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Hesitant cognitive uncertain information in aggregation and decision making

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Abstract

The concepts of cognitive interval information and cognitive uncertain information, which are two recently proposed types of uncertain information, have been extended in this work to the typical hesitant fuzzy environment. We introduce the notions of typical hesitant monopolar cognitive interval information and typical hesitant cognitive uncertain information. To facilitate their analysis, we define uncertainty degree functions and score functions for these concepts using extended aggregation operators. Furthermore, we reanalyze some decision models discussed in earlier literature using these newly proposed concepts to demonstrate their advantages and potential applications.

Keywords: Cognitive interval information, cognitive uncertain information, decision making, group decision making, information fusion, uncertain information.

1 Introduction

The classification of uncertain information encompasses various types, including probability information type, fuzzy information type, set-valued information type, and basic uncertain information type. Within these types, there are subtypes such as hesitant fuzzy granules [2, 3, 27] and interval information that can be categorized under the set-valued information type. Similarly, Z-number [31] and basic uncertain information (BUI) [18, 24], both in pair forms, should be classified under the basic uncertain information type. Additionally, interval fuzzy granule [14], intuitionistic fuzzy granule [1], and vague set granule [12] share a similar mathematical structure with interval information.

Some extensions and application scenarios of BUI have been developed in recent years [21, 22, 23]. Recently, a new type of uncertain information called cognitive uncertain type has emerged, which includes cognitive interval information (CII) [21], cognitive uncertain information (CUI) [15], interval cognitive interval information (ICII) [16], and interval cognitive uncertain information (ICUI) [16]. The granules of this type contain both evaluation values and acceptance areas (and unaccepted areas for CUI and ICUI), thereby enhancing intersubjective thinking, judgments, and compromise [16]. For instance, various decision models can be effectively designed to facilitate consensus building in group decision making.

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The interval extensions ICII and ICUI have proven to be successful expansions of CII and CUI, respectively. Considering the broad applicability of hesitant fuzzy sets, it is also meaningful in both theoretical studies and practical applications to extend the real-valued evaluation values in cognitive type information into typical hesitant fuzzy granules [3].

In the decision-making process, we often face various complex situations and problems, which are often accompanied by a great deal of uncertainty. In such cases, utilizing hesitant fuzzy information can bring certain advantages. By fully considering and utilizing hesitant fuzzy information, it is possible to better reflect the complexity and diversity of the real world in the decision-making process. Compared to simple and clear, high-certainty intelligence data, hesitant fuzzy information that includes more possibilities and potential factors can provide a more comprehensive, integrated, and highly credible reference basis.

However, due to the intricate structure of typical hesitant fuzzy granules, achieving this extension requires introducing additional tools such as extended aggregation operators for redefining uncertainty degree functions and score functions. Furthermore, it necessitates devising or rewriting decision models and related formulations.

The remaining sections of this work are structured as follows. Section 2 provides a review of cognitive interval information, cognitive uncertain information, interval cognitive interval information, and interval cognitive uncertain information. In Section 3, we formally define hesitant monopolar cognitive uncertain information and hesitant bipolar cognitive uncertain information while discussing their characteristics such as degenerations, uncertainty degrees, and score functions. Section 4 reanalyzes some decision models and related formulations to demonstrate the structural differences and advantages of our proposed extensions to cognitive uncertain information. Finally, in Section 5 we conclude and provide remarks on this work.

2 Cognitive uncertain information

The following section presents a comprehensive review of four variations of "cognitive uncertain information": cognitive interval information (CII), cognitive uncertain information (CUI), interval cognitive interval information (ICII), and interval cognitive uncertain information (ICUI).

The set of all intervals of the form $[a,b] \subseteq [0,1]$ is denoted by $\mathcal{I}$. The degenerated interval with the form $[a,a]$ is allowed and sometimes can be regarded as the real number a.

Definition 2.1. [15] *A cognitive interval information (CII) granule is with the pair form $(x,[a_1,a_2]) \in [0,1] \times \mathcal{I}$ such that $x \in [a_1,a_2]$. x is the evaluation value and $[a_1,a_2]$ is called the acceptance interval.*

Definition 2.2. [15] *A cognitive uncertain information (CUI) granule is with the triad form $(x,[a_1,a_2],[u_1,u_2]) \in [0,1] \times \mathcal{I} \times \mathcal{I}$ such that $x \in [a_1,a_2]$ and $[a_1,a_2] \subseteq [u_1,u_2]$. x is the evaluation value, $[a_1,a_2]$ is called the acceptance interval and $[0,1] \setminus [u_1,u_2]$ is called the unaccepted area.*

Definition 2.3. [16] *An interval (type) cognitive interval information (ICII) granule is with the pair form $([x_1,x_2],[a_1,a_2]) \in \mathcal{I}^2$ such that $[x_1,x_2] \subseteq [a_1,a_2]$. $[x_1,x_2]$ is the evaluation interval and $[a_1,a_2]$ is called the acceptance interval.*

Definition 2.4. [16] *An interval (type) cognitive uncertain information (ICUI) granule is with the triad form $([x_1,x_2],[a_1,a_2],[u_1,u_2]) \in \mathcal{I}^2$ such that $[x_1,x_2] \subseteq [a_1,a_2] \subseteq [u_1,u_2]$. $[x_1,x_2]$ is the evaluation interval, $[a_1,a_2]$ is called the acceptance interval and $[0,1] \setminus [u_1,u_2]$ is called the unaccepted area.*

For CII, CUI, ICII, and ICUI, the evaluation values or intervals can be provided by experts, evaluators, or other non-human sources of information. In the case of CII, CUI, ICII, and ICUI, an acceptance interval is often presented by an evaluator or expert who suggests a specific evaluation value or interval (for an object under evaluation) as x or $[x_1,x_2]$ but expresses uncertainty and lack of confidence in this given value or interval. Therefore, they may provide an interval $[a_1,a_2]$ (which includes the suggested evaluation value or interval) and accept any value (or interval) falling within this range that is provided by other sources of information.

In practical applications, CII, CUI, ICII, and ICUI can also be employed with alternative connotations. For instance, the evaluator offering a CII (or CUI, ICII, ICUI) granule may assert being "indeed certain" about the evaluation value x (or interval $[x_1,x_2]$), but due to compromise and tolerance or for facilitating group decision-making efficiency, they might agree to "accept" other evaluation values (or intervals) proposed by related evaluators with providing some other information of tolerance such as $[a_1,a_2]$ and $[u_1,u_2]$.

For a CII granule or an ICII granule, there is no related information, opinion, attitude or cognition about the values in the area $[0,1] \setminus [a_1,a_2]$, but this does not mean the values (or intervals) in $[0,1] \setminus [a_1,a_2]$ are unacceptable; for a CUI granule or an ICUI granule, there is also no related information, opinion, attitude or cognition about the values in the area $[u_1,u_2] \subseteq [a_1,a_2]$. Hence, whether or not considering the values in those areas is determined by decision makers and by the detailed decisional scenarios and backgrounds.

3 Hesitant cognitive uncertain information

In this section, we formally define the typical hesitant monopolar cognitive uncertain information and the typical hesitant cognitive uncertain information.

Definition 3.1. *A (typical) hesitant monopolar cognitive uncertain information (HMCUI) granule is with the pair form* $(\{x_r\}_{r=1}^k, [a_1, a_2]) \in (2^{[0,1]} \setminus \{\varnothing\}) \times \mathcal{I}$ *such that* $\{x_r\}_{r=1}^k$ *is finite and* $a_1 \leq x_1 < x_2 \cdots < x_{k-1} < x_k \leq a_2$. $\{x_r\}_{r=1}^k$ *is called the (typical) hesitant evaluation values and* $[a_1, a_2]$ *is called the accepted area.*

Remark 3.2. *Due to uncertainty, an evaluator may provide multiple potential evaluation values, and thus a typical hesitant fuzzy granule is employed as the representation of these evaluation values. Unlike both CUI and ICUI, HMCUI represents evaluations as finite sets that generally do not correspond to some real numbers, singletons or intervals unless certain degeneration occurs, which will be discussed later. The term "monopolar" is used to indicate that HMCUI does not define the associated "unaccepted area" as defined in the subsequent definition of typical hesitant cognitive uncertain information, which encompasses both accepted and unaccepted areas and therefore has a "bipolar" nature.*

Remark 3.3. *One HCUI granule can provide both the information of accepting other involved HCUI granules and the information of being accepted by other involved HCUI granules. For example, when all the hesitant evaluation values* $\{x_r\}_{r=1}^k$ *in one HCUI granule totally fall into the acceptance interval of another HCUI granule, it gains entire acceptance (or support in group decision making) from that HCUI granule (or that expert in group decision making); if only part of the hesitant evaluation values* $\{x_r\}_{r\in S}$ *(with* $S \subset \{1, ..., k\}$*) fall into the acceptance interval of another HCUI granule, then it gains only part of acceptance (or support). This characteristic of HMCUI makes it effective and flexible to be applied in group decision making.*

Definition 3.4. *A typical hesitant cognitive uncertain information (HCUI) granule is with the triad form* $(\{x_r\}_{r=1}^k, [a_1, a_2], [u_1, u_2]) \in (2^{[0,1]} \setminus \{\varnothing\}) \times \mathcal{I}^2$ *such that* $\{x_r\}_{r=1}^k$ *is finite and* $u_1 \leq a_1 \leq x_1 < x_2 \cdots < x_{k-1} < x_k \leq a_2 \leq u_2$. $\{x_r\}_{r=1}^k$ *is called the (typical) hesitant evaluation values,* $[a_1, a_2]$ *is called the acceptance interval and* $[0, 1] \setminus [u_1, u_2]$ *is called the unaccepted area.*

Remark 3.5. *In comparison to HMCUI, a HCUI granule has an additional area* $[0, 1] \setminus [u_1, u_2]$ *and any values within it must not be accepted. Therefore, apart from the consideration of whether the hesitant evaluation values* $\{x_r\}_{r=1}^k$ *are within the acceptance interval of another HCUI granule, we should also consider their inclusion relations to the unaccepted area of that HCUI granule. Hence, some related applications (such as in group decision making and aggregation) with HCUI will become possibly more involved but more practical than with HMCUI.*

Remark 3.6. *When* $k = 1$*, a HMCUI granule* $(\{x_r\}_{r=1}^k, [a_1, a_2])$ *becomes* $(\{x_r\}_{r=1}^1, [a_1, a_2]) = (\{x_1\}, [a_1, a_2])$ *and actually degenerates into CII granule* $(x_1, [a_1, a_2])$ *in both formal and practical senses. Similarly, when* $k = 1$*, a HCUI granule* $(\{x_r\}_{r=1}^k, [a_1, a_2], [u_1, u_2])$ *actually degenerates into CUI granule* $(x_1, [a_1, a_2], [u_1, u_2])$ *in both form and practice.*

Remark 3.7. *It should be noted that HMCUI and ICII are not mutually inclusive, and HCUI and ICUI are also not mutually inclusive. Only when* $k = 1$ *for HMCUI/HCUI and when* $[x_1, x_2]$ *represent degenerate intervals, HMCUI in actual coincides with ICII, and HCUI in actual coincides with ICUI.*

The structures of both HMCUI and HCUI contain complex uncertainties which can be measured by the following functions.

Definition 3.8. *The uncertainty degree of a HMCUI granule* $(\{x_r\}_{r=1}^k, [a_1, a_2])$ *is defined by*

$$U((\{x_r\}_{r=1}^k, [a_1, a_2])) = \lambda(x_k - x_1) + (1 - \lambda)(a_2 - a_1), \tag{1}$$

where $\lambda \in [0, 1]$ *is a parameter which can be determined according to the preferences of decision makers. The certainty degree of a HMCUI granule* $(\{x_r\}_{r=1}^k, [a_1, a_2])$ *is defined by* $1 - U((\{x_r\}_{r=1}^k, [a_1, a_2]))$.

Definition 3.9. *The uncertainty degree of a HCUI granule* $(\{x_r\}_{r=1}^k, [a_1, a_2], [u_1, u_2])$ *is defined by*

$$U((\{x_r\}_{r=1}^k, [a_1, a_2], [u_1, u_2])) = \lambda_1(x_k - x_1) + \lambda_2(a_2 - a_1) + \lambda_3(u_2 - u_1), \tag{2}$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ *is a normalized weight vector which can be determined according to the preferences of decision makers. The certainty degree of a HCUI granule* $(\{x_r\}_{r=1}^k, [a_1, a_2], [u_1, u_2])$ *is defined by* $1 - U((\{x_r\}_{r=1}^k, [a_1, a_2], [u_1, u_2]))$.

In the presence of uncertainty, an evaluator may provide multiple potential evaluation values, necessitating the application of a typical hesitant fuzzy granule. In certain data aggregation processes, the initial step involves determining weight vectors for assigning to input vectors and utilizing Yager preference-based weight allocation [28, 29, 30]. For instance, if a decision maker favors HMCUI or HCUI granules with lower degrees of uncertainty, it is advisable to assign higher weights to those granules in order to prepare for subsequent aggregation processes.

For two typical hesitant fuzzy granules, often they cannot be compared because they involve different numbers of evaluation values or even if they have the same numbers of evaluation values it is still cannot compare them in sense that not each ith largest value in the first granule is always larger (or smaller) than the ith largest value in the second granule for any i. Therefore, employing mean operators with inputs consisting of these evaluation values within hesitant fuzzy granules represents an ideal approach, as performing a mean operator on an input vector yields a single output real number bounded between the maximum and minimum of inputs, enabling comparison between any two real numbers.

Recall a real valued aggregation operator [13] (with dimension $n \in \mathbb{N}$) is a mapping $A : [0,1]^n \to [0,1]$ such that (i) $A(\mathbf{x}) \leq A(\mathbf{y})$ whenever $\mathbf{x} \leq \mathbf{y}$ (i.e., $x_i \leq y_i$ for all $i \in \{1, \cdots, n\}$), and (ii) $A(\mathbf{0}) = A(0, \cdots, 0) = 0$ and $A(\mathbf{1}) = A(1, \cdots, 1) = 1$. With convention, when $n = 1$ we define $A(x) = x$. Aggregation operators have been extensively applied in various applications [7, 8, 9, 10, 11], and their theory and extensions have been extensively researched over the past few decades [4, 5, 17, 20, 25, 26]. Based on the relationship between function values and inputs, aggregation operators can be classified into the following four classes [13]:

(I) an aggregation operator $A : [0,1]^n \to [0,1]$ is called conjunctive if $A(\mathbf{x}) \leq min(\mathbf{x})$ for any $\mathbf{x} \in [0,1]^n$, where $min(\mathbf{x}) = inf\{x_i\}_{i=1}^n$;

(II) an aggregation operator $A : [0,1]^n \to [0,1]$ is called disjunctive if $A(\mathbf{x}) \geq max(\mathbf{x})$ for any $\mathbf{x} \in [0,1]^n$, where $max(\mathbf{x}) = sup\{x_i\}_{i=1}^n$;

(III) an aggregation operator $A : [0,1]^n \to [0,1]$ is called mean if $min(\mathbf{x}) \leq A(\mathbf{x}) \leq max(\mathbf{x})$ for any $\mathbf{x} \in [0,1]^n$;

(IV) an aggregation operator $A : [0,1]^n \to [0,1]$ is called mixed (or hybrid) if it is neither conjunctive, nor disjunctive and nor mean, which contains, among others, compensation operators, uninorms, and nullnorms [6, 13].

Since the involved hesitant evaluation values $\{x_r\}_{r=1}^k$ may be with different numbers of values, then it is natural to introduce extended aggregation operators. Recall an extended aggregation operator [13] is a map $A : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ such that its restriction $A^{(n)} \triangleq A \mid_{[0,1]^n}$ to $[0,1]^n$ is an aggregation function on $[0,1]^n$ for any $n \in \mathbb{N}$.

Definition 3.10. *A score function for any HMCUI granule* $(\{x_r\}_{r=1}^k, [a_1, a_2])$, $Score : (2^{[0,1]} \setminus \{\varnothing\}) \times \mathcal{I} \to [0,1]$, *is defined by*

$$Score((\{x_r\}_{r=1}^k, [a_1, a_2])) = B(A^{(k)}((x_r)_{r=1}^k), a_1, a_2), \tag{3}$$

where $A^{(k)}$ *is the restriction of an extended mean operator* $A : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ *to* $[0,1]^k$ *and* $B : [0,1]^3 \to [0,1]$ *is any ternary mean operator.*

Definition 3.11. *A score function for any HCUI granule* $(\{x_r\}_{r=1}^k, [a_1, a_2], [u_1, u_2])$, $Score : (2^{[0,1]} \setminus \{\varnothing\}) \times \mathcal{I}^2 \to [0,1]$, *is defined by*

$$Score((\{x_r\}_{r=1}^k, [a_1, a_2], [u_1, u_2])) = B(A^{(k)}((x_r)_{r=1}^k), a_1, a_2, u_1, u_2), \tag{4}$$

where $A^{(k)}$ *is the restriction of an extended mean operator* $A : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ *to* $[0,1]^k$ *and* $B : [0,1]^5 \to [0,1]$ *is any mean operator of dimension five.*

Remark 3.12. *The involved (extended) aggregation operators can be selected based on the preferences of decision makers, enabling them to compare and evaluate alternatives in multi-criteria decision making using corresponding score functions.*

Example 3.13. *Suppose the extended mean function* $A : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ *with its restriction* $A^{(n)} \triangleq A \mid_{[0,1]^n}$ *satisfies* $A^{(n)}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n x_i$.

For the HMCUI granule $(\{x_r\}_{r=1}^3, [a_1, a_2]) = (\{0.5, 0.6, 0.7\}, [0.3, 0.9])$, suppose $B : [0,1]^3 \to [0,1]$ satisfies $B(\mathbf{y}) = B(y_1, y_2, y_3) = 0.2y_1 + 0.3y_2 + 0.5y_3$. Then from (3) we have

$Score((\{0.5, 0.6, 0.7\}, [0.3, 0.9])) = B(0.6, 0.3, 0.9) = 0.2(0.6) + 0.3(0.3) + 0.5(0.9) = 0.66.$

For the HCUI granule $(\{x_r\}_{r=1}^4, [a_1, a_2], [u_1, u_2]) = (\{0.3, 0.4, 0.5, 0.6\}, [0.3, 0.6], [0, 0.6])$, suppose $B : [0,1]^5 \to [0,1]$ satisfies $B(\mathbf{y}) = B(y_1, y_2, y_3, y_4, y_5) = 0.6y_1 + 0.2y_4 + 0.2y_5$. Then from (4) we have

$Score((\{0.3, 0.4, 0.5, 0.6\}, [0.3, 0.6], [0, 0.6])) = B(0.45, 0.3, 0.6, 0, 0.6) = 0.6(0.45) + 0.2(0) + 0.2(0.6) = 0.39.$

4 HMCUI and HCUI in evaluation, aggregation and decision making

For any HMCUI granule $(\{x_r\}_{r=1}^k, [a_1, a_2])$ and any HCUI granule $(\{x_r\}_{r=1}^k, [a_1, a_2], [u_1, u_2])$, the hesitant evaluation values $\{x_r\}_{r=1}^k$ are the main concerned data for comparing with or aggregating with the counterparts in other HMCUI or HCUI granules. To achieve this objective, it is appropriate to apply extended aggregation operators to hesitant evaluation values in order to obtain real values for evaluation and subsequent decision-making processes. In decision-making scenarios where multiple such real values are obtained from different HMCUI or HCUI granules, they can be further aggregated using weighted mean or other forms of aggregation for comprehensive or overall evaluation. Previous studies have highlighted certain advantages of CII, CUI, ICII, and ICUI in decision-making problems [15, 16]. Building upon similar backgrounds discussed in references [16], this section focuses on discussing weight allocation and aggregation methods specific to the corresponding HMCUI and HCUI environments.

4.1 Decision model 1

Suppose a decision maker presented a HMCUI granule $(\{x_r\}_{r=1}^k, [a_1, a_2])$ as the evaluation value for an object under evaluation, and he wants to have more different opinions from a group of n invited experts; each of the invited experts is allowed to have a typical hesitant fuzzy granule and they altogether can be denoted by a vector of typical hesitant fuzzy granules $(\{x_{ir}\}_{r=1}^{k_i})_{i=1}^n$. It is desired for the decision maker to consider the opinions of the experts but still under the main influence of his own cognition. However, different from the aggregation methods adopted in [15, 16], in HMCUI environment we may reasonably take the average of at most $k + \sum_{i=1}^n k_i$ real values; that is, we may consider $\{x_r\}_{r=1}^k$ and $\{x_{ir}\}_{r=1}^{k_i}$ whenever $x_{ir} \in [a_1, a_2]$ $(i \in \{1, \cdots, n\},\ r \in \{1, \cdots, k_i\})$. Therefore, with the influence of acceptance interval $[a_1, a_2]$, we may have the following reasonable aggregation result as a desired evaluation after the decision maker considers the opinions from those experts:

$$evaluation = \frac{\sum_{r=1}^k x_r + \sum_{i=1}^n \sum_{r \in S_i} r}{k + \sum_{i=1}^n |S_i|} \tag{5}$$

where $S_i = \{y \in \{x_{ir}\}_{r=1}^{k_i} : y \in [a_1, a_2]\}$ for any $i \in \{1, \cdots, n\}$ and $|S|$ denotes the cardinality of any finite set S.

Example 4.1. *For* $(\{x_r\}_{r=1}^3, [a_1, a_2]) = (\{0.6, 0.7, 0.8\}, [0.4, 0.9])$*,* $n = 4$ *with* $(\{x_{ir}\}_{r=1}^{k_i})_{i=1}^4 = (\{0.1, 0.5, 0.9\}, \{0.5\}, \{0.2, 0.3, 0.4\}, \{0.9, 1\})$*, we have* $S_1 = \{0.5, 0.9\}$*,* $S_2 = \{0.5\}$*,* $S_3 = \{0.4\}$*,* $S_4 = \{0.9\}$*. Then, using (5) we obtain*

$$evaluation = \frac{(0.6 + 0.7 + 0.8) + ((0.5 + 0.9) + (0.5) + (0.4) + (0.9))}{3 + (2 + 1 + 1 + 1)} = 0.6625.$$

4.2 Decision model 2

Similar to the background of decision model 2 in reference [16], still suppose a group of experts $\{E_i\}_{i=1}^n$ are invited and each of them is requested to offer an evaluation value that is a HCUI granule for an object under evaluation. That is, we may obtain a vector of n HCUI granules $((\{x_{ir}\}_{r=1}^{k_i}, [a_{i1}, a_{i2}], [u_{i1}, u_{i2}]))_{i=1}^n$. With the n different HCUI granules, a decision maker wants to obtain a single real value result as the final comprehensive evaluation used for later decision making. In this group decision making environment, the extent (i.e., S_{ij}) of expert E_j accepting the opinion (evaluation value) of expert E_i is determined by the structures of the HCUI granules $(\{x_{ir}\}_{r=1}^{k_i}, [a_{i1}, a_{i2}], [u_{i1}, u_{i2}])$ and $(\{x_{jr}\}_{r=1}^{k_j}, [a_{j1}, a_{j2}], [u_{j1}, u_{j2}])$. That is, if more x_{ir} fall into the acceptance interval $[a_{j1}, a_{j2}]$ and less into the unaccepted area $[0, 1] \setminus [u_{j1}, u_{j2}]$, then this acceptance extent S_{ij} will be larger and expert E_i can obtain more weight derived from expert E_j; after exhausting all $j \in \{1, \cdots, n\}$, $r_i = \sum_{j=1}^n S_{ij}$ denotes the overall acceptance extent gained for expert E_i. With the vector of overall acceptance extents $\mathbf{r} = (r_i)_{i=1}^n$ we may obtain a weight vector $\mathbf{w} = (w_i)_{i=1}^n$ by Yager preference induced weights allocation method and then take the weighted mean for those HCUI granules with the methods discussed previously and return a single desired result to use. That is, we may firstly use an extended aggregation operator $A : \bigcup_{n \in \mathbb{N}} [0, 1]^n \to [0, 1]$ to aggregate each vector of $(x_{ir})_{r=1}^{k_i}$ with $z_i = A^{(k_i)}((x_{ir})_{r=1}^{k_i})$ (with $A^{(n)} \triangleq A\mid_{[0,1]^n}$) and then take the weighted mean of the vector $\mathbf{z} = (z_i)_{i=1}^n$ with $\mathbf{w}$.

To calculate r_i (the overall acceptance extent for expert E_i) for each $i \in \{1, \cdots, n\}$, we need to obtain S_{ij} firstly. Note that if $x_{ir} \in [a_{j1}, a_{j2}]$ then it is reasonable for expert E_i to gain a bit of positive acceptance or support from expert E_j, and if $x_{ir} \in [0, 1] \setminus [u_{j1}, u_{j2}]$ then expert E_i should gain a bit of negative acceptance or disapproval. Therefore, for each $i, j \in \{1, \cdots, n\}$ we define

$$S_{ij} = 1 \cdot \frac{|P_{ij}|}{k_i} + (-1) \cdot \frac{|N_{ij}|}{k_i} = \frac{|P_{ij}| - |N_{ij}|}{k_i} \tag{6}$$

where $P_{ij} = \{y \in \{x_{ir}\}_{r=1}^{k_i} : y \in [a_{j1}, a_{j2}]\}$, $N_{ij} = \{y \in \{x_{ir}\}_{r=1}^{k_i} : y \in [0,1] \setminus [u_{j1}, u_{j2}]\}$. Note that we always have $S_{ii} = 1$ for any $i \in \{1, \cdots, n\}$.

With the vector of overall acceptance extents $\mathbf{r} = (r_i)_{i=1}^n \in [-1,1]^n$, an ideal method to generate the weight $\mathbf{w} = (w_i)_{i=1}^n$ is to adopt Yager preference induced weights allocation method. With a vector of overall acceptance extents $\mathbf{r} = (r_i)_{i=1}^n$, we can use a quantifier function Q to generate the desired weights for the experts. A (fuzzy) quantifier $Q : [0,1] \to [0,1]$ is a monotonic non-decreasing function with $Q(0) = 0$ and $Q(1) = 1$ [29]. For example, a concave quantifier function Q with $Q(t) = 1 - (1-t)^2$ can indicate some moderate extent of preference to the experts who get larger overall acceptance extent r_i. Then, given a quantifier Q, the weight vector $\mathbf{w} = (w_i)_{i=1}^n$ can be generated by the following generalized formulation of Yager's method [19].

$$w_i = \frac{Q(|S_{i1}|/n) - Q(|S_{i2}|/n)}{|S_{i3}|} \qquad (i = 1, \cdots, n) \tag{7}$$

where $S_{i1} = \{k \in \{1, \cdots, n\} : r_k \geq r_i\}$, $S_{i2} = \{k \in \{1, \cdots, n\} : r_k > r_i\}$ and $S_{i3} = \{k \in \{1, \cdots, n\} : r_k = r_i\}$.

Example 4.2. *In general, the value of a Chinese jade artifact increases with its age. A collector seeks to determine the value of an inherited jade piece as a benchmark for future sale. To assess its age, she has enlisted the expertise of four antique and archaeological specialists. Although they cannot provide an exact date, they estimate that it was produced between the 10th and 20th centuries AD. For evaluation purposes, we assume that the range [0,1] corresponds to the 11th-21st centuries, where any* $x \in [0,1]$ *represents the* $10 + 10x$*th century.*

Suppose a group of 4 antique experts (denoted by $\{E_i\}_{i=1}^4$) are invited and each of them is requested to provide an HCUI granule $(\{x_{ir}\}_{r=1}^{k_i}, [a_{i1}, a_{i2}], [u_{i1}, u_{i2}])$ for an object under evaluation. Assume $(\{x_{1r}\}_{r=1}^3, [a_{11}, a_{12}], [u_{11}, u_{12}]) = (\{0.5, 0.6, 0.7\}, [0.3, 0.9], [0.1, 1])$ (that is, this expert evaluates that the jade artifact is highly likely to date back to the Ming Dynasty (about 15th, 16th or 17th centuries AD), with a possibility (acceptance range for other experts' opinions) of originating from the Yuan (around 13th-14th centuries AD) or Qing Dynasties (including 18th-19th centuries AD); however, it is improbable to be a contemporary creation (20th century) or an artifact from the Tang Dynasty (existing before 11th century).

$$(\{x_{2r}\}_{r=1}^1, [a_{21}, a_{22}], [u_{21}, u_{22}]) = (\{0.7\}, [0.4, 1], [0.2, 1]),$$

$$(\{x_{3r}\}_{r=1}^4, [a_{31}, a_{32}], [u_{31}, u_{32}]) = (\{0.3, 0.4, 0.5, 0.6\}, [0.3, 0.6], [0, 0.6]),$$

$$(\{x_{4r}\}_{r=1}^5, [a_{41}, a_{42}], [u_{41}, u_{42}]) = (\{0.4, 0.5, 0.6, 0.7, 0.8\}, [0.2, 0.9], [0.2, 1]).$$

Suppose we choose a concave quantifier function Q with $Q(t) = 1 - (1-t)^2$ with moderate extent of preference to the experts who get larger overall acceptance extent r_i.

Then, from (6) we calculate:

$S_{11} = 1$, $S_{12} = 1 \cdot \frac{3}{3} + (-1) \cdot \frac{0}{3} = 1$, $S_{13} = 1 \cdot \frac{2}{3} + (-1) \cdot \frac{1}{3} = \frac{1}{3}$, $S_{14} = 1 \cdot \frac{3}{3} + (-1) \cdot \frac{0}{3} = 1$;
$S_{21} = 1$, $S_{22} = 1$, $S_{23} = -1$, $S_{24} = 1$;
$S_{31} = 1$, $S_{32} = 1 \cdot \frac{3}{4} + (-1) \cdot \frac{0}{4} = \frac{3}{4}$, $S_{33} = 1$, $S_{34} = 1$;
$S_{41} = 1$, $S_{42} = 1$, $S_{43} = 1 \cdot \frac{3}{5} + (-1) \cdot \frac{2}{5} = \frac{1}{5}$, $S_{44} = 1$.

Hence, $r_1 = \sum_{j=1}^4 S_{1j} = 3.33$, $r_2 = 2$, $r_3 = 3.75$, $r_4 = 3.2$.

With the obtained $\mathbf{r} = (r_i)_{i=1}^4 = (3.33, 2, 3.75, 3.2)$ and the preset quantifier function $Q(t) = 1 - (1-t)^2$, from (7) we have $w_1 = \frac{Q(2/4) - Q(1/4)}{1} = 0.3125$, $w_2 = 0.0625$, $w_3 = 0.4375$, $w_4 = 0.1875$.

Suppose the extended mean function $A : \bigcup_{n \in \mathbb{N}} [0,1]^n \to [0,1]$ with its restriction $A^{(n)} \triangleq A\mid_{[0,1]^n}$ satisfying $A^{(n)}(\mathbf{x}) \triangleq \frac{1}{n}\sum_{i=1}^n x_i$. Then, $z_1 = A^{(3)}((x_{1r})_{r=1}^3) = \frac{0.5+0.6+0.7}{3} = 0.6$, $z_2 = A^{(1)}((x_{2r})_{r=1}^1) = 0.7$, $z_3 = A^{(4)}((x_{3r})_{r=1}^4) = \frac{0.3+0.4+0.5+0.6}{4} = 0.45$, $z_4 = A^{(5)}((x_{4r})_{r=1}^5) = \frac{0.4+0.5+0.6+0.7+0.8}{5} = 0.6$.

Finally, take the weighted mean of the vector $\mathbf{z} = (0.6, 0.7, 0.45, 0.6)$ with $\mathbf{w} = (0.3125, 0.0625, 0.4375, 0.1875)$,and then we obtain

$$Evaluation = \sum_{i=1}^{4} w_i z_i = 0.540625,$$

which means, after being evaluated by various experts, the historical value of this jade artifact is approximately equivalent to that of 15th-century jade artifacts. Additionally, unlike extensive appraisal programs involving multiple experts with differing opinions engaging in repeated discussions and evaluations, our method does not necessitate mutual acquaintance among the experts.

We discuss some advantage of using the proposed method over hesitant fuzzy information in decision making. It is known that decision making processes often involve dealing with uncertain and imprecise information. In such scenarios, intuitionistic fuzzy sets have been widely used to represent and handle uncertainty by presenting acceptance and rejection information. Hesitant fuzzy sets have emerged as an alternative approach that allows decision makers to express their hesitancy or ambiguity towards different alternatives. However, hesitant fuzzy information cannot present further additional information such as acceptance and rejection information which are important in many decision making scenarios. In such contexts, HCUI presents itself as an advanced technique that offers several advantages over hesitant fuzzy set models HCUI can better capture and represent decision makers' cognitive behavior when faced with complex decision-making situations. Simply speaking, due to the acceptance and rejection areas it contains, HCUI consider the natures of both the intuitionistic fuzzy information and the hesitant fuzzy information.

5 Conclusions

CII, ICII, and HMCUI granules all consist of paired forms, where the first entry represents the main evaluation value (in different forms) and the second entry denotes the acceptance interval. As their bipolar extensions, CUI, ICUI, and HCUI granules adopt triad forms with an additional third entry representing the unaccepted area. HMCUI and HCUI can be considered as generalizations of CII and CUI respectively. Since hesitant fuzzy sets lack lattice structures, HMCUI and HCUI also do not possess such structures or defined order relations. Consequently, extended aggregation operators are employed to define score functions for HMCUI and HCUI in order to facilitate comparisons between their respective granules. Due to their distinct structures, decision models suitable for ICII and ICUI may no longer be applicable in HMCUI and HCUI environments. Therefore, corresponding reanalysis and reformulations have been made for these two models by incorporating extended aggregation operators along with Yager preference-induced weight allocation techniques. HCUI outperforms hesitant fuzzy set models in terms of accuracy, flexibility, interpretability, and computational efficiency. The potential utilization of HMCUI and HCUI extends beyond current decision-making problems; further extensions are also feasible which would contribute to enriching both aggregation theory and decision theory.

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References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Physica, Heidelberg (1999). https://doi.org/10.1007/978-3-7908-1870-3_1

[2] B. Bedregal, R. Reiser, H. Bustince, C. Lopez-Molina, V. Torra, *Aggregation functions for typical hesitant fuzzy elements and the action of automorphisms*, Information Sciences, **255** (2014), 82-99. https://doi.org/10.1016/j.ins.2013.08.024

[3] B. Bedregal, R. H. N. Santiago, H. Bustince, D. Paternain, R. Reiser, *Typical hesitant fuzzy negations*, International Journal of Intelligent Systems, **29** (2014), 525-543. https://doi.org/10.1002/int.21655

[4] M. Boczek, A. Hovana, O. Hutník, M. Kaluszka, *Hölder-Minkowski type inequality for generalized Sugeno integral*, Fuzzy Sets and Systems, **396** (2020), 51-71. https://doi.org/10.1016/j.fss.2020.01.005

[5] M. Boczek, A. Hovana, O. Hutník, M. Kaluszka, *New monotone measure-based integrals inspired by scientific impact problem*, European Journal of Operational Research, **290**(1) (2021), 346-357. https://doi.org/10.1016/j.ejor.2020.07.057

[6] T. Calvo, G. Mayor, R. Mesiar, *Aggregation operators: New trends and applications*, Springer Science and Business Media, 2002.

[7] J. P. Chang, Z. S. Chen, Z. J. Wang, L. Jin, W. Pedrycz, L. Martínez, M. J. Skibniewski, *Assessing spatial synergy between integrated urban rail transit system and urban form: A BULI-based MCLSGA model with the wisdom of crowds*, IEEE Transactions on Fuzzy Systems, **31**(2) (2023), 434-448. https://doi.org/10.1109/TFUZZ.2022.3185680

[8] Z. S. Chen, X. Zhang, W. Pedrycz, X. J. Wang, M. J. Skibniewski, *Bid evaluation in civil construction under uncertainty: A two-stage LSP-ELECTRE III-based approach*, Engineering Applications of Artificial Intelligence, **94** (2020), 103835. https://doi.org/10.1016/j.engappai.2020.103835

[9] Z. S. Chen, X. Zhang, R. M. Rodríguez, W. Pedrycz, L. Martínez, *Expertise-based bid evaluation for construction-contractor selection with generalized comparative linguistic ELECTRE III*, Automation in Construction, **125** (2021), 103578. https://doi.org/10.1016/j.autcon.2021.103578

[10] Z. S. Chen, X. Zhang, R. M. Rodríguez, W. Pedrycz, L. Martínez, M. J. Skibniewski, *Expertise-structure and risk-appetite-integrated two-tiered collective opinion generation framework for large-scale group decision making*, IEEE Transactions on Fuzzy Systems, **30**(12) (2022), 5496-5510. https://doi.org/10.1109/TFUZZ.2022.3179594

[11] Z. S. Chen, M. D. Zhou, K. S. Chin, A. Darko, X. J. Wang, W. Pedrycz, *Optimized decision support for BIM maturity assessment*, Automation in Construction, **149** (2023), 104808. https://doi.org/10.1016/j.autcon.2023.104808

[12] W. L. Gau, D. J. Buehrer, *Vague sets*, IEEE Transactions on Systems, Man, and Cybernetics, **23**(2) (1993), 610-614. https://doi.org/10.1109/21.229476

[13] M. Grabisch, J. L. Marichal, R. Mesiar, E. Pap, *Aggregation functions*, Cambridge University Press, 2009.

[14] I. Grattan-Guinness, *Fuzzy membership mapped onto intervals and many-valued quantities*, Mathematical Logic Quarterly, **22** (1976), 149-160. https://doi.org/10.1002/malq.19760220120

[15] L. Jin, Z. S. Chen, R. R. Yager, R. Langari, *Cognitive uncertain information with some properties and characteristics*, Journal of Intelligent and Fuzzy Systems, **44**(6) (2023), 9411-9418. https://doi.org/10.3233/JIFS-223119

[16] L. Jin, Z. S. Chen, R. R. Yager, R. Langari, *Interval type interval and cognitive uncertain information in information fusion and decision making*, International Journal of Computational Intelligence Systems, **16** (2023), 60. https://doi.org/10.1007/s44196-023-00227-4

[17] L. Jin, A. Kolesárová, R. Mesiar, *Semicopula based integrals*, Fuzzy Sets and Systems, **412** (2021), 106-119. https://doi.org/10.1016/j.fss.2021.01.004

[18] L. Jin, R. Mesiar, S. Borkotokey, M. Kalina, *Certainty aggregation and the certainty fuzzy measures*, International Journal of Intelligent Systems, **33** (2018), 759-770. https://doi.org/10.1002/int.21961

[19] L. Jin, R. Mesiar, R. Yager, *Ordered weighted averaging aggregation on convex poset*, IEEE Transactions on Fuzzy Systems, **27**(3) (2019), 612-617. https://doi.org/10.1109/TFUZZ.2019.2893371

[20] L. Jin, R. Mesiar, R. Yager, *Parameterized preference aggregation operators with improved adjustability*, International Journal of General Systems, **49**(8) (2020), 843-855. https://doi.org/10.1080/03081079.2020.1786822

[21] L. S. Jin, Y. Q. Xu, Z. S. Chen, R. Mesiar, R. R. Yager, *Relative basic uncertain information in preference and uncertain involved information fusion*, International Journal of Computational Intelligence Systems, **15** (2022), 12. https://doi.org/10.1007/s44196-022-00066-9

[22] L. Jin, R. R. Yager, Z. S. Chen, J. Špirková, R. Mesiar, *Interval and BUI type basic uncertain information in multi-sources evaluation and rules based decision making*, International Journal of General Systems, **52**(4) (2023), 443-454. https://doi.org/10.1080/03081079.2022.2142217

[23] G. Li, R. R. Yager, X. Zhang, R. Mesiar, H. Bustince, L. Jin, *Comprehensive rules-based and preferences induced weights allocation in group decision-making with BUI*, International Journal of Computational Intelligence Systems, **15** (2022), 54. https://doi.org/10.1007/s44196-022-00116-2

[24] R. Mesiar, S. Borkotokey, L. Jin, M. Kalina, *Aggregation under uncertainty*, IEEE Transactions on Fuzzy Systems, **26**(4) (2018), 2475-2478. https://doi.org/10.1109/TFUZZ.2017.2756828

[25] R. Mesiar, A. Kolesárová, S. Borkotokey, L. Jin, *Möbius product-based constructions of aggregation functions*, Fuzzy Sets and Systems, **448** (2022), 17-34. https://doi.org/10.1016/j.fss.2022.01.002

[26] G. Qian, E. Zhang, Z. Chen, R. Mesiar, R. R. Yager, L. Jin, *Consistent construction of evaluation threshold values and rules for heterogeneous linguistic input information*, International Journal of Computational Intelligence Systems, **14** (2021), 153. https://doi.org/10.1007/s44196-021-00003-2

[27] V. Torra, *Hesitant fuzzy sets*, International Journal of Intelligent Systems, **25**(6) (2010), 529-539. https://doi.org/10.1002/int.20418

[28] R. R. Yager, *On ordered weighted averaging aggregation operators in multicriteria decisionmaking*, IEEE Transactions on Systems, Man, and Cybernetics, **18**(1) (1988), 183-190. https://doi.org/10.1109/21.87068

[29] R. R. Yager, *Quantifier guided aggregation using OWA operators*, International Journal of Intelligent Systems, **11** (1996), 49-73. https://doi.org/10.1002/(SICI)1098-111X(199601)11:1<49::AID-INT3>3.0.CO;2-Z

[30] R. R. Yager, *Induced aggregation operators*, Fuzzy Sets and Systems, **137**(1) (2003), 59-69. https://doi.org/10.1016/S0165-0114(02)00432-3

[31] L. A. Zadeh, *A note on Z-numbers*, Information Sciences, **181**(14) (2011), 2923-2932. https://doi.org/10.1016/j.ins.2011.02.022