

ON APPROXIMATE CAUCHY EQUATION IN FELBIN'S TYPE FUZZY NORMED LINEAR SPACES

I. SADEQI, F. MORADLOU AND M. SALEHI

ABSTRACT. In this paper we study the Hyers-Ulam-Rassias stability of Cauchy equation in Felbin's type fuzzy normed linear spaces. As a result we give an example of a fuzzy normed linear space such that the fuzzy version of the stability problem remains true, while it fails to be correct in classical analysis. This shows how the category of fuzzy normed linear spaces differs from the classical normed linear spaces in general.

1. Introduction

In [7] Grantner et al. takes the fuzzy real number as a decreasing mapping from the real line to the unit interval or lattice in general. Lowen [18] applies the fuzzy real numbers as non-decreasing, left continuous mapping from the real line to the unit interval so that its supremum over \mathbb{R} is 1. Also fuzzy arithmetic operations on L-fuzzy real line were studied by Rodabaugh [35], where he showed that the binary addition is the only extension of addition to $\mathbb{R}(L)$. Hoehle [9] especially emphasized the role of fuzzy real numbers as modeling a fuzzy threshold softening the notion of Dedekind cut. Maturo [19], studied three possible types of alternative operations in the fuzzy numbers which, among these operations, there are additions and multiplications that satisfy the distributive property. In this paper a fuzzy real number is taken as a fuzzy normal and convex mapping from the real line to the unit interval.

The concept of the fuzzy metric space has been studied by Kaleva [14, 15] by using fuzzy number as a fuzzy set on the real axis. Kaleva also has recently showed that a fuzzy metric space can be embedded in a complete fuzzy metric space [16]. In [6], Felbin introduced the concept of fuzzy normed linear space (FNLS); Xiao and Zhu [39] studied its linear topological structures and some basic properties of a fuzzy normed linear space. It is known that theories of classical normed space and Menger probabilistic normed spaces are special cases of fuzzy normed linear spaces.

The stability problem of functional equations originated from a question of Ulam [38] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let (G_2, \diamond, d) be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

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for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \rightarrow G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable.

The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces.

In 1978, Th. M. Rassias [30] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded*. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [8] generalized the Rassias' result. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [5], [11]–[34]).

The authors in [20] showed that an approximately additive function can be approximated by an additive mapping in Samanta's type of fuzzy normed space. In this paper we study the Hyers–Ulam–Rassias stability problem in Febin's type fuzzy normed linear spaces. Also we show that $C(\Omega)$, in general, is a fuzzy normed linear space, while it is not classical normable when Ω is an open subset of \mathbb{R}^n .

2. Fuzzy Real Number

Let η be a fuzzy subset on \mathbb{R} , i.e., a mapping $\eta : \mathbb{R} \rightarrow [0, 1]$ associating with each real number t its grade of membership $\eta(t)$.

Definition 2.1. [6] A fuzzy subset η on \mathbb{R} is called a fuzzy real number, whose α -level set is denoted by $[\eta]_\alpha$, i.e., $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies two axioms:

(N1) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.

(N2) For each $\alpha \in (0, 1]$, $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha]$ where $-\infty < \eta^-_\alpha \leq \eta^+_\alpha < +\infty$.

The set of all fuzzy real numbers denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers.

The number $\bar{0}$ stands for the fuzzy real number as:

$$\bar{0}(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

Clearly, $\bar{0} \in F^*(\mathbb{R})$. Also the set of all real numbers can be embedded in $F(\mathbb{R})$ because if $r \in (-\infty, \infty)$, then $\bar{r} \in F(\mathbb{R})$ satisfies $\bar{r}(t) = \bar{0}(t - r)$.

Theorem 2.2. [2] Let $\{[a_\alpha, b_\alpha]; \alpha \in (0, 1]\}$ be a family of nested bounded closed intervals. Let $\eta : \mathbb{R} \rightarrow [0, 1]$ be a function defined by

$$\eta(t) = \sup\{\alpha \in (0, 1] : t \in [a_\alpha, b_\alpha]\}.$$

Then η is a fuzzy real number.

Theorem 2.3. [1] Let $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, be a family of non-empty intervals. If

(i) $[a_{\alpha_1}, b_{\alpha_1}] \supset [a_{\alpha_2}, b_{\alpha_2}]$ for all $0 < \alpha_1 \leq \alpha_2$.

(ii) $[\lim_{k \rightarrow \infty} a_{\alpha_k}, \lim_{k \rightarrow \infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$ whenever $\{\alpha_k\}$ is an increasing sequence in $(0, 1]$ converging to α .

Then the family $[a_\alpha, b_\alpha]$ represents the α -level sets of a fuzzy real number η such that $\eta(t) = \sup\{\alpha \in (0, 1] : t \in [a_\alpha, b_\alpha]\}$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+] = [a_\alpha, b_\alpha]$.

Definition 2.4. [6] Fuzzy arithmetic operations $\oplus, \ominus, \otimes, \odot$ on $F(\mathbb{R}) \times F(\mathbb{R})$ can be defined as:

$$(1) \quad (\eta \oplus \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(t-s)\}, \quad t \in \mathbb{R},$$

$$(2) \quad (\eta \ominus \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(s) \wedge \delta(s-t)\}, \quad t \in \mathbb{R},$$

$$(3) \quad (\eta \otimes \delta)(t) = \sup_{s \in \mathbb{R}, s \neq 0} \{\eta(s) \wedge \delta(t/s)\}, \quad t \in \mathbb{R},$$

$$(4) \quad (\eta \odot \delta)(t) = \sup_{s \in \mathbb{R}} \{\eta(st) \wedge \delta(s)\}, \quad t \in \mathbb{R}.$$

The additive and multiplicative identities in $F(\mathbb{R})$ are $\bar{0}$ and $\bar{1}$, respectively. Let $\ominus \eta$ be defined as $\bar{0} \ominus \eta$. It is clear that $\eta \ominus \delta = \eta \oplus (\ominus \delta)$.

Definition 2.5. [6] For $k \in \mathbb{R} \setminus \{0\}$, fuzzy scalar multiplication $k \odot \eta$ is defined as $(k \odot \eta)(t) = \eta(t/k)$ and $0 \odot \eta$ is defined to be $\bar{0}$.

Lemma 2.6. Let η, δ be fuzzy real numbers. Then

$$\forall t \in \mathbb{R}, \quad \eta(t) = \delta(t) \Leftrightarrow \forall \alpha \in (0, 1], \quad [\eta]_\alpha = [\delta]_\alpha.$$

Proof. The proof of (\Rightarrow) is clear. For (\Leftarrow) , suppose there exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) \neq \delta(t_0)$ and for all $\alpha \in (0, 1]$, $[\eta]_\alpha = [\delta]_\alpha$. Since $\eta(t_0) \neq \delta(t_0)$, we can assume that $\eta(t_0) < \delta(t_0)$ then there exists $\alpha \in (0, 1)$ such that $\eta(t_0) < \alpha < \delta(t_0)$. From $\alpha < \delta(t_0)$, we have $\delta_\alpha^- \leq t_0 \leq \delta_\alpha^+$ and since $\eta_\alpha^\pm = \delta_\alpha^\pm$ so $\eta(t_0) \geq \alpha$, which contradicts $\eta(t_0) < \alpha$. \square

Lemma 2.7. [6] Let $\eta, \delta \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\delta]_\alpha = [\delta_\alpha^-, \delta_\alpha^+]$. Then

$$(1) \quad [\eta \oplus \delta]_\alpha = [\eta_\alpha^- + \delta_\alpha^-, \eta_\alpha^+ + \delta_\alpha^+],$$

$$(2) \quad [\eta \ominus \delta]_\alpha = [\eta_\alpha^- - \delta_\alpha^+, \eta_\alpha^+ - \delta_\alpha^-],$$

$$(3) \quad [\eta \otimes \delta]_\alpha = [\eta_\alpha^- \delta_\alpha^-, \eta_\alpha^+ \delta_\alpha^+], \quad \eta, \delta \in F^*(\mathbb{R}),$$

$$(4) \quad [\bar{1} \odot \delta]_\alpha = [1/\delta_\alpha^+, 1/\delta_\alpha^-], \quad \delta_\alpha^- > 0.$$

Definition 2.8. Let η be a non-negative fuzzy real number and $p \neq 0$ be a real number. Define η^p as:

$$\eta^p(t) = \begin{cases} \eta(t^{\frac{1}{p}}), & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Set $\eta^p = \bar{1}$, in case $p = 0$.

We show that η^p is a non-negative fuzzy real number, i.e., $\eta^p \in F^*(\mathbb{R})$, $\forall p \in \mathbb{R}$. We need to investigate Conditions (N1) and (N2) in the definition of fuzzy real numbers.

For Condition (N1), since η is a fuzzy real number, there exists $t_0 \in [0, +\infty)$ such that $\eta(t_0) = 1$. Set $t' = t_0^p$. Then $\eta^p(t') = \eta((t')^{\frac{1}{p}}) = \eta((t_0^p)^{\frac{1}{p}}) = \eta(t_0) = 1$.

For Condition (N2), since for all $\alpha \in (0, 1]$, $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, we have

$$\begin{aligned} [\eta^p]_\alpha &= \{t ; \eta^p(t) \geq \alpha\} \\ &= \{t ; \eta(t^{\frac{1}{p}}) \geq \alpha\} = \{s^p ; \eta(s) \geq \alpha\} \\ &= (\{s ; \eta(s) \geq \alpha\})^p \\ &= ([\eta_\alpha^-, \eta_\alpha^+])^p. \end{aligned}$$

Therefore, η^p is a fuzzy real number. Also, it is clear that if $p > 0$ then $[\eta^p]_\alpha = [(\eta_\alpha^-)^p, (\eta_\alpha^+)^p]$ and if $p < 0$, then $[\eta^p]_\alpha = [(\eta_\alpha^+)^p, (\eta_\alpha^-)^p]$.

Note : Here by B^p we mean the set $\{x^p : x \in B\}$, where $B \subset \mathbb{R}$ and $p \in \mathbb{R}$.

Theorem 2.9. *Let η be a non-negative fuzzy real number and p, q be non-zero integers. Then*

- (1) $p > 0 \Rightarrow \eta^p = \otimes_{i=1}^p \eta$,
- (2) $p < 0 \Rightarrow \eta^p = \bar{1} \otimes (\otimes_{i=1}^{-p} \eta)$,
- (3) $\eta^p \otimes \eta^q = \eta^{p+q}$; $pq > 0$,
- (4) $(\eta^p)^q = \eta^{pq}$.

Proof. We prove this theorem by using Lemmas 2.6 and 2.7.

(1) Let $p > 0$ and $\alpha \in (0, 1]$. Then

$$[\eta^p]_\alpha = ([\eta_\alpha^-, \eta_\alpha^+])^p = [(\eta_\alpha^-)^p, (\eta_\alpha^+)^p] = \left[\prod_{i=1}^p \eta_\alpha^-, \prod_{i=1}^p \eta_\alpha^+ \right] = [\otimes_{i=1}^p \eta]_\alpha.$$

Hence $\eta^p = \otimes_{i=1}^p \eta$.

(2) For all $\alpha \in (0, 1]$ and $p < 0$ we have

$$\begin{aligned} [\eta^p]_\alpha &= ([\eta_\alpha^-, \eta_\alpha^+])^p = [(\eta_\alpha^+)^p, (\eta_\alpha^-)^p] \\ &= \left[\frac{1}{(\eta_\alpha^+)^{-p}}, \frac{1}{(\eta_\alpha^-)^{-p}} \right] = \left[\frac{1}{\prod_{i=1}^{-p} \eta_\alpha^+}, \frac{1}{\prod_{i=1}^{-p} \eta_\alpha^-} \right] \\ &= [\bar{1} \otimes (\otimes_{i=1}^{-p} \eta)]_\alpha. \end{aligned}$$

Thus $\eta^p = \bar{1} \otimes (\otimes_{i=1}^{-p} \eta)$.

(3) For all $\alpha \in (0, 1]$ and $p, q \geq 0$,

$$\begin{aligned} [\eta^p \otimes \eta^q]_\alpha &= [(\eta^p)_\alpha^-, (\eta^q)_\alpha^-, (\eta^p)_\alpha^+, (\eta^q)_\alpha^+] \\ &= [(\eta_\alpha^-)^p (\eta_\alpha^-)^q, (\eta_\alpha^+)^p (\eta_\alpha^+)^q] \\ &= [(\eta_\alpha^-)^{p+q}, (\eta_\alpha^+)^{p+q}] = [\eta^{p+q}]_\alpha. \end{aligned}$$

So $\eta^p \otimes \eta^q = \eta^{p+q}$. In case that $p, q < 0$, the proof is similar.

(4) For all $\alpha \in (0, 1]$ and $p, q \geq 0$,

$$[(\eta^p)^q]_\alpha = [(\eta^p)_\alpha^-, (\eta^p)_\alpha^+]^q = ([\eta_\alpha^-, \eta_\alpha^+])^{pq} = [\eta_\alpha^-, \eta_\alpha^+]^{pq} = [\eta^{pq}]_\alpha.$$

Hence $(\eta^p)^q = \eta^{pq}$. It is easy to check the other cases. \square

Definition 2.10. [6] Define a partial ordering \preceq in $F(\mathbb{R})$ by $\eta \preceq \delta$ if and only if $\eta_{\alpha}^{-} \leq \delta_{\alpha}^{-}$ and $\eta_{\alpha}^{+} \leq \delta_{\alpha}^{+}$ for all $\alpha \in (0, 1]$. The strict inequality in $F(\mathbb{R})$ is defined by $\eta \prec \delta$ if and only if $\eta_{\alpha}^{-} < \delta_{\alpha}^{-}$ and $\eta_{\alpha}^{+} < \delta_{\alpha}^{+}$ for all $\alpha \in (0, 1]$.

Definition 2.11. [39] Let X be a real linear space; L and R (respectively, left norm and right norm) be symmetric and non-decreasing mappings from $[0, 1] \times [0, 1]$ into $[0, 1]$ satisfying $L(0, 0) = 0$, $R(1, 1) = 1$. Then $\|\cdot\|$ is called a fuzzy norm and $(X, \|\cdot\|, L, R)$ is a fuzzy normed linear space (abbreviated to FNLS) if the mapping $\|\cdot\|$ from X into $F^*(\mathbb{R})$ satisfies the following axioms, where $[\|x\|]_{\alpha} = [\|x\|_{\alpha}^{-}, \|x\|_{\alpha}^{+}]$ for $x \in X$ and $\alpha \in (0, 1]$:

(A1) $\|x\| = \bar{0}$ if and only if $x = 0$,

(A2) $\|rx\| = |r| \odot \|x\|$ for all $x \in X$ and $r \in (-\infty, \infty)$,

(A3) For all $x, y \in X$:

(A3L) if $s \leq \|x\|_1^{-}$, $t \leq \|y\|_1^{-}$ and $s + t \leq \|x + y\|_1^{-}$, then

$\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$,

(A3R) if $s \geq \|x\|_1^{-}$, $t \geq \|y\|_1^{-}$ and $s + t \geq \|x + y\|_1^{-}$, then

$\|x + y\|(s + t) \leq R(\|x\|(s), \|y\|(t))$.

Lemma 2.12. [40] Let $(X, \|\cdot\|, L, R)$ be an FNLS, and suppose that

(R-1) $R(a, b) \leq \max(a, b)$,

(R-2) $\forall \alpha \in (0, 1]$, $\exists \beta \in (0, \alpha]$ such that $R(\beta, y) \leq \alpha$ for all $y \in (0, \alpha)$,

(R-3) $\lim_{a \rightarrow 0^+} R(a, a) = 0$.

Then $(R-1) \Rightarrow (R-2) \Rightarrow (R-3)$, but not conversely.

Lemma 2.13. [40] Let $(X, \|\cdot\|, L, R)$ be an FNLS. Then we have the following:

(1) If $R(a, b) \leq \max(a, b)$, then $\forall \alpha \in (0, 1]$, $\|x + y\|_{\alpha}^{+} \leq \|x\|_{\alpha}^{+} + \|y\|_{\alpha}^{+}$ for all $x, y \in X$.

(2) If (R-2) then for each $\alpha \in (0, 1]$ there is $\beta \in (0, \alpha]$ such that $\|x + y\|_{\alpha}^{+} \leq \|x\|_{\beta}^{+} + \|y\|_{\alpha}^{+}$ for all $x, y \in X$.

(3) If $\lim_{a \rightarrow 0^+} R(a, a) = 0$, then for each $\alpha \in (0, 1]$ there is $\beta \in (0, \alpha]$ such that $\|x + y\|_{\alpha}^{+} \leq \|x\|_{\beta}^{+} + \|y\|_{\beta}^{+}$ for all $x, y \in X$.

Lemma 2.14. Let $(X, \|\cdot\|, L, R)$ be an FNLS, and suppose that

(L-1) $L(a, b) \geq \min(a, b)$,

(L-2) $\forall \alpha \in (0, 1]$, $\exists \beta \in [\alpha, 1]$ such that $L(\beta, \gamma) \geq \alpha$ for all $\gamma \in [\alpha, 1]$,

(L-3) $\lim_{a \rightarrow 1^-} L(a, a) = 1$.

Then $(L-1) \Rightarrow (L-2) \Rightarrow (L-3)$.

Proof. Suppose that $L(a, b) \geq \min(a, b)$. Then for each $\alpha \in (0, 1]$ there exists $\beta = \frac{1+\alpha}{2} \in [\alpha, 1]$ such that

$$L(\beta, \gamma) = L\left(\frac{1+\alpha}{2}, \gamma\right) \geq \min\left(\frac{1+\alpha}{2}, \gamma\right) \geq \alpha, \quad \forall \gamma \in [\alpha, 1].$$

It means $(L-1) \Rightarrow (L-2)$. Suppose $(L-2)$ holds, then

$$\forall \varepsilon \in (0, 1), \exists \beta \in [\varepsilon, 1]; \quad L(\beta, \gamma) \geq \varepsilon, \quad \gamma \in [\varepsilon, 1].$$

Set $\gamma = \frac{1+\beta}{2}$. For all $\alpha \in [\frac{1+\beta}{2}, 1]$, we have

$$\varepsilon \leq L(\beta, \gamma) = L\left(\beta, \frac{1+\beta}{2}\right) < L(\alpha, \alpha).$$

Hence $\lim_{a \rightarrow 1^-} L(a, a) = 1$. \square

Lemma 2.15. *Let $(X, \|\cdot\|, L, R)$ be an FNLS. Then we have the following:*

(1) *If $L(a, b) \geq \min(a, b)$, then $\forall \alpha \in (0, 1]$, $\|x + y\|_{\alpha}^- \leq \|x\|_{\alpha}^- + \|y\|_{\alpha}^-$ for all $x, y \in X$.*

(2) *If $(L - 2)$ then for each $\alpha \in (0, 1]$ there is $\beta \in [\alpha, 1]$ such that $\|x + y\|_{\alpha}^- \leq \|x\|_{\beta}^- + \|y\|_{\alpha}^-$ for all $x, y \in X$.*

(3) *If $\lim_{a \rightarrow 1^-} L(a, a) = 1$, then for each $\alpha \in (0, 1]$ there is $\beta \in [\alpha, 1]$ such that $\|x + y\|_{\alpha}^- \leq \|x\|_{\beta}^- + \|y\|_{\beta}^-$ for all $x, y \in X$.*

Proof. (1) It is proved in [39]. (2) By $(L - 2)$, for each $\alpha \in (0, 1]$ there exist $\beta \in [\alpha, 1]$ such that $L(\beta, \gamma) \geq \alpha$ for all $\gamma \in [\alpha, 1]$. Suppose that $\|x + y\|_{\alpha}^- > \|x\|_{\beta}^- + \|y\|_{\alpha}^-$. Then there exists s, t such that $s = \|x\|_{\beta}^- \leq \|x\|_1^-$, $t = \|y\|_{\alpha}^- \leq \|y\|_1^-$ and $s + t = \|x\|_{\beta}^- + \|y\|_{\alpha}^- < \|x + y\|_{\alpha}^- \leq \|x + y\|_1^-$. Thus by (A3L),

$$\alpha > \|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t)) \geq L(\beta, \alpha) \geq \alpha,$$

which is a contradiction.

(3) By $\lim_{a \rightarrow 1^-} L(a, a) = 1$, for each $\alpha \in (0, 1]$ there exists $\beta \in [\alpha, 1]$ such that $L(\beta, \beta) \geq \alpha$. Suppose $\|x + y\|_{\alpha}^- > \|x\|_{\beta}^- + \|y\|_{\beta}^-$. Then there exist s, t such that $s = \|x\|_{\beta}^- \leq \|x\|_1^-$, $t = \|y\|_{\beta}^- \leq \|y\|_1^-$ and $s + t = \|x\|_{\beta}^- + \|y\|_{\beta}^- < \|x + y\|_{\alpha}^- \leq \|x + y\|_1^-$. Thus by (A3L),

$$\alpha > \|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t)) \geq L(\beta, \beta) \geq \alpha,$$

which is impossible. \square

Lemma 2.16. [39] *Let $(X, \|\cdot\|, L, R)$ be an FNLS. Then:*

(1) *If $R(a, b) \geq \max(a, b)$ and $\forall \alpha \in (0, 1]$, $\|x + y\|_{\alpha}^+ \leq \|x\|_{\alpha}^+ + \|y\|_{\alpha}^+$ for all $x, y \in X$ then (A3R).*

(2) *If $L(a, b) \leq \min(a, b)$ and $\forall \alpha \in (0, 1]$, $\|x + y\|_{\alpha}^- \leq \|x\|_{\alpha}^- + \|y\|_{\alpha}^-$ for all $x, y \in X$ then (A3L).*

Theorem 2.17. [36] *Let $(X, \|\cdot\|, L, R)$ be an FNLS and $\lim_{a \rightarrow 0^+} R(a, a) = 0$. Then $(X, \|\cdot\|, L, R)$ is a Hausdorff topological vector space, whose neighborhood base of origin θ is $\{N(\varepsilon, \alpha) : \varepsilon > 0, \alpha \in (0, 1]\}$, where $N(\varepsilon, \alpha) = \{x; \|x\|_{\alpha}^+ \leq \varepsilon\}$.*

Definition 2.18. [39] Let $(X, \|\cdot\|, L, R)$ be an FNLS and $\lim_{a \rightarrow 0^+} R(a, a) = 0$. A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ converges to $x \in X$, denoted by $\lim_{n \rightarrow \infty} x_n = x$, if $\lim_{n \rightarrow \infty} \|x_n - x\|_{\alpha}^+ = 0$ for every $\alpha \in (0, 1]$, and is called a Cauchy sequence if $\lim_{m, n \rightarrow \infty} \|x_m - x_n\|_{\alpha}^+ = 0$ for every $\alpha \in (0, 1]$. A subset $A \subseteq X$ is said to be complete if every Cauchy sequence in A , converges in A . The fuzzy normed space $(X, \|\cdot\|, L, R)$ is said to be a fuzzy Banach space if it is complete.

Theorem 2.19. [40] *Let $(X, \|\cdot\|, L, R)$ be an FNLS satisfying $(R - 2)$. Then:*

(1) *For each $\alpha \in (0, 1]$, $\|\cdot\|_{\alpha}^+$ is a continuous mapping from X into \mathbb{R} at $x \in X$.*

(2) *For any $n \in \mathbb{Z}^+$ and $\{x_i\}_{i=1}^n$ we have*

$$\forall \alpha \in (0, 1], \quad \exists \beta \in (0, \alpha]; \quad \left\| \sum_{i=1}^n x_i \right\|_{\alpha}^+ \leq \sum_{i=1}^n \|x_i\|_{\beta}^+.$$

Theorem 2.20. *Let $(X, \|\cdot\|, L, R)$ be an FNLS satisfying $(L - 2)$. Then:*

- (1) *For each $\alpha \in (0, 1]$, $\|\cdot\|_{\alpha}^{-}$ is a continuous mapping from X into \mathbb{R} at $x \in X$.*
- (2) *For any $n \in \mathbb{Z}^+$ and $\{x_i\}_{i=1}^n$ we have*

$$\forall \alpha \in (0, 1], \quad \exists \beta \in [\alpha, 1]; \quad \left\| \sum_{i=1}^n x_i \right\|_{\alpha}^{-} \leq \sum_{i=1}^n \|x_i\|_{\beta}^{-}.$$

Proof. The proof is the same as Theorem 2.19. □

Remark 2.21. Let $(X, \|\cdot\|_*)$ be a classical real normed linear space. Set $\|x\|(t) = \bar{0}(t - \|x\|_*)$, $L = \min$, $R = \max$. Then $(X, \|\cdot\|, L, R)$ is an FNLS induced by $(X, \|\cdot\|_*)$. So each classical real normed linear space can be considered as an FNLS. In the following we bring an example of a fuzzy normed linear space which is not a real normed space in the classical sense. Therefore, the spectrum of the category of fuzzy normed linear spaces is broader than the category of normed spaces. This is why the study of FNLS is of great importance.

Let Ω be a nonempty open set in an euclidean space. It is well known that Ω is a countable union of sets $K_n \neq \emptyset$ which can be chosen so that K_n lies in the interior of K_{n+1} ($n = 1, 2, 3, \dots$). The linear space $C(\Omega)$, the vector space of all complex valued continuous functions on Ω , topologized by the family of non-decreasing classical seminorms

$$p_n(f) = \sup\{|f(x)| ; x \in K_n\}.$$

In the following results, we will prove that $C(\Omega)$ is fuzzy normable, but in classical analysis, where $\Omega \subset \mathbb{R}^n$ is an open subset, it is not so.

Theorem 2.22. *The linear space $C(\Omega)$ is fuzzy normable in general.*

Proof. For $\alpha \in (0, 1]$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \alpha \leq \frac{1}{n}$. Let $f \in C(\Omega)$ define

$$\begin{aligned} f_{\alpha}^{-} &= \sup\{|f(x)| ; x \in K_1\}, \\ f_{\alpha}^{+} &= \sup\{|f(x)| ; x \in K_{n+1}\}. \end{aligned}$$

Then $\{[f_{\alpha}^{-}, f_{\alpha}^{+}] ; \alpha \in (0, 1]\}$ is a family of nested bounded closed intervals. We define the fuzzy norm $\|\cdot\|$ on $C(\Omega)$ as:

$$\forall t \in \mathbb{R} \quad , \quad \|f\|(t) = \sup\{\alpha \in (0, 1] \quad t \in [f_{\alpha}^{-}, f_{\alpha}^{+}]\}.$$

By Theorem 2.3, it is easy to see that for $f \in C(\Omega)$, $\|f\|$ is a fuzzy real number. Now we show that $(C(\Omega), \|\cdot\|, \min, \max)$ is a fuzzy normed space.

If $f = 0$, then $p_n(f) = 0$, $\forall n \in \mathbb{N}$ and so for all $\alpha \in (0, 1]$, $f_{\alpha}^{-} = f_{\alpha}^{+} = 0$. Therefore, if $t \neq 0$, $\|f\|(t) = 0$ and if $t = 0$, $\|f\|(0) = \sup\{\alpha \in (0, 1]; 0 \in [f_{\alpha}^{-}, f_{\alpha}^{+}]\} = 1$ and hence $\|f\| = \bar{0}$. Now, if $\|f\| = \bar{0}$, then for all $\alpha \in (0, 1]$, $\|f\|_{\alpha}^{-} = \|f\|_{\alpha}^{+} = 0$, and so $p_n(f) = 0$, $\forall n \in \mathbb{N}$. Therefore, for all $x \in \Omega$, $|f(x)| = 0$, thus $f = 0$. Hence the map defined satisfies Condition (A1).

For Condition (A2), if $f \in C(\Omega)$ and $r \neq 0$ then by the map defined and properties of classical seminorms, for every $\alpha \in (0, 1]$ there exists $n \in \mathbb{N}$ such that

$$\|rf\|_{\alpha}^{+} = p_{n+1}(rf) = |r|p_{n+1}(f) = |r|\|f\|_{\alpha}^{+},$$

$$\|rf\|_{\alpha}^{-} = p_1(rf) = |r|p_1(f) = |r|\|f\|_{\alpha}^{-}.$$

Therefore, by Lemma 2.6, $\|rf\| = |r| \odot \|f\|$.

For Condition (A3), since

$$p_n(f + g) \leq p_n(f) + p_n(g), \quad \forall n \in \mathbb{N},$$

for all $\alpha \in (0, 1]$ we have the following:

$$\begin{aligned} \|f + g\|_{\alpha}^{-} &\leq \|f\|_{\alpha}^{-} + \|g\|_{\alpha}^{-}, \\ \|f + g\|_{\alpha}^{+} &\leq \|f\|_{\alpha}^{+} + \|g\|_{\alpha}^{+}. \end{aligned}$$

Now by Lemma 2.16 and the last inequalities, if $L = \min$, then the map defined satisfies (A3L) and if $R = \max$, then the fuzzy norm satisfies (A3R). Therefore, $(C(\Omega), \|\cdot\|, \min, \max)$ is a fuzzy normed linear space. \square

3. Stability of the Cauchy Equation in Fuzzy Normed Linear Spaces

Using the results of the last section and an idea of Găvruta [8], we prove the fuzzy stability of Cauchy equation in the spirit of Hyers, Ulam and Rassias.

Theorem 3.1. *Let X be a linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R - 2)$. Let $f : X \rightarrow Y$ be a mapping for which there exists a function $\varphi : X \times X \rightarrow F^*(\mathbb{R})$ such that*

$$\sum_{i=0}^{\infty} \frac{1}{2^i} (\varphi(2^i x, 2^i y))_{\alpha}^{+} < \infty, \quad (1)$$

$$\|f(x + y) - f(x) - f(y)\| \preceq \varphi(x, y), \quad (2)$$

for all $x, y \in X$ and all $\alpha \in (0, 1]$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$\forall \alpha \in (0, 1], \exists \beta \in (0, \alpha], \text{ s.t. } \|f(x) - T(x)\|_{\alpha}^{+} \leq \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} (\varphi(2^i x, 2^i x))_{\beta}^{+}, \quad (3)$$

for all $x \in X$.

Proof. Take $y = x$ in (1), we get

$$\|f(2x) - 2f(x)\| \preceq \varphi(x, x), \quad (4)$$

for all $x \in X$. If we replace x in (4) by $2^n x$ and multiply both sides of (4) by $\frac{1}{2^{n+1}}$ in the fuzzy scalar multiplication sense, then we have

$$\left\| \frac{1}{2^{n+1}} f(2^{n+1}x) - \frac{1}{2^n} f(2^n x) \right\| \preceq \frac{1}{2^{n+1}} \odot \varphi(2^n x, 2^n x), \quad (5)$$

for all $x \in X$ and all non-negative integers $n \in \mathbb{N}$. By Lemma 2.19 and inequality (5), we conclude that for all $\alpha \in (0, 1]$ there exists $\beta \in (0, \alpha]$ such that

$$\begin{aligned} \left\| \frac{1}{2^{n+1}} f(2^{n+1}x) - \frac{1}{2^m} f(2^m x) \right\|_{\alpha}^{+} &\leq \sum_{i=m}^n \left\| \frac{1}{2^{i+1}} f(2^{i+1}x) - \frac{1}{2^i} f(2^i x) \right\|_{\beta}^{+} \\ &\leq \sum_{i=m}^n \frac{1}{2^{i+1}} (\varphi(2^i x, 2^i x))_{\beta}^{+}, \end{aligned} \quad (6)$$

for all $x \in X$ and all non-negative integers m and n with $n \geq m$. Now (1) and (6) imply that $\{\frac{1}{2^n} f(2^n x)\}$ is a fuzzy Cauchy sequence in Y for all $x \in X$. Since Y is a fuzzy Banach space, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges for all $x \in X$. So we can define the mapping $T : X \rightarrow Y$ by

$$T(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x),$$

for all $x \in X$. Letting $m = 0$ and passing the limit $n \rightarrow \infty$ in (3), by continuity of $\|\cdot\|_\alpha^+$ we get

$$\|f(x) - T(x)\|_\alpha^+ \leq \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} (\varphi(2^i x, 2^i x))_\beta^+, \tag{7}$$

for all $x \in X$. Therefore, we obtain (3). Now we show that T is additive and unique. Applying (1), (2) and the continuity of $\|\cdot\|_\alpha^+$, we have

$$\begin{aligned} \|T(x+y) - T(x) - T(y)\|_\alpha^+ &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)\|_\alpha^+ \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{2^n} (\varphi(2^n x, 2^n y))_\alpha^+ = 0, \end{aligned}$$

for all $x, y \in X$. Therefore, the mapping $T : X \rightarrow Y$ is additive.

To prove the uniqueness of T , let $T' : X \rightarrow Y$ be an additive mapping satisfying (3). Since by Lemma 2.19,

$$\begin{aligned} \|T(x) - T'(x)\|_\alpha^+ &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \|f(2^n x) - T'(2^n x)\|_\alpha^+ \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \sum_{i=0}^{\infty} \frac{1}{2^i} (\varphi(2^{n+i} x, 2^{n+i} x))_\beta^+ \\ &\leq \frac{1}{2} \lim_{n \rightarrow \infty} \sum_{i=n}^{\infty} \frac{1}{2^i} (\varphi(2^i x, 2^i x))_\beta^+ = 0, \end{aligned}$$

for all $x \in X$, $T = T'$. □

Remark 3.2. The above theorem is also true if $\|\cdot\|_\alpha^+$ is replaced by $\|\cdot\|_\alpha^-$ in (3) and the fuzzy Banach space Y satisfies $(L - 2)$ and $(R - 2)$.

The following theorem is an alternative result of Theorem 3.1.

Theorem 3.3. *Let X be a linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space such that $R(a, b) \leq \max(a, b)$ and $L(a, b) \geq \min(a, b)$. Let $f : X \rightarrow Y$ be a mapping for which there is a function $\varphi : X \times X \rightarrow F^*(\mathbb{R})$ satisfying (1) and (2) for all $x, y \in X$ and all $\alpha \in (0, 1]$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that*

$$\|f(x) - T(x)\| \preceq \bar{\varphi}(x, x), \tag{8}$$

for all $x \in X$, where $\bar{\varphi}(x, x)$ is a fuzzy real number generated by the families of nested bounded closed intervals $[a_\alpha, b_\alpha]$ such that

$$\begin{aligned} a_\alpha &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} (\varphi(2^i x, 2^i x))_\alpha^-, \\ b_\alpha &= \frac{1}{2} \sum_{i=0}^{\infty} \frac{1}{2^i} (\varphi(2^i x, 2^i x))_\alpha^+, \end{aligned}$$

for all $x \in X$.

Theorem 3.4. *Let X be a linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R-2)$. Let $f : X \rightarrow Y$ be a mapping for which there exists a function $\varphi : X \times X \rightarrow F^*(\mathbb{R})$ such that*

$$\sum_{i=1}^{\infty} 2^i \left(\varphi\left(\frac{x}{2^i}, \frac{y}{2^i}\right) \right)_{\alpha}^{+} < \infty, \quad (9)$$

$$\|f(x+y) - f(x) - f(y)\| \preceq \varphi(x, y), \quad (10)$$

for all $x, y \in X$ and all $\alpha \in (0, 1]$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$\forall \alpha \in (0, 1], \exists \beta \in (0, \alpha], \text{ s.t. } \|f(x) - T(x)\|_{\alpha}^{+} \leq \frac{1}{2} \sum_{i=1}^{\infty} 2^i \left(\varphi\left(\frac{x}{2^i}, \frac{x}{2^i}\right) \right)_{\beta}^{+}, \quad (11)$$

for all $x \in X$.

Proof. Take $y = x$ in (10), we get

$$\|f(2x) - 2f(x)\| \preceq \varphi(x, x), \quad (12)$$

for all $x \in X$. If we replace x in (12) by $\frac{x}{2^{n+1}}$ and multiply both sides of (12) by 2^n in the fuzzy scalar multiplication sense, then we have

$$\left\| 2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right) \right\| \preceq 2^n \odot \varphi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}\right), \quad (13)$$

for all $x \in X$ and all non-negative integers $n \in \mathbb{N}$. By Lemma 2.19 and the inequality (13), we conclude that for all $\alpha \in (0, 1]$ there exists $\beta \in (0, \alpha]$ such that

$$\begin{aligned} \left\| 2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\|_{\alpha}^{+} &\leq \sum_{i=m}^n \left\| 2^{i+1} f\left(\frac{x}{2^{i+1}}\right) - 2^i f\left(\frac{x}{2^i}\right) \right\|_{\beta}^{+} \\ &\leq \sum_{i=m}^n 2^i \left(\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right) \right)_{\beta}^{+}, \end{aligned} \quad (14)$$

for all $x \in X$ and all non-negative integers m and n with $n \geq m$. So by (9) and (14), the sequence $\{2^n f(\frac{x}{2^n})\}$ is a fuzzy Cauchy sequence in Y for all $x \in X$. Since Y is a fuzzy Banach space, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges for all $x \in X$. The rest of this proof is similar to the proof of Theorem 3.1. \square

The same discussion in Remark 3.2 does hold for the above theorem. Also the following theorem is an alternative result of Theorem 3.4.

Theorem 3.5. *Let X be a linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space such that $R(a, b) \leq \max(a, b)$ and $L(a, b) \geq \min(a, b)$. Let $f : X \rightarrow Y$ be a mapping for which there is a function $\varphi : X \times X \rightarrow F^*(\mathbb{R})$ satisfying (9) and (10) for all $x, y \in X$ and all $\alpha \in (0, 1]$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that*

$$\|f(x) - T(x)\| \preceq \bar{\varphi}(x, x), \quad (15)$$

for all $x \in X$, where $\bar{\varphi}(x, x)$ is a fuzzy real number generated by the families of nested bounded closed intervals $[a_\alpha, b_\alpha]$ such that

$$\begin{aligned} a_\alpha &= \frac{1}{2} \sum_{i=1}^{\infty} 2^i (\varphi(\frac{x}{2^i}, \frac{x}{2^i}))_\alpha^-, \\ b_\alpha &= \frac{1}{2} \sum_{i=1}^{\infty} 2^i (\varphi(\frac{x}{2^i}, \frac{x}{2^i}))_\alpha^+, \end{aligned}$$

for all $x \in X$.

Corollary 3.6. Let μ be a non-negative fuzzy real number and let X be a linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space such that $R(a, b) \leq \max(a, b)$ and $L(a, b) \geq \min(a, b)$. Suppose that the mapping $f : X \rightarrow Y$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \preceq \mu, \quad (16)$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \rightarrow Y$ satisfying

$$\|f(x) - T(x)\| \preceq \mu,$$

for all $x \in X$.

Proof. Let $\varphi(x, y) := \mu$ for all $x, y \in X$. By Theorem 3.1 we get the desired result. \square

Corollary 3.7. Let μ be a non-negative fuzzy real number and let p, q be non-negative real numbers such that $p, q > 1$ or $0 < p, q < 1$. Let X be a fuzzy normed linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R - 2)$. Suppose that the mapping $f : X \rightarrow Y$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\|_Y \preceq \mu \otimes (\|x\|_X^p \oplus \|y\|_X^q), \quad (17)$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$\forall \alpha \in (0, 1], \exists \beta \in (0, \alpha], \text{ s.t. } \|f(x) - T(x)\|_\alpha^+ \leq \mu_\beta^+ \left\{ \frac{(\|x\|_\beta^+)^p}{|2^p - 2|} + \frac{(\|x\|_\beta^+)^q}{|2^q - 2|} \right\},$$

for all $x \in X$.

Proof. The result follows from Theorems 3.1 and 3.4 by taking

$$\varphi(x, y) := \mu \otimes (\|x\|_X^p \oplus \|y\|_X^q),$$

for all $x, y \in X$. \square

Corollary 3.8. Let μ be a non-negative fuzzy real number and let p, q be non-negative real numbers such that $\lambda = p + q \in (0, 1) \cup (1, \infty)$. Let X be a fuzzy normed linear space and $(Y, \|\cdot\|, L, R)$ be a fuzzy Banach space satisfying $(R - 2)$. Suppose that the mapping $f : X \rightarrow Y$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\|_Y \preceq \mu \otimes \|x\|_X^p \otimes \|y\|_X^q, \quad (18)$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \rightarrow Y$ satisfying

$$\forall \alpha \in (0, 1], \exists \beta \in (0, \alpha], \text{ s.t. } \|f(x) - T(x)\|_\alpha^+ \leq \frac{(\|x\|_\alpha^+)^{\lambda} \mu_\alpha^+}{|2^\lambda - 2|},$$

for all $x \in X$.

Proof. The result follows from Theorems 3.1 and 3.4 by taking

$$\varphi(x, y) := \mu \otimes \|x\|_X^p \otimes \|y\|_X^q,$$

for all $x, y \in X$. □

Remark 3.9. In Section 2 we showed that $C(\Omega)$ is a fuzzy normed space but as we know it is not classical normable when Ω is an open subset of \mathbb{R}^n .

4. Conclusion

In this paper the fuzzy version of Hyers-Ulam-Rassias stability problems is studied. Also some conditions to approximate a function with an additive map is given. Furthermore, by an example it is shown that the spectrum of the fuzzy version of Hyers-Ulam-Rassias stability is broader than the classical case in general.

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I. SADEQI*, DEPARTMENT OF MATHEMATICS, SAHAND UNIVERSITY OF TECHNOLOGY, TABRIZ-IRAN

E-mail address: esadeqi@sut.ac.ir

F. MORADLOU, DEPARTMENT OF MATHEMATICS, SAHAND UNIVERSITY OF TECHNOLOGY, TABRIZ-IRAN

E-mail address: moradlou@sut.ac.ir

M. SALEHI, DEPARTMENT OF MATHEMATICS, SAHAND UNIVERSITY OF TECHNOLOGY, TABRIZ-IRAN

E-mail address: mit-paydar@yahoo.com

*CORRESPONDING AUTHOR