

NEW RESULTS ON THE EXISTING FUZZY DISTANCE MEASURES

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ABSTRACT. In this paper, we investigate the properties of some recently proposed fuzzy distance measures. We find out some shortcomings for these distances and then the obtained results are illustrated by solving several examples and compared with the other fuzzy distances.

1. Introduction

Applications of fuzzy numbers for indicating uncertain and vague information in decision making, linguistic controllers, expert systems, data mining, pattern recognition, etc. were stated in recently published articles [1, 2, 3, 4, 5, 11, 12, 15].

Many research articles have been published in order to construct the distance between fuzzy sets [10], fuzzy numbers [7, 8, 14, 17, 18]. Most of such efforts in the literature compute a distance measure between two fuzzy numbers as a crisp value. However, some researchers believe that the distance measure between fuzzy numbers should be fuzzy numbers instead of crisp numbers. To this end, as Voxman [16] said "if we are not certain about the numbers themselves how can we be "certain" about the distances among them", hence it is reasonable to use fuzzy distance measure instead of crisp distance between fuzzy numbers.

Recently, several authors have attempted to compute the fuzzy distance between fuzzy numbers. In [16], Voxman proposed a novel fuzzy distance measure between two arbitrary fuzzy numbers by using the extension principle of absolute distance. In [6], a new fuzzy distance measure is proposed based on the interval difference and also the metric properties are studied. This fuzzy distance has some shortcoming when the average of cores of compared fuzzy numbers is equal. This is the starting point to proposed some new result on the fuzzy distance measures. To this end, some improvement of fuzzy distance measure is proposed [6].

Consequently, Guha and Chakraborty in [9], improved the previous fuzzy distance measure by applying absolute value of fuzzy numbers which is obtained through the extension principle. Also, this modification has shortcoming again.

This paper is organized as follows:

In section 2, some basic concepts are provided. In section 3 some recently published fuzzy distance measures are considered and then some new results are given. In section 4, some illustrative examples are given and finally, Conclusion ends the paper in section 5.

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2. Basic Concepts

In this paper, we use the same notations and definitions of a generalized fuzzy number (GFN) as it was given in [6].

Definition 2.1. A generalized fuzzy number \tilde{A} represented by $\tilde{A} = (a_1, a_2; \beta, \gamma)$, i.e., (left point, right point, left spread, right spread), is a normalized convex fuzzy set on the real line \mathbb{R} if:

- (i) $\text{supp } \tilde{A} = \overline{\{x \mid \mu_{\tilde{A}}(x) > 0\}}$ is a closed and bounded interval, i.e. $[a_1 - \beta, a_2 + \gamma]$;
- (ii) $\mu_{\tilde{A}}$ is an upper semi-continuous function;
- (iii) $a_1 - \beta < a_1 \leq a_2 < a_2 + \gamma$; and
- (iv) the membership function is of the following form:

$$\mu_{\tilde{A}}(x) = \begin{cases} f(x), & x \in [a_1 - \beta, a_1], \\ 1, & x \in [a_1, a_2], \\ h(x), & x \in [a_2, a_2 + \gamma], \end{cases}$$

where f and h are the monotonic increasing and decreasing functions in $[a_1 - \beta, a_1]$ and $[a_2, a_2 + \gamma]$ respectively.

Note that, we have used the fuzzy zero, denoted by $\tilde{0}$, and its α -cut representation is defined by $\tilde{0}(\alpha) = [0, 0]$ for each $\alpha \in [0, 1]$.

Remark 2.2. Please notice that in Definition 2.1 the left and right spread should be non-zero, since it is assumed that $a_1 - \beta < a$ and $a_2 < a_2 + \gamma$. Also, these restrictions did not consider in [9] and then one can use a fuzzy number with zero spreads.

Let $\tilde{A} \in \mathbb{E}$, where \mathbb{E} denotes the class of all fuzzy numbers. Also, we state $[\tilde{A}]_{\alpha} = [A^L(\alpha), A^R(\alpha)]$ as the α -cut representation of \tilde{A} .

Remark 2.3. Moreover, the fuzzy numbers with segment core reduced to a single element as being a triangular fuzzy number which can be named as unimodal fuzzy number. However, since we refer our results to some previously published papers like Guha's paper [9] and Chakraborty et al.'s paper [6], we have used the same notations for definitions and results.

Remark 2.4. Please notice that we consider the Guha et al. approach [9] in normal case, since the other fuzzy distance measures are applied for normal fuzzy numbers.

3. Some Fuzzy Distance Measures

In this section, the Voxman [16], the Chakraborty et. al [6] and the Guha et al. [9] fuzzy distance measures are considered. Also, some new results are obtained for investigating the defects of fuzzy distance measures [6, 9, 16].

3.1. Voxman's Fuzzy Distance Measures [16]. Here, we briefly describe the fuzzy distance measure proposed by Voxman [16]. If the α -cut representations of $\tilde{A}_1 = (a_1, a_2; \beta_1, \gamma_1)$ and $\tilde{A}_2 = (a_3, a_4; \beta_2, \gamma_2)$ are $[A_1^L(\alpha), A_1^R(\alpha)]$ and $[A_2^L(\alpha), A_2^R(\alpha)]$

for all $\alpha \in [0, 1]$, respectively, then the α -cut representation of $\tilde{d}_{Voxman}(\tilde{A}_1, \tilde{A}_2)$ is $[L(\alpha), R(\alpha)]$ where

$$L(\alpha) = \begin{cases} \max\{A_2^L(\alpha) - A_1^R(\alpha), 0\}, & \text{if } \frac{A_1^L(1)+A_1^R(1)}{2} \leq \frac{A_2^L(1)+A_2^R(1)}{2}, \\ \max\{A_1^L(\alpha) - A_2^R(\alpha), 0\}, & \text{if } \frac{A_2^L(1)+A_2^R(1)}{2} \leq \frac{A_1^L(1)+A_1^R(1)}{2}, \end{cases}$$

and

$$R(\alpha) = \max\{A_1^R(\alpha) - A_2^L(\alpha), A_2^R(\alpha) - A_1^L(\alpha)\}. \quad (1)$$

Now, we emphasize on the case that the average of cores of considered fuzzy numbers are equal. To this end, let us consider the following result:

Theorem 3.1. *Let us consider $\frac{A_1^L(1)+A_1^R(1)}{2} = \frac{A_2^L(1)+A_2^R(1)}{2}$, then*

$$L(\alpha) = \max\{A_2^L(\alpha) - A_1^R(\alpha), 0\} = \max\{A_1^L(\alpha) - A_2^R(\alpha), 0\} = 0, \quad \forall \alpha \in [0, 1].$$

Proof. It is sufficient to show that the maximum value of $A_2^L(\alpha) - A_1^R(\alpha)$ and $A_1^L(\alpha) - A_2^R(\alpha)$ is less than or equal to zero for all $\alpha \in [0, 1]$. So, we prove that:

$$\begin{cases} A_2^L(\alpha) - A_1^R(\alpha) \leq 0, & \forall \alpha \in [0, 1], \\ A_1^L(\alpha) - A_2^R(\alpha) \leq 0, & \forall \alpha \in [0, 1]. \end{cases}$$

It is obvious that for all $\alpha \in [0, 1]$:

$$A_2^L(\alpha) - A_1^R(\alpha) \leq A_2^L(1) - A_1^R(1), \quad A_1^L(\alpha) - A_2^R(\alpha) \leq A_1^L(1) - A_2^R(1).$$

Now let us assume that $A_2^L(1) - A_1^R(1) > 0$. Then:

$$A_2^L(1) > A_1^R(1) \geq A_1^L(1) \Rightarrow A_2^L(1) > A_1^L(1).$$

Also, by the assumption of theorem we have:

$$A_1^L(1) - A_2^R(1) = A_2^L(1) - A_1^R(1) > 0 \Rightarrow A_1^L(1) > A_2^R(1) \geq A_2^L(1) \Rightarrow A_1^L(1) > A_2^L(1).$$

So, we should have $A_2^L(1) = A_1^L(1)$ (since, by using the assumption, we get $A_2^L(1) - A_1^R(1) > 0$ which leads to obtain the fact that $A_2^L(1) - A_1^L(1) > 0$ and also by using the last obtained result, we get $A_1^L(1) > A_2^L(1)$). Moreover, by applying the assumption of theorem, we get:

$$A_1^L(1) - A_1^R(1) = A_2^L(1) - A_1^R(1) > 0,$$

which is a contradiction. So the assumption of $A_2^L(1) - A_1^R(1) > 0$ is wrong and should be $A_2^L(1) - A_1^R(1) \leq 0$. Hence, we have $A_2^L(\alpha) - A_1^R(\alpha) \leq A_2^L(1) - A_1^R(1) \leq 0$. Also similar discussion is hold for $\max\{A_1^L(\alpha) - A_2^R(\alpha), 0\} = 0$, for all $\alpha \in [0, 1]$. \square

Now, we state a new result when the Voxman's fuzzy distance measure is applied. Intuitively, we expect that, the distance (each type) between two identical fuzzy numbers should be zero. However, the Voxman's approach has not such common property.

Theorem 3.2. *Let us consider $[\tilde{A}]_\alpha = [A^L(\alpha), A^R(\alpha)]$, then $[\tilde{d}_{Voxman}(\tilde{A}, \tilde{A})]_\alpha = [0, A^R(\alpha) - A^L(\alpha)]$, for all $\alpha \in [0, 1]$.*

Proof. Since the average of cores are equal, then by using Theorem 3.1 we obtain $L(\alpha) = 0$ for all $\alpha \in [0, 1]$. Also, by using Eq. (1) we get $R(\alpha) = A^R(\alpha) - A^L(\alpha)$. So, the α -cut representation of Voxman's fuzzy distance measure is obtained as follows:

$$\left[\tilde{d}_{Voxman}(\tilde{A}, \tilde{A}) \right]_{\alpha} = [0, A^R(\alpha) - A^L(\alpha)]. \quad \square$$

3.2. Chakraborty et al.'s Fuzzy Distance Measure [6]. Now, the Chakraborty et al.'s fuzzy distance measure [6] between two fuzzy numbers is reviewed. Let us consider two GFNs as $\tilde{A}_1 = (a_1, a_2; \beta_1, \gamma_1)$ and $\tilde{A}_2 = (a_3, a_4; \beta_2, \gamma_2)$. Therefore, the α -cut of \tilde{A}_1 and \tilde{A}_2 represents two intervals, respectively $\left[\tilde{A}_1 \right]_{\alpha} = [A_1^L(\alpha), A_1^R(\alpha)]$ and $\left[\tilde{A}_2 \right]_{\alpha} = [A_2^L(\alpha), A_2^R(\alpha)]$, for all $\alpha \in [0, 1]$.

Moreover, in [6] is employed the interval-difference operation for the intervals $[A_1^L(\alpha), A_1^R(\alpha)]$ and $[A_2^L(\alpha), A_2^R(\alpha)]$ to formulate the fuzzy distance between \tilde{A}_1 and \tilde{A}_2 . So, the distance between $\left[\tilde{A}_1 \right]_{\alpha}$ and $\left[\tilde{A}_2 \right]_{\alpha}$ for every $\alpha \in [0, 1]$ can be one of the following:

$$\begin{aligned} \text{either (a) } \left[\tilde{A}_1 \right]_{\alpha} - \left[\tilde{A}_2 \right]_{\alpha} \text{ if } \frac{A_1^L(1) + A_1^R(1)}{2} &\geq \frac{A_2^L(1) + A_2^R(1)}{2}, \\ \text{or (b) } \left[\tilde{A}_2 \right]_{\alpha} - \left[\tilde{A}_1 \right]_{\alpha} \text{ if } \frac{A_1^L(1) + A_1^R(1)}{2} &< \frac{A_2^L(1) + A_2^R(1)}{2}. \end{aligned}$$

To consider both notations together an indicator variable λ is used such that

$$\lambda \left(\left[\tilde{A}_1 \right]_{\alpha} - \left[\tilde{A}_2 \right]_{\alpha} \right) + (1 - \lambda) \left(\left[\tilde{A}_2 \right]_{\alpha} - \left[\tilde{A}_1 \right]_{\alpha} \right) = [d_{\alpha}^L, d_{\alpha}^R],$$

where

$$\lambda = \begin{cases} 1, & \text{if } \frac{A_1^L(1) + A_1^R(1)}{2} \geq \frac{A_2^L(1) + A_2^R(1)}{2}, \\ 0, & \text{if } \frac{A_2^L(1) + A_2^R(1)}{2} > \frac{A_1^L(1) + A_1^R(1)}{2}. \end{cases}$$

So,

$$\begin{aligned} d_{\alpha}^L &= \lambda \left[A_1^L(\alpha) - A_2^L(\alpha) + A_1^R(\alpha) - A_2^R(\alpha) \right] + \left[A_2^L(\alpha) - A_1^R(\alpha) \right], \\ d_{\alpha}^R &= \lambda \left[A_1^L(\alpha) - A_2^L(\alpha) + A_1^R(\alpha) - A_2^R(\alpha) \right] + \left[A_2^R(\alpha) - A_1^L(\alpha) \right]. \end{aligned}$$

Therefore, the fuzzy distance measure between \tilde{A}_1 and \tilde{A}_2 is defined by

$$\tilde{d}(\tilde{A}_1, \tilde{A}_2) = (d_{\alpha=1}^L, d_{\alpha=1}^R; \theta, \sigma), \quad (2)$$

where

$$\theta = d_{\alpha=1}^L - \max \left\{ \int_0^1 d_{\alpha}^L d\alpha, 0 \right\}, \quad \sigma = \int_0^1 d_{\alpha}^R d\alpha - d_{\alpha=1}^R.$$

3.2.1. Metric Properties. Also in [6], the following metric properties are studied as following:

- (a1) : $\tilde{d}(\tilde{A}_1, \tilde{A}_2) = (d_{\alpha=1}^L, d_{\alpha=1}^R; \theta, \sigma)$ is a positive fuzzy number;
- (a2) : $\tilde{d}(\tilde{A}_1, \tilde{A}_2) = \tilde{d}(\tilde{A}_2, \tilde{A}_1)$;
- (a3) : $\tilde{d}(\tilde{A}_3, \tilde{A}_2) \preceq \tilde{d}(\tilde{A}_1, \tilde{A}_2) + \tilde{d}(\tilde{A}_2, \tilde{A}_3)$.

Remark 3.3. In (a1), positive fuzzy number means that $d_{\alpha=1}^L - \theta > 0$. In [6] has not been proposed an ordering method for comparing two arbitrary fuzzy numbers. So, we believe that, the authors have considered the following ranking approach: Let us consider two arbitrary fuzzy numbers \widetilde{A}_1 and \widetilde{A}_2 then:

$$\widetilde{A}_1 \preceq \widetilde{A}_2 \text{ iff } \widetilde{A}_1 \subseteq \widetilde{A}_2.$$

So, by applying such ordering, the proposed fuzzy distance measure does not satisfy the triangular inequality.

There are some mistakes to produce the Chakraborty et al.'s fuzzy distance measure [6]. So, we investigate such defects.

Theorem 3.4. Let \widetilde{A}_1 and \widetilde{A}_2 be two arbitrary fuzzy numbers which fuzzy distance (2) is applied. If $\frac{A_1^L(1)+A_1^R(1)}{2} = \frac{A_2^L(1)+A_2^R(1)}{2}$, then $\theta = 0$ if both \widetilde{A}_1 and \widetilde{A}_2 are triangular fuzzy numbers and $\theta < 0$ if at least one of \widetilde{A}_1 or \widetilde{A}_2 are non triangular.

Proof. Suppose that \widetilde{A}_1 and \widetilde{A}_2 are triangular fuzzy numbers, then for proving $\theta = 0$, we show that:

$$d_{\alpha=1}^L = 0 \text{ and } \max \left\{ \int_0^1 d_{\alpha}^L d\alpha, 0 \right\} = 0.$$

Since, $\frac{A_1^L(1)+A_1^R(1)}{2} = \frac{A_2^L(1)+A_2^R(1)}{2}$, for $[\widetilde{A}_1]_{\alpha} - [\widetilde{A}_2]_{\alpha}$, we get:

$$[\widetilde{A}_1]_{\alpha} - [\widetilde{A}_2]_{\alpha} = (A_1^L(\alpha) - A_2^R(\alpha), A_1^R(\alpha) - A_2^L(\alpha)) \implies \begin{cases} d_{\alpha}^L = A_1^L(\alpha) - A_2^R(\alpha), \\ d_{\alpha}^R = A_1^R(\alpha) - A_2^L(\alpha), \end{cases}$$

Therefore, $d_{\alpha=1}^L = A_1^L(1) - A_2^R(1)$. By applying the assumption of theorem, it is obvious that $A_1^L(1) = A_1^R(1) = A_2^L(1) = A_2^R(1)$. So, $d_{\alpha=1}^L = A_1^L(1) - A_2^R(1) = A_1^L(1) - A_1^R(1) = 0$.

Also, we get

$$d_{\alpha}^L = A_1^L(\alpha) - A_2^R(\alpha) \leq d_{\alpha=1}^L = 0, (\forall \alpha \in [0, 1]) \implies \int_0^1 d_{\alpha}^L d\alpha \leq 0.$$

Therefore $\max \left\{ \int_0^1 d_{\alpha}^L d\alpha, 0 \right\} = 0$.

Now, suppose at least one of \widetilde{A}_1 or \widetilde{A}_2 is not triangular fuzzy number. To this end, suppose \widetilde{A}_1 is a triangular fuzzy number and \widetilde{A}_2 is a generalized non triangular fuzzy number, then to establish $\theta < 0$ we show that

$$d_{\alpha=1}^L < 0 \text{ and } \max \left\{ \int_0^1 d_{\alpha}^L d\alpha, 0 \right\} = 0.$$

Since, $\frac{A_1^L(1)+A_1^R(1)}{2} = \frac{A_2^L(1)+A_2^R(1)}{2}$, we consider for $[\widetilde{A}_1]_{\alpha} - [\widetilde{A}_2]_{\alpha}$ as follows. Firstly, it is obvious that $A_1^L(1) \neq A_2^R(1)$, otherwise by using the assumption of theorem we should have $A_2^L(1) = A_1^R(1)$ and this leads to \widetilde{A}_2 as a triangular fuzzy number, which is a contradiction.

Now let us consider $d_{\alpha=1}^L = A_1^L(1) - A_2^R(1) > 0$, then we should have

$$A_1^R(1) \geq A_1^L(1) > A_2^R(1) \geq A_2^L(1) \implies A_1^R(1) > A_2^L(1). \quad (3)$$

However, by hypothesis of theorem and the assumption $A_1^L(1) - A_2^R(1) > 0$, we have

$$\frac{A_1^L(1) + A_1^R(1)}{2} = \frac{A_2^L(1) + A_2^R(1)}{2} \implies A_1^R(1) < A_2^L(1). \quad (4)$$

Thus, by using equations (3)-(4), we get $A_1^R(1) = A_2^L(1)$ which contradicts $A_1^L(1) \neq A_2^R(1)$. So $d_{\alpha=1}^L = A_1^L(1) - A_2^R(1) < 0$.

Moreover,

$$d_{\alpha}^L = A_1^L(\alpha) - A_2^R(\alpha) \leq d_{\alpha=1}^L < 0, \quad (\forall \alpha \in [0, 1]) \implies \int_0^1 d_{\alpha}^L d\alpha < 0.$$

So, $\max \left\{ \int_0^1 d_{\alpha}^L d\alpha, 0 \right\} = 0$. Hence $\theta < 0$. Notice that similar discussion is valid when both A_1 and A_2 are generalized non triangular fuzzy numbers. \square

3.3. Guha et al.'s Fuzzy Distance Measure [9]. Now, we briefly review the new recently published fuzzy distance measure [9]. Let us consider the fuzzy distance measure [6] which is discussed in the previous section. The Guha's fuzzy distance measure depends on the previous one [6], strictly. So, we do not intend to repeat all the calculations, again. For more detail see [9].

The fuzzy distance measure proposed by Guha et al. [9] is denoted by \tilde{d}_{Guha} and is computed as follows:

$$\tilde{d}_{Guha}(\tilde{A}_1, \tilde{A}_2) = (d_{\alpha=1}^L, d_{\alpha=1}^R; \theta, \sigma), \quad (5)$$

where θ and σ are defined by the following way:

$$\begin{aligned} \theta &= d_{\alpha=1}^L - \max \left\{ \int_0^1 d_{\alpha}^L d\alpha, 0 \right\}, \\ \sigma &= \left| \int_0^1 d_{\alpha}^R d\alpha - d_{\alpha=1}^R \right|. \end{aligned}$$

By using the following result, similar to [16] we show that the valid result exists in the Guha et al. fuzzy distance measure [9], as follows:

Theorem 3.5. *Let us consider $\frac{A_1^L(1) + A_1^R(1)}{2} = \frac{A_2^L(1) + A_2^R(1)}{2}$, then $L(\alpha) = 0$ in the Guha et al.'s fuzzy distance measure (5) and then we conclude that $\theta = 0$.*

Proof. Based on the proposed fuzzy distance measure, $L(\alpha)$ is equal to $A_1^L(\alpha) - A_2^R(\alpha)$ or $A_2^L(\alpha) - A_1^R(\alpha)$. In both cases, the proof is completely similar to Theorem 3.1. Also, the proof of $\theta = 0$ is completely obvious. \square

This fuzzy distance measure has a significant shortcoming such that for two equal fuzzy numbers gives unreasonable result. We provide this defect by following result:

Theorem 3.6. Let us consider the fuzzy number $[\tilde{A}_1]_\alpha = [A_1^L(\alpha), A_1^R(\alpha)]$, then we have the following:

$$\tilde{d}_{Guha}(\tilde{A}_1, \tilde{A}_1) = (0, A_1^R(1) - A_1^L(1); 0, \sigma) \neq \tilde{0}.$$

Proof. By using the fuzzy distance measure \tilde{d}_{Guha} , the proof is obvious. \square

Moreover, in [9] some discussions are done in order to compare the ambiguity of the mentioned three fuzzy distance measures. Then, the following result is concluded:

$$Amb(\tilde{d}_{Guha}) \leq Amb(\tilde{d}_{Chakraborty}) \leq Amb(\tilde{d}_{Voxman}).$$

Now, we show through some illustrative example that this result (10) does not hold everywhere and therefore Proposition 1 in [9] is incorrect.

4. Examples

In this section, some illustrative examples are given to show the defects of the fuzzy distance measures [6, 9, 16].

Firstly, we show that the fuzzy distance measure [6] is not a generalized fuzzy number as it was given in Definition 2.1.

Example 4.1. Let us consider $[\tilde{A}_1]_\alpha = [1 + \alpha, 3 - \alpha]$ and $[\tilde{A}_2]_\alpha = [2\alpha, 4 - 2\alpha]$, then we get

$$[d_\alpha^L, d_\alpha^R] = [-3 + 3\alpha, 3 - 3\alpha] \Rightarrow \theta = 0, \sigma = 1.5 \Rightarrow \tilde{d}_{Chakraborty}(\tilde{A}_1, \tilde{A}_2) = (0, 0; 0, 1.5).$$

It is clear that in this example, left spread $\theta = 0$, which contradicts Definition 2.1. Since in Definition 2.1, spreads are assumed non zero. So this is a counterexample for fuzzy distance measure [6].

Example 4.2. Let us consider $[\tilde{A}_1]_\alpha = [\alpha, 3 - \alpha]$ and $[\tilde{A}_2]_\alpha = [\alpha, 4 - 2\alpha]$. Then, we get

$$[\tilde{A}_1]_\alpha - [\tilde{A}_2]_\alpha = [3\alpha - 4, 3 - 2\alpha] \Rightarrow \tilde{d}_{Chakraborty}(\tilde{A}_1, \tilde{A}_2) = (-1, 1; -1, 1).$$

It is clear that $\tilde{d}_{Chakraborty}(\tilde{A}_1, \tilde{A}_2) = (-1, 1; -1, 1)$ is not a fuzzy number, since $\theta < 0$. Also, consider another case as follows:

$$[\tilde{A}_2]_\alpha - [\tilde{A}_1]_\alpha = [2\alpha - 3, 4 - 3\alpha] \Rightarrow \tilde{d}_{Chakraborty}(\tilde{A}_2, \tilde{A}_1) = (-1, 1; -1, 1.5).$$

Obviously, $\tilde{d}_{Chakraborty}(\tilde{A}_2, \tilde{A}_1) = (-1, 1; -1, 1.5)$ is not fuzzy number, again, since $\theta < 0$.

Moreover, properties (a1) – (a2) does not hold. So, it is another counterexample for fuzzy distance measure [6].

However, if we neglect the non-zero condition for the spreads of fuzzy number in Definition 2.1, the defects still remain.

Example 4.3. Let us consider $[\widetilde{A}_1]_\alpha = [\alpha - 8, 6 - \alpha]$ and $[\widetilde{A}_2]_\alpha = [\alpha - 7, -6]$. So, $[d_\alpha^L, d_\alpha^R] = [\alpha - 2, 13 - 2\alpha] \Rightarrow \theta = -1, \sigma = 1 \Rightarrow \widetilde{d}_{Chacraborty}(\widetilde{A}_1, \widetilde{A}_2) = (-1, 11; -1, 1)$.

Clearly, $\widetilde{d}_{Chacraborty}(\widetilde{A}_1, \widetilde{A}_2) = (-1, 11; -1, 1)$ is not fuzzy number.

The influences of such defects of fuzzy distance measure are appeared in the triangular inequality when one of spreads is zero.

Example 4.4. Let us consider fuzzy numbers $[\widetilde{A}_1]_\alpha = [\alpha, 1]$, $[\widetilde{A}_2]_\alpha = [1, 2 - \alpha]$ and $[\widetilde{A}_3]_\alpha = [1, 3 - \alpha]$.

It is easy to verify that the averages of cores of fuzzy numbers \widetilde{A}_1 and \widetilde{A}_2 are identical and equal to 1. Therefore, we consider the interval-difference $[\widetilde{A}_1]_\alpha - [\widetilde{A}_2]_\alpha = [2\alpha - 2, 0]$ which gives $\widetilde{d}_{Chacraborty}(\widetilde{A}_1, \widetilde{A}_2) = (0, 0; 0, 0)$. Also,

$$\widetilde{d}_{Chacraborty}(\widetilde{A}_2, \widetilde{A}_3) = (0, 1; 0, 0.5), \quad \widetilde{d}_{Chacraborty}(\widetilde{A}_1, \widetilde{A}_3) = (0, 1; 0, 1).$$

So, we have

$$\widetilde{d}_{Chacraborty}(\widetilde{A}_1, \widetilde{A}_3) \not\subseteq \widetilde{d}_{Chacraborty}(\widetilde{A}_1, \widetilde{A}_2) + \widetilde{d}_{Chacraborty}(\widetilde{A}_2, \widetilde{A}_3).$$

Also, we investigate some shortcoming of fuzzy distance measure [9].

Example 4.5. Let us consider $[\widetilde{A}_1]_\alpha = [1 + \alpha, 3 - \alpha]$, then

$$\widetilde{d}_{Guha}(\widetilde{A}_1, \widetilde{A}_1) = (0, 0; 0, 1),$$

which is not a satisfactory result. Since we expect that the distance between two identical fuzzy numbers should be zero. However, using the Voxman's fuzzy distance measure, leads to

$$\widetilde{d}_{Voxman}(\widetilde{A}_1, \widetilde{A}_1) = (0, 0; 0, 2) \text{ which is an unreasonable result, too.}$$

Example 4.6. Let us consider two fuzzy numbers $[\widetilde{A}_1]_\alpha = [-1 + \alpha, 0]$ and $[\widetilde{A}_2]_\alpha = [0, 1 - \alpha]$.

Since, $\frac{A_1^L(1) + A_1^R(1)}{2} \geq \frac{A_2^L(1) + A_2^R(1)}{2}$, then by using $[\widetilde{A}_1]_\alpha - [\widetilde{A}_2]_\alpha$, we have the following:

$$\widetilde{d}_{Chacraborty}(\widetilde{A}_1, \widetilde{A}_2) = (0, 0; 0, 0), \quad \widetilde{d}_{Guha}(\widetilde{A}_1, \widetilde{A}_2) = (0, 0; 0, 1), \quad \widetilde{d}_{Voxman}(\widetilde{A}_1, \widetilde{A}_2) = (0, 0; 0, 2).$$

Then, by using definition of ambiguity of fuzzy numbers, $Amb(\widetilde{A}) = \int_0^1 \alpha (A^R(\alpha) - A^L(\alpha)) d\alpha$, we get

$$\begin{aligned} Amb(\widetilde{d}_{Chacraborty}(\widetilde{A}_1, \widetilde{A}_2)) &= 0, \\ Amb(\widetilde{d}_{Guha}(\widetilde{A}_1, \widetilde{A}_2)) &= \frac{1}{6}, \\ Amb(\widetilde{d}_{Voxman}(\widetilde{A}_1, \widetilde{A}_2)) &= \frac{1}{3}, \end{aligned}$$

which is contradicts Proposition 1 in [9].

5. Conclusion

In this paper, we discussed the Voxman's fuzzy distance measure [16], Chakraborty et. al [6] and Guha et. al [9] fuzzy distance measure. In [16], the extension principle has been applied to obtain fuzzy distance. We found a new shortcoming on fuzzy distance measure [16].

When the average of cores of two fuzzy numbers are equal, then $L(\alpha)$ is zero in Voxman's fuzzy distance, but the fuzzy distance proposed by Chakraborty et.al. [6] dose not have such property. It was the starting point of investigating the new results on three recently published fuzzy distance measures. To this end, some theorems are given to show the defects of existing fuzzy distance approaches.

Consequently, Guha et al. [9] improved the previous fuzzy distance [6] by adding some restriction on the sign of $L(\alpha)$. We show by some example that such modification has shortcoming too.

As a conclusion, it seems that this factor, the average of cores of fuzzy numbers, is not a sufficient condition to obtain a reasonable fuzzy distance measure.

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