utilizing network intuitionistic fuzzy efficiencies had dramatically increased. We'll focus on various types of data uncertainty and the extended Markowitz model in upcoming investigations. Future research can explore the integration of various types of data uncertainty, such as imprecise and vague information, into the network intuitionistic fuzzy efficiencies model. Additionally, investigations should focus on extending the Markowitz model to incorporate additional factors that affect investment decisions, such as environmental, social, and governance (ESG) criteria. These advancements would enhance the accuracy and reliability of the decision-making process in portfolio management and provide more comprehensive insights for investors. Furthermore, future research should also consider the dynamic nature of financial markets and incorporate time-series analysis to capture the evolving trends and patterns in asset returns and risk. This could involve developing models that adapt to changing market conditions and incorporate real-time data updates. Additionally, exploring the application of machine learning and artificial intelligence techniques, such as deep learning and neural networks, can provide more robust and accurate predictions for portfolio management. Moreover, incorporating alternative data sources, such as social media sentiment analysis or satellite imagery, can offer valuable insights into market dynamics and help in identifying new investment opportunities. Finally, conducting empirical studies with a larger sample size and across different industries can validate the effectiveness and generalizability of the proposed models and strategies. For future research, it is strongly recommended to investigate the following:

1. Integration of Machine Learning Techniques: The researchers could explore the integration of machine learning techniques, such as Biogeography-based Optimization of Artificial Neural Network (BBO-ANN), into the hybrid model. By incorporating machine learning algorithms, the model could potentially improve its accuracy in selecting the optimal stock portfolio. This could involve training the model on historical stock data to learn patterns and make more informed portfolio decisions.

2. Risk Management Strategies: The focus could be shifted towards incorporating advanced risk management strategies into the hybrid model. This could involve developing type 2 fuzzy logic-based rules and algorithms to dynamically adjust the portfolio composition based on the changing market conditions and risk factors. The researchers could explore methods to handle various types of risks, such as market risk, credit risk, or liquidity risk, within the framework of the hybrid model.

These suggestions aim to enhance the existing hybrid model by incorporating additional techniques or addressing specific aspects of portfolio optimization. Further research in these directions could potentially lead to improved decision-making capabilities and more robust stock portfolio selection under uncertainty.

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