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Clifford's order based on non-commutative operations

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Abstract

Based on the classical works of Clifford inducing partial order from semigroups, recently, Gupta and Jayaram explored the order $\sqsubseteq_F$ from an associative operation F through *local left identity* (**LLI**). Inspired by their works, we further present an order $\sqsubseteq_F^*$ obtained from non-commutative operation F which has the *local right identity* (**LRI**) since the non-commutativity of F implies that the local left and right identity may be different for each element, which means that both orders may not coincide in the same domain. Firstly, we determine an equivalent characterization for two orders induced by non-commutative operation F. Secondly, we investigate both orders induced by semi-t-operators and deeply study their properties. Finally, we characterize both orders obtained from semi-uninorm (resp. semi-nullnorm) under the condition that semi-uninorm (resp. semi-nullnorm) is locally continuous.

Keywords: Partial order, aggregation operator, poset, semi-t-operator, semi-uninorm.

1 Introduction

In recent years, an order induced by fuzzy logic connectives has attracted increasing attention due to its potential applications in theory of algebra. Order-theoretic exploration of algebras has become a vital research topic in the field of theory of algebra, which can be dated back to the classical works of semigroups of Clifford[3], Nambooripad [29] and Mitsch[28]. Clifford's work declared that a semigroup could be expressed as an ordinal sum of subsemigroups if *Clifford's relation* forms a total order. In light of the idea of the *Clifford's order*, a series of related researches from fuzzy logic connectives are proposed, such as Karaçal and Kesicioğlu [12] introduced partial order defined by means of t-norms on bounded lattices, Kesicioğlu and Karaçal [6] gave a relation $\preceq_U$ obtained from a uninorm U on a bounded lattice, Li and Su [20] described the linearly ordered index set for an ordinal sum of semigroups yielding a uninorm. Meanwhile, Aşıcı [1] presented a relation $\preceq_N$ induced by a nullnorm N on bounded lattices. Later on, a growing number of scholars attached importance to order based on fuzzy logic connectives (see [2, 11, 13, 14, 15, 16, 17, 22, 23, 24, 31]). In a more general sense, Mesiarová-Zemánková [26] studied natural partial order based on a commutative, associative and idempotent function, she further gave a representation of non-commutative, idempotent, associative functions by pair-orders [27]. Interestingly, regarding order-theoretic exploration of algebras, Gupta and Jayaram[9] showed that an importation algebra can impose an order on the same underlying set. More concretely, they proposed a novel algebra called an importation algebra, which induces a partial order on its underlying set, by employing an implicative-type operation.

1.1 Motivation for this work

It is noteworthy that Gupta and Jayaram [10, 11] and Nanavati and Jayaram [30] gave the order $\sqsubseteq_F$ induced by a binary operation $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ as follows.

$$x \sqsubseteq_F y \;\Leftrightarrow\; F(\ell, y) = x \text{ for some } \ell \in \mathbb{P}.$$

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They characterized the order $\sqsubseteq_F$ through the local left identity property (**LLI**). However, inspired by their work, we can find a class of binary operations $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ that have a local right neutral element for F and may not have a local left neutral element for every element $x \in \mathbb{P}$. For instance, one can consider $F : [0,1] \times [0,1] \to [0,1]$ as follows:

$$F(x, y) = \begin{cases} x & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Clearly, it follows from the definition of F that $x \in (0,1]$ has not local left neutral element. Indeed, assume that there exists an $\ell_x \in L$ such that $F(\ell_x, x) = x$, then there are two cases:

(i) If $\ell_x \neq x$, then $F(\ell_x, x) = \ell_x$, a contradiction.

(ii) If $\ell_x = x$, then $F(\ell_x, x) = 0$, a contradiction.

Hence, F does not yield (**LLI**). It is interesting to note that each $x \in [0,1]$ has local right neutral element for F, i.e., $F(x, \ell) = x$ for $x \neq \ell$. Accordingly, it is very desirable to explore a new order $\sqsubseteq_F^*$ induced by a non-commutative operation F with local right neutral elements (**LRI**) on $\mathbb{P}$. For $x, y \in \mathbb{P}$, a relation $\sqsubseteq_F^*$ is defined as:

$$x \sqsubseteq_F^* y \iff F(y, \ell) = x \text{ for some } \ell \in \mathbb{P}.$$

Observe that, non-commutative fuzzy logic connectives play an indispensable role in mathematics and science. Drygaś [5] presented the semi-t-operators by omitting commutative law from t-operators. Recently, semi-t-operators have been intensively investigated in all aspects, including the distributivity of semi-t-operators [32], migrativity property and the structure for semi-t-operators on bounded lattices in [7, 33]. Semi-t-operators are significant not only from a theoretical point of view but also for applications, specially, when the inputs do not have the same weight in application fields since the commutativity of semi-t-operator is lost. Moreover, semi-t-operators and semi-nullnorms be able to model pseudo-operations in pseudo-analysis. Hopefully, orders from these non-commutative operations can be used to develop pseudo-analysis.

1.2 Contributions of this work

Inasmuch as the non-commutativity of F means that $\sqsubseteq_F$ may not coincide with $\sqsubseteq_F^*$ in the same domain. Orders induced by non-commutative operations are not known yet in the literature, our strategy is to fill this gap. The main contributions of this work are as follows.

(i) We define a new order $\sqsubseteq_F^*$ for non-commutative operations F and discuss the relationship between the orders $\sqsubseteq_F$ and $\sqsubseteq_F^*$ in $\mathbb{P}$. Furthermore, we give a sufficient and necessary condition that both orders are equivalent when F is locally commutative.

(ii) For semi-t-operator $\mathbb{F}$ on $[0,1]$ with $\mathbb{F}(0,1) = a$ and $\mathbb{F}(1,0) = b$, whether $\sqsubseteq_{\mathbb{F}}^*$ (resp. $\sqsubseteq_{\mathbb{F}}$) is a partial order or not depends heavily on the order relation between a and b. In this sense, we completely characterize the orders $\sqsubseteq_{\mathbb{F}}$ and $\sqsubseteq_{\mathbb{F}}^*$ on $[0,1]$.

(iii) We obtain the orders from semi-uninorms and semi-nullnorms under condition that semi-uninorms and semi-nullnorms are locally continuous.

1.3 Outline of this submission

In Section 2 we provide the necessary background material. In Section 3, we present some examples to compare the orders $\sqsubseteq_F$ and $\sqsubseteq_F^*$ and deeply study its properties. In Section 4, we investigate the orders $\sqsubseteq_{\mathbb{F}}$ and $\sqsubseteq_{\mathbb{F}}^*$ obtained from semi-t-operators $\mathbb{F}$ on $[0,1]$ with $\mathbb{F}(0,1) = a$ and $\mathbb{F}(1,0) = b$. More precisely: it follows that $\sqsubseteq_{\mathbb{F}}^*$ (resp. $\sqsubseteq_{\mathbb{F}}$) is a partial order when $a \leqslant b$ (resp. $a \geqslant b$). In Section 5, we characterize the orders $\sqsubseteq_{\mathbb{U}}^*$ and $\sqsubseteq_{\mathbb{U}}$ from semi-uninorm (resp. semi-nullnorm) $\mathbb{U}$ under condition that $\mathbb{U}$ is locally continuous. A conclusion is drawn in Section 6.

2 Preliminaries

This section contains a short overview of the basic notions that are essential for the presented research. Throughout the paper we use 'iff' for 'if and only if'.

In what follows, we always assume that the set $\mathbb{P} \neq \emptyset$. A *poset* is a structure $(\mathbb{P}, \leqslant)$ where $\leqslant$ is a partial order (reflexive, antisymmetric and transitive relation) on $\mathbb{P}$.

A *lattice* is a poset $(L, \leqslant)$ in which every two elements subset $\{x, y\}$ has the greatest lower bound, meet, denoted by $x \wedge y$, and the least upper bound, join, denoted by $x \vee y$ in [8]. Moreover, $x \parallel y$ denotes that x is *incomparable* with y, and $x \nparallel y$ denotes that $x \geqslant y$ or $x \leqslant y$.

Definition 2.1. [19] *Let $(L, \leqslant, 0, 1)$ be a bounded lattice and $n \in L$ be fixed. A mapping $F : L^n \to L$ is called an (n-ary) aggregation operator on L whenever it is increasing, i.e., if $F(\boldsymbol{x}) \leqslant F(\boldsymbol{y})$ whenever $\boldsymbol{x} \leqslant \boldsymbol{y}$ ($x_1 \leqslant y_1, \cdots, x_n \leqslant y_n$) and it satisfies boundary conditions $F(0, 0, \cdots, 0) = 0$ and $F(1, 1, \cdots, 1) = 1$.*

Definition 2.2. *Let F be a binary aggregation operator on L. F is said to be*

- ***semi-t-norm** (resp. **semi-t-conorm**), if it has the neutral element 1 (resp. 0) (see [18]).*
- ***t-norm** (resp. **t-conorm**), if it is a semi-t-norm (resp. semi-t-conorm) which is commutative and associative (see [18]).*
- ***semi-t-operator**, if it is associative such that the operations F_0, F_1, F^0, F^1 all are continuous, where $F_0(x) = F(0, x)$, $F_1(x) = F(1, x)$, $F^0(x) = F(x, 0)$, $F^1(x) = F(x, 1)$ (see [5]).*
- ***left-uninorm** (resp. **right-uninorm**), if it has the left (resp. right) neutral element e_L (resp. e_R), i.e., $F(e_L, x) = x$ (resp. $F(x, e_R) = x$) for all $x \in L$ (see [34]).*
- ***pseudo-uninorm**, if it is a left-uninorm and right-uninorm with the neutral element $e = e_L = e_R$ (see [34]).*
- ***semi-nullnorm**, if it has a zero element $z \in L$ such that $F(0, x) = F(x, 0) = x$ for all $x \leqslant z$ and $F(1, x) = F(x, 1) = x$ for all $x \geqslant z$ (see [5]).*

For convenience, there are slight differences between some terminologies in the following lemma and those in the original articles [10, 30].

Definition 2.3. [10, 30] *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$. F is said to satisfy the*

(i) **Local Left Identity** (**LLI**), *if for each $x \in \mathbb{P}$, there is an $\ell_x \in \mathbb{P}$, depending on each x, such that $F(\ell_x, x) = x$.*

(ii) **Right Quasi-Projection** (**RQP**), *if for any $x, y, z \in \mathbb{P}$, $F(x, F(y, z)) = z$ implies $F(y, z) = z$.*

(iii) **Generalized Right Quasi-Projection** (**GRQP**), *if for any $x, y, z, w \in \mathbb{P}$, $F(x, F(y, z)) = w$ implies that there exists an $\ell \in \mathbb{P}$ fulfilling $F(\ell, z) = w$.*

Lemma 2.4. [10, 30] *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$. A relation $\sqsubseteq_F$ is defined as*

$$x \sqsubseteq_F y \text{ for } x, y \in \mathbb{P}, \text{ if } F(\ell, y) = x \text{ for some } \ell \in \mathbb{P}.$$

Then $(\mathbb{P}, \sqsubseteq_F)$ is a poset iff F satisfies (**LLI**), (**RQP**) *and* (**GRQP**).

Lemma 2.5. [10] *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ be associative. Then $(\mathbb{P}, \sqsubseteq_F)$ is a poset iff F fulfills* (**LLI**) *and* (**RQP**).

3 Order based on non-commutative operations

Inspired by the works on obtaining order from Gupta and Jayaram[10] and Nanavati and Jayaram[30], we can introduce the following Definitions 3.1 and 3.2, as well as Theorems 3.3 and 3.4.

Definition 3.1. *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$. F is said to satisfy the*

(i) **Local Right Identity** (**LRI**), *if for each $x \in \mathbb{P}$, there is an $\ell_x \in \mathbb{P}$, depending on each x, such that $F(x, \ell_x) = x$.*

(ii) **Left Quasi-Projection** (**LQP**), *if for any $x, y, z \in \mathbb{P}$, $F(F(x, y), z) = x$ implies $F(x, y) = x$.*

(iii) **Generalized Left Quasi-Projection** (**GLQP**), *if for any $x, y, z, w \in \mathbb{P}$, $F(F(x, y), z) = w$ implies that there exists an $\ell \in \mathbb{P}$ satisfying $F(x, \ell) = w$.*

Definition 3.2. *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$. A relation $\sqsubseteq_F^*$ is defined as*

$$x \sqsubseteq_F^* y \text{ for } x, y \in \mathbb{P}, \text{ if } F(y, \ell) = x \text{ for some } \ell \in \mathbb{P}.$$

Theorem 3.3. *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$. Then $(\mathbb{P}, \sqsubseteq_F^*)$ is a poset iff F satisfies* (**LRI**), (**LQP**) *and* (**GLQP**).

Theorem 3.4. *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ be associative. Then $(\mathbb{P}, \sqsubseteq_F^*)$ is a poset iff F satisfies* (**LRI**) *and* (**LQP**).

Proposition 3.5. *Let $F : [\alpha, \beta] \times [\alpha, \beta] \to [\alpha, \beta]$ for $[\alpha, \beta] \subseteq [0, 1]$. If $F(p, q) \geqslant p$ for any $p, q \in [\alpha, \beta]$ or $F(p, q) \leqslant p$ for any $p, q \in [\alpha, \beta]$, then F fulfills* (**LQP**) *on $[\alpha, \beta]^2$.*

Proof. Assume that $F(F(x, y), z) = x$ but $F(x, y) = r \neq x$. Then it follows from $F(p, q) \geqslant p$ for any $p, q \in [\alpha, \beta]$ that $r = F(x, y) > x$. Thus, one has that $F(F(x, y), z) = F(r, z) \geqslant r > x$, contrary to the fact that $F(F(x, y), z) = x$. Consequently, $F(F(x, y), z) = x$ implies $F(x, y) = x$, i.e., F fulfills (**LQP**). Analogously, one can obtain the proof for $F(p, q) \leqslant p$ for any $p, q \in [\alpha, \beta]$. □

It is natural to ask whether F can induce both orders $\sqsubseteq_F$ and $\sqsubseteq_F^*$ in $\mathbb{P}$? In other words, do both $\sqsubseteq_F$ and $\sqsubseteq_F^*$ exist simultaneously in $\mathbb{P}$? In fact, the answer is positive for some classical operations F, for example, t-norm[18], t-conorm[18] and uninorm[35] as well as nullnorm[25] etc.

Theorem 3.6. *Let $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$. If F is commutative, then $\sqsubseteq_F$ and $\sqsubseteq_F^*$ coincide on $\mathbb{P}$.*

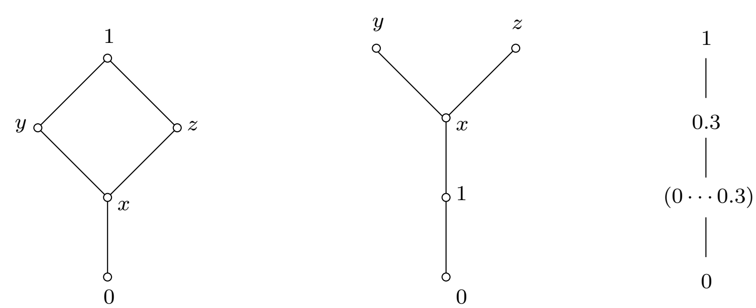
Proof. Straightforward. □

The following example illustrates the case when $\sqsubseteq_F^*$ is a partial order but $\sqsubseteq_F$ is not a partial order in $\mathbb{P}$ as well as the opposite case.

Example 3.7. (i) *Consider the lattice $(L = \{0, x, y, z, 1\}, \leqslant)$ visualized in Fig. 1 (a). A right uninorm U_1 on L with two right neutral elements y, z as shown in Table 1.*

Table 1: Table 1. A right uninorm U_1 on L

U_1	0	x	y	z	1
0	0	0	0	0	0
x	0	x	x	x	1
y	0	x	y	y	1
z	0	x	z	z	1
1	0	1	1	1	1



(*a*) *The lattice L.* (*b*) *The order $\sqsubseteq_{U_1}^*$ on L* (*c*) *The order $\sqsubseteq_{\mathbb{C}}$ on* $[0, 1]$

Fig. 1. Graphical representation of the orders in Example 3.7.

*Apparently, U_1 does not satisfy **(RQP)** since $U_1(z, U_1(y, z)) = z$ but $U_1(y, z) = y \neq z$ from Definition 2.3. Thus, $\sqsubseteq_{U_1}$ is not a partial order on L from Lemma 2.4. However, it follows by Theorem 3.3 that $(L, \sqsubseteq_{U_1}^*)$ is a poset since U_1 satisfies* (**LRI**), (**LQP**) *and* (**GLQP**). *Especially, $(L, \sqsubseteq_{U_1}^*)$ is a meet semi-lattice but not a lattice (see Fig. 1 (b)).*

Recall that a semi-copula $\mathbb{C}$ is a binary operation on L which is increasing in each variable and $\mathbb{C}(x, 1) = \mathbb{C}(1, x) = x$ for all $x \in L$

(ii) *Consider a semi-copula* $\mathbb{C}$ *as follows:*

$$\mathbb{C}(x,y)=\begin{cases}0 & \text{if } (x,y)\in[0,0.3]\times[0,1),\\ \min(x,y) & \text{otherwise.}\end{cases}$$

It is trivial to check that $\mathbb{C}$ *satisfies* (**LLI**), (**RQP**) *and* (**GRQP**), *that is,* $\sqsubseteq_{\mathbb{C}}$ *is a partial order (see Fig. 1 (c), where* $(0\ldots 0.3)$ *indicates that these elements between* 0 *and* 0.3 *are not comparable) but the relation* $\sqsubseteq_{\mathbb{C}}^{*}$ *fails to be an order on* $[0,1]$ *since* $\mathbb{C}$ *does not satisfy* (**LRI**).

Even though both orders exist, one has that $\sqsubseteq_F \neq \sqsubseteq_F^*$ for a non-commutative operation F in general as is demonstrated by Remark 3.1.

Remark 3.1. *Consider the lattice* $(L=\{0,x,y,z,e,1\},\leqslant)$ *visualized in Fig. 2. Define a binary operation* U_2 *on* L *described in Table 2.*

Table 2: Table 2. Binary operation U_2 on L.

U_2	0	x	y	z	e	1
0	0	0	0	0	0	0
x	0	0	x	0	x	x
y	0	0	y	0	y	y
z	0	x	x	z	z	z
e	0	x	y	z	e	1
1	0	x	y	z	1	1

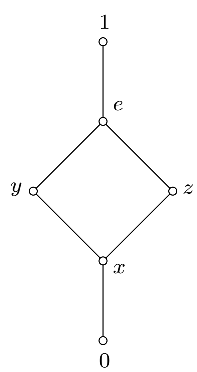
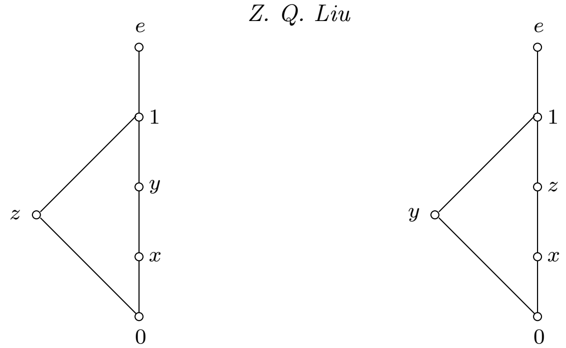


Fig. 2. The lattice L.

It is obvious that U_2 *is a pseudo-uninorm with neutral element* e *(see* [34]*). Note that the order* $\sqsubseteq_{U_2}$ *does not coincide with the order* $\sqsubseteq_{U_2}^*$ *on* L *by Lemma 2.4 and Theorem 3.3 since* $x \nparallel y$ *with respect to* $\sqsubseteq_{U_2}$ *but* $x \parallel y$ *with respect to* $\sqsubseteq_{U_2}^*$ *(see Fig. 3 (a) and (b)).*



(a) *The order* $\sqsubseteq_{U_2}$ *on* L (b) *The order* $\sqsubseteq^*_{U_2}$ *on* L

Fig. 3. Graphical representation of the orders in Remark 3.1.

Interestingly, we can find a non-commutative operation F on $\mathbb{P}$ such that $\sqsubseteq_F$ coincides with $\sqsubseteq^*_F$ on $\mathbb{P}$.

Example 3.8. *Let* $\mathbb{P} = \{0, x, y, z, 1\}$ *be a linearly ordered set with* $0 < x < y < z < 1$*. Define a binary operation* F *on* $\mathbb{P}$ *given in Table 3. Obviously,* F *is non-commutative. Furthermore, it is easy to check that* $\sqsubseteq_F$ *and* $\sqsubseteq^*_F$ *coincide on* $\mathbb{P}$*. One observes that* $\sqsubseteq_F$ *and* $\sqsubseteq^*_F$ *coincide with the basic order on* P*.*

Table 3: Table 3. The operation F in Example 3.8.

F	0	x	y	z	1
0	0	0	0	0	0
x	0	x	x	x	x
y	0	0	x	y	y
z	0	x	y	z	z
1	0	x	y	z	1

(a) *The order* $\sqsubseteq_F$ (b) *The order* $\sqsubseteq^*_F$

Fig. 4. Graphical representation of the orders in Example 3.8.

Hence, from Example 3.8, one can see that the commutativity of F is quite strong in Theorem 3.6. In what follows, we relax the commutativity of F such that $\sqsubseteq_F$ and $\sqsubseteq^*_F$ are equivalent on $\mathbb{P}$. To do this, the following definition is useful.

Definition 3.9. *A binary operation* $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ *is said to satisfy*

- **Localised Commutativity** (**LC**)*, if for each* $x, y \in \mathbb{P}$*, there exists* $z \in \mathbb{P}$ *such that* $F(y, x) = F(x, z)$*.*

Example 3.10. *Let* $P = \{a, b, c\}$*. Define a binary operation* $F : P \times P \to P$ *shown in Table 4. Clearly,* F *is not commutative. One can observe that* F *fulfills* (**LC**) *on* P*.*

Theorem 3.11. *Let* $F : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$*. If* F *satisfies* (**LC**)*, then* $\sqsubseteq_F$ *is a partial order iff* $\sqsubseteq^*_F$ *is a partial order. Moreover, the orders* $\sqsubseteq_F$ *and* $\sqsubseteq^*_F$ *coincide on* $\mathbb{P}$*.*

Proof. Assume that $\sqsubseteq_F$ is a partial order, we shall prove that $\sqsubseteq^*_F$ is a partial order and vice versa.

Table 4: Table 4.

F	a	b	c
a	a	b	c
b	c	b	b
c	b	c	c

- **Reflexivity**. For $x \in \mathbb{P}$ and the reflexivity of $\sqsubseteq_F$, there exists an $\ell_1 \in \mathbb{P}$ satisfying $F(\ell_1, x) = x$. Thus, it follows by Definition 3.9 that there exists an $\ell_2 \in \mathbb{P}$ such that $F(x, \ell_2) = F(\ell_1, x) = x$, i.e., $x \sqsubseteq_F^* x$ for $x \in \mathbb{P}$.

- **Antisymmetry**. If $x \sqsubseteq_F^* y$ and $y \sqsubseteq_F^* x$ for $x, y \in \mathbb{P}$, then there are $\ell_1, \ell_2 \in \mathbb{P}$ such that $F(y, \ell_1) = x$ and $F(x, \ell_2) = y$. It follows by hypothesis that there exist two elements $\ell_3, \ell_4 \in \mathbb{P}$ such that $F(\ell_3, y) = x$ and $F(\ell_4, x) = y$, which means $x \sqsubseteq_F y$ and $y \sqsubseteq_F x$. Thus, it holds that $x = y$ from the antisymmetry of $\sqsubseteq_F$.

- **Transitivity**. If $x \sqsubseteq_F^* y$ and $y \sqsubseteq_F^* z$ for $x, y, z \in \mathbb{P}$, then there exist two elements $\ell_1, \ell_2 \in \mathbb{P}$ satisfying $F(y, \ell_1) = x$ and $F(z, \ell_2) = y$. Furthermore, from the hypothesis, we have $F(\ell_3, y) = F(y, \ell_1) = x$ and $F(\ell_4, z) = F(z, \ell_2) = y$ for some $\ell_3, \ell_4 \in \mathbb{P}$, which implies $x \sqsubseteq_F y$ and $y \sqsubseteq_F z$. Thus, it holds that $x \sqsubseteq_F z$ by the transitivity of $\sqsubseteq_F$, i.e., $x = F(\ell_5, z) = F(z, \ell_6)$ for $\ell_5, \ell_6 \in \mathbb{P}$. Consequently, $x \sqsubseteq_F^* z$.

Consequently, one concludes that if $\sqsubseteq_F$ is a partial order, then $\sqsubseteq_F^*$ is a partial order.

Conversely, assume that $\sqsubseteq_F^*$ is a partial order, one can exchange the roles of $\sqsubseteq_F$ and $\sqsubseteq_F^*$ in the above proof, it is easy to check that if $\sqsubseteq_F^*$ is a partial order, then $\sqsubseteq_F$ is a partial order.

To sum up, one obtains that $\sqsubseteq_F$ coincides with $\sqsubseteq_F^*$ on $\mathbb{P}$. □

4 Order from semi-t-operators

In particular, with regard to a semi-t-operator $\mathbb{F}$, in some applications, the commutativity of $\mathbb{F}$ is not desired. In this section, we continue the investigation of both orders $\sqsubseteq_{\mathbb{F}}$ and $\sqsubseteq_{\mathbb{F}}^*$ based on semi-t-operator F.

Lemma 4.1. [5] *Let $\mathbb{F} : [0,1]^2 \to [0,1]$ with $\mathbb{F}(0,1) = a$ and $\mathbb{F}(1,0) = b$. Opertion $\mathbb{F}$ is a semi-t-operator iff an associative semi-t-norm $T_{\mathbb{F}}$ and an associative semi-t-conorm $S_{\mathbb{F}}$ exist such that*

(a)

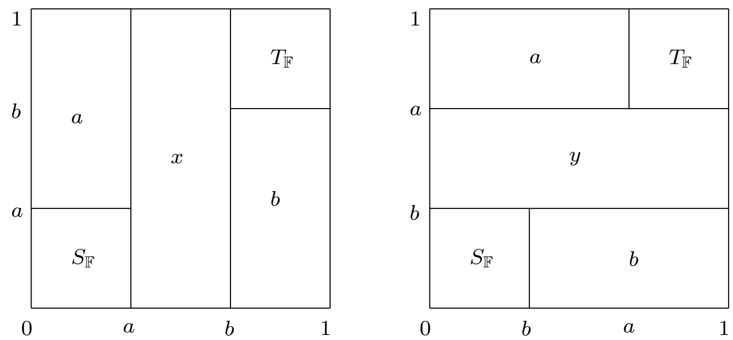
$$\mathbb{F}(x,y) = \begin{cases} aS_{\mathbb{F}}\left(\frac{x}{a}, \frac{y}{a}\right) & \text{if } (x,y) \in [0,a]^2, \\ b + (1-b)T_{\mathbb{F}}\left(\frac{x-b}{1-b}, \frac{y-b}{1-b}\right) & \text{if } (x,y) \in [b,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1], \\ b & \text{if } (x,y) \in [b,1] \times [0,b], \\ x & \text{otherwise} \end{cases}$$

for $a \leqslant b$ (see Fig. 5 (i)).

(b)

$$\mathbb{F}(x,y) = \begin{cases} bS_{\mathbb{F}}\left(\frac{x}{b}, \frac{y}{b}\right) & \text{if } (x,y) \in [0,b]^2, \\ a + (1-a)T_{\mathbb{F}}\left(\frac{x-a}{1-a}, \frac{y-a}{1-a}\right) & \text{if } (x,y) \in [a,1]^2, \\ a & \text{if } (x,y) \in [0,a] \times [a,1], \\ b & \text{if } (x,y) \in [b,1] \times [0,b], \\ y & \text{otherwise} \end{cases}$$

for $b \leqslant a$ (see Fig. 5 (ii)).



$(i)\ a \leqslant b$ $(ii)\ b \leqslant a$

Fig. 5. Structures of $\mathbb{F}$ in Lemma 4.1.

Theorem 4.2. *Let $\mathbb{F}$ be a semi-t-operator on $[0,1]$ with $\mathbb{F}(0,1)=a$ and $\mathbb{F}(1,0)=b$.*

(a) *If $a \leqslant b$, then $\sqsubseteq^*_{\mathbb{F}}$ is a partial order given in Definition 3.2, that is, $([0,1],\sqsubseteq^*_{\mathbb{F}})$ is a poset.*

(b) *If $a \geqslant b$, then $\sqsubseteq_{\mathbb{F}}$ is a partial order given in Lemma 2.4, that is, $([0,1],\sqsubseteq_{\mathbb{F}})$ is a poset.*

Proof. We only prove the statement (a), the proof of the statement (b) is dually given. It suffices from the associativity of semi-t-operator $\mathbb{F}$ to prove that $\sqsubseteq^*_{\mathbb{F}}$ is a partial order if $\mathbb{F}$ satisfies (**LRI**) and (**LQP**) from Theorem 3.4.

Firstly, since $\mathbb{F}$ is a semi-t-operator with $a \leqslant b$, then $\mathbb{F}(x,0)=x$ for any $x \in [0,a]$, $\mathbb{F}(x,y)=x$ for any $x \in [a,b]$ and $\mathbb{F}(x,1)=x$ for any $x \in [b,1]$. Thus, each element $x \in [0,1]$ has local right neutral element for $\mathbb{F}$, i.e., (**LRI**) is valid by Definition 3.1 (i).

Now, in order to prove $\mathbb{F}$ fulfilling (**LQP**), we distinguish five cases.

(i) If $(x,y) \in [0,a]^2$, then since $[0,a]^2$ is closed under $\mathbb{F}$ there is $F(x,y) \in [0,a]$.

- If $z \in [0,a]$, due to $\mathbb{F}(\alpha,\beta)=aS_{\mathbb{F}}\left(\frac{\alpha}{a},\frac{\beta}{a}\right) \geqslant a\max\left(\frac{\alpha}{a},\frac{\beta}{a}\right) \geqslant \alpha$ for any $\alpha,\beta \in [0,a]$. It follows by Proposition 3.5 that $\mathbb{F}$ satisfies (**LQP**).
- If $z \in [a,1]$ and $\mathbb{F}(\mathbb{F}(x,y),z)=x$, then $x=\mathbb{F}(\mathbb{F}(x,y),z)=a$ since $\mathbb{F}(x,y) \in [0,a]$. Thus, $\mathbb{F}(x,y)=\mathbb{F}(a,y)=a=x$.

(ii) If $(x,y) \in [0,a] \times [a,1]$, then $\mathbb{F}(x,y)=a$ by Lemma 4.1. Thus, for any $z \in [0,1]$ and $\mathbb{F}(\mathbb{F}(x,y),z)=x$, we have $x=\mathbb{F}(\mathbb{F}(x,y),z)=\mathbb{F}(a,z)=a$. Thus, $\mathbb{F}(x,y)=\mathbb{F}(a,y)=a=x$.

(iii) If $(x,y) \in [a,b] \times [0,1]$, then $\mathbb{F}(x,y)=x$ from Lemma 4.1. Thus, it is trivial that $\mathbb{F}(\mathbb{F}(x,y),z)=x$ implies $\mathbb{F}(x,y)=x$ for any $z \in [0,1]$.

(iv) If $(x,y) \in [b,1] \times [0,b]$, then $\mathbb{F}(x,y)=b$.

- If $z \in [0,b]$ and $\mathbb{F}(\mathbb{F}(x,y),z)=x$, then $x=\mathbb{F}(\mathbb{F}(x,y),z)=\mathbb{F}(b,z)=b$. Thus, $\mathbb{F}(x,y)=\mathbb{F}(b,y)=b=x$.
- If $z \in [b,1]$ and $\mathbb{F}(\mathbb{F}(x,y),z)=x$, then $x=\mathbb{F}(\mathbb{F}(x,y),z)=\mathbb{F}(b,z)=b+(1-b)T_{\mathbb{F}}\left(\frac{b-b}{1-b},\frac{z-b}{1-b}\right)=b$. Thus, $\mathbb{F}(x,y)=\mathbb{F}(b,y)=b=x$.

(v) If $(x,y) \in [b,1]^2$, then since $[b,1]^2$ is closed under $\mathbb{F}$ there is $F(x,y) \in [b,1]$.

- If $z \in [0,b]$ and $\mathbb{F}(\mathbb{F}(x,y),z)=x$, then $x=\mathbb{F}(\mathbb{F}(x,y),z)=b$ since $\mathbb{F}(x,y) \in [b,1]$. Furthermore, $\mathbb{F}(x,y)=\mathbb{F}(b,y)=b+(1-b)T_{\mathbb{F}}\left(\frac{b-b}{1-b},\frac{y-b}{1-b}\right)=b=x$.
- If $z \in [b,1]$, due to $\mathbb{F}(\alpha,\beta)=b+(1-b)T_{\mathbb{F}}\left(\frac{\alpha-b}{1-b},\frac{\beta-b}{1-b}\right) \leqslant b+(1-b)\min\left(\frac{\alpha-b}{1-b},\frac{\beta-b}{1-b}\right) \leqslant \alpha$ for any $\alpha,\beta \in [b,1]$. It follows by Proposition 3.5 that $\mathbb{F}$ satisfies (**LQP**).

To sum up, $\mathbb{F}$ fulfills (**LQP**) on $[0,1]$.

Therefore, (**LRI**) and (**LQP**) imply that $([0,1],\sqsubseteq^*_{\mathbb{F}})$ is a poset by Theorem 3.4. □

Remark 4.1. *Unfortunately, the relation $\sqsubseteq^*_{\mathbb{F}}$ (resp. $\sqsubseteq_{\mathbb{F}}$) is not a partial order again for semi-t-operator $\mathbb{F}$ if $a > b$ (resp. $a < b$) as is demonstrated by the following statements.*

(i) *If $a > b$, then the relation $\sqsubseteq_{\mathbb{F}}^*$ is not a partial order on $[0,1]$. As a matter of fact, if $\mathbb{F}(\mathbb{F}(x,y),z) = x$ for $x, y \in [b,a]$ with $x \neq y$ and $z \in [0,b]$, then $\mathbb{F}(x,y) = y$ from Lemma 4.1 (b). Hence, $x = \mathbb{F}(\mathbb{F}(x,y),z) = \mathbb{F}(y,z) = b$, i.e., $x = b$. However, $\mathbb{F}(x,y) = \mathbb{F}(b,y) = y \neq x$, that is, $\mathbb{F}$ does not fulfill (**LQP**). Consequently, $\sqsubseteq_{\mathbb{F}}^*$ does not give rise to a partial order by Theorem 3.4.*

(ii) *If $a < b$, then the relation $\sqsubseteq_{\mathbb{F}}$ is not a partial order on $[0,1]$. Indeed, if $\mathbb{F}(x,\mathbb{F}(y,z)) = z$ for $x \in [b,1]$ and $y, z \in [a,b]$ with $y \neq z$, then it holds that $\mathbb{F}(y,z) = y$ by Lemma 4.1 (a). Thus, $\mathbb{F}(x,\mathbb{F}(y,z)) = \mathbb{F}(x,y) = b$, i.e., $z = b$. But $\mathbb{F}(y,z) = \mathbb{F}(y,b) = y \neq z$, i.e., $\mathbb{F}$ does not fulfill (**RQP**). Therefore, $\sqsubseteq_{\mathbb{F}}$ is not a partial order by Lemma 2.5.*

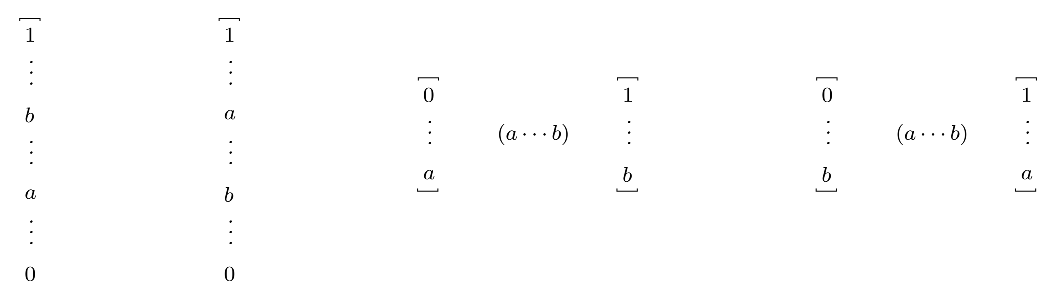
Definition 4.3. [7] *Let $\mathbb{F}$ be a semi-t-operator on $[0,1]$ with $\mathbb{F}(0,1) = a$ and $\mathbb{F}(1,0) = b$. A relation $\preceq_{\mathbb{F}}$ is defined as*

$$x \preceq_{\mathbb{F}} y :\Leftrightarrow \begin{cases} \text{if} & (x,y) \in [0, a\wedge b]^2 \text{ and there exists } k \in [0, a \wedge b] \text{ such that } \mathbb{F}(x,k) = y \text{ or,} \\ \text{if} & (x,y) \in [a \vee b, 1]^2 \text{ and there exists } \ell \in [a \vee b, 1] \text{ such that } \mathbb{F}(y,\ell) = x \text{ or,} \\ \text{if} & (x,y) \in [0,1]^2 \setminus \{[0, a\wedge b]^2 \cup [a \vee b, 1]^2\} \text{ and } x \leqslant y. \end{cases}$$

Fang and Hu [7] verified that $\preceq_{\mathbb{F}}$ is a partial order on $[0,1]$. Nevertheless, the orders $\preceq_{\mathbb{F}}$ and $\sqsubseteq_{\mathbb{F}}^*$ (resp. $\sqsubseteq_{\mathbb{F}}$) do not coincide in general.

Remark 4.2. *Let $\mathbb{F}$ be a semi-t-operator on $[0,1]$.*

- *If $a \leqslant b$ and $x, y \in [0,a]$, then $x \preceq_{\mathbb{F}} y$ iff $y \sqsubseteq_{\mathbb{F}}^* x$; if $x, y \in [b,1]$, then $x \preceq_{\mathbb{F}} y$ iff $x \sqsubseteq_{\mathbb{F}}^* y$ (see Figs. 6 (i) and (iii)).*
- *If $a \geqslant b$ and $x, y \in [0,b]$, then $x \preceq_{\mathbb{F}} y$ iff $x \sqsubseteq_{\mathbb{F}} y$; if $x, y \in [a,1]$, then $x \preceq_{\mathbb{F}} y$ iff $y \sqsubseteq_{\mathbb{F}} x$ (see Figs. 6 (ii) and (iv)).*
- *If x and y are strictly between a and b, then $x \parallel y$ with respect to $\sqsubseteq_{\mathbb{F}}^*$ (resp. $\sqsubseteq_{\mathbb{F}}$) (see Figs. 6 (iii) and (iv)).*
- *If $a = b$, then semi-t-operator $\mathbb{F}$ degenerates into a nullnorm N. In particular, $\sqsubseteq_{\mathbb{F}}^*$ coincides with $\sqsubseteq_{\mathbb{F}}$ by Theorem 3.6 (see Fig. 7).*



(i) $\preceq_{\mathbb{F}}$ $(a \leqslant b)$ *(ii)* $\preceq_{\mathbb{F}}$ $(a \geqslant b)$ *(iii) The order* $\sqsubseteq_{\mathbb{F}}^*$ *on* $[0,1]$ *(iv) The order* $\sqsubseteq_{\mathbb{F}}$ *on* $[0,1]$

Fig. 6. Hasse diagram of the orders $\preceq_{\mathbb{F}}$, $\sqsubseteq_{\mathbb{F}}^*$ and $\sqsubseteq_{\mathbb{F}}$.

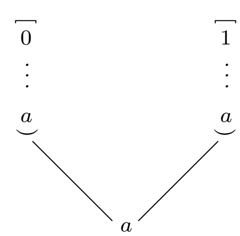


Fig. 7. The orders $\sqsubseteq_{\mathbb{F}}^*$ and $\sqsubseteq_{\mathbb{F}}$ on $[0,1]$ with $a = b$.

5 Order from semi-uninorms and semi-nullnorms

In this section, we continue the same topic by focusing on semi-uninorms and semi-nullnorms.

5.1 Order induced by semi-uninorms

In this subsection, we begin by recalling the definition of semi-uninorm $\mathbb{U}$. Following this we give a condition such that $\sqsubseteq_{\mathbb{U}}^*$ (resp. $\sqsubseteq_{\mathbb{U}}$) induced by semi-uninorm $\mathbb{U}$ is a partial order on $[0,1]$.

Definition 5.1. [21] *An operation $\mathbb{U}:[0,1]\times[0,1]\to[0,1]$ is called a semi-uninorm if it is increasing in each variable and it has a neutral element $e\in[0,1]$, i.e., $\mathbb{U}(e,x)=\mathbb{U}(x,e)=x$ for all $x\in[0,1]$.*

Lemma 5.2. [21] *An operation $\mathbb{U}:[0,1]\times[0,1]\to[0,1]$ is a semi-uninorm with neutral element $e\in(0,1)$ iff*

$$\mathbb{U}(x,y)=\begin{cases} eT_{\mathbb{U}}\left(\frac{x}{e},\frac{y}{e}\right) & \text{if } (x,y)\in[0,e]^2,\\ e+(1-e)S_{\mathbb{U}}\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y)\in[e,1]^2,\\ \Omega(x,y) & \text{otherwise,}\end{cases} \tag{1}$$

where $T_{\mathbb{U}}$ and $S_{\mathbb{U}}$ are semi-t-norm and semi-t-conorm, respectively, and $\Omega(x,y):[0,e)\times(e,1]\cup(e,1]\times[0,e)$ is increasing and satisfies $\min(x,y)\leqslant\Omega(x,y)\leqslant\max(x,y)$ for all $(x,y)\in[0,e)\times(e,1]\cup(e,1]\times[0,e)$.

$\mathbf{U}_e^{\min}$ (resp. $\mathbf{U}_e^{\max}$) denotes the family of all semi-uninorms with neutral element $e\in(0,1)$ satisfying $\mathbb{U}(1,x)=\mathbb{U}(x,1)=x$ (resp. $\mathbb{U}(0,x)=\mathbb{U}(x,0)=x$) for all $x\in[0,e)$ (resp. $x\in(e,1]$).

Lemma 5.3. [4] *Let $\mathbb{U}$ be a semi-uninorm on $[0,1]$ with neutral element $e\in(0,1)$.*

(a) $\mathbb{U}\in\mathbf{U}_e^{\min}$ *iff* $\Omega(x,y)=\min(x,y)$ *in Eq. (1).*

(b) $\mathbb{U}\in\mathbf{U}_e^{\max}$ *iff* $\Omega(x,y)=\max(x,y)$ *in Eq. (1).*

Theorem 5.4. *Let $\mathbb{U}$ be a semi-uninorm on $[0,1]$ with neutral element $e\in(0,1)$.*

(a) *If $\mathbb{U}\in\mathbf{U}_e^{\min}$ (resp. $\mathbb{U}\in\mathbf{U}_e^{\max}$) and both $T_{\mathbb{U}}$ and $S_{\mathbb{U}}$ are continuous in the second component, then $\sqsubseteq_{\mathbb{U}}^*$ is a partial order given in Definition 3.2, that is, $([0,1],\sqsubseteq_{\mathbb{U}}^*)$ is a poset.*

(b) *If $\mathbb{U}\in\mathbf{U}_e^{\min}$ (resp. $\mathbb{U}\in\mathbf{U}_e^{\max}$) and both $T_{\mathbb{U}}$ and $S_{\mathbb{U}}$ are continuous in the first component, then $\sqsubseteq_{\mathbb{U}}$ is a partial order given in Lemma 2.4, that is, $([0,1],\sqsubseteq_{\mathbb{U}})$ is a poset.*

Proof. We prove only the first statement for $\mathbb{U}\in\mathbf{U}_e^{\min}$, the proofs of the remaining statements are nearly identical.

Firstly, it follows by Definition 5.1 that $\mathbb{U}(x,e)=x$ for any $x\in[0,1]$. Thus, $\mathbb{U}$ fulfills (**LRI**) by Definition 3.1.

Next, we prove that $\mathbb{U}(\mathbb{U}(x,y),z)=x$ implies $\mathbb{U}(x,y)=x$ for any $x,y,z\in[0,1]$, that is, $\mathbb{U}$ fulfills (**LQP**). We have four cases.

(i) If $(x,y)\in[0,e]^2$, then since $[0,e]^2$ is closed under $\mathbb{U}$ there is $\mathbb{U}(x,y)\in[0,e]$.

- If $z\in[0,e]$, since $\mathbb{U}\mid_{[0,e]^2}$ is a semi-t-norm, then it follows by Proposition 3.5 that $\mathbb{U}$ satisfies (**LQP**).
- If $z\in[e,1]$ and $\mathbb{U}(\mathbb{U}(x,y),z)=x$, then $x=\mathbb{U}(\mathbb{U}(x,y),z)=\min(\mathbb{U}(x,y),z)=\mathbb{U}(x,y)$ by Lemma 5.3 (a), i.e., $\mathbb{U}(x,y)=x$.

(ii) If $(x,y)\in[e,1]^2$, then since $[e,1]^2$ is closed under $\mathbb{U}$ there is $\mathbb{U}(x,y)\in[e,1]$.

- If $z\in[0,e)$ and $\mathbb{U}(\mathbb{U}(x,y),z)=x$, then $x=\mathbb{U}(\mathbb{U}(x,y),z)=z$, a contradiction. From this we conclude that $\mathbb{U}(\mathbb{U}(x,y),z)\neq x$ for all $(x,y)\in[e,1]^2$ and $z\in[0,e)$. Thus, the antecedent of (**LQP**) is not true.
- If $z\in[e,1]$, because $\mathbb{U}\mid_{[e,1]^2}$ is a semi-t-conorm, then it follows by Proposition 3.5 that $\mathbb{U}$ fulfills (**LQP**).

(iii) If $(x,y)\in[0,e]\times[e,1]$, then $\mathbb{U}(x,y)=x$.

- If $z\in[0,x)$ and $\mathbb{U}(\mathbb{U}(x,y),z)=x$, then $x=\mathbb{U}(\mathbb{U}(x,y),z)=\mathbb{U}(x,z)\leqslant z$, a contradiction. Thus, the antecedent of (**LQP**) is not valid.
- If $z\in[x,1]$ and $\mathbb{U}(\mathbb{U}(x,y),z)=x$, then $x=\mathbb{U}(\mathbb{U}(x,y),z)=\mathbb{U}(x,z)$, i.e., $\mathbb{U}(x,z)=x$.

(iv) If $(x,y)\in[e,1]\times[0,e]$, then $\mathbb{U}(x,y)=y$.

- If $z\in[0,e)$ and $\mathbb{U}(\mathbb{U}(x,y),z)=x$, then $x=\mathbb{U}(\mathbb{U}(x,y),z)=\mathbb{U}(y,z)\leqslant z$, a contradiction. Thus, the antecedent of (**LQP**) is not true.

- If $z \in [e,1]$ and $\mathbb{U}(\mathbb{U}(x,y),z) = x$, then $x = \mathbb{U}(\mathbb{U}(x,y),z) = \mathbb{U}(y,z) = \min(y,z) = y$, which means $x = y = e$, thus, $\mathbb{U}(x,y) = x$.

Cases (i), (ii), (iii) and (iv) imply that F fulfills (**LQP**).

Now we will prove (**GLQP**) for semi-uninorm $\mathbb{U}$, i.e., $\mathbb{U}(\mathbb{U}(x,y),z) = w$ for any $x,y,z,w \in [0,1]$ means that there exists a $t \in [0,1]$ satisfying $\mathbb{U}(x,t) = w$. Next, we have four cases.

(i) If $w = \mathbb{U}(\mathbb{U}(x,y),z)$ with $\mathbb{U}(x,y) \in [0,e]$ and $z \in [0,e]$, then we have either $(x,y) \in [0,e]^2 \cup [0,e]\times[e,1] \cup [e,1]\times[0,e]$.

- If $(x,y) \in [0,e]^2$ and $z \in [0,e]$, then $w = \mathbb{U}(\mathbb{U}(x,y),z) \leqslant \mathbb{U}(x,y) \leqslant x$, the continuity of $T_{\mathbb{U}}$ in the second variable implies the continuity of the vertical section $\mathbb{U}(x,\cdot)$ for $x \in [0,e]$, which means that there exists a $p \in [0,e]$ such that $w = \mathbb{U}(x,p)$ on $[0,e]^2$.
- If $(x,y) \in [0,e]\times[e,1]$ and $z \in [0,e]$, then it follows from $w = \mathbb{U}(\mathbb{U}(x,y),z)$ that $w = \mathbb{U}(x,z)$ since $\mathbb{U}(x,y) = x$ by Lemma 5.3 (a).
- If $(x,y) \in [e,1] \times [0,e]$ and $z \in [0,e]$ satisfying $w = \mathbb{U}(\mathbb{U}(x,y),z)$, then $w = \min(x,w) = \mathbb{U}(x,w)$ since $x \in [e,1]$ and $w \in [0,e]$ by Lemma 5.3 (a).

(ii) If $w = \mathbb{U}(\mathbb{U}(x,y),z)$ with $\mathbb{U}(x,y) \in [0,e]$ and $z \in [e,1]$, then $w = \mathbb{U}(\mathbb{U}(x,y),z) = \min(\mathbb{U}(x,y),z) = \mathbb{U}(x,y)$, i.e., $\mathbb{U}(x,y) = w$.

(iii) If $w = \mathbb{U}(\mathbb{U}(x,y),z)$ with $\mathbb{U}(x,y) \in [e,1]$ and $z \in [0,e]$, then $x,y \in [e,1]$ by Lemma 5.3 (a). Thus, it holds that $w = \mathbb{U}(\mathbb{U}(x,y),z) = \min(\mathbb{U}(x,y),z) = z$. From this one concludes that $w = \min(x,w) = \mathbb{U}(x,w)$, i.e., $\mathbb{U}(x,w) = w$.

(iv) If $w = \mathbb{U}(\mathbb{U}(x,y),z)$ with $\mathbb{U}(x,y) \in [e,1]$ and $z \in [e,1]$, then $w = \mathbb{U}(\mathbb{U}(x,y),z) \geqslant \mathbb{U}(x,y) \geqslant x$, the continuity of $S_{\mathbb{U}}$ in the second variable implies that there exists a $q \in [0,e]$ such that $w = \mathbb{U}(x,q)$ on $[e,1]^2$.

Cases (i), (ii), (iii) and (iv) imply that F fulfills (**GLQP**).

To sum up, (**LRI**), (**LQP**) and (**GLQP**) imply that $([0,1], \sqsubseteq_{\mathbb{U}}^*)$ is a poset by Theorem 3.3. □

5.2 Order obtained from semi-nullnorms

In complete analogy to Subsection 5.1 one gets the following results for semi-nullnorms.

Lemma 5.5. [4] *An operation* $\mathbb{N} : [0,1] \times [0,1] \to [0,1]$ *is a semi-nullnorm with zero element* $k \in (0,1)$ *iff*

$$\mathbb{N}(x,y) = \begin{cases} kS_{\mathbb{N}}\left(\frac{x}{k}, \frac{y}{k}\right) & \text{if } (x,y) \in [0,k]^2, \\ k + (1-k)T_{\mathbb{N}}\left(\frac{x-k}{1-k}, \frac{y-k}{1-k}\right) & \text{if } (x,y) \in [k,1]^2, \\ k & \text{otherwise,} \end{cases} \tag{2}$$

where $S_{\mathbb{N}}$ *and* $T_{\mathbb{N}}$ *are semi-t-conorm and semi-t-norm, respectively.*

Since $\mathbb{N}$ is non-commutative, similarly to the proof of Theorem 5.4, Lemma 5.5 leads to the following Theorem 5.6.

Theorem 5.6. *Let* $\mathbb{N}$ *be a semi-nullnorm on* $[0,1]$ *with zero element* $k \in (0,1)$.

(a) *If both* $T_{\mathbb{N}}$ *and* $S_{\mathbb{N}}$ *are continuous in the second component, then* $\sqsubseteq_{\mathbb{N}}^*$ *is a partial order given in Definition 3.2, that is,* $([0,1], \sqsubseteq_{\mathbb{N}}^*)$ *is a poset.*

(b) *If both* $T_{\mathbb{N}}$ *and* $S_{\mathbb{N}}$ *are continuous in the first component, then* $\sqsubseteq_{\mathbb{N}}$ *is a partial order given in Lemma 2.4, that is,* $([0,1], \sqsubseteq_{\mathbb{N}})$ *is a poset.*

The following example shows that the local continuity of semi-uninorm $\mathbb{U}$ (resp. semi-nullnorm $\mathbb{N}$) is indispensable.

Example 5.7. *Consider the semi-uninorm* $\mathbb{U}$ *(resp. semi-nullnorm* $\mathbb{N}$*) with neutral element (resp. zero element)* $\frac{1}{2}$*, where semi-t-norm* $\mathbb{T}_{\mathbb{U}}$ *(resp.* $\mathbb{T}_{\mathbb{N}}$*) and semi-t-conorm* $\mathbb{S}_{\mathbb{U}}$ *(resp.* $\mathbb{S}_{\mathbb{N}}$*) are given as follows:*

$$\mathbb{T}_{\mathbb{U}}(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,\frac{1}{2}]^2, \\ \frac{1}{2}x & \text{if } x,y) \in [0,\frac{1}{2}] \times [\frac{1}{2}, 1-x], \\ \frac{1}{3}y & \text{if } (x,y) \in [\frac{1}{2},1] \times [0,1-x] \\ \min(x,y) & \text{otherwise} \end{cases}$$

and

$$\mathbb{S}_{\mathbb{U}}(x,y)=\begin{cases}1 & \text{if } (x,y)\in[\frac{1}{2},1]^2,\\ \min(1.01y,1) & \text{if } x,y)\in[0,\frac{1}{2}]\times[1-x,1],\\ \min(1.03x,1) & \text{if } (x,y)\in[\frac{1}{2},1]\times[1-x,\frac{1}{2}]\\ \max(x,y) & \text{otherwise }.\end{cases}$$

It is clear that both $\mathbb{T}$ and $\mathbb{S}$ are non-commutative and non-associative. Moreover, it is trivial that $\mathbb{U}$ is not locally continuous, i.e., $\mathbb{U}$ is neither continuous in the first variable nor in the second variable. We claim that both $\sqsubseteq^*_{\mathbb{U}}$ and $\sqsubseteq_{\mathbb{U}}$ are not partial orders. Indeed, for $x=\frac{3}{8}$, $y=\frac{1}{3}$ and $z=\frac{1}{6}$, a simple calculation shows that $\mathbb{U}(\mathbb{U}(x,y),z)=\mathbb{U}(\mathbb{U}(\frac{3}{8},\frac{1}{3}),\frac{1}{6})=\frac{1}{18}$. Nevertheless, there does not exist an $\ell\in[0,1]$ fulfilling $\mathbb{U}(\frac{3}{8},\ell)=\frac{1}{18}$. As a matter of fact, if $\ell\leqslant\frac{1}{8}$, then $\mathbb{U}(\frac{3}{8},\ell)=\frac{1}{2}\mathbb{T}_{\mathbb{U}}(\frac{3}{4},2\ell)=\frac{1}{2}\cdot\frac{1}{3}\cdot(2\ell)=\frac{\ell}{3}\leqslant\frac{1}{24}$, which means $\mathbb{U}(\frac{3}{8},\ell)<\frac{1}{18}$ for $\ell\leqslant\frac{1}{8}$; If $\frac{1}{8}<\ell\leqslant\frac{1}{2}$, then $\mathbb{U}(\frac{3}{8},\ell)=\frac{1}{2}\mathbb{T}_{\mathbb{U}}(\frac{3}{4},2\ell)=\frac{1}{2}\cdot\min\left(\frac{3}{4},2\ell\right)>\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$, which means $\mathbb{U}(\frac{3}{8},\ell)>\frac{1}{8}$ for $\ell>\frac{1}{8}$. Therefore, (**GLQP**) *is not valid, it follows by Theorem 3.3 that $\sqsubseteq^*_{\mathbb{U}}$ is not a partial order. Analogously, one can verify that $\sqsubseteq_{\mathbb{U}}$ is not a partial order by Lemma 2.4.*

6 Conclusion

We mainly investigated both orders $\sqsubseteq_F$ and $\sqsubseteq^*_F$ for non-commutative operations F on $\mathbb{P}$. More specifically:

(i) We defined a new order $\sqsubseteq^*_F$ and discussed the relationship between the order $\sqsubseteq_F$ and $\sqsubseteq^*_F$ in $\mathbb{P}$. Furthermore, we gave a sufficient and necessary condition that the orders $\sqsubseteq_F$ and $\sqsubseteq^*_F$ coincide when F is locally commutative on $\mathbb{P}$.

(ii) For semi-t-operator $\mathbb{F}$ on $[0,1]$ with $\mathbb{F}(0,1)=a$ and $\mathbb{F}(1,0)=b$.

- If $a\leqslant b$, then $([0,1],\sqsubseteq^*_{\mathbb{F}})$ is a poset; If $a>b$, then $([0,1],\sqsubseteq^*_{\mathbb{F}})$ is not a poset.
- If $a\geqslant b$, then $([0,1],\sqsubseteq_{\mathbb{F}})$ is a poset; If $a<b$, then $([0,1],\sqsubseteq_{\mathbb{F}})$ is not a poset.

(iii) We obtained the orders $\sqsubseteq^*_{\mathbb{U}}$ and $\sqsubseteq^*_{\mathbb{N}}$ induced by semi-uninorms $\mathbb{U}$ and semi-nullnorms $\mathbb{N}$ under condition that semi-uninorms and semi-nullnorms are locally continuous, respectively.

The orders based on non-commutative fuzzy logic connectives of this paper built a bridge between fuzzy logic connectives and order-theoretic structrues. These results will be very significant for order-theoretic exploration of algebras.

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