Iranian Journal of Fuzzy Systems

Volume 21, Number 4, (2024), pp. 61-80

Original paper

Convergence structures in L-concave spaces

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Abstract

Considering a complete residuated lattice L as the lattice background, the concept of (preconcave, concave) L-convergence spaces via L-ordered co-Scott closed sets is introduced and its diagonal axioms are proposed. It is shown that concave L-convergence spaces are isomorphic to strong L-concave spaces in a categorical viewpoint. Also, it is proved that a preconcave L-convergence space satisfies the Kowalsky diagonal axiom if and only if it is concave, and an L-convergence space satisfies the Fischer diagonal axiom if and only if it is concave.

Keywords: L-convex space, L-concave space, L-convergence space, L-ordered co-Scott closed set, diagonal axiom.

1 Introduction

The theory of convexity plays an important role in mathematics and convexity exists in many mathematical environments, such as vector spaces [30], graphs [29], metric spaces [14], lattices [31] and so on. However, the concept of convex subsets has different forms in different environments. That is to say, in each specific environment, convex subsets have their own characters. In order to better study the basic properties of convexities, each specific environment cannot be qualified to be the most suitable setting. This leads to the concept of so-called convex structures [30] (also called abstract or axiomatic convex structures). To some extent, the concept of convex structures is obtained by applying the axiomatic approach to the concrete convexities in specific environments.

Since Zadeh introduced the notion of fuzzy sets, fuzzy set theory has been applied to both of real-life applications and theoretical research. From the theoretical aspect, the theory of fuzzy topology as well as some related theories, such as fuzzy metric [25, 26], fuzzy convergence [3, 6, 16, 19, 37, 38], fuzzy order [2, 34, 35] and so on, have received wide attention in recent years. As a topology-like structure, convex structures possess some similar characters of topologies and hence are also discussed in the fuzzy case. To date, convex structures have been generalized to different types of fuzzy convex structures. Rosa [24] first introduced the notion of fuzzy convexity spaces and then Maruyama [18] proposed the concept of L-fuzzy convexity spaces, where L denotes a completely distributive lattice. Adopting the terminology of fuzzy topology, both of fuzzy convexity spaces in the sense of Rosa and Maruyama are called L-convex structures nowadays. In the setting of L-convex structures, Pang et al. [20, 21, 22] presented several characterizations of L-convex structures. In a different way, Shi and Xiu [27] provided a new fuzzification method to propose the concept of M-fuzzifying convex structures. Recently, Shi and Xiu [28] proposed the notion of (L, M)-fuzzy convex structures, which can include L-convex structures and M-fuzzifying convex structures as special cases. Up to now, the theory of fuzzy convex structures has captured more and more attention.

As we all know, concavity as a dual concept of convexity plays an important role in mathematics. Combined with fuzzy set theory, concave structrues have also been extended to the fuzzy case. Based on a completely distributive lattice L, Pang [22] proved that L-concave structures can be characterized by L-interior operators and L-neighborhood systems. Xiu [33] introduced fuzzy point based convergence structures in L-concave spaces and pointed that the category

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Received: May 2024; Revised: August 2024; Accepted: August 2024.

https://doi.org/10.22111/ijfs.2024.48804.8608

of concave L-convergence spaces is isomorphic to that of L-concave spaces. Wu and Shi [32] introduced the notion of L-concave bases and discussed its relationship with L-concave structures. From the aspect of lattice background, it is worth choosing a more general lattice to define fuzzy concave structures. Based on a complete residuated lattice L, Li and Wang [17] proposed the concepts of stratified L-concave spaces, L-ordered neighborhood spaces and L-ordered interior spaces and showed that the resulting categories are isomorphic. Zhang and Pang [36] proposed a fuzzy convergence structure in a strong L-concave space and proved that the category of strong L-concave spaces can be embedded in the category of L-convergence spaces as a reflective subcategory. This demonstrates that convergence structures are useful tools to characterize L-concave structures. Moreover, it is well known that diagonal axioms defined by a compression operator can used to describe convergence structures [4, 10, 11, 12, 15]. By this motivation, we will propose the concept of L-convergence structures in the framework of L-concave spaces and study its relationship with L-concave structures. Then we will propose two kinds of diagonal axioms for an L-convergence structure, called the Kowalsky diagonal axiom and the Fischer diagonal axiom, and investigate their mutual relationships.

The content is organized as follows. In Section 2, we recall some necessary concepts and notations. In Section 3, we introduce the notion of L-ordered co-Scott closed sets. In Section 4, we propose the notion of L-convergence spaces via L-ordered co-Scott closed sets and illustrate its relationship with strong L-concave spaces. In Section 5, we will propose the Kowalsky diagonal axiom and the Fischer diagonal axiom to describe L-convergence spaces.

2 Preliminaries

In this paper, if not otherwise specified, $(L, *, \top)$ is always a complete residuated lattice [1]. That is, L is a complete lattice with the top element $\top$ and the bottom element $\perp$ and $*$ is a binary operation on L such that

(i) $(L, *, \top)$ is a commutative monoid;

(ii) $*$ distributes over arbitrary joins, i.e.,
$$\alpha * \Big(\bigvee_{i\in I} \beta_i\Big) = \bigvee_{i\in I} \alpha * \beta_i,$$
for any $\alpha \in L$ and $\{\beta_i\}_{i\in I} \subseteq L$.

Since the binary operation $*$ distributes over arbitrary joins, the map $\alpha * (-) : L \longrightarrow L$ has a right adjoint $\alpha \to (-) : L \longrightarrow L$ given by $\alpha \to \beta = \bigvee\{\gamma \in L \mid \alpha * \gamma \leq \beta\}$. The binary operation $\to$ is called the implication with respect to $*$. Some basic properties of the binary operations $*$ and $\to$ are collected in the following proposition, which can be found in many works, for instance [1, 9].

Proposition 2.1. *Let $(L, *, \top)$ be a complete residuated lattice. Then*

(I1) $\perp * \alpha = \perp$ *and* $\top \to \alpha = \alpha$;

(I2) $\alpha \to \beta = \top \Longleftrightarrow \alpha \leq \beta$;

(I3) $\alpha * (\alpha \to \beta) \leq \beta$ *and* $(\alpha \to \beta) * (\beta \to \gamma) \leq \alpha \to \gamma$;

(I4) $\alpha \to (\beta \to \gamma) = \beta \to (\alpha \to \gamma) = (\alpha * \beta) \to \gamma$;

(I5) $(\bigvee_{j\in J} \alpha_j) \to \beta = \bigwedge_{j\in J}(\alpha_j \to \beta)$;

(I6) $\alpha \to (\bigwedge_{j\in J} \beta_j) = \bigwedge_{j\in J}(\alpha \to \beta_j)$;

(I7) $\alpha \leq \beta \Rightarrow \gamma \to \alpha \leq \gamma \to \beta$ *and* $\alpha \to \gamma \geq \beta \to \gamma$.

For a nonempty set X, let $\mathcal{P}(X)$ denote the power set of X. An L-subset on X is a map from X to L, and the family of all L-subsets on X will be denoted by L^X, called the L-power set of X. We do not distinguish an element $\alpha \in L$ and the constant map $\alpha_X : X \longrightarrow L$ such that $\alpha_X(x) = \alpha$ for all $x \in X$. All algebraic operations on L can be extended to the L-power set L^X pointwisely. That is, for all $A, B \in L^X$, $\alpha \in L$ and $x \in X$,

(i) $(A \vee B)(x) = A(x) \vee B(x)$;

(ii) $(A \wedge B)(x) = A(x) \wedge B(x)$;

(iii) $(A * B)(x) = A(x) * B(x)$ and $(\alpha * A)(x) = \alpha * A(x)$;

(iv) $(A \to B)(x) = A(x) \to B(x)$, $(\alpha \to B)(x) = \alpha \to B(x)$ and $(B \to \alpha)(x) = B(x) \to \alpha$.

We say a nonempty family $\{A_j\}_{j\in J} \subseteq L^X$ is a directed (resp. co-directed) subset of L^X if for each $A_{j_1}, A_{j_2} \in \{A_j\}_{j\in J}$, there exists $A_{j_3} \in \{A_j\}_{j\in J}$ such that $A_{j_1} \leq A_{j_3}$ and $A_{j_2} \leq A_{j_3}$ (resp. $A_{j_3} \leq A_{j_1}$ and $A_{j_3} \leq A_{j_2}$). We usually use the symbols $\{A_j\}_{j\in J} \subseteq^{dir} L^X$ (resp. $\{A_j\}_{j\in J} \subseteq^{cdir} L^X$) to denote that $\{A_j\}_{j\in J}$ is a directed (resp. co-directed) subset of L^X. Let $f : X \longrightarrow Y$ be an ordinary map. Define $f^{\to} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ by $f^{\to}(A)(y) = \bigvee_{f(x)=y} A(x)$ for $A \in L^X$ and $y \in Y$, and $f^{\leftarrow}(B) = B \circ f$ for $B \in L^Y$ [23].

Definition 2.2. [3] *The map $\mathcal{S}(-,-) : L^X \times L^X \longrightarrow L$ defined by*

$$\forall A, B \in L^X, \mathcal{S}(A,B) = \bigwedge_{x\in X} \Big(A(x) \to B(x)\Big),$$

is called the lattice-valued inclusion order between L-subsets on X.

Proposition 2.3. [1, 3] *Let $\mathcal{S}(-,-)$ be the lattice-valued inclusion order between L-subsets on X. Then for $A, B, C, D \in L^X$, the following statements hold*

(1) $A \leq B \Longleftrightarrow \mathcal{S}(A,B) = \top$;

(2) $\mathcal{S}(A,B) * \mathcal{S}(B,C) \leq \mathcal{S}(A,C)$ *and hence* $\mathcal{S}(B,C) \leq \mathcal{S}(A,B) \to \mathcal{S}(A,C)$;

(3) $\mathcal{S}(A,B) * \mathcal{S}(C,D) \leq \mathcal{S}(A * C, B * D)$ *and* $\mathcal{S}(A,B) \wedge \mathcal{S}(C,D) \leq \mathcal{S}(A \wedge C, B \wedge D)$;

(4) $A \leq B$ *implies* $\mathcal{S}(C,A) \leq \mathcal{S}(C,B)$ *and* $\mathcal{S}(B,C) \leq \mathcal{S}(A,C)$.

Proposition 2.4. [3] *Let $f : X \longrightarrow Y$ be a map. Then for $A, B \in L^X, C, D \in L^Y$, it holds that*

$$\mathcal{S}(A,B) \leq \mathcal{S}(f^{\to}(A), f^{\to}(B)) \quad \text{and} \quad \mathcal{S}(C,D) \leq \mathcal{S}(f^{\leftarrow}(C), f^{\leftarrow}(D)).$$

Definition 2.5. [36] *A subset $\mathcal{C}$ of L^X is called a strong L-concave structure on X if it satisfies*

(SLCS1) $\perp_X, \top_X \in \mathcal{C}$;

(SLCS2) $\{A_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{C}$ *implies* $\bigvee_{\lambda\in\Lambda} A_\lambda \in \mathcal{C}$;

(SLCS3) *If* $\{A_j\}_{j\in J} \subseteq \mathcal{C}$ *is nonempty and codirected, then* $\bigwedge_{j\in J} A_j \in \mathcal{C}$;

(SLCSs) $\alpha * A \in \mathcal{C}$ *for each* $A \in \mathcal{C}$ *and* $\alpha \in L$;

(SLCScs) $\alpha \to A \in \mathcal{C}$ *for each* $A \in \mathcal{C}$ *and* $\alpha \in L$.

For a strong L-concave structure $\mathcal{C}$ on X, the pair $(X, \mathcal{C})$ is called a strong L-concave space.

A map $f : (X, \mathcal{C}) \longrightarrow (Y, \mathcal{D})$ between two strong L-concave spaces is called L-concavity-preserving provided that $f^{\leftarrow}(D) \in \mathcal{C}$ for each $D \in \mathcal{D}$.

It is easy to check that all strong L-concave spaces and their L-concavity-preserving maps form a category, denoted by **SLSC**.

3 L-ordered co-Scott closed sets

In this section, we will propose the notion of L-ordered co-Scott closed sets on L^X and study its properties. This will provide a basic tool to define convergence structures in the next section.

Definition 3.1. *A map $\mathcal{F} : L^X \longrightarrow L$ is called an L-ordered co-Scott closed set on L^X if it satisfies*

(LCSC1) $\mathcal{F}(\top_X) = \top$;

(LCSC2) $\mathcal{S}(A,B) \leq \mathcal{F}(A) \to \mathcal{F}(B)$ *for each* $A, B \in L^X$;

(LCSC3) $\bigwedge_{j\in J} \mathcal{F}(A_j) \leq \mathcal{F}(\bigwedge_{j\in J} A_j)$ *for each* $\{A_j\}_{j\in J} \subseteq^{cdir} L^X$;

(LCSCs) $\alpha * \mathcal{F}(A) \leq \mathcal{F}(\alpha * A)$ *for each* $A \in L^X$ *and* $\alpha \in L$;

(LCSCcs) $\alpha \to \mathcal{F}(A) \leq \mathcal{F}(\alpha \to A)$ *for each* $A \in L^X$ *and* $\alpha \in L$.

For convenience, let $\mathcal{C}_L(X)$ denote all L-ordered co-Scott closed sets on L^X.

Remark 3.2. *In domain theory, a subset* $\mathbb{F} \subseteq \mathcal{P}(X)$ *is called Scott closed [7] if it satisfies the following properties: (SC1)* $A \in \mathbb{F}$ *and* $B \subseteq A$ *imply* $B \in \mathbb{F}$*; (SC2)* $\{A_j\}_{j\in J} \subseteq^{dir} \mathbb{F}$ *implies* $\bigcup_{j\in J} A_j \in \mathbb{F}$*. The dual of a Scott closed set is called a co-Scott closed set,* i.e., *a subset* $\mathbb{F} \subseteq \mathcal{P}(X)$ *is called co-Scott closed if it satisfies the following properties: (CSC1)* $A \in \mathbb{F}$ *and* $A \subseteq B$ *imply* $B \in \mathbb{F}$*; (CSC2)* $\{A_j\}_{j\in J} \subseteq^{cdir} \mathbb{F}$ *implies* $\bigcap_{j\in J} A_j \in \mathbb{F}$*. If* $L = \{\perp, \top\}$*, then an* L*-ordered co-Scott closed set in Definition 3.1 exactly reduces to a co-Scott closed set in the classical case.*

An order on $\mathcal{C}_L(X)$ can be defined by $\mathcal{F} \leq \mathcal{G}$ if and only if $\mathcal{F}(A) \leq \mathcal{G}(A)$ for each $A \in L^X$.

Example 3.3. *Let* X *be a nonempty set.*

(1) *Define a map* $[x] : X \longrightarrow L$ *by* $[x](A) = A(x)$ *for each* $A \in L^X$ *and* $x \in X$*. Then* $[x] \in \mathcal{C}_L(X)$*.*

(2) *Let* $f : X \longrightarrow Y$ *be a map and* $\mathcal{F} \in \mathcal{C}_L(X)$*. Then the map* $f^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$ *defined by* $f^{\Rightarrow}(\mathcal{F})(B) = \mathcal{F}(f^{\leftarrow}(B))$ *for each* $B \in L^Y$*, is an* L*-ordered co-Scott closed set on* L^Y*, which is called the image of* $\mathcal{F}$ *under* f *in [8].*

(3) *For a family of* L*-ordered co-Scott closed sets* $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{C}_L(X)$*, define* $\bigwedge_{\lambda\in\Lambda} \mathcal{F}_\lambda : L^X \longrightarrow L$ *by*

$$\forall A \in L^X, \Big(\bigwedge_{\lambda\in\Lambda} \mathcal{F}_\lambda\Big)(A) = \bigwedge_{\lambda\in\Lambda} \mathcal{F}_\lambda(A).$$

Obviously, $\bigwedge_{\lambda\in\Lambda}\mathcal{F}_\lambda \in \mathcal{C}_L(X)$*.*

Proposition 3.4. *Let* $f : X \longrightarrow Y$ *be a map and* $\mathcal{G} \in \mathcal{C}_L(Y)$*. Define* $f^{\Leftarrow}(\mathcal{G}) : L^X \longrightarrow L$ *by*

$$f^{\Leftarrow}(\mathcal{G})(A) = \bigvee_{f^{\leftarrow}(B)\leq A} \mathcal{G}(B),$$

for each $A \in L^X$*. If* L *satisfies that* $\alpha \to (\cdot)$ *is Scott continuous (*i.e., $\alpha \to (\cdot)$ *preserves directed joins) for each* $\alpha \in L$*, then* $f^{\Leftarrow}(\mathcal{G}) \in \mathcal{C}_L(X)$*.*

Proof. It suffices to verify that $f^{\Leftarrow}(\mathcal{G})$ satisfies (LCSC1)–(LCSC3), (LCSCs) and (LCSCcs).

For (LCSC1), it is straightforward and is omitted.

For (LCSC2), take any $A, C \in L^X$. For each $B \in L^X$, if $f^{\leftarrow}(B) \leq A$, then $f^{\leftarrow}(\mathcal{S}(A, C) * B) \leq C$. This implies

$$\begin{aligned} \mathcal{S}(A,C) * f^{\Leftarrow}(\mathcal{G})(A) &= \mathcal{S}(A,C) * \bigvee_{f^{\leftarrow}(B)\leq A} \mathcal{G}(B) = \bigvee_{f^{\leftarrow}(B)\leq A} \mathcal{S}(A,C) * \mathcal{G}(B) \leq \bigvee_{f^{\leftarrow}(B)\leq A} \mathcal{G}(\mathcal{S}(A,C) * B) \\ &\leq \bigvee_{f^{\leftarrow}(\mathcal{S}(A,C)*B)\leq C} \mathcal{G}(\mathcal{S}(A,C) * B) \leq \bigvee_{f^{\leftarrow}(D)\leq C} \mathcal{G}(D) = f^{\Leftarrow}(\mathcal{G})(C). \end{aligned}$$

For (LCSC3), let $\{A_j\}_{j\in J} \subseteq^{cdir} L^X$. Since $f^{\leftarrow} : L^Y \longrightarrow L^X$ preserves arbitrarily joins, there exists a right adjoint $f^\star : L^X \longrightarrow L^Y$ of $f^{\leftarrow}$ and $\{f^\star(A_j)\}_{j\in J} \subseteq^{cdir} L^Y$. Then

$$\begin{aligned} \bigwedge_{j\in J} f^{\Leftarrow}(\mathcal{G})(A_j) &= \bigwedge_{j\in J} \bigvee_{f^{\leftarrow}(B_j)\leq A_j} \mathcal{G}(B_j) = \bigwedge_{j\in J} \bigvee_{B_j\leq f^\star(A_j)} \mathcal{G}(B_j) \leq \bigwedge_{j\in J} \mathcal{G}(f^\star(A_j)) \\ &\leq \mathcal{G}\Big(\bigwedge_{j\in J} f^\star(A_j)\Big) = \mathcal{G}\Big(f^\star\Big(\bigwedge_{j\in J} A_j\Big)\Big) \leq \bigvee_{f^{\leftarrow}(B)\leq \bigwedge_{j\in J} A_j} \mathcal{G}(B) = f^{\Leftarrow}(\mathcal{G})\Big(\bigwedge_{j\in J} A_j\Big). \end{aligned}$$

For (LCSCs), if $f^{\leftarrow}(B) \leq A$, then $f^{\leftarrow}(\alpha * B) \leq \alpha * A$. So

$$\begin{aligned} \alpha * f^{\Leftarrow}(\mathcal{G})(A) &= \alpha * \bigvee_{f^{\leftarrow}(B)\leq A} \mathcal{G}(B) = \bigvee_{f^{\leftarrow}(B)\leq A} \alpha * \mathcal{G}(B) \leq \bigvee_{f^{\leftarrow}(B)\leq A} \mathcal{G}(\alpha * B) \\ &\leq \bigvee_{f^{\leftarrow}(\alpha*B)\leq \alpha*A} \mathcal{G}(\alpha * B) \leq \bigvee_{f^{\leftarrow}(C)\leq \alpha*A} \mathcal{G}(C) = f^{\Leftarrow}(\mathcal{G})(\alpha * A). \end{aligned}$$

For (LCSCcs), if $f^{\leftarrow}(B) \leq A$, then $f^{\leftarrow}(\alpha \to B) \leq \alpha \to A$. So

$$\begin{aligned}\alpha \to f^{\Leftarrow}(\mathcal{G})(A) &= \alpha \to \bigvee_{f^{\leftarrow}(B)\leq A} \mathcal{G}(B) = \bigvee_{f^{\leftarrow}(B)\leq A} \alpha \to \mathcal{G}(B) \quad (\alpha \to (\cdot) \text{ is Scott continuous}) \\ &\leq \bigvee_{f^{\leftarrow}(B)\leq A} \mathcal{G}(\alpha \to B) \leq \bigvee_{f^{\leftarrow}(\alpha\to B)\leq \alpha\to A} \mathcal{G}(\alpha \to B) = \bigvee_{f^{\leftarrow}(C)\leq \alpha\to A} \mathcal{G}(C) = f^{\Leftarrow}(\mathcal{G})(\alpha \to A).\end{aligned}$$

□

The above proposition actually gives another example of L-ordered co-Scott closed set. Similar to Example 3.3 (2), the L-ordered co-Scott closed set $f^{\Leftarrow}(\mathcal{G})$ is called the inverse image of $\mathcal{G}$ under f.

Next, we provide an example of L-ordered co-Scott closed sets from the aspect of strong L-concave spaces.

Proposition 3.5. *Suppose that L satisfies that $\alpha \to (\cdot)$ is Scott continuous (*i.e., *$\alpha \to (\cdot)$ preserves directed joins) for each $\alpha \in L$. Let $(X, \mathcal{C})$ be a strong L-concave space and define $\mathcal{N}^x_{\mathcal{C}} : L^X \longrightarrow L$ by*

$$\mathcal{N}^x_{\mathcal{C}}(A) = \bigvee_{B\in\mathcal{C}, B\leq A} B(x),$$

for each $A \in L^X$. Then $\mathcal{N}^x_{\mathcal{C}} \in \mathcal{C}_L(X)$.

Proof. It suffices to verify that $\mathcal{N}^x_{\mathcal{C}}$ satisfies (LCSC1)–(LCSC3), (LCSCs) and (LCSCcs).

For (LCSC1), it is straightforward and is omitted.

For (LCSC2), take any $A, B \in L^X$. For each $C \in \mathcal{C}$ and $C \leq A$, we have $\mathcal{S}(A, B) * C \in \mathcal{C}$ and $\mathcal{S}(A, B) * C \leq \mathcal{S}(A, B) * A \leq B$. Then

$$\begin{aligned}\mathcal{S}(A, B) * \mathcal{N}^x_{\mathcal{C}}(A) &= \mathcal{S}(A, B) * \bigvee_{C\in\mathcal{C}, C\leq A} C(x) \\ &= \bigvee_{C\in\mathcal{C}, C\leq A} \mathcal{S}(A, B) * C(x) \\ &\leq \bigvee_{\mathcal{S}(A,B)*C\in\mathcal{C}, \mathcal{S}(A,B)*C\leq B} \mathcal{S}(A, B) * C(x) \\ &\leq \bigvee_{D\in\mathcal{C}, D\leq B} D(x) \\ &= \mathcal{N}^x_{\mathcal{C}}(B).\end{aligned}$$

For (LCSC3), take any $\{A_j\}_{j\in J} \subseteq^{cdir} L^X$. Then $\bigvee\{B_j \in \mathcal{C} \mid B_j \leq A_j\}_{j\in J} \subseteq^{cdir} \mathcal{C}$ and $\bigvee\{B_j \in \mathcal{C} \mid B_j \leq A_j\}_{j\in J} \leq A_j$. It follows that $\bigwedge_{j\in J} \bigvee_{B_j\in\mathcal{C}, B_j\leq A_j} B_j \in \mathcal{C}$ and $\bigwedge_{j\in J} \bigvee_{B_j\in\mathcal{C}, B_j\leq A_j} B_j \leq \bigwedge_{j\in J} A_j$. So

$$\begin{aligned}\bigwedge_{j\in J} \mathcal{N}^x_{\mathcal{C}}(A_j) &= \bigwedge_{j\in J} \bigvee_{B_j\in\mathcal{C}, B_j\leq A_j} B_j(x) \\ &\leq \bigvee_{C\in\mathcal{C}, C\leq \bigwedge_{j\in J} A_j} C(x) \\ &= \mathcal{N}^x_{\mathcal{C}}\Big(\bigwedge_{j\in J} A_j\Big).\end{aligned}$$

For (LCSCs), take any $A \in L^X$ and $\alpha \in L$. Then

$$\begin{aligned}\alpha * \mathcal{N}^x_{\mathcal{C}}(A) &= \alpha * \bigvee_{B\in\mathcal{C}, B\leq A} B(x) \\ &= \bigvee_{B\in\mathcal{C}, B\leq A} \alpha * B(x) \\ &\leq \bigvee_{\alpha*B\in\mathcal{C}, \alpha*B\leq\alpha*A} \alpha * B(x) \\ &\leq \bigvee_{C\in\mathcal{C}, C\leq\alpha*A} C(x) \\ &= \mathcal{N}^x_{\mathcal{C}}(\alpha * A).\end{aligned}$$

For (LCSCcs), take any $A \in L^X$ and $\alpha \in L$. Then

$$\begin{aligned}\alpha \to \mathcal{N}^x_{\mathcal{C}}(A) &= \alpha \to \bigvee_{B\in\mathcal{C}, B\le A} B(x)\\ &= \bigvee_{B\in\mathcal{C}, B\le A} \alpha \to B(x) \quad (\alpha \to (\cdot) \text{ is Scott continuous})\\ &\le \bigvee_{\alpha\to B\in\mathcal{C}, \alpha\to B\le \alpha\to A} \alpha * B(x)\\ &\le \bigvee_{C\in\mathcal{C}, C\le \alpha\to A} C(x)\\ &= \mathcal{N}^x_{\mathcal{C}}(\alpha \to A).\end{aligned}$$

□

4 L-convergence spaces

In this section, we introduce the notion of L-convergence structures via L-oredered co-Scott closed sets and establish its relationship with strong L-concave structures.

Definition 4.1. *A map* $\lim : \mathcal{C}_L(X) \longrightarrow L^X$ *is called an L-convergence structure on X if it satisfies*

(LCS1) $\forall x \in X, \lim[x](x) = \top$;

(LCS2) $\mathcal{S}(\mathcal{F},\mathcal{G}) * \lim \mathcal{F}(x) \le \lim \mathcal{G}(x)$ *for each* $\mathcal{F},\mathcal{G} \in \mathcal{C}_L(X)$.

For an L-convergence structure $\lim$ *on X, the pair* $(X,\lim)$ *is called an L-convergence space.*

A map $f : (X,\lim^X) \longrightarrow (Y,\lim^Y)$ between two L-convergence spaces is called continuous provided that $\lim^X \mathcal{F}(x) \le \lim^Y f^{\Rightarrow}(\mathcal{F})(f(x))$ for each $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$.

It is easy to check that L-convergence spaces and their continuous maps form a category, denoted by **LCS**.

Theorem 4.2. *The category* **LCS** *is a topological category over* **Set**.

Proof. We only note that for a set X, the initial structure $\lim^X$ on X with respect to a class $\{(X_\lambda,\lim^{X_\lambda})\}_{\lambda\in\Lambda}$ of L-convergence spaces and a family $\{f_\lambda : X \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ of maps, is determined by

$$\lim^X \mathcal{F}(x) = \bigwedge_{\lambda\in\Lambda} \lim^{X_\lambda} f_\lambda^{\Rightarrow}(\mathcal{F})(f_\lambda(x)),$$

for each $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$. □

In an L-convergence space $(X,\lim)$, a special L-ordered co-Scott closed set can be defined in the following way.

Proposition 4.3. *Let* $(X,\lim)$ *be an L-convergence space and $x \in X$. Define* $\mathcal{N}^x_{\lim} : L^X \longrightarrow L$ *by*

$$\mathcal{N}^x_{\lim}(A) = \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big(\lim \mathcal{F}(x) \to \mathcal{F}(A)\Big),$$

for each $A \in L^X$. Then $\mathcal{N}^x_{\lim} \in \mathcal{C}_L(X)$.

Proof. It suffices to verify that $\mathcal{N}^x_{\lim}$ satisfies (LCSC1)–(LCSC3), (LCSCs) and (LCSCcs). (LCSC1), (LCSCs) and (LCSCcs) are straightforward, so we only need to prove (LCSC2) and (LCSC3).

For (LCSC2), take any $A, B \in L^X$. Then

$$\begin{aligned}
\mathcal{S}(A,B) * \mathcal{N}^x_{\lim}(A) &= \mathcal{S}(A,B) * \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big(\lim\mathcal{F}(x) \to \mathcal{F}(A)\Big) \\
&\leq \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big(\mathcal{S}(A,B) * (\lim\mathcal{F}(x) \to \mathcal{F}(A))\Big) \\
&\leq \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big(\lim\mathcal{F}(x) \to (\mathcal{S}(A,B) * \mathcal{F}(A))\Big) \\
&\leq \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big(\lim\mathcal{F}(x) \to \mathcal{F}(B)\Big) \\
&= \mathcal{N}^x_{\lim}(B).
\end{aligned}$$

For (LCSC3), let $\{A_j\}_{j\in J} \subseteq^{cdir} L^X$. Then

$$\begin{aligned}
\bigwedge_{j\in J} \mathcal{N}^x_{\lim}(A_j) &= \bigwedge_{j\in J} \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big(\lim\mathcal{F}(x) \to \mathcal{F}(A_j)\Big) \\
&= \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big(\lim\mathcal{F}(x) \to \bigwedge_{j\in J} \mathcal{F}(A_j)\Big) \\
&\leq \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big(\lim\mathcal{F}(x) \to \mathcal{F}\Big(\bigwedge_{j\in J} A_j\Big)\Big) \\
&= \mathcal{N}^x_{\lim}\Big(\bigwedge_{j\in J} A_j\Big).
\end{aligned}$$

□

By equipping additional requirements on (LCS1) and (LCS2), we can obtain special types of L-convergence structures.

Definition 4.4. *An L-convergence space $(X, \lim)$ is called preconcave if it satisfies*

$$\textbf{(Lcp)} \quad \lim\mathcal{F}(x) = \mathcal{S}(\mathcal{N}^x_{\lim}, \mathcal{F}),$$

for each $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$.

Now let us give some characterizations of preconcave L-convergence spaces. To this end, the following lemma is necessary.

Lemma 4.5. *Let $\mathcal{F} \in \mathcal{C}_L(X)$ and $\alpha \in L$. Then $\alpha \to \mathcal{F} \in \mathcal{C}_L(X)$.*

Proof. It suffices to verify that $\alpha \to \mathcal{F}$ satisfies that (LCSC1)–(LCSC3), (LCSCs) and (LCSCcs). (LCSC1), (LCSC2) and (LCSCs) are straightforward, so we only need to prove (LCSC3) and (LCSCcs).

For (LCSC3), let $\{A_j\}_{j\in J} \subseteq^{cdir} L^X$. Then

$$\begin{aligned}
\alpha * \bigwedge_{j\in J} \Big(\alpha \to \mathcal{F}\Big)(A_j) &= \alpha * \bigwedge_{j\in J} \Big(\alpha \to \mathcal{F}(A_j)\Big) \\
&= \alpha * \Big(\alpha \to \bigwedge_{j\in J} \mathcal{F}(A_j)\Big) \\
&\leq \bigwedge_{j\in J} \mathcal{F}(A_j) \\
&\leq \mathcal{F}\Big(\bigwedge_{j\in J} A_j\Big).
\end{aligned}$$

This implies that $\bigwedge_{j\in J}(\alpha\to\mathcal{F})(A_j)\le(\alpha\to\mathcal{F})(\bigwedge_{j\in J}A_j)$ holds.

For (LCSCcs), take any $\beta\in L$ and $A\in L^X$. Then

$$\begin{array}{rcl}\alpha*(\beta\to(\alpha\to\mathcal{F}))(A) & = & \alpha*(\beta\to(\alpha\to\mathcal{F}(A)))\\ & = & \alpha*(\alpha\to(\beta\to\mathcal{F}(A)))\\ & \le & \beta\to\mathcal{F}(A)\\ & \le & \mathcal{F}(\beta\to A).\end{array}$$

This implies that $\beta\to(\alpha\to\mathcal{F})(A)\le(\alpha\to\mathcal{F})(\beta\to A)$ holds. □

For an L-convergence space $(X,\lim)$, we consider the following axioms:

(Lcn) For each $x\in X$, $\lim\mathcal{N}^x_{\lim}(x)=\top$;

(Lcq) For each $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}\subseteq\mathcal{C}_L(X)$, $\bigwedge_{\lambda\in\Lambda}\lim\mathcal{F}_\lambda=\lim(\bigwedge_{\lambda\in\Lambda}\mathcal{F}_\lambda)$ and $\lim(\alpha\to\mathcal{F})=\alpha\to\lim\mathcal{F}$.

Proposition 4.6. *Let $(X,\lim)$ be an L-convergence space. Then* **(Lcn)** $\Longleftrightarrow$ **(Lcp)** $\Longleftrightarrow$ **(Lcq)**.

Proof. **(Lcn)** $\Longrightarrow$ **(Lcp)**: On the one hand,

$$\mathcal{S}(\mathcal{N}^x_{\lim},\mathcal{F})=\mathcal{S}(\mathcal{N}^x_{\lim},\mathcal{F})*\lim\mathcal{N}^x_{\lim}(x)\le\lim\mathcal{F}(x).$$

On the other hand,

$$\begin{array}{rcl}\mathcal{S}(\mathcal{N}^x_{\lim},\mathcal{F}) & = & \displaystyle\bigwedge_{A\in L^X}\Big(\mathcal{N}^x_{\lim}(A)\to\mathcal{F}(A)\Big)\\ & = & \displaystyle\bigwedge_{A\in L^X}\Big(\Big(\bigwedge_{\mathcal{G}\in\mathcal{C}_L(X)}\Big(\lim\mathcal{G}(x)\to\mathcal{G}(A)\Big)\Big)\to\mathcal{F}(A)\Big)\\ & \ge & \displaystyle\bigwedge_{A\in L^X}\Big(\Big(\lim\mathcal{F}(x)\to\mathcal{F}(A)\Big)\to\mathcal{F}(A)\Big)\\ & \ge & \lim\mathcal{F}(x).\end{array}$$

This proves that $\lim\mathcal{F}(x)=\mathcal{S}(\mathcal{N}^x_{\lim},\mathcal{F})$.

(Lcp) $\Longrightarrow$ **(Lcq)**: Take any $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}\subseteq\mathcal{C}_L(X)$. Then

$$\lim\Big(\bigwedge_{\lambda\in\Lambda}\mathcal{F}_\lambda\Big)=\mathcal{S}\Big(\mathcal{N}^x_{\lim},\bigwedge_{\lambda\in\Lambda}\mathcal{F}_\lambda\Big)=\bigwedge_{\lambda\in\Lambda}\mathcal{S}\Big(\mathcal{N}^x_{\lim},\mathcal{F}_\lambda\Big)=\bigwedge_{\lambda\in\Lambda}\lim\mathcal{F}_\lambda(x),$$

and

$$\lim(\alpha\to\mathcal{F})(x)=\mathcal{S}(\mathcal{N}^x_{\lim},\alpha\to\mathcal{F})=\alpha\to\mathcal{S}(\mathcal{N}^x_{\lim},\mathcal{F})=\alpha\to\lim\mathcal{F}(x).$$

(Lcq) $\Longrightarrow$ **(Lcn)**: By **(Lcq)**, we have

$$\begin{array}{rcl}\lim\mathcal{N}^x_{\lim}(x) & = & \displaystyle\lim\Big(\bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)}\Big(\lim\mathcal{F}(x)\to\mathcal{F}\Big)\Big)(x)\\ & = & \displaystyle\bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)}\Big(\lim(\lim\mathcal{F}(x)\to\mathcal{F})(x)\Big)\\ & = & \displaystyle\bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)}\Big(\lim\mathcal{F}(x)\to\lim\mathcal{F}(x)\Big)\\ & = & \top.\end{array}$$

□

For an L-convergence space $(X,\lim)$, define $\widehat{\mathcal{N}_{\lim}}:L^X\longrightarrow L^X$ by

$$\widehat{\mathcal{N}_{\lim}}(A)(x)=\mathcal{N}^x_{\lim}(A),$$

for each $A\in L^X$ and $x\in X$. Then we have

Proposition 4.7. *Let $(X, \lim)$ be an L-convergence space and $x \in X$. Then $\mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}} \in \mathcal{C}_L(X)$.*

Proof. It suffices to verify that $\mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}}$ satisfies (LCSC1)–(LCSC3), (LCSCs) and (LCSCcs).

For (LCSC1), since $\widehat{\mathcal{N}_{\lim}}(\top_X) = \top_X$, we have $\mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}}(\top_X) = \top$.

For (LCSC2), take any $A, B \in L^X$. Then

$$\begin{array}{rcll} \mathcal{S}(A,B) * \mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}}(A) & = & \mathcal{S}(A,B) * \mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(A)) & \\ & \leq & \mathcal{S}(\widehat{\mathcal{N}_{\lim}}(A), \widehat{\mathcal{N}_{\lim}}(B)) * \mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(A)) & \text{(by Proposition 2.4)} \\ & \leq & \mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(B)) & \\ & = & \mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}}(B). & \end{array}$$

For (LCSC3), let $\{A_j\}_{j\in J} \subseteq^{cdir} L^X$. Take any $A_{j_1}, A_{j_2} \in \{A_j\}_{j\in J}$. Then there exists $A_{j_3} \in \{A_j\}_{j\in J}$ such that $A_{j_3} \leq A_{j_1}, A_{j_2}$. Since $\mathcal{N}_{\lim}^x(A_{j_3}) \leq \mathcal{N}_{\lim}^x(A_{j_1}) \wedge \mathcal{N}_{\lim}^x(A_{j_2})$ for each $x \in X$, we have $\widehat{\mathcal{N}_{\lim}}(A_{j_3}) \leq \widehat{\mathcal{N}_{\lim}}(A_{j_1}) \wedge \widehat{\mathcal{N}_{\lim}}(A_{j_2})$. This shows that $\left\{\widehat{\mathcal{N}_{\lim}}(A_j)\right\}_{j\in J} \subseteq^{cdir} L^X$. Since

$$\bigwedge_{j\in J} \widehat{\mathcal{N}_{\lim}}(A_j)(x) = \bigwedge_{j\in J} \mathcal{N}_{\lim}^x(A_j) \leq \mathcal{N}_{\lim}^x\Big(\bigwedge_{j\in J} A_j\Big) = \widehat{\mathcal{N}_{\lim}}\Big(\bigwedge_{j\in J} A_j\Big)(x),$$

for each $x \in X$, we have $\bigwedge_{j\in J} \widehat{\mathcal{N}_{\lim}}(A_j) \leq \widehat{\mathcal{N}_{\lim}}\Big(\bigwedge_{j\in J} A_j\Big)$. So

$$\begin{array}{rcl} \bigwedge_{j\in J} \mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}}(A_j) & = & \bigwedge_{j\in J} \mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(A_j)) \\ & \leq & \mathcal{N}_{\lim}^x\Big(\bigwedge_{j\in J} \widehat{\mathcal{N}_{\lim}}(A_j)\Big) \\ & \leq & \mathcal{N}_{\lim}^x\Big(\widehat{\mathcal{N}_{\lim}}\Big(\bigwedge_{j\in J} A_j\Big)\Big) \\ & = & \mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}}\Big(\bigwedge_{j\in J} A_j\Big). \end{array}$$

For (LCSCs), since $\alpha * \widehat{\mathcal{N}_{\lim}}(A) \leq \widehat{\mathcal{N}_{\lim}}(\alpha * A)$, we have

$$\begin{array}{rcl} \alpha * \mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}}(A) & = & \alpha * \mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(A)) \\ & \leq & \mathcal{N}_{\lim}^x(\alpha * \widehat{\mathcal{N}_{\lim}}(A)) \\ & \leq & \mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(\alpha * A)) \\ & = & \mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}}(\alpha * A). \end{array}$$

For (LCSCcs), since $\alpha \to \widehat{\mathcal{N}_{\lim}}(A) \leq \widehat{\mathcal{N}_{\lim}}(\alpha \to A)$, we have

$$\begin{array}{rcl} \alpha \to \mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}}(A) & = & \alpha \to \mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(A)) \\ & \leq & \mathcal{N}_{\lim}^x(\alpha \to \widehat{\mathcal{N}_{\lim}}(A)) \\ & \leq & \mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(\alpha \to A)) \\ & = & \mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}}(\alpha \to A). \end{array}$$

□

Next we introduce the concept of concave L-convergence spaces and illustrate its relationship with strong L-concave spaces.

Definition 4.8. *A preconcave L-convergence space $(X, \lim)$ is called concave if it satisfies*

$$\textbf{(Lct)} \quad \mathcal{N}_{\lim}^x \leq \mathcal{N}_{\lim}^x \circ \widehat{\mathcal{N}_{\lim}}.$$

The full subcategory of **LCS** consisting of concave L-convergence spaces is denoted by **CLCS**.

Proposition 4.9. *Let $(X, \lim)$ be a concave L-convergence space. Define $\mathcal{C}^{\lim} \subseteq L^X$ by*

$$\mathcal{C}^{\lim} = \{A \in L^X \mid \forall \mathcal{F} \in \mathcal{C}_L(X), \forall x \in X,\ A(x) * \lim \mathcal{F}(x) \leq \mathcal{F}(A)\}.$$

Then $(X, \mathcal{C}^{\lim})$ is a strong L-concave space.

Proof. It suffices to verify that $\mathcal{C}^{\lim}$ satisfies (SLCS1)–(SLCS3), (SLCSs) and (SLCScs).

For (SLCS1), it is straightforward and is omitted.

For (SLCS2), if $\{A_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{C}^{\lim}$, that is,

$$\forall \mathcal{F} \in \mathcal{C}_L(X), \forall x \in X,\ A_\lambda(x) * \lim \mathcal{F}(x) \leq \mathcal{F}(A_\lambda),$$

for each $\lambda \in \Lambda$, then

$$\Big(\bigvee_{\lambda\in\Lambda} A_\lambda(x)\Big) * \lim \mathcal{F}(x) = \bigvee_{\lambda\in\Lambda} \Big(A_\lambda(x) * \lim \mathcal{F}(x)\Big) \leq \bigvee_{\lambda\in\Lambda} \mathcal{F}(A_\lambda) \leq \mathcal{F}\Big(\bigvee_{\lambda\in\Lambda} A_\lambda\Big).$$

For (SLCS3), if $\{A_j\}_{j\in J} \subseteq^{cdir} \mathcal{C}^{\lim}$, that is,

$$\forall \mathcal{F} \in \mathcal{C}_L(X), \forall x \in X,\ A_j(x) * \lim \mathcal{F}(x) \leq \mathcal{F}(A_j),$$

for each $j \in J$, then

$$\Big(\bigwedge_{j\in J} A_j(x)\Big) * \lim \mathcal{F}(x) \leq \bigwedge_{j\in J} \Big(A_j(x) * \lim \mathcal{F}(x)\Big) \leq \bigwedge_{j\in J} \mathcal{F}(A_j) \leq \mathcal{F}\Big(\bigwedge_{j\in J} A_j\Big).$$

For (SLCSs), if $\alpha \in L$ and $A \in \mathcal{C}^{\lim}$, then for each $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$, we have

$$(\alpha * A)(x) * \lim \mathcal{F}(x) = \alpha * \Big(A(x) * \lim \mathcal{F}(x)\Big) \leq \alpha * \mathcal{F}(A) \leq \mathcal{F}(\alpha * A).$$

For (SLCScs), if $\alpha \in L$ and $A \in \mathcal{C}^{\lim}$, then for each $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$, we have

$$(\alpha \to A)(x) * \lim \mathcal{F}(x) \leq \alpha \to \Big(A(x) * \lim \mathcal{F}(x)\Big) \leq \alpha \to \mathcal{F}(A) \leq \mathcal{F}(\alpha \to A).$$

□

By Proposition 3.5, we have $A \in \mathcal{C}$ if and only if $\mathcal{N}_{\mathcal{C}}^x(A) = A(x)$ for each $x \in X$. For a strong L-concave space $(X, \mathcal{C})$, define $\widehat{\mathcal{N}_{\mathcal{C}}} : L^X \longrightarrow L^X$ by

$$\widehat{\mathcal{N}_{\mathcal{C}}}(A)(x) = \mathcal{N}_{\mathcal{C}}^x(A),$$

for each $A \in L^X$ and $x \in X$. Then, we have

Lemma 4.10. *Let $(X, \mathcal{C})$ be a strong L-concave space and $x \in X$. Then*

$$\mathcal{N}_{\mathcal{C}}^x(A) = \mathcal{N}_{\mathcal{C}}^x(\widehat{\mathcal{N}_{\mathcal{C}}}(A)),$$

for each $A \in L^X$.

Proof. By the definition of $\mathcal{N}_{\mathcal{C}}^x$, we know that $\mathcal{N}_{\mathcal{C}}^x(A) \leq A(x)$ for each $A \in L^X$. Since

$$\widehat{\mathcal{N}_{\mathcal{C}}}(A)(x) = \mathcal{N}_{\mathcal{C}}^x(A) = \bigvee_{B\in\mathcal{C}, B\leq A} B(x),$$

it follows that

$$\widehat{\mathcal{N}_{\mathcal{C}}}(A) = \bigvee_{B\in\mathcal{C}, B\leq A} B \in \mathcal{C}.$$

So

$$\mathcal{N}_{\mathcal{C}}^x(A) = \widehat{\mathcal{N}_{\mathcal{C}}}(A)(x) = \mathcal{N}_{\mathcal{C}}^x(\widehat{\mathcal{N}_{\mathcal{C}}}(A)).$$

□

Lemma 4.11. *Let $(X,\mathcal{C})$ be a strong L-concave space. Define a map $\lim^{\mathcal{C}} : \mathcal{C}_L(X) \longrightarrow L^X$ by*

$$\lim^{\mathcal{C}}\mathcal{F}(x) = \bigwedge_{A\in\mathcal{C}} \Big(A(x) \to \mathcal{F}(A)\Big),$$

for each $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$. Then $(X, \lim^{\mathcal{C}})$ is an L-convergence space and $A(x) \le \mathcal{N}^x_{\lim^{\mathcal{C}}}(\widehat{\mathcal{N}_{\lim^{\mathcal{C}}}}(A))$ for each $A \in \mathcal{C}$.

Proof. It is straightforward to check that $(X, \lim^{\mathcal{C}})$ is an L-convergence space, so we only need to prove that $A(x) \le \mathcal{N}^x_{\lim^{\mathcal{C}}}(\widehat{\mathcal{N}_{\lim^{\mathcal{C}}}}(A))$ for each $A \in \mathcal{C}$.

Take any $A \in \mathcal{C}$ and $x \in X$. Then

$$\begin{aligned}
\widehat{\mathcal{N}_{\lim^{\mathcal{C}}}}(A)(x) &= \mathcal{N}^x_{\lim^{\mathcal{C}}}(A)\\
&= \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big(\lim^{\mathcal{C}}\mathcal{F}(x) \to \mathcal{F}(A)\Big)\\
&= \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big(\bigwedge_{B\in\mathcal{C}} \Big(B(x) \to \mathcal{F}(B)\Big) \to \mathcal{F}(A)\Big)\\
&\ge \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big((A(x) \to \mathcal{F}(A)) \to \mathcal{F}(A)\Big)\\
&\ge A(x).
\end{aligned}$$

It follows that $A \le \widehat{\mathcal{N}_{\lim^{\mathcal{C}}}}(A)$. Furthermore, we have

$$A(x) \le \widehat{\mathcal{N}_{\lim^{\mathcal{C}}}}(A)(x) = \mathcal{N}^x_{\lim^{\mathcal{C}}}(A) \le \mathcal{N}^x_{\lim^{\mathcal{C}}}(\widehat{\mathcal{N}_{\lim^{\mathcal{C}}}}(A)).$$

□

Proposition 4.12. *Let $(X,\mathcal{C})$ be a strong L-concave space. Define a map $\lim^{\mathcal{C}} : \mathcal{C}_L(X) \longrightarrow L^X$ as Lemma 4.11. Then $(X, \lim^{\mathcal{C}})$ is a concave L-convergence space.*

Proof. By Lemma 4.11, it suffices to verify that $\lim^{\mathcal{C}}$ satisfies **(Lcq)** and **(Lct)**.

For **(Lcq)**, take any $x \in X$, $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{C}_L(X)$. Then

$$\begin{aligned}
\lim^{\mathcal{C}}\Big(\bigwedge_{\lambda\in\Lambda}\mathcal{F}_\lambda\Big)(x) &= \bigwedge_{A\in\mathcal{C}} \Big(A(x) \to \bigwedge_{\lambda\in\Lambda}\mathcal{F}_\lambda(A)\Big)\\
&= \bigwedge_{\lambda\in\Lambda}\bigwedge_{A\in\mathcal{C}} \Big(A(x) \to \mathcal{F}_\lambda(A)\Big)\\
&= \bigwedge_{\lambda\in\Lambda} \lim^{\mathcal{C}}\mathcal{F}_\lambda(x).
\end{aligned}$$

Take any $\alpha \in L$ and $\mathcal{F} \in \mathcal{C}_L(X)$. Then

$$\begin{aligned}
(\alpha \to \lim^{\mathcal{C}}\mathcal{F})(x) &= \alpha \to \bigwedge_{A\in\mathcal{C}} \Big(A(x) \to \mathcal{F}(A)\Big)\\
&= \bigwedge_{A\in\mathcal{C}} \Big(\alpha \to (A(x) \to \mathcal{F}(A))\Big)\\
&= \bigwedge_{A\in\mathcal{C}} \Big(A(x) \to (\alpha \to \mathcal{F})(A)\Big)\\
&= \lim^{\mathcal{C}}(\alpha \to \mathcal{F})(x).
\end{aligned}$$

For **(Lct)**, take any $x \in X$. Then by Lemma 4.11, we have

$$\lim^{\mathcal{C}}\mathcal{N}^x_{\lim^{\mathcal{C}}} \circ \widehat{\mathcal{N}_{\lim^{\mathcal{C}}}}(x) = \bigwedge_{A\in\mathcal{C}} \Big(A(x) \to \mathcal{N}^x_{\lim^{\mathcal{C}}}(\widehat{\mathcal{N}_{\lim^{\mathcal{C}}}}(A))\Big) = \top.$$

By **(Lcp)**, it follows that $\mathcal{N}^x_{\lim^{\mathcal{C}}} \le \mathcal{N}^x_{\lim^{\mathcal{C}}} \circ \widehat{\mathcal{N}_{\lim^{\mathcal{C}}}}$. □

Proposition 4.13.

(1) If $f : (X, \lim^X) \longrightarrow (Y, \lim^Y)$ is continuous, then $f : (X, \mathcal{C}^{\lim^X}) \longrightarrow (Y, \mathcal{C}^{\lim^Y})$ is L-concavity-preserving;

(2) If $f : (X, \mathcal{C}^X) \longrightarrow (Y, \mathcal{C}^Y)$ is L-concavity-preserving, then $f : (X, \lim^{\mathcal{C}^X}) \longrightarrow (Y, \lim^{\mathcal{C}^Y})$ is continuous.

Proof. (1) It suffices to verify that $f^{\leftarrow}(B) \in \mathcal{C}^{\lim^X}$ for each $B \in \mathcal{C}^{\lim^Y}$. Take any $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$. Then

$$\begin{aligned} f^{\leftarrow}(B)(x) * \lim^X \mathcal{F}(x) &= B(f(x)) * \lim^X \mathcal{F}(x) \\ &\leq B(f(x)) * \lim^Y f^{\Rightarrow}(\mathcal{F})(f(x)) \\ &\leq f^{\Rightarrow}(\mathcal{F})(B) \\ &= \mathcal{F}(f^{\leftarrow}(B)). \end{aligned}$$

This shows that $f^{\leftarrow}(B) \in \mathcal{C}^{\lim^X}$. Thus, we know that $f : (X, \mathcal{C}^{\lim^X}) \longrightarrow (Y, \mathcal{C}^{\lim^Y})$ is L-concavity-preserving.

(2) It suffices to verify that $\lim^{\mathcal{C}^X} \mathcal{F}(x) \leq \lim^{\mathcal{C}^Y} f^{\Rightarrow}(\mathcal{F})(f(x))$ for each $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$. In fact,

$$\begin{aligned} \lim^{\mathcal{C}^Y} f^{\Rightarrow}(\mathcal{F})(f(x)) &= \bigwedge_{B \in \mathcal{C}^Y} \Big(B(f(x)) \to f^{\Rightarrow}(\mathcal{F})(B) \Big) \\ &\geq \bigwedge_{f^{\leftarrow}(B) \in \mathcal{C}^X} \Big(f^{\leftarrow}(B)(x) \to \mathcal{F}(f^{\leftarrow}(B)) \Big) \\ &\geq \bigwedge_{A \in \mathcal{C}^X} \Big(A(x) \to \mathcal{F}(A) \Big) \\ &= \lim^{\mathcal{C}^X} \mathcal{F}(x). \end{aligned}$$

This shows that $f : (X, \lim^{\mathcal{C}^X}) \longrightarrow (Y, \lim^{\mathcal{C}^Y})$ is continuous. □

Lemma 4.14. *Let* $(X, \lim)$ *be a concave L-convergence space. Then*

(1) $\mathcal{N}^x_{\lim} \leq \mathcal{N}^x_{\mathcal{C}^{\lim}}$;

(2) $\bigwedge_{A \in \mathcal{C}^{\lim}} \Big(A(x) \to \mathcal{F}(A) \Big) \leq \bigwedge_{B \in L^X} \Big(\mathcal{N}^x_{\mathcal{C}^{\lim}}(B) \to \mathcal{F}(B) \Big)$.

Proof. (1) Take any $A \in L^X$, $x \in X$ and $\mathcal{F} \in \mathcal{C}_L(X)$. On the one hand, we have

$$\widehat{\mathcal{N}_{\lim}}(A)(x) = \bigwedge_{\mathcal{G} \in \mathcal{C}_L(X)} \Big(\lim \mathcal{G}(x) \to \mathcal{G}(A) \Big) \leq A(x).$$

On the other hand, we have

$$\begin{aligned} \widehat{\mathcal{N}_{\lim}}(A)(x) * \lim \mathcal{F}(x) &= \mathcal{N}^x_{\lim}(A) * \lim \mathcal{F}(x) \\ &\leq \mathcal{N}^x_{\lim}(\widehat{\mathcal{N}_{\lim}}(A)) * \lim \mathcal{F}(x) \\ &= \bigwedge_{\mathcal{G} \in \mathcal{C}_L(X)} \Big(\lim \mathcal{G}(x) \to \mathcal{G}(\widehat{\mathcal{N}_{\lim}}(A)) \Big) * \lim \mathcal{F}(x) \\ &\leq (\lim \mathcal{F}(x) \to \mathcal{F}(\widehat{\mathcal{N}_{\lim}}(A))) * \lim \mathcal{F}(x) \\ &\leq \mathcal{F}(\widehat{\mathcal{N}_{\lim}}(A)). \end{aligned}$$

This implies that $\widehat{\mathcal{N}_{\lim}}(A) \in \mathcal{C}^{\lim}$ and $\widehat{\mathcal{N}_{\lim}}(A) \leq A$. By the definition of $\mathcal{N}^x_{\mathcal{C}^{\lim}}$, we have $\mathcal{N}^x_{\lim} \leq \mathcal{N}^x_{\mathcal{C}^{\lim}}$.

(2) Take any $B \in L^X$. Then by Lemma 4.10, we have

$$\begin{aligned} \mathcal{N}^x_{\mathcal{C}^{\lim}}(B) * \bigwedge_{A \in \mathcal{C}^{\lim}} \Big(A(x) \to \mathcal{F}(A) \Big) &= \mathcal{N}^x_{\mathcal{C}^{\lim}}(B) * \bigwedge_{A \in \mathcal{C}^{\lim}} \Big(\mathcal{N}^x_{\mathcal{C}^{\lim}}(A) \to \mathcal{F}(A) \Big) \\ &= \mathcal{N}^x_{\mathcal{C}^{\lim}}(\widehat{\mathcal{N}_{\mathcal{C}^{\lim}}}(B)) * \bigwedge_{A \in \mathcal{C}^{\lim}} \Big(\mathcal{N}^x_{\mathcal{C}^{\lim}}(A) \to \mathcal{F}(A) \Big) \\ &\leq \mathcal{N}^x_{\mathcal{C}^{\lim}}(\widehat{\mathcal{N}_{\mathcal{C}^{\lim}}}(B)) * \Big(\mathcal{N}^x_{\mathcal{C}^{\lim}}(\widehat{\mathcal{N}_{\mathcal{C}^{\lim}}}(B)) \to \mathcal{F}(\widehat{\mathcal{N}_{\mathcal{C}^{\lim}}}(B)) \Big) \\ &\leq \mathcal{F}(\widehat{\mathcal{N}_{\mathcal{C}^{\lim}}}(B)) \\ &\leq \mathcal{F}(B). \end{aligned}$$

This implies that $\bigwedge_{A\in\mathcal{C}^{\lim}}\Big(A(x)\to\mathcal{F}(A)\Big)\le\bigwedge_{B\in L^X}\Big(\mathcal{N}^x_{\mathcal{C}^{\lim}}(B)\to\mathcal{F}(B)\Big)$. $\square$

Proposition 4.15. *Suppose that $(X,\mathcal{C})$ is a strong L-concave space and $(X,\lim)$ is a concave L-convergence space. Then $\mathcal{C}^{\lim^{\mathcal{C}}}=\mathcal{C}$ and $\lim^{\mathcal{C}^{\lim}}=\lim$.*

Proof. For $\mathcal{C}^{\lim^{\mathcal{C}}}=\mathcal{C}$, take any $A\in L^X$. On the one hand, we have

$$\begin{aligned} A\in\mathcal{C} &\Longrightarrow \forall\mathcal{F}\in\mathcal{C}_L(X),\forall x\in X,\ A(x)*\bigwedge_{B\in\mathcal{C}}\Big(B(x)\to\mathcal{F}(B)\Big)\le\mathcal{F}(A)\\ &\Longleftrightarrow \forall\mathcal{F}\in\mathcal{C}_L(X),\forall x\in X,\ A(x)*\lim{}^{\mathcal{C}}\mathcal{F}(x)\le\mathcal{F}(A)\\ &\Longleftrightarrow A\in\mathcal{C}^{\lim^{\mathcal{C}}}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} A\in\mathcal{C}^{\lim^{\mathcal{C}}} &\Longleftrightarrow \forall\mathcal{F}\in\mathcal{C}_L(X),\forall x\in X,\ A(x)*\lim{}^{\mathcal{C}}\mathcal{F}(x)\le\mathcal{F}(A)\\ &\Longleftrightarrow \forall\mathcal{F}\in\mathcal{C}_L(X),\forall x\in X,\ A(x)*\bigwedge_{B\in\mathcal{C}}\Big(B(x)\to\mathcal{F}(B)\Big)\le\mathcal{F}(A)\\ &\Longrightarrow \forall x\in X,\ A(x)*\bigwedge_{B\in\mathcal{C}}\Big(B(x)\to\mathcal{N}^x_{\mathcal{C}}(B)\Big)\le\mathcal{N}^x_{\mathcal{C}}(A)\\ &\Longleftrightarrow \forall x\in X,\ A(x)\le\mathcal{N}^x_{\mathcal{C}}(A)\\ &\Longleftrightarrow A\in\mathcal{C}. \end{aligned}$$

This shows that $\mathcal{C}^{\lim^{\mathcal{C}}}=\mathcal{C}$.

For $\lim^{\mathcal{C}^{\lim}}=\lim$, take any $\mathcal{F}\in\mathcal{C}_L(X)$ and $x\in X$. On the one hand, we have

$$\begin{aligned} \lim{}^{\mathcal{C}^{\lim}}\mathcal{F}(x) &= \bigwedge_{A\in\mathcal{C}^{\lim}}\Big(A(x)\to\mathcal{F}(A)\Big)\\ &= \bigwedge_{\forall\mathcal{G}\in\mathcal{C}_L(X),\forall y\in X,\ A(y)*\lim\mathcal{G}(y)\le\mathcal{G}(A)}\Big(A(x)\to\mathcal{F}(A)\Big)\\ &= \bigwedge_{\forall\mathcal{G}\in\mathcal{C}_L(X),\forall y\in X,\ \lim\mathcal{G}(y)\le A(y)\to\mathcal{G}(A)}\Big(A(x)\to\mathcal{F}(A)\Big)\\ &\ge \lim\mathcal{F}(x). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \lim{}^{\mathcal{C}^{\lim}}\mathcal{F}(x) &= \bigwedge_{A\in\mathcal{C}^{\lim}}\Big(A(x)\to\mathcal{F}(A)\Big)\\ &\le \mathcal{S}(\mathcal{N}^x_{\mathcal{C}^{\lim}},\mathcal{F})\quad\text{(by Lemma 4.14 (2))}\\ &\le \mathcal{S}(\mathcal{N}^x_{\lim},\mathcal{F})\quad\text{(by Lemma 4.14 (1))}\\ &= \mathcal{S}(\mathcal{N}^x_{\lim},\mathcal{F})*\lim\mathcal{N}^x_{\lim}(x)\\ &\le \lim\mathcal{F}(x). \end{aligned}$$

This shows that $\lim^{\mathcal{C}^{\lim}}=\lim$. $\square$

By Propositions 4.9, 4.12, 4.13 and 4.15, we obtain the main result in this section.

Theorem 4.16. **CLCS** *is isomorphic to* **SLCS**.

5 Diagonal axioms for L-convergence spaces

In this section, we discuss the diagonal axioms for L-convergence spaces. Concretely, we first introduce a compression operator for an L-ordered co-Scott closed set. Then we present the Kowalsky diagonal axiom and the Fischer diagonal axiom, and show that a preconcave L-convergence spaces satisfies the Kowalsky diagonal axiom **(Lck)** if and only if it is concave, and an L-convergence spaces satisfies the Fischer diagonal axiom **(SLcf)** if and only if it is concave.

5.1 Compression operators

In Remark 3.2, we recall the definition of co-Scott closed sets on $\mathcal{P}(X)$ that can be considered as special case of L-ordered co-Scott closed sets on L^X. For convenience, let $\mathcal{C}_2(X)$ denote the set of all co-Scott closed sets on $\mathcal{P}(X)$.

Proposition 5.1. *Let X be a nonempty set, $\Theta \in \mathcal{C}_2(\mathcal{C}_2(X))$. Define $\kappa_X(\Theta) \subseteq \mathcal{P}(X)$ by*

$$A \in \kappa_X(\Theta) \Longleftrightarrow A^\sharp \in \Theta,$$

for each $A \in \mathcal{P}(X)$, where $A^\sharp = \{\mathbb{F} \in \mathcal{C}_2(X) \mid A \in \mathbb{F}\}$. Then $\kappa_X(\Theta) \in \mathcal{C}_2(X)$.

Proof. It suffices to verify that $\kappa_X(\Theta)$ satisfies (CSC1) and (CSC2).

For (CSC1), let $A \in \kappa_X(\Theta)$ and $A \subseteq B$. Then we have $A^\sharp \in \Theta$ and $A^\sharp \subseteq B^\sharp$. This implies that $B^\sharp \in \Theta$. It follows that $B \in \kappa_X(\Theta)$.

For (CSC2), take each $\{A_j\}_{j\in J} \subseteq^{cdir} \kappa_X(\Theta)$. Then $A_j^\sharp \in \Theta$ for each $j \in J$. By the co-directedness of $\{A_j\}_{j\in J}$, we have $\{A_j^\sharp\}_{j\in J} \subseteq^{cdir} \Theta$. This implies that $\bigcap_{j\in J} A_j^\sharp \in \Theta$. Since

$$\begin{aligned}
\mathbb{F} \in \bigcap_{j\in J} A_j^\sharp &\Longleftrightarrow \forall\, j \in J,\ \mathbb{F} \in A_j^\sharp \\
&\Longleftrightarrow \forall\, j \in J,\ A_j \in \mathbb{F} \\
&\Longleftrightarrow \bigcap_{j\in J} A_j \in \mathbb{F} \\
&\Longleftrightarrow \mathbb{F} \in \Big(\bigcap_{j\in J} A_j\Big)^\sharp,
\end{aligned}$$

we have $\left(\bigcap_{j\in J} A_j\right)^\sharp \in \Theta$. This shows that $\bigcap_{j\in J} A_j \in \kappa_X(\Theta)$. □

Adopting the terminology in general topology, κ_X in the above proposition is called a compression operator with respect to co-Scott closed set on $\mathcal{P}(X)$. Correspondingly, $\kappa_X(\Theta)$ is called the diagonal co-Scott closed set of Θ. Now let us generalize these two concepts to the lattice-valued case.

Proposition 5.2. *Let X be a nonempty set, $\theta \in \mathcal{C}_L(\mathcal{C}_L(X))$. Define $\kappa_X(\theta) : L^X \longrightarrow L$ by*

$$\kappa_X(\theta)(A) = \theta(e_A),$$

where $e_A : \mathcal{C}_L(X) \longrightarrow L$ is defined by $e_A(\mathcal{F}) = \mathcal{F}(A)$ for each $\mathcal{F} \in \mathcal{C}_L(X)$. Then $\kappa_X(\theta) \in \mathcal{C}_L(X)$.

Proof. It suffices to verify that $\kappa_X(\theta)$ satisfies (LCSC1)–(LCSC3), (LCSCs) and (LCSCcs).

For (LCSC1), since $e_{\top_X}(\mathcal{F}) = \mathcal{F}(\top_X) = \top_{\mathcal{C}_L(X)}(\mathcal{F})$ for each $\mathcal{F} \in \mathcal{C}_L(X)$, we have $e_{\top_X} = \top_{\mathcal{C}_L(X)}$. So

$$\kappa_X(\theta)(\top_X) = \theta(e_{\top_X}) = \theta(\top_{\mathcal{C}_L(X)}) = \top.$$

For (LCSC2), since $\mathcal{S}(A,B) \leq \mathcal{F}(A) \to \mathcal{F}(B)$ for each $\mathcal{F} \in \mathcal{C}_L(X)$, we have $\mathcal{S}(A,B) \leq \mathcal{S}(e_A, e_B)$. So

$$\mathcal{S}(A,B) * \kappa_X(\theta)(A) = \mathcal{S}(A,B) * \theta(e_A) \leq \mathcal{S}(e_A,e_B) * \theta(e_A) \leq \theta(e_B) = \kappa_X(\theta)(B).$$

For (LCSC3), let $\{A_j\}_{j\in J} \subseteq^{cdir} L^X$. Then $\{e_{A_j}\}_{j\in J} \subseteq^{cdir} L^{\mathcal{C}_L(X)}$ and $\bigwedge_{j\in J} e_{A_j} \leq e_{\bigwedge_{j\in J} A_j}$. So

$$\bigwedge_{j\in J} \kappa_X(\theta)(A_j) = \bigwedge_{j\in J} \theta(e_{A_j}) \leq \theta\Big(\bigwedge_{j\in J} e_{A_j}\Big) \leq \theta(e_{\bigwedge_{j\in J} A_j}) = \kappa_X(\theta)\Big(\bigwedge_{j\in J} A_j\Big).$$

For (LCSCs), since $\alpha * e_A \leq e_{\alpha * A}$, we have

$$\alpha * \kappa_X(\theta)(A) = \alpha * \theta(e_A) \leq \theta(\alpha * e_A) \leq \theta(e_{\alpha * A}) = \kappa_X(\theta)(\alpha * A).$$

For (LCSCcs), since $\alpha \to e_A \leq e_{\alpha \to A}$, we have

$$\alpha \to \kappa_X(\theta)(A) = \alpha \to \theta(e_A) \leq \theta(\alpha \to e_A) \leq \theta(e_{\alpha \to A}) = \kappa_X(\theta)(\alpha \to A).$$

□

Note that $\kappa_X(\theta)$ in the above proposition is the diagonal L-ordered co-Scott closed set of θ. In particular, by Proposition 5.2, we have the following corollary.

Corollary 5.3. *Let X be a nonempty set, $\mathcal{F} \in \mathcal{C}_L(X)$ and $\sigma_{(\cdot)} : X \longrightarrow \mathcal{C}_L(X)$ be maps. Then $\kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F})) \in \mathcal{C}_L(X)$.*

Remark 5.4. *Take any $A \in L^X$. Then we have $\kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))(A) = \sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F})(e_A) = \mathcal{F}(\sigma_{(\cdot)}^{\Leftarrow}(e_A)) = \mathcal{F}(e_A \circ \sigma_{(\cdot)}) = \mathcal{F}(\sigma_{(\cdot)}(A))$. It follows that $\kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F})) = \mathcal{F}(\sigma_{(\cdot)})$.*

5.2 Diagonal axioms

In [13], Kowalsky defined a diagonal axiom for convergence structures in general topology, which can be used to ensure a pretopological convergence structure to be topological. For simplicity, this axiom is denoted by **(K)**. Concretely, for each filter F and a family of filters $\{\sigma_y \mid y \in X\}$ on X,

$$\textbf{(K)} \quad \mathrm{F} \longrightarrow x,\ \sigma_y \longrightarrow y\ (\forall y \in X) \implies \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathrm{F})) \longrightarrow x,$$

where $\mathrm{F} \longrightarrow x$ means that F converges to x. Now, we adopt this approach to propose two lattice-valued counterparts of Kowalsky diagonal axioms for an L-convergence space.

(Lck) $\forall \sigma_{(\cdot)} : X \longrightarrow \mathcal{C}_L(X)$ and $\forall \mathcal{F} \in \mathcal{C}_L(X)$:

$$\lim \mathcal{F}(x) * \bigwedge_{y \in X} \lim \sigma_y(y) \leq \lim \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))(x).$$

(Lck′) $\forall\ \sigma_{(\cdot)} : X \longrightarrow \mathcal{C}_L(X)$ with $\forall y \in X, \lim \sigma_y(y) = \top$ and $\forall \mathcal{F} \in \mathcal{C}_L(X)$:

$$\lim \mathcal{F}(x) \leq \lim \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))(x).$$

Proposition 5.5. *For a preconcave L-convergence space $(X, \lim)$,* **(Lct)** $\Longleftrightarrow$ **(Lck)** $\Longleftrightarrow$ **(Lck′)**.

Proof. **(Lct)** $\Longrightarrow$ **(Lck)** : Let $\alpha = \bigwedge_{y \in X} \lim \sigma_y(y)$. Then for each $y \in X$, we have

$$\alpha \leq \lim \sigma_y(y) = \bigwedge_{A \in L^X} \Big(\mathcal{N}_{\lim}^y(A) \to \sigma_y(A)\Big).$$

Now fix an L-set $A \in L^X$. It follows that $\alpha * \mathcal{N}_{\lim}^y(A) \leq \sigma_y(A)$. That is, $\alpha * \widehat{\mathcal{N}_{\lim}}(A) \leq \sigma_{(\cdot)}(A)$. For each $\mathcal{F} \in \mathcal{C}_L(X)$, we have

$$\alpha * \mathcal{F}(\widehat{\mathcal{N}_{\lim}}(A)) \leq \mathcal{F}(\alpha * \widehat{\mathcal{N}_{\lim}}(A)) \leq \mathcal{F}(\sigma_{(\cdot)}(A)).$$

Then

$$\begin{aligned}
\lim \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))(x) &= \bigwedge_{A \in L^X} \Big(\mathcal{N}_{\lim}^x(A) \to \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))(A)\Big) \\
&= \bigwedge_{A \in L^X} \Big(\mathcal{N}_{\lim}^x(A) \to \mathcal{F}(\sigma_{(\cdot)}(A))\Big) \\
&\geq \bigwedge_{A \in L^X} \Big(\mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(A)) \to \mathcal{F}(\sigma_{(\cdot)}(A))\Big) \\
&\geq \bigwedge_{A \in L^X} \Big(\mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(A)) \to \Big(\alpha * \mathcal{F}(\widehat{\mathcal{N}_{\lim}}(A)\Big)\Big) \\
&\geq \bigwedge_{A \in L^X} \Big(\alpha * \Big(\mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(A)) \to \mathcal{F}(\widehat{\mathcal{N}_{\lim}}(A)\Big)\Big) \\
&\geq \alpha * \bigwedge_{A \in L^X} \Big(\mathcal{N}_{\lim}^x(\widehat{\mathcal{N}_{\lim}}(A)) \to \mathcal{F}(\widehat{\mathcal{N}_{\lim}}(A)\Big) \\
&\geq \alpha * \bigwedge_{A \in L^X} \Big(\mathcal{N}_{\lim}^x(A) \to \mathcal{F}(A\Big) \\
&= \alpha * \lim \mathcal{F}(x) \\
&= \bigwedge_{y \in X} \lim \sigma_y(y) * \lim \mathcal{F}(x).
\end{aligned}$$

(Lck) $\Longrightarrow$ **(Lck′)** : It is straightforward and is omitted.
(Lck′) $\Longrightarrow$ **(Lct)** : Let $\mathcal{F}=\mathcal{N}^x_{\lim}$ and $\sigma_y=\mathcal{N}^y_{\lim}$ for each $y\in Y$. By **(Lck′)**, we have

$$\begin{aligned}\top &= \lim\mathcal{N}^x_{\lim}(x)\\ &\leq \lim\kappa(\sigma^{\Rightarrow}_{(\cdot)}(\mathcal{N}^x_{\lim}))(x)\\ &= \bigwedge_{A\in L^X}\Big(\mathcal{N}^x_{\lim}(A)\to\kappa(\sigma^{\Rightarrow}_{(\cdot)}(\mathcal{N}^x_{\lim}))(A)\Big)\\ &= \bigwedge_{A\in L^X}\Big(\mathcal{N}^x_{\lim}(A)\to\mathcal{N}^x_{\lim}(\sigma_{(\cdot)}(A))\Big)\\ &= \bigwedge_{A\in L^X}\Big(\mathcal{N}^x_{\lim}(A)\to\mathcal{N}^x_{\lim}(\widehat{\mathcal{N}_{\lim}}(A))\Big).\end{aligned}$$

This implies that $\mathcal{N}^x_{\lim}\leq\mathcal{N}^x_{\lim}\circ\widehat{\mathcal{N}_{\lim}}$. □

In general topology, Cook and Fischer [5] gave a stronger version of the Kowalsky diagonal condition (denoted by **(F)**) via diagonal filters, which is called the Fischer diagonal axiom. The Fischer diagonal axiom can be used to ensure a convergence structure to be topological.

(F) Let Λ be a nonempty set, $\psi:\Lambda\longrightarrow X$ be a map and $\{\sigma_\lambda\mid\lambda\in\Lambda\}$ be a family of filters on X. Then for each filter F on Λ and $x\in X$,

$$\psi^{\Rightarrow}(\mathrm{F})\longrightarrow x,\sigma_\lambda\longrightarrow\psi(\lambda)\ (\forall\lambda\in\Lambda)\Longrightarrow\kappa(\sigma^{\Rightarrow}_{(\cdot)}(\mathrm{F}))\longrightarrow x.$$

Considering the lattice-valued counterpart of the Fischer diagonal axiom, we give the following Fischer diagonal axiom for an L-convergence space.

(Lcf) Let Λ be a nonempty set, $\psi:\Lambda\longrightarrow X$ and $\sigma_{(\cdot)}:\Lambda\longrightarrow\mathcal{C}_L(X)$ be maps. Then for each $\mathcal{F}\in\mathcal{C}_L(\Lambda)$ and $x\in X$,

$$\lim\psi^{\Rightarrow}(\mathcal{F})(x)*\bigwedge_{\lambda\in\Lambda}\lim\sigma_\lambda(\psi(\lambda))\leq\lim\kappa(\sigma^{\Rightarrow}_{(\cdot)}(\mathcal{F}))(x).$$

Next, let us discuss the relationship between two diagonal axioms, preconcave axioms and concave axioms.

Proposition 5.6. *For an L-convergence space,* **(Lcq)**+**(Lck)** $\Longrightarrow$ **(Lcf)**.

Proof. Let Λ be a nonempty set, $\mathcal{F}\in\mathcal{C}_L(\Lambda)$, $\psi:\Lambda\longrightarrow X$ and $\sigma_{(\cdot)}:\Lambda\longrightarrow\mathcal{C}_L(X)$ be maps. Define $\{\mathcal{F}_y\mid y\in X\}$ by

$$\mathcal{F}_y=\begin{cases}\bigwedge_{\lambda\in\psi^{\leftarrow}(y)}\sigma_\lambda, & y\in\psi(\Lambda),\\ [y], & y\in X/\psi(\Lambda).\end{cases}$$

Take any $A\in L^X$ and $\lambda\in\Lambda$. Then

$$\psi^{\leftarrow}(\mathcal{F}_{(\cdot)}(A))(\lambda)=\mathcal{F}_{(\cdot)}(A)(\psi(\lambda))=\mathcal{F}_{\psi(\lambda)}(A)\leq\sigma_\lambda(A)=\sigma_{(\cdot)}(A)(\lambda).$$

It follows that $\psi^{\leftarrow}(\mathcal{F}_{(\cdot)}(A))\leq\sigma_{(\cdot)}(A)$. So we have $\psi^{\Rightarrow}(\mathcal{F})(\mathcal{F}_{(\cdot)}(A))=\mathcal{F}(\psi^{\leftarrow}(\mathcal{F}_{(\cdot)}(A)))\leq\mathcal{F}(\sigma_{(\cdot)}(A))$. This implies that $\psi^{\Rightarrow}(\mathcal{F})(\mathcal{F}_{(\cdot)})\leq\mathcal{F}(\sigma_{(\cdot)})$. Furthermore, we have

$$\begin{aligned}\bigwedge_{y\in X}\lim\mathcal{F}_y(y) &= \Big(\bigwedge_{y\in\psi(\Lambda)}\lim\mathcal{F}_y(y)\Big)\wedge\Big(\bigwedge_{y\in X/\psi(\Lambda)}\lim\mathcal{F}_y(y)\Big)\\ &= \Big(\bigwedge_{y\in\psi(\Lambda)}\lim\bigwedge_{\lambda\in\psi^{\leftarrow}(y)}\sigma_\lambda(y)\Big)\wedge\Big(\bigwedge_{y\in X/\psi(\Lambda)}\lim[y](y)\Big)\\ &= \Big(\bigwedge_{y\in\psi(\Lambda)}\bigwedge_{\lambda\in\psi^{\leftarrow}(y)}\lim\sigma_\lambda(y)\Big)\wedge\Big(\bigwedge_{y\in X/\psi(\Lambda)}\lim[y](y)\Big)\quad(\text{by }\mathbf{(Lcq)})\\ &= \bigwedge_{\lambda\in\Lambda}\lim\sigma_\lambda(\psi(\lambda)).\end{aligned}$$

Take any $x \in X$. Then

$$\begin{aligned}
\lim \psi^{\Rightarrow}(\mathcal{F})(x) * \bigwedge_{\lambda\in\Lambda} \lim \sigma_\lambda(\psi(\lambda)) &= \lim \psi^{\Rightarrow}(\mathcal{F})(x) * \bigwedge_{y\in X} \lim \mathcal{F}_y(y) \\
&\leq \lim \kappa(\mathcal{F}_{(\cdot)}^{\Rightarrow}(\psi^{\Rightarrow}(\mathcal{F})))(x) \quad (\text{by } \mathbf{(Lck)}) \\
&= \lim \psi^{\Rightarrow}(\mathcal{F})(\mathcal{F}_{(\cdot)})(x) \quad (\text{by Remark 5.4}) \\
&\leq \lim \mathcal{F}(\sigma_{(\cdot)})(x) \\
&= \lim \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))(x).
\end{aligned}$$

□

In the following example, we will show **(Lcf)** does not imply **(Lcq)** for an L-convergence space.

Example 5.7. *Let L be the linearly ordered frame $(\{\bot, \alpha, \top\}, \wedge, \top)$ with $\bot < \alpha < \top$. Assume $X = \{x, y\}$. For each $\mathcal{F} \in \mathcal{C}_L(X)$ and $z \in X$, let*

$$\lim \mathcal{F}(z) = \begin{cases} \top, & \text{if } \mathcal{F} \geq [z], \\ \alpha, & \text{otherwise.} \end{cases}$$

It is easy to verify that $(X, \lim)$ is an L-convergence space that satisfies **(Lcf)**. *However, $(X, \lim)$ does not satisfy* **(Lcq)**. *Let $\mathcal{F} = [x]$. Then $(\alpha \to \lim[x])(y) = \top$. Since $\alpha \to [x] \ngeq [y]$, we have $\lim(\alpha \to [x])(y) = \alpha$. Hence, $\alpha \to \lim[x] \neq \lim(\alpha \to [x])$.*

Since **(Lcf)** does not imply **(Lcq)**, an L-convergence space satisfying **(Lcf)** did not to be concave. Hence, we consider the following axiom **(SLcf)** as a stronger version of **(Lcf)**.

(SLcf) Let Λ be a nonempty set, $\psi : \Lambda \longrightarrow X$ and $\sigma_{(\cdot)} : \Lambda \longrightarrow \mathcal{C}_L(X)$ be maps. For each $\mathcal{F} \in \mathcal{C}_L(\Lambda)$ and $x \in X$, if $\lim \sigma_\lambda(\psi(\lambda)) = \top$ and $\sigma_\lambda \leq [\psi(\lambda)]$ for each $\lambda \in \Lambda$, then $\lim \psi^{\Rightarrow}(\mathcal{F})(x) = \mathcal{S}(\mathcal{N}^x_{\lim}, \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F})))$.

In order to characterize concave L-convergence structures by means of **(Lcf)**, we first introduce a weaker version axiom of **(Lcf)**.

(WLcf) Let Λ be a nonempty set, $\psi : \Lambda \longrightarrow X$ and $\sigma_{(\cdot)} : \Lambda \longrightarrow \mathcal{C}_L(X)$ be maps. For each $\mathcal{F} \in \mathcal{C}_L(\Lambda)$, if $\lim \sigma_\lambda(\psi(\lambda)) = \top$ for each $\lambda \in \Lambda$, then for each $x \in X$, $\lim \psi^{\Rightarrow}(\mathcal{F})(x) \leq \lim \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))(x)$.

Proposition 5.8. *For an L-convergence space $(X, \lim)$,* **(WLcf)** + **(Lcp)** $\Rightarrow$ **(Lcf)**.

Proof. It is clear that **(WLcf)** implies **(Lck′)**, so it follows from Propositions 4.6, 5.5 and 5.6 that **(WLcf)** + **(Lcp)** $\Rightarrow$ **(Lcf)**. □

Proposition 5.9. *For an L-convergence space $(X, \lim)$,* **(SLcf)** $\Longleftrightarrow$ **(Lcf)** + **(Lcp)**.

Proof. **(SLcf)** $\Longrightarrow$ **(Lcp)** : Let $\psi = id_X$ and $\sigma_{(\cdot)} : X \longrightarrow \mathcal{C}_L(X)$ be given by $\sigma_x = [x]$. Then $\kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F})) = \mathcal{F}$. So we have

$$\lim \mathcal{F}(x) = \lim \psi^{\Rightarrow}(\mathcal{F})(x) = \mathcal{S}(\mathcal{N}^x_{\lim}, \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))) = \mathcal{S}(\mathcal{N}^x_{\lim}, \mathcal{F}).$$

(SLcf) $\Longrightarrow$ **(Lcf)** : It suffices to verify that **(SLcf)** $\Longrightarrow$ **(WLcf)**. Let $\sigma_{(\cdot)}$ and ψ be maps satisfying the condition in **(WLcf)**. Define a map $\overline{\sigma_{(\cdot)}} : \Lambda \longrightarrow \mathcal{C}_L(X)$ by $\overline{\sigma_\lambda} = \sigma_\lambda \wedge [\psi(\lambda)]$. Then we have

$$\begin{aligned}
\lim \overline{\sigma_\lambda}(\psi(\lambda)) &= \lim(\sigma_\lambda \wedge [\psi(\lambda)])(\psi(\lambda)) \\
&= \lim \sigma_\lambda(\psi(\lambda)) \wedge \lim[\psi(\lambda)](\psi(\lambda)) \ (\text{by } \mathbf{(Lcq)}) \\
&= \top,
\end{aligned}$$

and $\overline{\sigma_\lambda} \leq [\psi(\lambda)]$ for each $\lambda \in \Lambda$. Hence, by **(SLcf)**, we have

$$\lim \psi^{\Rightarrow}(\mathcal{F})(x) = \mathcal{S}(\mathcal{N}^x_{\lim}, \kappa(\overline{\sigma_{(\cdot)}}^{\Rightarrow}(\mathcal{F}))) = \lim \kappa(\overline{\sigma_{(\cdot)}}^{\Rightarrow}(\mathcal{F}))(x) \leq \lim \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))(x).$$

(Lcf) + **(Lcp)** $\Longrightarrow$ **(SLcf)** : Let $\sigma_{(\cdot)}$ and ψ be maps satisfying the condition in **(SLcf)**. On the one hand, by **(Lcf)**, we have

$$\lim \psi^{\Rightarrow}(\mathcal{F})(x) \leq \lim \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))(x) = \mathcal{S}(\mathcal{N}^x_{\lim}, \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))).$$

On the other hand, we have $\sigma_\lambda(A) \leq [\psi(\lambda)](A) = A \circ \psi(\lambda)$ for each $A \in L^X$ and $\lambda \in \Lambda$. It follows that $\sigma_{(\cdot)}(A) \leq A \circ \psi$. Hence

$$\kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))(A) = \mathcal{F}(\sigma_{(\cdot)}(A)) \leq \mathcal{F}(A \circ \psi) = \psi^{\Rightarrow}(\mathcal{F})(A).$$

By **(Lcp)**, we have

$$\mathcal{S}(\mathcal{N}_{\lim}^{x}, \kappa(\sigma_{(\cdot)}^{\Rightarrow}(\mathcal{F}))) \leq \mathcal{S}(\mathcal{N}_{\lim}^{x}, \psi^{\Rightarrow}(\mathcal{F})) = \lim \psi^{\Rightarrow}(\mathcal{F})(x).$$

□

Now we obtain the main result in this subsection.

Theorem 5.10. *For an L-convergence space $(X, \lim)$,* **(SLcf)** $\Longleftrightarrow$ **(Lcp)** + **(Lck)**. *That is, an L-convergence space $(X, \lim)$ is concave if and only if it satisfies the axiom* **(SLcf)**.

Proof. Since (**Lcf**) implies (**Lck**), it follows from Propositions 5.9 that the necessity holds. By Propositions 5.6 and 5.9, it follows that the sufficiency holds. □

Acknowledgement

The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the referees. This work is supported by the Natural Science Foundation of China (Nos. 12071033, 12271036, 12471428) and Beijing Institute of Technology Science and Technology Innovation Plan Cultivation Project (No. 2021CX01030).

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