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Original paper

Basic fuzzy logics and weak associative uninorms

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Abstract

Micanorm-based logics with a weak form of associativity are introduced and their completeness results are addressed. More concretely, first the basic wa$_t$-uninorm logic $\mathbf{WA_tBUL}$ and its axiomatic extensions are introduced as $[0, t]$-continuous wa$_t$-uninorm analogues of the logics based the $[0, 1)$-continuous uninorms. Next algebraic structures characterizing the logics are introduced along with algebraic completeness results. Third, wa$_t$-uninorms are introduced as uninorms with weak t-associativity instead of associativity and associated properties are discussed. Finally, by virtue of Yang–style construction, it is verified that the logics based on wa$_t$-uninorms are complete on unit real interval $[0, 1]$, i.e., so called *standard* complete.

Keywords: Fuzzy logic, t-norm, wa$_t$-uninorm, uninorm, micanorm.

1 Introduction

The multiplication and addition of natural numbers are associative, whereas their division and subtraction are not. Associated with this, we may think of *partially* associative operations as operations between fully associative and fully non-associative operations. In fact, many such operations have been introduced. Especially, operations that satisfy weak forms of associativity have been addressed extensively. The following are most famous examples in mathematics: quasigroups [16, 22], hypergroups [3], power associativity in abstract algebra [1, 25, 27], hyperstructures [2, 11, 18, 21, 28, 29, 30], weakly associative lattices [9, 12, 24] and weakly associative relation algebras [15, 17, 26].

In substructural fuzzy logic, logics based on more general structures have been introduced. T-norm-based logics [7, 8, 13, 14], uninorm-based logics [10, 19, 20, 31, 32], micanorm-based logics [4, 33, 38, 41, 42, 43] and mianorm-based logics [4, 34, 36, 39, 40] are examples. One interesting fact is that t-norm-based and uninorm-based logics require associativity as a structural rule, whereas micanorm-based and mianorm-based logics do not. Note that uninorms are micanorms with associativity. Then we may similarly consider binary operations satisfying weak forms of associativity as operations between uninorms and micanorms.

In fact, Yang [35, 37] introduced such operations as weak associative uninorms and dealt with corresponding logics. Especially he [37] introduced the fuzzy logics based on some particular weak t-associative uninorms (simply wa$_t$-uninorms), where the index 't' represents identity, $\mathbf{WA_tBL}$, $\mathbf{WA_t}$**Ł**, $\mathbf{WA_tG}$, and $\mathbf{WA_t}\Pi$ as a non-associative generalization of the continuous t-norm-based logics **BL** (Basic fuzzy logic), **Ł** (Łukasiewicz logic), **G** (Gödel logic) and Π (Product logic), respectively, and showed that they are standard complete, i.e., complete over $[0, 1]$ (real unit interval). Those logics are $[0, t]$-continuous wa$_t$-uninorm-based analogues of the continuous t-norm-based logics.

Notice that Gabbay and Metcalfe [10] introduced basic uninorm logics **BUL** (Basic uninorm logic), **IBUL** (Involutive BUL), **CBUL** (Cancellative BUL) and **CRL** (Cross ratio logic) as uninorm-based analogues of continuous t-norm-based logics. Those logics are based on $[0, 1)$-continuous uninorms. Related to this fact, to provide a wa$_t$-uninorm generalization of the logics was given an open problem by Yang [37]. The problem can be divided into the following two questions.

- Q1: Can wa$_t$-uninorm analogues of the basic uninorm logics be introduced?

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- Q2: Do those logics (if exist) form a $[0,t]$-continuous wa_t-uninorm-based generalization of the $[0,1)$-continuous uninorm-based logics?

Notice also that a continuous t-norm $\diamond$ has the properties

P1: if $o \leq l$, then $o = l \diamond (l \to o)$, for all $o, l \in [0,1]$; and
P2: if $o \leq l \leq m$ and l is idempotent, then $o = o \diamond m$, for all $o, l, m \in [0,1]$ (see [13]),

whereas a $[0,1)$-continuous uninorm $\diamond$ has their restricted ones

P1′: if $o \leq l < 1$, then $o = l \diamond (l \to o)$, for all $o, l \in [0,1]$; and
P2′: if u is $1 \to t$ and $o \leq u \leq l$, then $o = o \diamond l$, for all $o, l \in [0,1]$ (see [10]).

We in general call the properties P1 and P1′ *division* and *restricted division* properties, respectively. The element u in P2′ plays the role of l in P2.

It would be interesting to mention that $[0,t]$-continuous wa_t-uninorms introduced in [37] have the property P1′ but need not have P2′. Note that as an idempotent element wa_t-uninorms need not require u. Then the following arises as a natural question.

- Q3: Does a wa_t-uninorm with the idempotent element u have the property P2′ if u is added to a $[0,t]$-continuous wa_t-uninorm?

As an answer to all of the three questions, we introduce basic wa_t-uninorm logics. More exactly, Sect. 2 introduces the basic wa_t-uninorm logic $\mathbf{WA_tBUL}$ and its axiomatic extensions as a $[0,t]$-continuous wa_t–uninorm-based generalization, and addresses related algebraic semantics. Sect. 3 deals with some $[0,t]$-continuous wa_t-uninorms as algebraic structures for the logics over $[0,1]$. Sect. 4 provides standard completeness results for the logics by virtue of a construction of Yang–style in [33, 37].

We finally notice that a wa_t-uninorm has a subalgebra isomorphic to a t-norm. This implies at least two things. One is that since wa_t-uninorms have subalgebras isomorphic to t-norms, logically interesting t-norms such as Łukasiewicz, Gödel and Product t-norms and their applications to fuzzy logic and fuzzy sets can be still considered in wa_t-uninorms. The other is that since wa_t-uninorms are non-associative operations, properties of such operations and their applications such as compensation behavior and full reinforcement can be investigated in wa_t-uninorms.

2 Logics and algebraic semantics

Henceforth, the notion of *basic wa_t-uninorm-based logics* denotes substructural fuzzy logics being characterized by algebraic semantics, where the connective pair $(\&, \to)$ is interpreted by a basic wa_t-uninorm and its residuum. For these logics, this section consists of four steps: 1st. A language $\mathcal{L}$ for the logics is provided; 2nd. On $\mathcal{L}$, the logics intended are introduced using a deductive consequence relation $\vdash$ and related notions such as proof are considered; 3rd. Algebraic structures to characterize the logics are introduced and a related semantic consequence relation $\models$ is defined along with related notions such as tautology and model; 4th. Using completeness theorem, it is verified that the deductive and semantic consequences are equivalent to each other.[1]

We first introduce a language for basic wa_t-uninorm-based logics. A language $\mathcal{L}$ for these logics is a countable propositional language with a set of formulas *For* built inductively by a set of variables *Var*, binary connectives $\vee$ (disjunction), $\wedge$ (conjunction), & (fusion), $\to$ (implication), and constants $\top, \mathbf{t}, \mathbf{f}, \bot$. As the additional connectives and constant, it further has $\leftrightarrow$ (biimplication), $\sim$ (negation), $\neg$ (negation) and U given by:

df1. $R \leftrightarrow P := (R \to P) \wedge (P \to R)$,
df2. $\sim R := R \to \mathbf{f}$,
df3. $\neg R := R \to \bot$, and
df4. $U := \top \to \mathbf{t}$.

We also simplify $R \wedge \mathbf{t}$ and $R \wedge U$ by $R_{\mathbf{t}}$ and R_U, respectively. An *$\mathcal{L}$-substitution* is a map $s : \mathit{For} \to \mathit{For}$ such that $s(\#(R_1, \ldots, R_n) = \#(s(R_1), \ldots, s(R_n))$ for all n-ary connective $\#$ and all formulas $R_1, \ldots, R_n \in \mathit{For}$. We denote the formulas by uppercase Latin letters $R, P, Q, \ldots$, and their sets by uppercase Greek letters $\Gamma, \Sigma, \ldots$.

We next present axiomatizations for the logics using a deductive consequence relation $\vdash$.[2] (For the notion of logics in general as the same kind of objects as axiom systems, see [6].)

[1] For substructural fuzzy logics and algebraic semantics in general, see [4, 5, 6].

[2] More exactly, the relation $\vdash$ is introduced as a *consecution relation* since it is not necessary to be a proof or derivation relation from premises to conclusion in a logic. For this relation, see [6, 23].

Definition 2.1. 1. *The basic weak t-associative uninorm logic* $\boldsymbol{WA_tBUL}$ *consists of the following axioms and rules:*

$R \to R$ *(self-implication, SI)*
$(R \wedge P) \to R,\ (R \wedge P) \to P$ *($\wedge$-elimination, $\wedge$-E)*
$((R \to P) \wedge (R \to Q)) \to (R \to (P \wedge Q))$ *($\wedge$-introduction, $\wedge$-I)*
$R \to (R \vee P),\ P \to (R \vee P)$ *($\vee$-introduction, $\vee$-I)*
$((R \to Q) \wedge (P \to Q)) \to ((R \vee P) \to Q)$ *($\vee$-elimination, $\vee$-E)*
$\bot \to R$ *(ex falso quodlibet, EF)*
$(R\&P) \to (P\&R)$ *(&-commutativity, &-C)*
$(\boldsymbol{t} \to R) \leftrightarrow R$ *(push and pop, PP)*
$R \to (P \to (P\&R))$ *(&-adjunction, &-Adj)*
$(R_{\boldsymbol{t}}\&P_{\boldsymbol{t}}) \to (R \wedge P)$ *(&$\wedge$)*
$(P\&(R\&(R \to (P \to Q)))) \to Q$ *(residuation, Res$'$)*
$(R \to ((R\&(R \to P))\&(P \to Q))) \to (R \to Q)$ *(transitivity, T$'$)*
$((Z\&W) \to (Z\&(W\&(R \to P)_{\boldsymbol{t}}))) \vee (Z' \to (W' \to ((W'\&Z')\&(P \to R)_{\boldsymbol{t}})))$ *(prelinearity, PL)*
$(R_{\boldsymbol{t}}\&(P_{\boldsymbol{t}}\&Q_{\boldsymbol{t}})) \leftrightarrow ((R_{\boldsymbol{t}}\&P_{\boldsymbol{t}})\&Q_{\boldsymbol{t}})$ *(weak $\boldsymbol{t}$-associativity, wAS$_{\boldsymbol{t}}$)*
$(\boldsymbol{t} \to R) \vee (R \to P) \vee (P \to (R\&(R \to P)))$ *(weak $\boldsymbol{t}$-restricted divisibility, RDIV$^w_{\boldsymbol{t}}$)*
$R_U \to (U\&R_U)$ *(U-restricted identity, U-RI)*
$(U \vee R_{\boldsymbol{t}}) \to ((U \vee R_{\boldsymbol{t}})\&(U \vee R_{\boldsymbol{t}}))$ *($^{\boldsymbol{t}}_U$-idempotence, ID$^{\boldsymbol{t}}_U$)*
$R \to P,\ R \vdash P$ *(modus ponens, mp)*
$R \vdash R_{\boldsymbol{t}}$ *($\boldsymbol{t}$-adjunction, adj$_{\boldsymbol{t}}$)*
$R \vdash (Z\&W) \to (Z\&(W\&R))$ *(&-associativity, &-as)*
$R \vdash Z \to (W \to ((W\&Z)\&R))$ *($\to$-associativity, $\to$-as)*

2. *The following are logics extending* $\boldsymbol{WA_tBUL}$*:*

- $\boldsymbol{WA_tIBUL}$ *(Involutive* $\boldsymbol{WA_tBUL}$*) is* $\boldsymbol{WA_tBUL}$ *plus*
  $\sim\sim R \to R$ *(double negation elimination, DNE); and*
  $(\neg\neg R)_U \to R$ *(U-double negation elimination, DNE$_U$).*
- $\boldsymbol{WA_tCBUL}$ *(Cancellative* $\boldsymbol{WA_tBUL}$*) is* $\boldsymbol{WA_tBUL}$ *plus*
  $(R \wedge \neg R) \to \bot$ *(Gödel negation, GN);*
  $(R \to P) \to (\neg P \to \neg R)$ *(contraposition, CP); and*
  $(\neg\neg R)_U \to ((R_U \to (R_U\&P))_U \to P)$ *(weak U-restricted cancellation, RCAN^w_U).*
- $\boldsymbol{WA_tGBUL}$ *(Restricted idempotent* $\boldsymbol{WA_tBUL}$*) is* $\boldsymbol{WA_tBUL}$ *plus*
  $(R\&R)_U \leftrightarrow R_U$ *(U-restricted idempotence, RID$_U$).*
- $\boldsymbol{WA_tCRL}$ *(Cross ratio* $\boldsymbol{WA_tBUL}$*) is* $\boldsymbol{WA_tIBUL}$ *plus*
  $\boldsymbol{t} \leftrightarrow \boldsymbol{f}$ *(fixed-point, FP).*

Remark 2.2. 1. *We obtain the* $[0,t]$*-continuous wa$_t$-uninorm logic* $\boldsymbol{WA_tBL}$ *by dropping (U-RI) and (ID$^{\boldsymbol{t}}_U$) from* $\boldsymbol{WA_tBUL}$*, the wa$_t$-uninorm logic* $\boldsymbol{WA_tMUL}$ *by eliminating (RDIV$^w_{\boldsymbol{t}}$) from* $\boldsymbol{WA_tBL}$*, and the micanorm logic* $\boldsymbol{MICAL}$*, described as* SL^ℓ_e *in* [4, 5, 6]*, by dropping (wAS$_{\boldsymbol{t}}$) from* $\boldsymbol{WA_tMUL}$ (see [33, 35, 37]).

2. *The systems* $\boldsymbol{WA_tBUL}$*,* $\boldsymbol{WA_tIBUL}$*,* $\boldsymbol{WA_tCBUL}$*, and* $\boldsymbol{WA_tCRL}$ *are* $[0,t]$*-continuous wa$_t$-uninorm analogues of the* $[0,1)$*-continuous uninorm logics* $\boldsymbol{BUL}$*,* $\boldsymbol{IBUL}$*,* $\boldsymbol{CBUL}$*, and* $\boldsymbol{CRL}$*, respectively, which are obtained as follows* (see [10])*:*

- *Basic uninorm logic* $\boldsymbol{BUL}$ *is* $\boldsymbol{UL}$ *plus df1 to df4 and:*
  $U \leftrightarrow (U\&U)$ *(U-idempotence, U-ID).*
  $(\top \to R) \vee (R \to (P \wedge U)) \vee (P \to (R\&(R \to P)))$ *(restricted divisibility, RDIV).*
- $\boldsymbol{IBUL}$ *(Involutive* $\boldsymbol{BUL}$*) is* $\boldsymbol{BUL}$ *plus (DNE).*
- $\boldsymbol{CBUL}$ *(Cancellative* $\boldsymbol{BUL}$*) is* $\boldsymbol{BUL}$ *plus*
  $(R \to \bot) \vee (\top \to R) \vee ((R \to (R\&P)) \to P)$ *(restricted cancellation, RCAN).*
- $\boldsymbol{CRL}$ *(Cross ratio logic) is* $\boldsymbol{IBUL}$ *plus (FP).*

Definition 2.3. *WA$_t$Ls* $= \{\boldsymbol{WA_tBUL}, \boldsymbol{WA_tIBUL}, \boldsymbol{WA_tCBUL}, \boldsymbol{WA_tGBUL}, \boldsymbol{WA_tCRL}\}$.

We define a proof as a deductive consequence relation $\vdash$, which is a relation between sets of formulas and formulas. A *proof* in a theory Σ (a set of formulas) on WA$_{\boldsymbol{t}}$L ($\in$ WA$_{\boldsymbol{t}}$Ls) is defined as a sequence of formulas whose elements are either members of Σ, axioms of WA$_{\boldsymbol{t}}$L, or are derived from previous elements of the sequence by rules of WA$_{\boldsymbol{t}}$L. By $\Sigma \vdash_{WA_{\boldsymbol{t}}L} R$, we mean that R is *provable* in Σ on WA$_{\boldsymbol{t}}$L.

A p-formula is constructed by $Var \cup \{p\}$ and a p-substitution is a substitution in the extended language. Let R be a formula, Z be a p-formula and s be a p-substitution such that $s(p) = R$ and $s(q) = q$ for $q \in Var$. Let $Z(R)$ denote the formula $s(Z)$, which is in the original set of variables. For a set Σ of p-formulas, Σ^* is defined as the least set of p-formulas, where $p \in \Sigma^*$ and $Z(P) \in \Sigma^*$ for all $P \in \Sigma^*$ and all $Z \in \Sigma$, and $\Pi(\Sigma)$ is the least set of p-formulas containing $\Sigma \cup \{\mathbf{t}\}$ and closed under &. Let bDT, the set of basic deduction terms, be a set of p-formulas. A substructural logic L is almost mp-based on bDT if (i) the set bDT is closed under every p-substitution s, $s(p) = p$, (ii) L is an axiomatic system with the rules (mp) and those from $\{\langle R, P(R)\rangle : R \in For$ of L and $P \in bDT\}$, and (iii) for all $Q \in bDT$ and all formulas R and P, one has some $Q_1, Q_2 \in bDT^*$ such that $\vdash_L Q_1(R \to P) \to (Q_2(R) \to Q(P))$ (see ([4, 5, 6, 33]).

Since $\mathrm{WA}_{\mathbf{t}}\mathrm{L}$ is an almost mp-based logic, we have the following deduction theorem for $\mathrm{WA}_{\mathbf{t}}\mathrm{L}$.

Theorem 2.4. [4, 5, 6] *Given a set $\Gamma \cup \{R, P\}$ of formulas, it holds:*

$$\Sigma, R \vdash_{WA_tL} P \text{ if and only if } \Gamma \vdash_{WA_tL} Q(R) \to P \text{ for some } Q \in \Pi(bDT^*).$$

We then introduce algebraic structures corresponding to $\mathrm{WA}_{\mathbf{t}}\mathrm{L}$ and define a semantic consequence relation $\models$. For convenience, the notations '$\top$,' '$\bot$' are ambiguously used as both constants and special elements, and similarly the notations '$\neg$,' '$\sim$,' '$\to$,' '$\vee$,' and '$\wedge$' as both connectives and operators.

Definition 2.5. 1. [35] *A bounded residuated lattice-ordered pointed commutative groupoid with identity is a structure $(S, \bot, \top, f, t, \to, \star, \vee, \wedge)$ such that:*

- *$(S, \bot, \top, \vee, \wedge)$ is a bounded lattice, where the least and greatest elements are $\bot$ and $\top$, respectively.*
- *$(S, \star, t)$ is a commutative groupoid with identity.*
- *f is a special element in S.*
- *$l \leq o \to m$ if and only if $o \star l \leq m$, for all $o, l, m \in S$ (residuation).*

2. [37] *Let o_t be $o \wedge t$. A residuated* wa_t-monoidal lattice *is a bounded residuated lattice-ordered pointed commutative groupoid with identity that satisfies:*

- *$o_t \star (l_t \star m_t) = (o_t \star l_t) \star m_t$, for all $o, l, m \in S$ ($wAS_{\mathbf{t}}{}^{\mathcal{S}}$).*

3. [37] *A* $\mathrm{WA}_t\mathrm{MUL}$-algebra *is a residuated wa_t-monoidal lattice satisfying:*

- *$t \leq ((m \star p) \to (m \star (p \star (o \to l)_t))) \vee (m' \to (p' \to ((p' \star m') \star (l \to o)_t)))$, for all $o, l, m, p, m', p' \in A$ ($PL^{\mathcal{S}}$).*

Definition 2.6. *(WA_tL-algebras) Let $\sim o$, $\neg o$, and u be $o \to f$, $o \to \bot$ and $\top \to t$, respectively.*

1. *A* $\mathrm{WA}_t\mathrm{BUL}$-algebra *is a WA_tMUL-algebra satisfying:*
   - *$t \leq (t \to o) \vee (o \to l) \vee (l \to (o \star (o \to l)))$, for all $o, l \in S$ ($RDIV_{\mathbf{t}}^{w\mathcal{S}}$),*
   - *$o_u \leq (u \star o_u)$, for all $o \in S$ (U-$RI^{\mathcal{S}}$), and*
   - *$(u \vee o_t) \leq ((u \vee o_t) \star (u \vee o_t))$, for all $o \in S$ ($ID_U^{\mathbf{t}\,\mathcal{S}}$).*
2. *A* $\mathrm{WA}_t\mathrm{IBUL}$-algebra *is a WA_tBUL-algebra satisfying:*
   - *$t \leq \sim\sim o \to o$, for all $o \in S$ ($DNE^{\mathcal{S}}$), and*
   - *$t \leq (\neg\neg o)_u \to o$, for all $o \in S$ ($DNE_U{}^{\mathcal{S}}$)*
3. *A* $\mathrm{WA}_t\mathrm{CBUL}$-algebra *is a WA_tBUL-algebra satisfying:*
   - *$t \leq (o \wedge \neg o) \to \bot$ for all $o \in S$ ($GN^{\mathcal{S}}$)*
   - *$t \leq (o \to l) \to (\neg l \to \neg o)$ for all $o, l \in S$ ($CP^{\mathcal{S}}$)*
   - *$t \leq (\neg\neg o)_u \to ((o_u \to (o_u \star l))_u \to l)$ for all $o, l \in S$ ($RCAN_U^{w\mathcal{S}}$).*
4. *A* $\mathrm{WA}_t\mathrm{GBUL}$-algebra *is a WA_tBUL-algebra satisfying*
   - *$(o \star o)_u = o_u$, for all $o \in S$ ($RID_U{}^{\mathcal{S}}$).*
5. *A* $\mathrm{WA}_t\mathrm{CRL}$-algebra *is a WA_tIBUL-algebra satisfying*
   - *$t = f$ ($FP^{\mathcal{S}}$).*

All of these algebras are called $\mathrm{WA}_t\mathrm{L}$-algebras.

Let $\mathcal{S}$ be a $\mathrm{WA}_t\mathrm{L}$-algebra, $\mathcal{K}$ be a class of $\mathrm{WA}_t\mathrm{L}$-algebras, R and P be formulas and Γ be a theory. We then define several notions for the semantic consequence relation for $\mathbf{WA}_t\mathbf{L}$. An $\mathcal{S}$-*interpretation* is a homomorphism $It : For \to \mathcal{S}$ such that $It(\bot) = \bot$, $It(\top) = \top$, $It(\mathbf{t}) = t$, $It(\mathbf{f}) = f$, $It(R \to P) = It(R) \to It(P)$, $It(R\&P) = It(R) \star It(P)$, $It(R \vee P) = It(R) \vee It(P)$, $It(R \wedge P) = It(R) \wedge It(P)$ (and so $It(\sim R) = \sim It(R)$, $It(\neg R) = \neg It(R)$ and $It(U) = u$). R is an t-*tautology* in $\mathcal{S}$, an $\mathcal{S}$-*tautology* for simplicity, if $t \leq It(R)$ for every $\mathcal{S}$-interpretation It. An $\mathcal{S}$-interpretation It is an $\mathcal{S}$-*model* of Γ if $t \leq It(R)$ for all $R \in \Gamma$. The class of $\mathcal{S}$-models of Γ is referred to by $Mod(\Gamma, \mathcal{S})$ R is a *semantic consequence* of Γ over $\mathcal{K}$, $\Gamma \models_{\mathcal{K}} R$, if $Mod(\Gamma, \mathcal{S}) = Mod(\Gamma \cup \{R\}, \mathcal{S})$ for each $\mathcal{S} \in \mathcal{K}$. $\mathcal{S}$ is a $\boldsymbol{WA_tL}$-*algebra* if and only if $\Gamma \vdash_{WA_tL} R$, $\mathrm{WA}_{\mathbf{t}}\mathrm{L} \in \mathrm{WA}_{\mathbf{t}}\mathrm{Ls}$, entails that $\Gamma \models_{\{\mathcal{S}\}} R$, $\mathcal{S}$ a corresponding $\mathrm{WA}_t\mathrm{L}$-algebra. *MOD*(WA_tL) and *MOD*^{l}(WA_tL) refer to the class of $\mathbf{WA_tL}$-algebras and the class of linearly ordered $\mathbf{WA_tL}$-algebras, respectively. For simplicity, $\Gamma \models_{WA_tL} R$ and $\Gamma \models^l_{WA_tL} R$ express $\Gamma \models_{MOD(WA_tL)} R$ and $\Gamma \models_{MOD^l(WA_tL)} R$, respectively.

We finally prove completeness for $\mathrm{WA}_{\mathbf{t}}\mathrm{L}$, which shows that the two consequence relations $\vdash$ and $\models$ on $\mathrm{WA}_{\mathbf{t}}\mathrm{L}$ are equivalent to each other.

Theorem 2.7. 1. (*Strong completeness*) *Suppose that* Γ *is a theory over* $WA_tL \in \{\boldsymbol{WA_tBUL}, \boldsymbol{WA_tGBUL}\}$ *and* R *be a formula.* $\Gamma \vdash_{WA_tL} R$ *if and only if* $\Gamma \models_{WA_tL} R$ *if and only if* $\Gamma \models^l_{WA_tL} R$.

2. (*Finite strong completeness*) *Let* Γ *be a finite theory over* $WA_tL \in WA_tLs$ *and* R *be a formula.* $\Gamma \vdash_{WA_tL} R$ *if and only if* $\Gamma \models_{WA_tL} R$ *if and only if* $\Gamma \models^l_{WA_tL} R$.

Proof. The claims are directly obtained by Theorem 3.1.8 in [6]. □

3 Basic weak-t-associative uninorms

We henceforth express $\bot$ and $\top$ on $[0, 1]$ by 0 *and* 1, respectively. As usual, we call a $\mathrm{WA}_t\mathrm{L}$-algebra on $[0, 1]$ *standard*. This section consists of two steps: 1st. Basic notions of *weak-t*-associative uninorms (wa_t-uninorms for simplicity) as standard $\mathrm{WA}_t\mathrm{L}$-algebras are introduced. 2nd. It is verified that such wa_t-uninorms form $\mathrm{WA}_t\mathrm{L}$-algebra on $[0, 1]$.

The operator $\diamond$ is a micanorm in standard $\mathrm{WA}_t\mathrm{L}$-algebras. A *micanorm* is a function $\diamond : [0, 1]^2 \to [0, 1]$ satisfying: for all $o, l, m \in [0, 1]$ and for some $t \in [0, 1]$,

(i) $o \diamond l = l \diamond o$ (commutativity),
(ii) $t \diamond o = o = o \diamond t$ (identity), and
(iii) $o \leq l$ entails $o \diamond m \leq l \diamond m$ and $m \diamond o \leq m \diamond l$ (isotonicity), see [33].

A *weak-t*-associative uninorm (wa_t-uninorm for simplicity) is a micanorm satisfying:

(wAS_t) $o, l, m \leq t$ entails $o \diamond (l \diamond m) = (o \diamond l) \diamond m$, see [37].

Note that a *uninorm* is a micanorm satisfying: for all $o, l, m \in [0, 1]$,

(AS) $o \diamond (l \diamond m) = (o \diamond l) \diamond m$,

and a *t-norm* is a uninorm satisfying $t = 1$. We call a micanorm $\diamond$ satisfying $0 \diamond 1 = 1 \diamond 0 = 0$ *conjunctive*, and a micanorm $\diamond$ satisfying $0 \diamond 1 = 1 \diamond 0 = 1$ *disjunctive* (see [10]).

Note also that we can get the implication functions using residual operators of micanorms and wa_t-uninorms. We call a micanorm $\diamond$ *residuated* whenever we have a function $\to: [0, 1]^2 \to [0, 1]$ such that it satisfies (residuation), called the *residuum* of $\diamond$. Given a micanorm (wa_t-uninorm resp) $\diamond$, *implication* $\to$ can be defined as follows: for all $o, l \in [0, 1]$,

$$o \to l := sup\{m \in [0, 1] : o \diamond m \leq l\}.$$

Over $[0, 1]$, the operation $\diamond$ of a UL-algebra is a conjunctive uninorm with its residuum $\to$ and identity t, and conversely a residuated uninorm $\diamond$ arises a UL-algebra (see [19]). Similarly for a MICAL-algebra on $[0, 1]$ and a residuated micanorm (see [33]).

Let the wa_t-uninorms to characterize $\mathrm{WA}_t\mathrm{L}$-algebras be *basic* wa_t-uninorms. We notice that an arbitrary residuated $[0, 1)$-continuous uninorm $\diamond$ arises a BUL-algebra on $[0, 1]$ (see [10]). Analogously, some residuated $[0, t]$-continuous wa_t-uninorm $\diamond$ arises a $\mathrm{WA}_t\mathrm{L}$-algebra on $[0, 1]$. We verify it.

A negation function $\nu : [0, 1] \to [0, 1]$ is in general defined as a non-increasing function which satisfies $\nu(1) = 0$ and $\nu(0) = 1$. Since this definition does not work for uninorms in general[3], we instead define a non-increasing function as a *negation*. A negation ν is called: *conjunctive* if it satisfies:

[3]For instance, a non-increasing function ν needs not satisfy $\nu(1) = 0$ in a uninorm.

• (conj) $\nu(0) = 1$;

fixed-pointed if it satisfies:

• (fp) $\nu(id) = id$;

super-involutive if it satisfies:

• (s-inv) $\nu^2(o) \geq o$ for all $o \in [0, 1]$;

weak if it is super-involutive and satisfies:

• (w) $\nu(0) = 1$ and $\nu(1) = 0$;

involutive if it is weak and satisfies:

• (inv) $\nu^2(o) = o$ for all $o \in [0, 1]$.

It is known that involutive negations on $[0, 1]$ are isomorphic to each other (see [8]). Thus for all $o \in [0, 1]$, an involutive negation $\nu(o)$ is definable by $1 - o$. The negation defined by $1 - o$ is called *standard*. An involutive negation ν on $[0, u]$ is called *u-involutive*.

WA$_t$L-algebras on $[0, 1]$ can be obtained from residuated $[0, t]$-continuous wa$_t$-uninorms satisfying:

(^t_u-ID) $u \leq o \leq t$ entails $o = (o \diamond o)$, for all $o \in [0, 1]$, and
(u-RI) $o \leq u$ entails $o = (u \diamond o)$, for all $o \in [0, 1]$.

We call wa$_t$-uninorms satisfying these conditions ^t_u*-idempotent restricted* u*-identical* wa$_t$-uninorms. These wa$_t$-uninorms form basic wa$_t$-uninorms.

Proposition 3.1. *Let u be $1 \to t$.*

1. *A structure $([0, 1], 0, 1, t, f, u, \diamond, max, min, \to)$ is a standard WA$_t$BUL-algebra if $\diamond$ is a residuated $[0, t]$-continuous ^t_u-idempotent restricted u-identical wa$_t$-uninorm with identity t.*

2. *A structure $([0, 1], 0, 1, t, f, u, \nu_1, \nu_2, \diamond, max, min, \to)$ is a standard WA$_t$IBUL-algebra if $\diamond$ is a residuated $[0, t]$-continuous ^t_u-idempotent restricted u-identical wa$_t$-uninorm with identity t, an involutive negation ν_1 and a u-involutive negation ν_2.*

3. *Suppose that ν is a negation satisfying (GN$^{\mathcal{S}}$), i.e., Gödel negation. A structure $([0, 1], 0, 1, t, f, u, \nu, \diamond, max, min, \to)$ is a standard WA$_t$CBUL-algebra if $\diamond$ is a residuated $[0, t]$-continuous ^t_u-idempotent restricted u-identical wa$_t$-uninorm with identity t and Gödel negation ν, and is strictly isotone on $[0, u]$.*

4. *A structure $([0, 1], 0, 1, t, f, u, \diamond, max, min, \to)$ is a standard WA$_t$GBUL-algebra if $\diamond$ is a residuated $[0, t]$-continuous $[0, t]$-idempotent wa$_t$-uninorm with identity t.*

5. *A structure $([0, 1], 0, 1, \frac{1}{2}, \frac{1}{2}, u, \nu_1, \nu_2, \diamond, max, min, \to)$ is a standard WA$_t$CRL-algebra if $\diamond$ is a residuated $[0, t]$-continuous ^t_u-idempotent restricted u-identical wa$_t$-uninorm with a standard negation ν_1, a u-involutive negation ν_2 and identity $t = \frac{1}{2} = \nu_1(t)$.*

Proof. 1. Let $\diamond$ be a residuated $[0, t]$-continuous ^t_u-idempotent restricted u-identical wa$_t$-uninorm with t as identity. Since one is capable of obtaining a WA$_t$BL-algebra from an arbitrary residuated $[0, t]$-continuous wa$_t$-uninorm (see [37]), we just need to verify that (U-RI$^{\mathcal{S}}$) and (ID$^{\mathbf{t}}_U{}^{\mathcal{S}}$) hold. For ($U$-RI$^{\mathcal{S}}$), one has to show that: for all $o \in [0, 1]$,

$$min\{o, u\} \leq u \diamond min\{o, u\}.$$

Let $o \leq u$. Since $\diamond$ is restricted u-identical, we obtain $o \leq u \diamond o$ by (u-RI). Otherwise, $min\{o, u\} = u$ and so $u \leq u \diamond u$ by (u-RI). Hence the claim holds.

For (ID$^{\mathbf{t}}_U{}^{\mathcal{S}}$), one has to show that: for all $o \in [0, 1]$,

$$max\{u, min\{o, t\}\} \leq max\{u, min\{o, t\}\} \diamond max\{u, min\{o, t\}\}.$$

Let $u \leq o \leq t$. Then $max\{u, min\{o, t\}\} = o$. Also, since $\diamond$ is ^t_u-idempotent, we obtain $o = o \diamond o$ and so $o \leq o \diamond o$. Let $t < o$. Then $max\{u, min\{o, t\}\} = t$ and so $t \leq t \diamond t$. Otherwise, $max\{u, min\{o, t\}\} = u$. Then, since $\diamond$ is ^t_u-idempotent, we obtain $u = u \diamond u$ and so $u \leq u \diamond u$. Hence the claim holds.

2. Let $\diamond$ be a residuated $[0, t]$-continuous ^t_u-idempotent restricted u-identical wa$_t$-uninorm with identity t, an involutive negation ν_1 and a u-involutive negation ν_2. We need to further verify that (DNE$^{\mathcal{S}}$) and (DNE$_U{}^{\mathcal{S}}$) hold. For (DNE$^{\mathcal{S}}$), one has to show that: for all $o \in [0, 1]$,

$$t \leq \nu_1^2(o) \rightarrow o.$$

Since ν_1 is involutive, i.e., $\nu_1^2(o) = o$, we have $\nu_1^2(o) \leq o$ and so $t \leq \nu_1^2(o) \rightarrow o$ by (residuation).

For ($\mathrm{DNE}_U{}^{\mathcal{S}}$), one has to show that: for all $o \in [0,1]$,

$$t \leq min\{\nu_2^2(o), u\} \rightarrow o.$$

Let $o \leq u$. Since ν_2 is u-involutive, we have $min\{\nu_2^2(o), u\} = \nu_2^2(o) = o$. Then we further obtain $\nu_2^2(o) \leq o$ and so $t \leq \nu_2^2(o) \rightarrow o$ by (residuation). Otherwise, i.e., if $u < o$, then $u = u \diamond t < o$ and so $t \leq u \rightarrow o$ by (residuation). Moreover we have $min\{\nu_2^2(o), u\} = u$. Hence the claim holds.

3. Let $\diamond$ be a residuated $[0,t]$-continuous ${}^t_u$-idempotent restricted u-identical wa$_t$-uninorm with identity t and Gödel negation ν, and is strictly isotone on $[0,u]$. We need to further verify that ($\mathrm{GN}^{\mathcal{S}}$), ($\mathrm{CP}^{\mathcal{S}}$) and ($\mathrm{RCAN}_U{}^{\mathcal{S}}$) hold.

For ($\mathrm{GN}^{\mathcal{S}}$), one has to show that: for all $o \in [0,1]$,

$$t \leq min\{o, \nu(o)\} \rightarrow o.$$

By (residuation), we may instead show that $min\{o, \nu(o)\} \leq o$. If $o = 0$, then clearly $min\{o, \nu(o)\} \leq o$. Otherwise, $\nu(o) = 0$ and so $min\{o, \nu(o)\} \leq o$.

For ($\mathrm{CP}^{\mathcal{S}}$), one has to show that: for all $o, l \in [0,1]$,

$$t \leq (o \rightarrow l) \rightarrow (\nu(l) \rightarrow \nu(o)).$$

Since the implication $\rightarrow$ is transitive, it holds that

$$t \leq (o \rightarrow l) \rightarrow ((l \rightarrow 0) \rightarrow (o \rightarrow 0)).$$

Hence the claim holds by df3.

For ($\mathrm{RCAN}_U^{w\mathcal{S}}$), one has to show that: for all $o, l \in [0,1]$,

$$t \leq min\{\neg\neg o, t\} \rightarrow (min\{(min\{o,u\} \rightarrow (min\{o,u\} \diamond l)), u\} \rightarrow l).$$

Let $o = 0$. We have $\neg\neg o = 0$. Thus the claim holds. Let $o \neq 0$. We obtain $\neg\neg o = 1$ and so $min\{\neg\neg o, t\} = t$. Then, by (residuation) (twice), it suffices to verify

$$(\alpha)\ min\{min\{o,u\} \rightarrow (min\{o,u\} \diamond l), u\} \leq l.$$

Here, we consider the case $o \leq u$. If $u < l$, then (α) holds because $min\{o \rightarrow (o \diamond l), u\} \leq l$. Let $l < u$. Then, since $o \diamond l \leq min\{o, l\} \leq l$ and $\diamond$ is strict isotone on $[0,u]$, (α) also holds.

4. Let $\diamond$ be a residuated $[0,t]$-continuous $[0,t]$-idempotent wa$_t$-uninorm with identity t. We need to further verify that (U-$\mathrm{RI}^{\mathcal{S}}$), ($\mathrm{ID}^{\mathbf{t}}_U{}^{\mathcal{S}}$) and ($\mathrm{RID}_U{}^{\mathcal{S}}$) hold. For ($U$-$\mathrm{RI}^{\mathcal{S}}$), one may verify ($u$-RI). Let $o \leq u$. Then since $\diamond$ is $[0,t]$-idempotent, we have

$$o = o \diamond o \leq u \diamond o \leq t \diamond o = o,$$

and so $o = (u \diamond o)$. Hence (u-RI) holds.

For ($\mathrm{ID}^{\mathbf{t}}_U{}^{\mathcal{S}}$), one may verify (${}^t_u$-ID). Let $u \leq o \leq t$. Then since $\diamond$ is $[0,t]$-idempotent, we have $o = o \diamond o$. Hence (${}^t_u$-ID) holds.

For ($\mathrm{RID}_U{}^{\mathcal{S}}$), one has to show that: for all $o \in [0,1]$,

$$(o \diamond o)_u = o_u.$$

If $u \leq o$, then $(o \diamond o)_u = u = o_u$. Otherwise, $(o \diamond o)_u = o = o_u$ since $\diamond$ is $[0,t]$-idempotent. Hence the claim holds.

5. Let $\diamond$ be a residuated $[0,t]$-continuous ${}^t_u$-idempotent restricted u-identical wa$_t$-uninorm with a standard negation ν_1, a u-involutive negation ν_2 and identity $t = \frac{1}{2} = \nu_1(t)$. Since $t = \frac{1}{2} = \nu_1(t) = f$, the claim directly follows from 2. □

4 Standard completeness

Standard completeness results are established for $\mathrm{WA_tL}$ using a construction of Yang–style in [33, 37]. The idea for standard completeness is this: Countable or finite linearly ordered $\mathrm{WA_tL}$-algebras are embeddable into dense linearly ordered ones, and these algebras again embeddable into standard algebras. This section consists of four steps: 1st. Standard completeness of $\mathbf{WA}_t\mathbf{BUL}$ is provided based on the above idea. 2nd. This proof is applied to its extensions. 3rd. It is verified that WA_tL-algebras on $[0, 1]$ have subalgebras isomorphic to corresponding t-norms. 4th. It is verified that standard WA_tL-algebras are given on restricted υ-identical $^{\iota}_{\upsilon}$-idempotent $[0, t]$-continuous wa_t-uninorms.

Notice first that linearly ordered countable or finite **MICAL**-algebras can be embedded into a standard algebra. (For the shake of easiness, we add the 'less than or equal to' relation symbol "$\leq$.")

Fact 4.1. [33] *For each linearly ordered countable or finite **MICAL**-algebra $\boldsymbol{S} = (S, \leq_S, \bot, \top, f, t, \to, \star, \vee, \wedge)$, one can construct a countable ordered set T, a binary operation $\ominus$ on T, and a function i from S into T satisfying:*

1. *T is a dense ordered set and has a minimum $\bigwedge$, a maximum $\bigvee$ and special elements ρ and ι.*
2. *$(T, \ominus, \precsim, \iota)$ is a commutative, isotone linearly ordered groupoid with identity.*
3. *$\ominus$ is left-continuous and conjunctive over the order topology on $(T, \precsim)$.*
4. *i is an embedding function of the structure $(S, \leq_S, \bot, \top, f, t, \to, \star, \vee, \wedge)$ into $(T, \precsim, \bigwedge, \bigvee, \rho, \iota, \ominus, max, min)$, and for all $o, l \in S$, $i(o \to l)$ is the residuum of $i(o)$ and $i(l)$ in $(T, \precsim, \bigwedge, \bigvee, \rho, \iota, \ominus, max, min)$.*

We first establish standard completeness for $\mathbf{WA_tBUL}$.

Proposition 4.2. *For each countable or finite, linearly ordered $\boldsymbol{WA}_t\boldsymbol{BUL}$-algebra $\boldsymbol{S} = (S, \leq_S, \bot, \top, f, t, u, \to, \star, \vee, \wedge)$, one can construct a countable ordered set T, a binary operation $\ominus$ on T, and a function i from S into T satisfying:*

1. *T is densely ordered and has a minimum $\bigwedge$, a maximum $\bigvee$, and special elements ι, ρ and υ.*
2. *$(T, \ominus, \precsim, \iota, \upsilon)$ is a linearly ordered, isotonic, commutative $^{\iota}_{\upsilon}$-idempotent restricted υ-identical weak ι-associative groupoid with identity ι.*
3. *$\ominus$ is left-continuous and conjunctive over the order topology on $(T, \precsim)$, and continuous over $\{o \in T : \bigwedge \precsim o \precsim \iota\}$.*
4. *i is an embedding function of the structure $(S, \leq_S, \top, \bot, f, t, u, \wedge, \vee, \star)$ into $(T, \precsim, \bigvee, \bigwedge, \rho, \iota, \upsilon, min, max, \ominus)$, and for all $\sigma, \theta \in S$, $i(\sigma \to \theta)$ is the residuum of $i(\sigma)$ and $i(\theta)$ in $(T, \precsim, \bigvee, \bigwedge, \rho, \iota, \upsilon, min, max, \ominus)$.*

Proof. For easiness, let S be a subset of $[0, 1] \cap \mathbf{Q}$ with countable or finite elements, where 0 and 1 are the least and greatest elements, each of which corresponds to $\bot$ and $\top$, respectively. Let

$$T = \{(\sigma, o) : \sigma \in S \setminus \{0\ (= \bot)\} \text{ and } o \in \mathbf{Q} \cap (0, \sigma]\} \cup \{(0, 0)\}.$$

For $(\sigma, o), (\theta, l) \in T$, we define:

$$(\sigma, o) \precsim (\theta, l) \text{ if and only if either } \sigma <_S \theta, \text{ or } \sigma =_S \theta \text{ and } o \leq l.$$

Let ρ, ι and υ be (f, f), (t, t) and (u, u), respectively. Since $\mathbf{WA}_t\mathbf{BUL}$-algebras are **MICAL**-algebras, we just need to verify that $(T, \ominus, \precsim, \iota, \upsilon)$ is a $^{\iota}_{\upsilon}$-idempotent restricted υ-identical weak ι-associative groupoid with identity in 2 and $\ominus$ is continuous over $\{o \in T : \bigwedge \precsim o \precsim \iota\}$ in 3. For convenience, the index S in $\leq_S$ and $=_S$ is dropped unless specified otherwise.

Define, for $(\sigma, o), (\theta, l) \in T$:

$$(\sigma, o) \ominus (\theta, l) = \begin{cases} max\{(\sigma, o), (\theta, l)\} & \text{if } \sigma \star \theta = \sigma \vee \theta, \sigma \neq \theta, \text{ and} \\ & (\sigma, o) \precsim \iota \text{ or } (\theta, l) \precsim \iota; \\ min\{(\sigma, o), (\theta, l)\} & \text{if } \sigma \star \theta = \sigma \wedge \theta, \text{ and} \\ & (\sigma, o) \precsim \iota \text{ or } (\theta, l) \precsim \iota; \\ (\sigma \star \theta, \sigma \star \theta) & \text{otherwise.} \end{cases}$$

Let $T|_\iota := \{(\sigma, o) \in T : (0, 0) \precsim (\sigma, o) \precsim (t, t)\}$. For weak ι-associativity and continuity on $T|_\iota$ of $\ominus$, see Proposition 3 in [37]. For restricted υ-identity, we have to show (u-RI), i.e.,

$$\text{if } (\sigma, o) \precsim \upsilon, \text{ then } (\sigma, o) = \upsilon \ominus (\sigma, o).$$

Let $(\sigma, o) \precsim \upsilon = (u, u)$. Then $\sigma \leq u$ and so $\sigma = u \star \sigma$. Thus we have $(\sigma, o) = min\{(u, u), (\sigma, o)\} = \upsilon \ominus (\sigma, o)$. Hence the claim holds.

For $^{\iota}_{\upsilon}$-idempotence, we have to show ($^{t}_{u}$-ID), i.e.,

$$\text{if } \upsilon \precsim (\sigma, o) \precsim \iota, \text{ then } (\sigma, o) = (\sigma, o) \ominus (\sigma, o).$$

Let $\upsilon \precsim (\sigma, o) \precsim \iota$. Then $u \leq \sigma \leq t$ and so $\sigma = \sigma \star \sigma$. Thus we have $(\sigma, o) = min\{(\sigma, o), (\sigma, o)\} = (\sigma, o) \ominus (\sigma, o)$. Hence the claim holds.

The remaining conditions can be proved as in Proposition 2 in [33]. □

Proposition 4.3. *Each dense countable linearly ordered* $\boldsymbol{WA_tBUL}$*-algebra is embeddable into a standard* $\boldsymbol{WA_tBUL}$*-algebra.*

Proof. Let first $T, \mathbf{S}$, etc. be given as in Proposition 4.2. Since $(T, \precsim)$ is a dense linearly ordered countable set with least and greatest elements, $(T, \precsim)$ is order isomorphic to $([0, 1] \cap \mathbf{Q}, \leq)$. Also, let f be such an ordered isomorphism, and for $o, l \in [0, 1]$, $o \ominus' l = f(f^{-1}(o) \ominus f^{-1}(l))$ and, for all $o \in S$, $i'(o) = f(i(o))$. Then if the conditions 1, 2, 3, and 4 hold, then the structure $([0, 1] \cap \mathbf{Q},\ \leq,\ 0,\ 1,\ \rho,\ \iota,\ \upsilon,\ \ominus')$ and i' satisfy conditions 1, 2, 3, and 4 of Proposition 4.2 whenever the structure $(T, \precsim, \bigwedge, \bigvee,\ \rho,\ \iota,\ \upsilon,\ \ominus)$ and i do the same. Thus we may assume $T = [0, 1] \cap \mathbf{Q}$, $\precsim = \leq$ and $\ominus = \ominus'$.

Define: for all $o, l \in [0, 1]$,

$$o \hat{\ominus} l = sup_{v \in T: v \leq o} sup_{w \in T: w \leq l} v \ominus w.$$

Since $\mathbf{WA}_t\mathbf{BUL}$-algebras are **MICAL**-algebras, we just need to verify that $\hat{\ominus}$ is restricted υ-identical, $^{\iota}_{\upsilon}$-idempotent, weak ι-associative, and continuous over $[0, \iota]$. For weak ι-associativity and continuity over $[0, \iota]$ of $\hat{\ominus}$, see Proposition 4 in [37]. We prove restricted υ-identity and $^{\iota}_{\upsilon}$-idempotence of $\hat{\ominus}$.

For restricted υ-identity of $\hat{\ominus}$, we need to verify that: for all $o \in T$,

$$\text{if } o \precsim \upsilon, \text{ then } o = \upsilon \hat{\ominus} o.$$

Let $o \precsim \upsilon$. Then we have $o = \upsilon \ominus o$ and so

$$sup_{r \in T: r \leq o} r = sup_{s \in T: s \leq \upsilon} sup_{r \in T: r \leq o} s \ominus r.$$

Hence the claim holds.

For $^{\iota}_{\upsilon}$-idempotence of $\hat{\ominus}$, we need to verify that: for all $o \in T$,

$$\text{if } \upsilon \precsim o \precsim \iota, \text{ then } o = o \hat{\ominus} o.$$

Let $\upsilon \precsim o \precsim \iota$. Then we have $o = o \ominus o$ and so

$$sup_{r \in T: r \leq o} r = sup_{r \in T: r \leq o} sup_{r \in T: r \leq o} r \ominus r.$$

Hence the claim holds. □

Theorem 4.4. (*Standard completeness*) *Let* Γ *be a theory over* $\boldsymbol{WA_tBUL}$ *and* R *be a formula. For* $\boldsymbol{WA_tBUL}$*, the following are equivalent:*

1. $\Gamma \vdash_{\boldsymbol{WA_tBUL}} R$.

2. *For every standard* $\boldsymbol{WA_tBUL}$*-algebra and interpretation* It*, if* $It(P) \geq \iota$ *for all* $P \in \Gamma$*, then* $It(R) \geq \iota$.

Proof. $1 \to 2$: This follows from the definition.
$2 \to 1$: Let R be a formula such that $\Gamma \nvdash_{\mathbf{WA_tBUL}} R$, $\mathbf{S}$ a linearly ordered $\mathbf{WA}_t\mathbf{BUL}$-algebra, and It an interpretation in $\mathbf{S}$ such that $It(P) \geq t$ for all $P \in \Gamma$ and $It(R) < t$. Let i' be the embedding of $\mathbf{S}$ into the standard $\mathbf{WA}_t\mathbf{BUL}$-algebra as in Proposition 4.2. Then, $i' \ominus It$ is an interpretation into the standard $\mathbf{WA}_t\mathbf{BUL}$-algebra such that $i' \ominus It(P) \geq \iota$ and yet $i' \ominus It(R) < \iota$. □

The proof of standard completeness for $\mathbf{WA_tBUL}$ can be analogously applied to the other logics.

Theorem 4.5. *Let* Γ *be a theory over* $\boldsymbol{WA_tGBUL}$ *and a finite theory over each of* $\boldsymbol{WA_tIBUL}$*,* $\boldsymbol{WA_tCBUL}$ *and* $\boldsymbol{WA_tCRL}$*. For* $WA_tL \in \{\boldsymbol{WA_tIBUL}, \boldsymbol{WA_tCBUL}, \boldsymbol{WA_tGBUL}, \boldsymbol{WA_tCRL}\}$*, the following are equivalent:*

1. $\Gamma \vdash_{WA_tL} R$.

2. *For every standard $\boldsymbol{WA}_t\boldsymbol{L}$-algebra and interpretation It, if $It(P) \geq \iota$ for all $P \in \Gamma$, then $It(R) \geq \iota$.*

Proof. First, the definition of $\ominus$ for $\mathbf{WA_tCBUL}$ is the same as in the proof of Proposition 4.2. Thus, for $\mathbf{WA_tCBUL}$, we need to prove the conditions ($\mathrm{GN}^{\mathcal{S}}$), ($\mathrm{CP}^{\mathcal{S}}$) and ($\mathrm{RCAN}_U^{w\mathcal{S}}$). For a set T, Gödel negation $\neg_G$ is defined as follows: for $(\sigma, o) \in T$,

$$\neg_G(\sigma, o) = \begin{cases} \bigwedge & \text{if } (\sigma, o) \neq \bigwedge \\ \bigvee & \text{otherwise} \end{cases}$$

For ($\mathrm{GN}^{\mathcal{S}}$), we need to show: for $(\sigma, o) \in T$,

$$min\{(\sigma, o), \neg_G(\sigma, o)\} = \bigwedge.$$

We consider the case $(\sigma, o) \neq \bigwedge$. From the definition, it is immediate that $\neg_G(\sigma, o) = \bigwedge$. Hence the claim holds.

For ($\mathrm{CP}^{\mathcal{S}}$), we need to show: for $(\sigma, o), (\theta, l) \in T$,

$$\iota \precsim ((\sigma, o) \to (\theta, l)) \to (\neg_G(\theta, l) \to \neg_G(\sigma, o)).$$

We instead show that

$$(\sigma, o) \to (\theta, l) \precsim \neg_G(\theta, l) \to \neg_G(\sigma, o).$$

For $(\tau, o), (\sigma, l) \in T$, we may define $(\tau, o) \to (\sigma, l)$ as $sup\{(\theta, m) \in T : (\tau, o) \ominus (\theta, m) \leq (\sigma, l)\}$. Let $(\sigma, o) = \bigwedge$. Then since $\neg_G(\sigma, o) = \bigvee$ and so $\neg_G(\theta, l) \to \neg_G(\sigma, o) = \bigvee$, the claim holds. Let $(\sigma, o) \neq \bigwedge$. If $(\theta, l) = \bigwedge$, then $(\sigma, o) \to (\theta, l) = \bigwedge$ and so the claim holds. Otherwise, $\neg_G(\sigma, o) = \neg_G(\theta, l) = \bigwedge$ and so $\neg_G(\theta, l) \to \neg_G(\sigma, o) = \bigvee$. Hence the claim holds.

For ($\mathrm{RCAN}_U^{w\mathcal{S}}$), we need to show: for $(\sigma, o), (\theta, l) \in T$,

$$\iota \precsim min\{\neg_G\neg_G(\sigma, o), \iota\} \to (min\{(min\{(\sigma, o), \upsilon\} \to (min\{(\sigma, o), \upsilon\} \ominus (\theta, l))), \upsilon\} \to (\theta, l)).$$

We instead show that

$$min\{\neg_G\neg_G(\sigma, o), \iota\} \precsim min\{(min\{(\sigma, o), \upsilon\} \to (min\{(\sigma, o), \upsilon\} \ominus (\theta, l))), \upsilon\} \to (\theta, l).$$

Let $(\sigma, o) = \bigwedge$. Then $\neg_G\neg_G(\sigma, o) = \bigwedge$ and so $min\{\neg_G\neg_G(\sigma, o), \iota\} = \bigwedge$. Hence the claim holds. Let $(\sigma, o) \neq \bigwedge$. Then we have $\neg_G\neg_G(\sigma, o) = \bigvee$ and so $min\{\neg_G\neg_G(\sigma, o), \iota\} = \iota$. Thus we need to verify

$$\text{(A)} \quad min\{(min\{(\sigma, o), \upsilon\} \to (min\{(\sigma, o), \upsilon\} \ominus (\theta, l))), \upsilon\} \precsim (\theta, l).$$

We must consider the case $(\theta, l) \prec \upsilon$ since $min\{(min\{(\sigma, o), \upsilon\} \to (min\{(\sigma, o), \upsilon\} \ominus (\theta, l))), \upsilon\} \precsim \upsilon$. Note that $(\sigma_u \to (\sigma_u \star \theta))_u \leq \theta$. Thus one can assure that (A) holds.

For $\mathbf{WA_tGBUL}$, we need to prove ($\mathrm{RID}_U{}^{\mathcal{S}}$), i.e., $(\sigma, o) = (\sigma, o) \ominus (\sigma, o)$ for $(\sigma, o) \precsim \upsilon$. Note that $\sigma \star \sigma = \sigma$. This ensures $(\sigma, o) = min\{(\sigma, o), (\sigma, o)\} = (\sigma, o) \ominus (\sigma, o)$.

For $\mathbf{WA_tIBUL}$, first take S as in the proof of Proposition 4.2. For each $\sigma \in S$, let σ^+ denote the successor of σ in case it exists, otherwise $\sigma^+ = \sigma$. Notice that, since the negation in S, defined as $\sim \sigma := \sigma \to f$, is involutive, one has that $\sigma = (\sim \theta)^+$ if and only if $\theta = (\sim \sigma)^+$; moreover, if $\sigma < \sigma^+$, then $(\sim (\sigma^+))^+ = \sim \sigma$. Let us take V below instead of the T, which is defined in the proof of Proposition 4.2. Let $(V, \precsim)$ be the linearly ordered set defined by

$$\begin{gathered} V = \{(\sigma, \sigma) : \sigma \in S\} \cup \\ \{(\sigma, o) : \exists \sigma' \in S \text{ such that } \sigma = \sigma'^+ > \sigma', \text{ and } o \in \mathbf{Q} \cap (0, \sigma)\}, \end{gathered}$$

and $\precsim$, which is the corresponding lexicographic order as above. Certainly $(V, \precsim)$ is a subset of the ordered set $(T, \precsim)$ defined in Proposition 4.2 having the same bounds and special elements ι, ρ and υ. Note that V is closed under $\ominus$. Certainly, as in the proof of Proposition 4.2, one can verify the condition 1 (with V instead of T) in Proposition 4.2.

A new operation $\odot$ on V, based on $\ominus$, is define as follows:

$$(\sigma, o) \odot (\theta, l) = \begin{cases} max\{(\sigma + \theta - u, o + l - u), \bigwedge\} & \text{if } \sigma, \theta \leq u, \\ min\{\partial, (\sigma, o) \ominus (\theta, l)\} & \text{if } \sigma > t \text{ or } \theta > t, \text{ and either} \\ & \sigma = (\sim \theta)^+ \text{ and } \frac{c}{d} + \frac{c'}{d'} \leq 1, \\ & \text{where } o = \sigma\frac{c}{d} \text{ and } l = \theta\frac{c'}{d'}, \text{ or} \\ & \sigma < (\sim \theta)^+; \\ (\sigma, o) \ominus (\theta, l) & \text{otherwise.} \end{cases}$$

We need to verify that the conditions ($\mathrm{DNE}^{\mathcal{S}}$) and ($\mathrm{DNE}_U{}^{\mathcal{S}}$) hold. For ($\mathrm{DNE}^{\mathcal{S}}$), see Theorem 5 in [33]. Let $\neg(\sigma, o)$ be $(\sigma \to 0, o \to 0)$ for $(\sigma, o) \in T$. For ($\mathrm{DNE}_U{}^{\mathcal{S}}$), we have to show: for $(\sigma, o) \in T$,

$$\iota \precsim min\{\neg\neg(\sigma, o), \upsilon\} \to (\sigma, o).$$

We instead show that

$$\text{(B)}\quad min\{\neg\neg(\sigma, o), \upsilon\} \precsim (\sigma, o).$$

We need to consider the case $(\sigma, o) \precsim \upsilon$. Let $\neg(\tau, o)$ be $(u - \tau, u - o)$ for $(\tau, o) \in V$ such that $(\tau, o) \precsim \upsilon$. Then we have $\neg(\sigma, o) = (u - \sigma, u - o)$ and so $\neg\neg(\sigma, o) = (u - (u - \sigma), u - (u - o)) = (\sigma, o)$. Hence (B) holds.

For $\mathbf{WA_tCRL}$, first note that the set V and operation $\odot$ are the same as in $\mathbf{WA_tIBUL}$. We need to verify that the condition ($\mathrm{FP}^{\mathcal{S}}$) holds. For ($\mathrm{FP}^{\mathcal{S}}$), we have to show:

$$\iota = \rho.$$

This is immediate since $t = f$ and so $(t, t) = (f, f)$.

The proof of the remaining for standard completeness of $\mathrm{WA_tL}$ is almost the same as in $\mathbf{WA_tBUL}$. □

Let us call a linearly ordered algebra a *chain*. We note that every $\mathrm{WA_t}$BL-chain ($\mathrm{WA_t}$Ł-chain, $\mathrm{WA_t}\prod$-chain and $\mathrm{WA_t}$G-chain resp) contains a BL-chain (Ł-chain, P-chain and G-chain resp) (see Proposition 5 in [37]). Analogously, we show that every $\mathrm{WA_t}$BUL-chain ($\mathrm{WA_t}$IBUL-chain, $\mathrm{WA_t}$CBUL-chain and $\mathrm{WA_t}$GBUL-chain resp) contains a BL-chain (Ł-chain, P-chain and G-chain resp). Let S be the universe and $S_u := \{o : \bot \leq o \leq u\}$. Define $o \rightarrow_u l := (o \rightarrow l) \wedge u$.

Proposition 4.6. 1. *For every WA_tBUL-chain, $(S_u, \wedge, \vee, \star, \rightarrow_u, \bot, u)$ is a BL-chain.*

2. *For every WA_tIBUL-chain, $(S_u, \wedge, \vee, \star, \rightarrow_u, \bot, u)$ is a Ł-chain.*
3. *For every WA_tCBUL-chain, $(S_u, \wedge, \vee, \star, \rightarrow_u, \bot, u)$ is a P-chain.*
4. *For every WA_tGBUL-chain, $(S_u, \wedge, \vee, \star, \rightarrow_u, \bot, u)$ is a G-chain.*

Proof. 1. If $o, l \in S_u$, then it is clear that $o \wedge l, o \vee l, o \star l, o \rightarrow_u l \in S_u$. Moreover, $(S_u, \wedge, \vee, \bot, u)$ is a bounded lattice and $(S_u, \star, u)$ is a commutative monoid. It is routine to verify prelinearity and residuation properties. We verify divisibility property. One has to show: for all $o, l \in S_u$,

$$u = (o \rightarrow_u l) \vee (l \rightarrow_u (o \star (o \rightarrow_u l))).$$

Let $o \leq l$. Then since $u = o \rightarrow_u l$ by (residuation), we are done. Let $o > l$. Then, since $l < o \leq t$, we have $t \leq l \rightarrow (o \star (o \rightarrow l))$ by ($\mathrm{RDIV}_{\mathbf{t}}^{w\mathcal{S}}$) and so $u \leq l \rightarrow (o \star (o \rightarrow l))$. Since $l < o = o \star u$ by ($\mathrm{U\text{-}RI}^{\mathcal{S}}$) and so $o \rightarrow l < u$, we have $o \rightarrow l = o \rightarrow_u l$ and so $u \leq l \rightarrow_u (o \star (o \rightarrow_u l))$. Hence $u = l \rightarrow_u (o \star (o \rightarrow_u l))$.

2. By 1, we need to verify that the negation is involutive. Let us define $\neg_u o = o \rightarrow_u \bot$. Then, for all $o, l \in S_u$, we can define $o \star l := \neg_u(o \rightarrow_u \neg_u l)$ and $o \rightarrow_u l := \neg_u(o \star \neg_u l)$. Since $\neg_u(o \rightarrow_u \bot) = o \star u = o$, we have $\neg_u\neg_u o = o$.

3. Let $\neg_G$ be the $\neg_u$ satisfying ($\mathrm{GN}^{\mathcal{S}}$). We need to verify cancellation property, i.e., for all $o, l \in S_u$,

$$u = \neg_G\neg_G o \rightarrow_u ((o \rightarrow_u (o \star l)) \rightarrow_u l).$$

Suppose $o = \bot$. We have $\neg_G\neg_G o = \bot$ and so the claim holds. Otherwise, $\neg_G\neg_G o = u$. Then we have to verify

$$u = (o \rightarrow_u (o \star l)) \rightarrow_u l.$$

Let $o \leq o \star l$. By ($\mathrm{RCAN}_U^{w\mathcal{S}}$), we obtain $u \leq ((o \rightarrow (o \star l)) \wedge u) \rightarrow l$, and so $u = (o \rightarrow_u (o \star l)) \rightarrow_u l$. Otherwise, we get $t \leq ((o \rightarrow (o \star l)) \wedge u) \rightarrow l$ by ($\mathrm{RCAN}_U^{w\mathcal{S}}$). Since $o \star l < o = o \star u$ by ($\mathrm{U\text{-}RI}^{\mathcal{S}}$), we further obtain $o \rightarrow (o \star l) = o \rightarrow_u (o \star l)$ and so $u = (o \rightarrow_u (o \star l)) \rightarrow_u l$. Therefore the claim holds.

4. We further need to verify ($\mathrm{RID}_U{}^{\mathcal{S}}$). Since $o \star o = o \leq u$ for all $o, l \in S_u$, we have $o \star o = (o \star o) \wedge u = o \wedge u$. Hence $\star$ satisfies ($\mathrm{RID}_U{}^{\mathcal{S}}$). □

We finally verify that each standard $\mathrm{WA_t}$L-algebra is given over a restricted υ-identical ${}^{\iota}_{\upsilon}$-idempotent $[0, t]$-continuous wa_t-uninorm.

Proposition 4.7. *Let $\mathcal{S} = ([0, 1], 1, 0, \iota, \rho, \upsilon, min, max, \ominus, \rightarrow)$ be a WA_tL-algebra. The operation $\ominus$ is $[0, \iota]$-continuous, restricted υ-identical and ${}^{\iota}_{\upsilon}$-idempotent.*

Proof. Let $\mathcal{S}$ be a $WA_{\mathbf{t}}L$-algebra. For $[0, \iota]$-continuity of $\ominus$, we first notice that a wa$_t$-uninorm $\ominus$ is residuated if and only if it is conjunctive and left-continuous (see [33]). Hence we need to verify that $\ominus$ is right-continuous on $[0, \iota]$. Suppose that $o, l \in [0, \iota]$ and $(o_i)_{0 \leq i}$ is a non-increasing sequence in $[0, \iota]$ so that $o = inf_i o_i$. If $m = inf_i(o_i \ominus l)$, then $o \ominus l \leq m$. Thus, one has to verify $o \ominus l \geq m$. Notice that the monoid operation of a standard BL-algebra (P-algebra, Ł-algebra, G-algebra resp) forms a continuous t-norm. Hence $\ominus$ is continuous on $[0, u]$ by Proposition 4.6. Moreover, since $\ominus = min$ on $[\upsilon, \iota]$ by ($\mathrm{ID}_U^{\mathbf{t}}{}^{\mathcal{S}}$) and min is continuous, $\ominus$ is continuous on $[\upsilon, \iota]$. Therefore, $\ominus$ is continuous on $[0, \iota]$.

For the restricted υ-identity of the operation $\ominus$, let $o \in [0, 1]$ be such that $o \leq \upsilon$. Then $\upsilon \ominus o = o$ by (U-RI$^{\mathcal{S}}$). Hence it is restricted υ-identical.

For the $^{\iota}_{\upsilon}$-idempotence of the operation $\ominus$, let $o \in [0, 1]$ be such that $\upsilon \leq o \leq \iota$. Then $o \ominus o = o$ by ($\mathrm{ID}_U^{\mathbf{t}}{}^{\mathcal{S}}$). Hence it is $^{\iota}_{\upsilon}$-idempotent. □

Theorem 4.8. 1. (*Finite standard completeness*) *Let* Γ *be a finite theory over* WA_tL ($\in Ls$) *and* R *a formula.* $\Gamma \vdash_{WA_tL} R$ *if and only if for each standard* $[0, \iota]$*-continuous* $^{\iota}_{\upsilon}$*-idempotent restricted* υ*-identical* $\boldsymbol{WA_tL}$*-algebra and interpretation* It*, if* $It(P) \geq \iota$ *for each* $P \in \Gamma$*, then* $It(R) \geq \iota$.

2. (*Standard completeness*) *Let* Γ *be a theory over* $WA_tL \in \{\boldsymbol{WA_tBUL}, \boldsymbol{WA_tGBUL}\}$ *and* R *a formula.* $\Gamma \vdash_{WA_tL} R$ *if and only if for each standard* $[0, \iota]$*-continuous* $^{\iota}_{\upsilon}$*-idempotent restricted* υ*-identical* $\boldsymbol{WA_tL}$*-algebra and interpretation* It*, if* $It(P) \geq \iota$ *for each* $P \in \Gamma$*, then* $It(R) \geq \iota$.

Proof. 1 and 2 follow from Propositions 4.2 and 4.7, and Theorems 4.4 and 4.5. □

5 Concluding remarks

Micanorm-based logics with a weak form of associativity, denoted by basic wa$_t$-uninorm logics, were introduced as a weak associative generalization of basic uninorm logics introduced by Gabbay and Metcalfe [10] and it was verified that those logics are algebraically complete. After related uninorms with the weak associativity instead of associativity were introduced as $[0, t]$-continuous wa$_t$-uninorms, it was proved that the logics are standard complete on these wa$_t$-uninorms.

Notice that Yang [37] introduced such a generalization of continuous t-norm-based logics and left the work in this paper as an open problem. This is the answer to the problem. Furthermore, it will provide an insight for algebraic applications based on weak associative logics.

We may think of other weak associative generalizations of basic uninorm logics. For instance, t-associative and strong t-associative generalizations can be introduced as in [35]. Moreover, a weak u-associative generalization can be dealt with in place of the weak t-associative generalization addressed in this paper. (Note that the basic uninorm logics were introduced as logics based on $[0, 1)$-continuous uninorms, which have the element u, see [10].) To study such systems and related semantics is a future work. Moreover, as a problem it remains to investigate applications such as weak t-associative compensation behavior and reinforcement in wa$_t$-uninorms.

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