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On the global optimal solutions of continuous FRE programming problems

A. Ghodousian [1] and S. Zal [2]

[1] *Faculty of Engineering Science, College of Engineering, University of Tehran, P.O.Box 11365-4563, Tehran, Iran*
[1,2] *Department of Engineering Science, College of Engineering, University of Tehran, Tehran, Iran*

a.ghodousian@ut.ac.ir, sara.zal@ut.ac.ir

Abstract

This paper presents some novel theoretical results as well as practical algorithms and computational procedures on continuous fuzzy relational equations programming problems. The fuzzy relational programming problem is a minimization (maximization) problem with a linear objective function subject to fuzzy relational equalities or inequalities defined by certain algebraic operations. In the literature, the commonly seen frameworks for such optimization models are to assume that the operation takes minimum t-norm, strict continuous t-norms (e.g., product t-norm), nilpotent continuous t-norms (e.g., Łukasiewicz t-norm) or Archimedean continuous t-norms. Based on new concepts called partial solution sets, the current paper considers this problem in the most general case where the fuzzy relational equality constraints are defined by an arbitrary continuous t-norm and capture some special characteristics of its feasible domain and the optimal solutions. It is shown that the current generalized results are automatically reduced to (apparently) different ones that hold for special operators when continuous t-norm is replaced by strict, nilpotent or Archimedean continuous t-norm. Also, the relationship between the results derived here and those of previous publications regarding this subject is also discussed. Finally, the proposed algorithm is outlined and illustrated by a numerical example where the continuous fuzzy relational equations is defined by Mayor-Torrens operator that is not an Archimedean t-norm (and then, neither strict nor nilpotent).

Keywords: Fuzzy relational equations, strict t-norms, nilpotent t-norms, Archimedean t-norms, continuous t-norms, global optimization, linear optimization.

1 Introduction

The theory of fuzzy relational equations (FRE) as a generalized version of Boolean relation equations was first proposed by Sanchez (based upon the max-min composition) and was applied to problems related to the medical diagnosis [37]. The properties and applications of FREs based on the max-min composition were further investigated by Czogala, Drewniak, Pedrycz [5], Prevot [35], and Higashi and Klir [20]. On the other hand, the fundamental result for FRE with max-product composition goes back to Pedrycz [31] and was further studied in [1, 25]. Pedrycz categorized and extended two ways of the generalizations of FRE in terms of sets under discussion and various operations which are taken into account [30]. It was pointed out by Pedrycz in [32] that FREs play a significant role as a platform for the uniform development of techniques in fuzzy sets. Since then, the resolution of FREs over lattices has been a theme of continuous interest in fuzzy inference and fuzzy systems theory and many theoretical and applied improvements have been investigated [21, 29, 33, 34, 39, 44]. For example, Klement et al. [21] presented the basic analytical and algebraic properties of triangular norms and important classes of fuzzy operators' generalization such as Archimedean, strict and nilpotent t-norms. In [34] the author demonstrates how problems of interpolation and approximation of fuzzy functions are converted with solvability of systems of FRE. From a practical point of view, the majority of fuzzy inference

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systems can be implemented using FREs [39]. Generally, when rules of inference are applied and their corresponding consequences are known, the problem of determining antecedents is simplified and mathematically reduced to solving an FRE [29]. Nowadays, it is well known that many of the issues associated with body knowledge can be treated as FRE problems [33]. Because of important applications in various practical fields, many scholars have focused on the theoretical research of FRE, including its resolution approach and some specific optimization problems with FRE constraints.

The solvability identification and finding a set of solutions belong to the topic of fuzzy relational equations which play an important role in the fuzzy modeling and have been widely studied by many researchers. For example, Higashi and Klir [20] proposed several alternative general schemes for solving FREs with max-min composition and compared these schemes from a computational viewpoint. In [1], the authors extended the study of an inverse solution of an FRE system with max-product composition. They provided theoretical results for determining the complete solution sets as well as the conditions for the existence of resolutions. In [26] the authors studied the same max-product FREs and presented some special characteristics of the feasible solution set, two procedures for reducing the original problem and some considerations for decomposing the problem into sub-problems. Compared to the regular linear programming problems, this linear optimization problem subject to FRE has a very different nature. When the solution set of FRE is non-empty, then it is generally a non-convex set that can be completely determined by one maximum solution and a finite number of minimal solutions [5, 20] (for max-min FRE), [1, 7] (for max-product FRE). Due to the non-convexity of its feasible domain defined by FRE, designing an efficient solution procedure for solving such problems is not a trivial job [10] (for max-min), [26] (for max-product). Indeed, because the solution set is non-convex, traditional linear programming methods, such as the simplex and interior point algorithms, cannot be applied. Another bottleneck point is concerned with detecting the minimal solutions for FREs. Since the total number of minimal solutions has a combinatorial nature in terms of the problem size, an efficient solution procedure is always in demand [26]. It is not easy to obtain all minimal solutions for a large scale problem because the number of minimal solutions may increase exponentially as of the problem size increases. Chen and Wang [3] presented an algorithm for obtaining the logical representation of all minimal solutions and deduced that a polynomial-time algorithm with the ability to find all minimal solutions of FRE (with max-min composition) may not exist. Also, Markovskii showed that solving max-product FRE is closely related to the covering problem which is a type of NP-hard problem [28]. In fact, the same result holds true for more general t-norms instead of the minimum and product operators [3, 23]. Over the past decades, the solvability of FRE which is defined using different max-t compositions has been investigated by many researchers [13, 15, 17, 40].

The optimization problem subject to FRE and FRI is one of the most interesting and on-going research topics amongst similar problems [2, 12, 13, 14, 15, 16, 17, 18, 19, 24, 36, 38, 41, 48]. Many methods were designed based on the translation of the main problem into a 0-1 integer linear programming problem which is then solved using the branch and bound techniques [2, 10, 18, 19, 26, 42, 43]. On the contrary, other algorithms benefit the resolution of the feasible region, some necessary and sufficient conditions for the optimality and simplification processes [13, 15, 38]. Fang and Li converted a linear optimization problem subjected to FRE constraints with max-min operation into an integer programming problem and solved it by a branch-and-bound method using a jump-tracking technique [10]. Wu et al. worked on the improvement of the method employed by Fang and Li; this was done by decreasing the search domain by deriving an initial upper bound for the optimal objective value in the branch-and-bound part [43]. They also presented a simplification process by three rules which resulted from a necessary condition [42]. Chang and Shieh presented new theoretical results concerning the linear optimization problem constrained by fuzzy max–min relation equations [2]. They improved an upper bound on the optimal objective value, some rules for simplifying the problem and proposed a rule for reducing the solution tree. Linear optimization problem was further investigated by numerous scholars focusing on max-product operation [18, 26]. Loetamonphong and Fang defined two sub-problems by separating negative from non-negative coefficients in the objective function, and then obtained an optimal solution by combining the optimal solutions of the two sub-problems [26]. Moreover, there is a growing interest in more general research on FRE with max-t-norm; this was done through the replacement of max-min and max-product compositions with different fuzzy compositions such as max-average composition [41], max-Lukasiewicz composition [13], max-Dubois-Prade composition [15], max-strict-t-norm composition [44] or max-Archimedean t-norm [19, 38, 39].

Recently, many interesting forms of generalizations of the linear programming applied to the system of fuzzy relations have been introduced, and developed based on composite operations used in FRE, fuzzy relations used in the definition of the constraints, some developments on the objective function of the problems and other ideas [6, 8, 11, 17, 22, 24, 45, 46]. For example, Wu et al. represented an efficient method to optimize a linear fractional programming problem under FRE with max-Archimedean t-norm composition [45]. Dempe and Ruziyeva generalized the fuzzy linear optimization problem by considering fuzzy coefficients [6]. In addition, Dubey et al. studied linear programming problems involving interval uncertainty modeled using intuitionistic fuzzy set [8]. The linear optimization of bipolar FRE was also the

focus of study carried out by some researchers where FRE was defined with max-min (with application in public awareness of the products for a supplier) [11], max-product [4] and max-Lukasiewicz composition [22, 24, 47]. In [11], the concept of bipolar FRE was first proposed with max-min composition where the constraints are expressed as $\max_{j=1}^{n}\left\{\max\left\{\min\{a_{ij}^{+}, x_j\}, \min\{a_{ij}^{-}, 1-x_j\}\right\}\right\}$ for $i = 1, ..., m$, where $a_{ij}^{+}, a_{ij}^{-}, x_j \in [0,1]$. Similarly, in [22], the authors introduced a linear optimization problem subjected to a system of bipolar FRE defined as $x(A^{+}, A^{-}, b) = \{x \in [0,1]^m \ : \ x \circ A^{+} \vee \tilde{x} \circ A^{-} = b\}$ where $\tilde{x}_i = 1 - x_i$ for each component of $\tilde{x} = (\tilde{x}_i)_{1\times m}$ and the notations "$\vee$" and "$\circ$" denote max operation and the max-Lukasiewicz composition, respectively. They translated the original problem into a 0-1 integer linear problem. In separate work, the foregoing bipolar linear optimization problem was solved by an analytical method based on the resolution and some structural properties of the feasible region (using a necessary condition for characterizing an optimal solution and a simplification process for reducing the problem) [24]. Yang [47] studied bipolar max-Lukasiewicz FRE and showed that the complete solution set of the system is fully determined by finite conservative bipolar paths. The optimization problem applied to various versions of FRI is widely available in the literature as well [12, 14, 48, 46]. Yang [46] studied the optimal solution of minimizing a linear objective function subject to a FRI where the constraints defined as $\sum_{j=1}^{n} \min\{\, a_{ij}, x_j \,\} \geq b_i$ for $i = 1, ..., m$. Also, in [48], the authors introduced the latticized linear programming problem subject to max-product fuzzy relation inequalities with application in the optimization management model for wireless communication emission base stations.

In this paper, we study the following optimization problem in which the constraints are formed as fuzzy relational equations defined by an arbitrary continuous t-norm:

$$\begin{aligned} \min \quad & z = c^T x \\ & A \,\omega\, x = b \\ & x \in [0,1]^n \end{aligned} \tag{1}$$

where matrix $A = (a_{ij})_{m\times n}$ is a fuzzy matrix such that $0 \leq a_{ij} \leq 1$, $\forall i \in I$ and $\forall j \in J$, $I = \{\, 1, 2, .., m \,\}$ and $J = \{\, 1, 2, .., n \,\}$. Also, $b = (b_i)_{m\times 1}$ is an m–dimensional fuzzy vector in $[0,1]^m$ (i.e., $0 \leq b_i \leq 1, \forall i \in I$). Moreover, c is a vector in R^n and $A \,\omega\, x = b$ denotes a system of fuzzy max-ω composition where ω is an arbitrary continuous t-norm. By these notations, if a_i is the i'th row of matrix A, then the constraint part of this problem is to find a set of solution vectors $x = (x_1, x_2, ..., x_n)$ such that:

$$a_i \,\omega\, x = \max_{j\in J}\{\omega\,(a_{ij}, x_j)\} = b_i, \quad i \in I. \tag{2}$$

Based on some properties of continuous functions, a new concept is initially introduced that is called partial solution set in this paper. The existence of the partial solution sets is proved, and then their structural properties are derived. It is shown that each partial solution set is a closed convex cell if the t-norm involved in the FRE is continuous. Partial solution sets provide useful insights into the characteristics of the feasible region of the problem in the most general case where the fuzzy relational equalities are defined by continuous t-norms. Some necessary and sufficient conditions are presented for determining the feasibility of the problem. Moreover, two basic procedures are provided with the aim of simplifying the current optimization problems. These procedures enable us to derive an efficient algorithm for solving the main problem.

The remainder of the paper is organized as follows. In Section 2, partial solution sets and their properties are discussed. Subsequently, the solution sets of each single equation is investigated, and then the feasible region of Problem (1) is determined as a union of the finite number of closed convex cells. Some necessary and sufficient conditions are derived for analyzing the feasibility of Problem (1). As a consequence, an algorithm is presented to determine the feasible solution set of the problem. In Section 3, two simplification operations are introduced to accelerate the resolution of the problem. Moreover, a necessary and sufficient condition (based on the simplification operations) is derived to identify the feasibility of the problem. Problem (1) is resolved through optimization of the linear objective function which is considered in Section 4. In addition, the existence of an optimal solution is proved if Problem (1) is not empty. The previously obtained results are summarized as an algorithm and, finally in Section 5 an illustration of this algorithm is provided using an example where the FRE constraints are defined by Mayor-Torrens operator that is a continuous non Archimedean t-norm.

2 Feasibility, solvability and fundamental results

This section describes the basic definitions and structural properties concerning the feasible region of Problem (1) where ω is generally defined as an arbitrary continuous t-norm. For the sake of simplicity, let $S(a_i, b_i)$ denote the feasible solution set of i'th equation, that is, $S(a_i, b_i) = \{\, x \in [0,1]^n \ : \ a_i \,\omega\, x = b_i \,\}$. Also, let $S(A, b) = \{\, x \in$

$[0,1]^n \ : \ A\,\omega\,x = b\,\}$ be the feasible solution set for Problem (1). Based on the foregoing notations, it is clear that $S(A,b) = \bigcap_{i\in I} S(a_i, b_i)$. For $x, y \in S(A,b)$, we say $x \leq y$ if and only if $x_j \leq y_j$, $\forall j \in J$. It is well-known that the operator "$\leq$" forms a partial-order relation on $[0,1]^n$, and hence $([0,1]^n, \leq)$ becomes a lattice with ω and max as its meet and join, respectively. In this way, $(S(A,b)\,, \leq)$ is a sub-poset of $([0,1]^n, \leq)$. Over this lattice, one can explore the concepts of maximum and minimal solutions. To be more precise, $\overline{x} \in S(A,b)$ is the maximum solution, if $x \leq \overline{x}$, $\forall x \in S(A,b)$. On the other hand, $\underline{x} \in S(A,b)$ is called a minimal solution, if $x \leq \underline{x}$ implies $x = \underline{x}$ for any $x \in S(A,b)$.Throughout this paper, for any vectors y and z in $[0,1]^n$, the notation $[y,z]$ is used to denote the cell consisting of all points x whose coordinates satisfy the inequalities $y_j \leq x_j \leq z_j$ for any $j \in J$.

2.1 Partial solution sets and their properties

Based on the equality $S(A,b) = \bigcap_{i\in I} S(a_i, b_i)$, we note that the set of the solutions $x \in [0,1]^n$ satisfying $A\,\omega\,x = b$ (i.e., all solutions $x \in S(A,b)$) is dependent on the feasible solution set of each equation $a_i\,\omega\,x = b_i$, $\forall i \in I$. Similarly, from (2), the feasible solution set of each single equation $a_i\,\omega\,x = b_i$ (i.e., $S(a_i, b_i)$) in turn depends on that of partial equations $\omega\,(a_{ij}, x) = b_i$, $\forall j \in J$, where $x \in [0,1]$. Motivated by these considerations, it is reasonable to analyze the partial equations $\omega\,(a_{ij}, x) = b_i$ $(i \in I)$ in the first place. In this paper, the solution sets of the partial equations are referred as Partial Solution Sets (PSS). As it will be shown in the next sections, the notion of PSS plays a central role in the feasibility, solvability and simplification of the system $A\,\omega\,x = b$. For ease of future reference and before going into in-depth description of the PSS properties, the concept of the partial solution sets is formally given in the following definition.

Definition 2.1. *For each $i \in I$ and each $j \in J$, the partial solution set Ω_{ij} is defined as $\Omega_{ij} = \{x \in [0,1] \ : \ \omega\,(a_{ij}\,, x) = b_i\}$. Also, if $\Omega_{ij} \neq \emptyset$, we define $\underline{x}_{ij} = \inf\ \Omega_{ij}$ and $\bar{x}_{ij} = \sup\ \Omega_{ij}$.*

Remark 2.2. *If $\Omega_{ij} \neq \emptyset$, then both $\underline{x}_{ij} = \inf\ \Omega_{ij}$ and $\bar{x}_{ij} = \sup\ \Omega_{ij}$ exist. This consequence is actually resulted from the fact that $\Omega_{ij} \subseteq [0,1]$ ($\forall i \in I$ and $\forall j \in J$), and the least-upper-bound property of $\mathbb{R}$ (i.e., the set of all real numbers).*

Lemma 2.3. *Let $\Omega_{ij} \neq \emptyset$. Then, Ω_{ij} is a closed set.*

Proof. Suppose that x' is a limit point of Ω_{ij} and $\{x_n\}_{n=1}^{\infty}$ is any convergent sequence of points x_n contained in Ω_{ij} with limit point x', i.e., $\{x_n\}_{n=1}^{\infty} \subseteq \Omega_{ij}$ and $\lim\limits_{n\to\infty} x_n = x'$. Let $f(x) = \omega\,(a_{ij}\,, x) - b_i$. Since ω is a continuous t-norm, f is a continuous function. Therefore, we have $\omega\,(a_{ij}\,, x') - b_i = f(x') = f(\lim\limits_{n\to\infty} x_n) = \lim\limits_{n\to+\infty} f(x_n) = 0$ (because,$f(x_n) = 0$ for $n = 1, 2, ...$). Thus, $\omega\,(a_{ij}\,, x') = b_i$ that implies $x' \in \Omega_{ij}$. □

From Lemma 2.3, each PSS is a closed subset of $[0,1]$. The following lemma shows that any PSS is also a connected set.

Lemma 2.4. *Let $\Omega_{ij} \neq \emptyset$. Then, Ω_{ij} is a connected subset of $[0,1]$.*

Proof. Since Ω_{ij} is a closed subset of $[0,1]$ (Definition 2.1 and Lemma 2.3), we have $\underline{x}_{\,ij} \in \Omega_{ij}$ and $\bar{x}_{ij} \in \Omega_{ij}$ (Definition 2.1). To complete the proof, it is sufficient to show that $x \in \Omega_{ij}$, $\forall x \in [\underline{x}_{\,ij}, \bar{x}_{ij}]$. But, given any $x \in [\underline{x}_{\,ij}, \bar{x}_{ij}]$, the monotonicity property of t-norms implies $\omega\,(a_{ij}\,, \underline{x}_{\,ij}) \leq \omega\,(a_{ij}\,, x) \leq \omega\,(a_{ij}\,, \bar{x}_{ij})$. Hence, since $\omega\,(a_{ij}\,, \underline{x}_{\,ij}) = \omega\,(a_{ij}\,, \bar{x}_{ij}) = b_i$, it is concluded that $\omega\,(a_{ij}\,, x) = b_i$, i.e., $x \in \Omega_{ij}$. □

As an important consequence, the following corollary is resulted from Lemmas 2.3 and 2.4.

Corollary 2.5. *Let $\Omega_{ij} \neq \emptyset$. Then, $\Omega_{ij} = [\,\underline{x}_{ij}, \bar{x}_{ij}]$, i.e., each PSS is a closed subinterval of $[0,1]$ with the minimum solution $\underline{x}_{ij}$ and the maximum solution $\bar{x}_{ij}$.*

Similar to Corollary 2.5, it has been proved that Ω_{ij} is a closed convex set if ω is any continuous non-decreasing operator [14].

Definition 2.6. *For each $i \in I$, we define $J_i = \{j \in J \ : \ \Omega_{ij} \neq \emptyset\}$. Also, for each $j \in J$, we define $I_j = \{i \in I \ : \ \Omega_{ij} \neq \emptyset\}$.*

According to Definition 2.6, we clearly have $j \in J_i$ if and only if $i \in I_j$. The following lemma and its corollaries give the basic properties of Ω_{ij} for a general case where ω is an arbitrary continuous t-norm.

Lemma 2.7. *Let $i \in I$ and $j \in J$. Then, $\Omega_{ij} \neq \emptyset \Leftrightarrow a_{ij} \geq b_i$.*

Proof. Assume that $a_{ij} \geq b_i$. Since ω is a t-norm, we have $\omega(a_{ij}, 0) = 0$ and $\omega(a_{ij}, 1) = a_{ij}$ [9]. So, if $b_i = 0$, then the former equality implies that $0 \in \Omega_{ij}$, and if $a_{ij} = b_i$, from the latter equality we have $1 \in \Omega_{ij}$. Otherwise, let $a_{ij} > b_i > 0$ and consider the continuous function $f(x) = \omega(a_{ij}, x) - b_i$. In this case, $f(0) = -b_i < 0$ and $f(1) = a_{ij} - b_i > 0$. Hence, based on the intermediate value theorem for continuous functions, there exists some $c \in (0,1)$ such that $f(c) = 0$. Therefore, $\omega(a_{ij}, c) = b_i$ that means $c \in \Omega_{ij}$. Conversely, suppose that $a_{ij} < b_i$. Since ω is a t-norm, then $\omega(a_{ij}, x) \leq \min\{a_{ij}, x\}, \forall x \in [0,1]$ [9]. So, $\omega(a_{ij}, x) \leq \min\{a_{ij}, x\} \leq a_{ij} < b_i, \forall x \in [0,1]$. Therefore, $\omega(a_{ij}, x) < b_i$, $\forall x \in [0,1]$, that means $\Omega_{ij} = \emptyset$. □

Corollary 2.8. *From Definition 2.6 and Lemma 2.7, we have* $J_i = \{j \in J : a_{ij} \geq b_i\}$, $\forall i \in I$. *Similarly,* $I_j = \{i \in I : a_{ij} \geq b_i\}$, $\forall j \in J$.

Corollary 2.9. *Let* $i \in I$. *If* $j \notin J_i$, *then* $\omega(a_{ij}, x) < b_i$, $\forall x \in [0,1]$.

Proof. From the monotonicity and identity laws of t-norms, we have $\omega(a_{ij}, x) \leq \omega(a_{ij}, 1) = a_{ij} < b_i$, $\forall x \in [0,1]$, where the last inequality follows from Corollary 2.8 and the assumption that $j \notin J_i$. □

Corollary 2.10. *Let* $i \in I$ *and* $j \in J_i$. *Then, (a)* $x > \bar{x}_{ij} \Leftrightarrow \omega(a_{ij}, x) > b_i$, *(b)* $x < \underline{x}_{ij} \Leftrightarrow \omega(a_{ij}, x) < b_i$, *where* $\underline{x}_{ij} = \min \Omega_{ij}$ *and* $\bar{x}_{ij} = \max \Omega_{ij}$ *(see Corollary 2.5).*

Proof. (a) Let $x > \bar{x}_{ij}$. So, from Corollary 2.5 we have $x \notin \Omega_{ij}$ that means $\omega(a_{ij}, x) \neq b_i$. On the other hand, $\omega(a_{ij}, x) \geq \omega(a_{ij}, \bar{x}_{ij}) = b_i$, from the monotonicity property of t-norms. Therefore, $\omega(a_{ij}, x) \geq b_i$ and $\omega(a_{ij}, x) \neq b_i$ that imply $\omega(a_{ij}, x) > b_i$. Conversely, let $\omega(a_{ij}, x) > b_i$. So, we must have $x > \bar{x}_{ij}$, because otherwise the inequality $x \leq \bar{x}_{ij}$ and the monotonicity property necessitate that $\omega(a_{ij}, x) \leq \omega(a_{ij}, \bar{x}_{ij}) = b_i$. (b) The proof is quite similar. □

Table A1 (see Appendix A) presents the thirteen commonly used continuous t-norms in literature. A more detailed description of the t-norms and their interesting properties are found in [9, 21]. Moreover, based on Definition 2.6 we know that each partial solution set Ω_{ij} is empty if and only if $j \notin J_i$. In the case that $j \in J_i$, the values of $\bar{x}_{ij}$ and $\underline{x}_{ij}$ are listed in Tables A2 and A3, respectively, for all the t-norms of Table A1.

2.2 Solution sets of the single equations

In this part, the single equation $a_i \,\omega\, x = \max_{j \in J}\{\omega(a_{ij}, x_j)\} = b_i$ and its feasible solution set (i.e., $S(a_i, b_i)$) are studied. For each $i \in I$, it is shown that the fundamental properties of $S(a_i, b_i)$ can be derived based on the properties of its corresponding partial solution sets (i.e. Ω_{ij}, $j \in J$). First, a new notation is introduced, that is useful to illustrate some properties.

Definition 2.11. *Let* $i \in I$ *and* $j \in J$. *Define* $\Phi_{ij} = \{x \in [0,1] : \omega(a_{ij}, x) \leq b_i\}$.

Remark 2.12. *From Definitions 2.1 and 2.11,* $\Omega_{ij} \subseteq \Phi_{ij}$ *for any* $i \in I$ *and*$j \in J$. *Also, from Corollaries 2.9 and 2.10, and the fact that* $\omega(a_{ij}, 0) = 0$ *for each t-norm, we have*

$$\Phi_{ij} = \begin{cases} [0,1] & j \notin J_i \\ [0, \bar{x}_{ij}] & j \in J_i, \end{cases} \tag{3}$$

where $\bar{x}_{ij} = \max \Omega_{ij}$.

Lemma 2.13 provides necessary and sufficient conditions for the consistency of the i'th constraint of Problem (1).

Lemma 2.13. *For a fixed* $i \in I$, $S(a_i, b_i) \neq \emptyset$ *if and only if the following two conditions are satisfied:*
(a) $\bigcup_{j \in J} \Omega_{ij} \neq \emptyset$, (b) *for each* $j \in J$, *if* $\Omega_{ij} = \emptyset$, *then* $\Phi_{ij} \neq \emptyset$.

Proof. Assume that conditions (a) and (b) are satisfied. From the condition (a) there exists some $j_0 \in J$ such that $\Omega_{ij_0} \neq \emptyset$ (i.e., $j_0 \in J_i$). Define vector $x = (x_1, x_2, ..., x_n)$ as follows. If $j \in J_i$ (then $\Omega_{ij} \neq \emptyset$), select some $x'_j \in \Omega_{ij}$ and set $x_j = x'_j$; otherwise, if $j \notin J_i$ (then $\Omega_{ij} = \emptyset$, and therefore $\Phi_{ij} \neq \emptyset$ by condition (b)), select some $x''_j \in \Phi_{ij}$ and set $x_j = x''_j$. By this definition, $\omega(a_{ij}, x_j) = \omega(a_{ij}, x'_j) = b_i$, $\forall j \in J_i$ (particularly, $j_0 \in J_i$ and hence $\omega(a_{ij_0}, x_{j_0}) = b_i$), and $\omega(a_{ij}, x_j) = \omega(a_{ij}, x''_j) < b_i$, $\forall j \notin J_i$. Therefore, we have

$$\max_{j=1}^{n} \{\omega(a_{ij}, x_j)\} = \max\left\{\max_{j \notin J_i} \{\omega(a_{ij}, x''_j)\}, \max_{j \in J_i} \{\omega(a_{ij}, x'_j)\}\right\} = \max_{j \in J_i} \{\omega(a_{ij}, x'_j)\} = b_i,$$

which means that $x \in S(a_i, b_i)$. Conversely, let $S(a_i, b_i) \neq \emptyset$ and $x \in S(a_i, b_i)$. Thus, $\max_{j=1}^{n} \{\omega(a_{ij}, x_j)\} = b_i$ which implies that both $\omega(a_{ij}, x_j) \leq b_i$ (i.e., $x_j \in \Phi_{ij}$), $\forall j \in J$, and $\omega(a_{ij_0}, x_{j_0}) = b_i$ (i.e., $x_{j_0} \in \Omega_{ij_0}$) for at least some $j_0 \in J$. The former requires $\Phi_{ij} \neq \emptyset$, $\forall j \in J$ (whether Ω_{ij} is empty or not), and the latter requires $\Omega_{ij_0} \neq \emptyset$, that implies $\bigcup_{j \in J} \Omega_{ij} \neq \emptyset$. □

Remark 2.14. *Based on the proof of Lemma 2.13, we can equivalently express this lemma as follows. There exists some vector* $x = (x_1, x_2, ..., x_n)$ *such that* $\max_{j=1}^{n} \{ \omega(a_{ij}, x_j) \} = b_i$ *(i.e.,* $x \in S(a_i, b_i)$*) if and only if the following two conditions hold:*
(a) *for at least some* $j_0 \in J_i$, $\omega(a_{ij_0}, x_{j_0}) = b_i$ *(or equivalently,* $x_{j_0} \in \Omega_{ij_0}$*).*
(b) *for each* $j \in J - \{j_0\}$, $\omega(a_{ij}, x_j) \leq b_i$ *(or equivalently,* $x_j \in \Phi_{ij}$*).*

Lemma 2.13 provides the most general conditions that are applicable for all the continuous t-norms. If ω is the minimum t-norm, Remark 2.14 is reduced to the necessary and sufficient condition used by relations (6) and (7) in [20]. As other special examples, we can refer to the necessary conditions stated by Lemma 2.3 in [10] (for minimum) and Lemma 2.3 in [26] (for product).
It is clear that the condition (a) of Lemma 2.13 is satisfied if and only if $J_i \neq \emptyset$(Definition 2.6). On the other hand, from relation (3) it can be easily concluded that $\Phi_{ij} \neq \emptyset$, $\forall j \in J$. For these reasons, Lemma 2.13 can be strengthened through deletion of condition (b). These results have been summarized in the following lemma.

Lemma 2.15. *Let* $i \in I$. $S(a_i, b_i) \neq \emptyset$ *if and only if* $J_i \neq \emptyset$.

Lemma 2.15 gives a simple necessary and sufficient condition for the feasibility of $S(a_i, b_i)$, $\forall i \in I$. It is to be noted that this lemma additionally provides a necessary condition for the feasibility of Problem (1).

Definition 2.16. *Suppose that* $i \in I$ *and* $S(a_i, b_i) \neq \emptyset$ *(hence,* $J_i \neq \emptyset$ *from Lemma 2.15). We define vector* $\bar{x}(i) = (\bar{x}(i)_1, \bar{x}(i)_2, ..., \bar{x}(i)_n)$ *where the components are defined as follows:*

$$\bar{x}(i)_j = \begin{cases} \bar{x}_{ij} & j \in J_i \\ 1 & j \notin J_i \end{cases}, \forall j \in J \tag{4}$$

Also, for each $j \in J_i$, *we define* $\underline{x}(i,j) = (\underline{x}(i,j)_1, \underline{x}(i,j)_2, ..., \underline{x}(i,j)_n)$ *such that*

$$\underline{x}(i,j)_k = \begin{cases} \underline{x}_{ij} & k = j \\ 0 & k \neq j \end{cases}, \forall k \in J \tag{5}$$

The feasible region of the i'th relational equation ($i \in I$) is characterized by means of the next theorem.

Theorem 2.17. *Let* $i \in I$. *If* $S(a_i, b_i) \neq \emptyset$, *then* $S(a_i, b_i) = \bigcup_{j \in J_i} \{ x \in [0,1]^n \ : \ \underline{x}(i,j)_k \leq x_k \leq \bar{x}(i)_k, \forall k \in J \}$.

Proof. Based on Remark 2.14, $x \in S(a_i, b_i)$ iff $x_{j_0} \in \Omega_{ij_0}$ for some $j_0 \in J_i$ and $x_j \in \Phi_{ij}$ for each $j \in J - \{j_0\}$. But, $x_{j_0} \in \Omega_{ij_0}$ means $x_{j_0} \in [\underline{x}_{ij_0}, \bar{x}_{ij_0}]$ (Corollary 2.5), and $x_j \in \Phi_{ij}$ means $x_j \in [0, \bar{x}_{ij}]$ when $j \in J_i$, and $x_j \in [0,1]$ when $j \notin J_i$ (Remark 2.12). Therefore, from Definition 2.16, $x \in \{ x \in [0,1]^n \ : \ \underline{x}(i,j_0)_k \leq x_k \leq \bar{x}(i)_k, \forall k \in J \}$. Consequently, $x \in S(a_i, b_i)$ if and only if $x \in \{ x \in [0,1]^n \ : \ \underline{x}(i,j_0)_k \leq x_k \leq \bar{x}(i)_k, \forall k \in J \}$ for some $j_0 \in J_i$. Now, the result follows from the fact that $\{ x \in [0,1]^n \ : \ \underline{x}(i,j_0)_k \leq x_k \leq \bar{x}(i)_k, \forall k \in J \} \subseteq \bigcup_{j \in J_i} \{ x \in [0,1]^n \ : \ \underline{x}(i,j)_k \leq x_k \leq \bar{x}(i)_k, \forall k \in J \}$. □

Theorem 2.17 also determines the upper and lower bounds for the feasible solution set of the i'th relational equality. In fact, for each $i \in I$, $S(a_i, b_i)$ has the unique maximum $\bar{x}(i)$, and the finite number of minimal solutions $\underline{x}(i,j)$ ($\forall j \in J_i$). If ω is the minimum t-norm, Theorem 2.17 is reduced to the well-known result stated by Lemma 3 in [20].

Remark 2.18. *By relation* (5) *and Theorem 2.17, it follows that each element* $j \in J_i$ *is associated with the minimal solution* $\underline{x}(i,j)$. *Conversely, each minimal solution* $\underline{x}(i,j)$ *can be also interpreted as the solution generated by* $j \in J_i$.

2.3 Solution set of the continuous FRE

In this subsection, we investigate the structural properties of $S(A,b)$, that is, the feasible solution set of the system $A \,\omega\, x = b$ where $x \in [0,1]^n$. First, a lemma is given, that is directly resulted from $S(A,b) = \bigcap_{i \in I} S(a_i, b_i)$ and Theorem 2.17. Then, some definitions are presented, which are used to characterize the feasible region of Problem (1).

Lemma 2.19. *Let* $S(A,b) \neq \emptyset$. *Then,* $S(A,b) = \bigcap_{i \in I} \bigcup_{j \in J_i} [\underline{x}(i,j), \bar{x}(i)]$.

Definition 2.20. *Let* $e : I \to \bigcup_{i \in I} J_i$ *be a function from* I *to* $\bigcup_{i \in I} J_i$ *such that* $e(i) \in J_i$, $\forall i \in I$. *Also, let* E *be the set of all the functions* e. *Moreover, for a given* $e \in E$, *we define* $I_j(e) = \{ i \in I \ : \ e(i) = j \}$.

According to Definition 2.20, the cardinality of E is equal to $|E| = \prod_{i=1}^{m} |J_i|$. Also, each $e \in E$ can be equivalently considered as the set of ordered pairs
$\{(1, j_1), (2, j_2), ..., (m, j_m)\}$ in which $e(i) = j_i \in J_i$, $\forall i \in I$. However, for the sake of simplicity, this set is usually represented only in terms of the second entries j_1, j_2,..., j_m and denoted by the notation $e = [j_1, j_2, ..., j_m]$ as a finite sequence [14, 20].

Definition 2.21. *We define vector $\bar{x} = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ whose j'th component is defined as $\bar{x}_j = \min_{i\in I}\{\bar{x}(i)_j\}$, where $\bar{x}(i)$ is the maximum solution of $S(a_i, b_i)$ $(i \in I)$. Also, for each $e \in E$, we define $\underline{x}(e) = (\underline{x}(e)_1, \underline{x}(e)_2, ..., \underline{x}(e)_n)$, where $\underline{x}(e)_j = \max_{i\in I}\{\underline{x}(i, e(i))_j\}$, $\forall j \in J$, and $\underline{x}(i, e(i))$ is the minimal solution of $S(a_i, b_i)$, that is generated by $e(i)$ (see Remark 2.18). Furthermore, denote the set of all the solutions $\underline{x}(e)$, where $e \in E$, by S_E, i.e., $S_E = \{\underline{x}(e) : e \in E\}$.*

Remark 2.22. *From Definitions 2.20 and 2.21, it is clear that the cardinality of S_E is equal to the number of all the functions $e \in E$. More precisely, we have $|S_E| = |E| = \prod_{i=1}^{m} |J_i|$.*

Based on Lemma 2.19 and Definition 2.21, we have the following theorem in which the bounds of the feasible solution set for Problem (1) are attained. As described in Theorem 2.23, by using these obtained bounds, the entire feasible region can then be found in the most familiar way.

Theorem 2.23. *Let $S(A, b) \neq \emptyset$. Then, $S(A, b) = \bigcup_{e\in E}[\underline{x}(e), \bar{x}]$.*

Proof. The proof is obtained by the following equalities

$$\begin{aligned} S(A,b) &= \bigcap_{i\in I}\bigcup_{j\in J_i}[\underline{x}(i,j), \bar{x}(i)] = \bigcup_{e\in E}\bigcap_{i\in I}[\underline{x}(i,e(i)), \bar{x}(i)] = \bigcup_{e\in E}[\max_{i\in I}\{\underline{x}(i,e(i))\}, \min_{i\in I}\{\bar{x}(i)\}] \\ &= \bigcup_{e\in E}[\underline{x}(e), \bar{x}] \end{aligned},$$

where the first and the last equalities are attained from Lemma 2.19 and Definition 2.21, respectively. □

Special cases of Theorem 2.23 can be frequently found in the literature when ω is the minimum or product t-norm. For the minimum t-norm, in order to find set $S(A, b)$, Prevot [35] proposed an algorithm that was based on essentially the same argument as Theorem 1 in [20]. It should be mentioned, however, that Prevot's algorithm generates all the solutions in the special case that all b_i $(i \in I)$ are distinct and $a_{ij} \neq b_i$ for any $i \in I$ and $j \in J$. Basically, Scheme (I) in [20] for the generation of all the solutions can be viewed as a generalization of the procedure by Prevot. similar to what is presented in [20], for the product t-norm, in [1] the authors showed $x \in S(A, b)$ if and only if there exists some $\underline{x} \in \underline{S}(A, b)$ such that $\underline{x} \leq x \leq \bar{x}$.

By Theorem 2.23, $S(A, b)$ is equal to the union of a finite number of multi-dimensional intervals $[\underline{x}(e), \bar{x}]$ in which $e \in E$ and $[\underline{x}(e), \bar{x}]$ means all points $x \in [0, 1]^n$ such that $\underline{x}(e) \leq x \leq \bar{x}$. Specially, $\bar{x}$ belongs to all of these intervals. Moreover, for a fixed $e \in E$, the interval $[\underline{x}(e), \bar{x}]$ is meaningful (or equivalently, $\underline{x}(e)$ is a feasible solution) if and only if $\underline{x}(e) \leq \bar{x}$. As it will be shown in Section 5, set S_E often contains many infeasible solutions $\underline{x}(e)$, i.e., solutions $\underline{x}(e)$ such that $\underline{x}(e) \not\leq \bar{x}$. Consequently, by removing infeasible intervals $[\underline{x}(e), \bar{x}]$, the feasible region of Problem (1) is attained as the set of all points between one maximum solution, $\bar{x}$, and a finite number of feasible solutions $\underline{x}(e)$. These considerations lead to the following corollaries:

Corollary 2.24. $S(A, b) \neq \emptyset \Leftrightarrow \bar{x} \in S(A, b)$.

As a special case, Sanchez [37] proved that $S(A, b) \neq \emptyset \Leftrightarrow \bar{x} \in S(A, b)$ where FRE was defined by max-min composition. In [20], the authors provided another way of proving this theorem (part (ii) of Theorem 2.17).

Corollary 2.25. *(a) Let $\underline{S}(A, b)$ denote the set of all the minimal solutions of $S(A, b)$. Then, $\underline{S}(A, b) \subseteq S_E$. (b) $S(A, b) = \bigcup_{\underline{x}\in \underline{S}(A,b)}[\underline{x}, \bar{x}]$.*

Proof. (a) let $\underline{x} \in \underline{S}(A, b)$. Hence, by Theorem 2.23, there exists some feasible interval $[\underline{x}(e), \bar{x}]$ (i.e., $e \in E$, $\underline{x}(e) \in S(A, b)$ and $\underline{x}(e) \leq \bar{x}$) such that $\underline{x} \in [\underline{x}(e), \bar{x}]$. So, we must have $\underline{x} = \underline{x}(e)$; because otherwise, from $\underline{x}(e) \leq \underline{x}$, $\underline{x}(e) \neq \underline{x}$ and the definition of the minimal solution, we will have $\underline{x}(e) \notin S(A, b)$ which is a contradiction. (b) Suppose that for a given $e' \in E$, $\underline{x}(e')$ is feasible and $\underline{x}(e') \notin \underline{S}(A, b)$. Since $\underline{x}(e')$ is not a minimal solution, there exists some feasible solution, say x', such that $x' \leq \underline{x}(e')$. From Theorem 2.23, we have $x' \in [\underline{x}(e''), \bar{x}]$ for some $e'' \in E$. Therefore, $[\underline{x}(e'), \bar{x}] \subseteq [\underline{x}(e''), \bar{x}]$, and hence $S(A, b) = \bigcup_{e\in E-\{e'\}}[\underline{x}(e), \bar{x}]$. By repeating this argument the result follows. □

Sanchez presented some theorems for existence and determination of solutions of certain basic FRE [37]. However, the solution obtained in that work is only the maximum solution $\bar{x}$ derived from the max-min composition. Czogala et al. [5] tried to prove the proposition (Theorem 8 in their paper) that $x \in S(A, b)$ if and only if there exists some

$\underline{x} \in \underline{S}(A,b)$ such that $\underline{x} \leq x \leq \bar{x}$. However, they did not succeed in proving it completely. In [20], the authors proved this proposition (for the minimum t-norm) by different approaches (Theorems 1 (i) and 4 in [20]).
According to Theorem 2.23, the problem of solving FREs is reduced to finding the maximum solution, $\bar{x}$, and the minimal solutions $\underline{x}(e)$ ($e \in E$). Subsequently, the following theorem shows the essential role of the partial solution sets Ω_{ij} in the resolution of FREs; indeed, vectors $\bar{x}$ and $\underline{x}(e)$ ($e \in E$) can be attained in terms of the maximum and minimum solutions of Ω_{ij}(i.e., $\bar{x}_{ij}$ and $\underline{x}_{ij}$ for $i \in I$ and $j \in J$).

Theorem 2.26. *For each $j \in J$, let I_j be as defined in Definition 2.6. Then,*

$$\bar{x}_j = \begin{cases} \min\limits_{i \in I_j} \{\bar{x}_{ij}\} & I_j \neq \emptyset \\ 1 & I_j = \emptyset \end{cases} , \forall j \in J \tag{6}$$

Also, for each $e \in E$, we have

$$\underline{x}(e)_j = \begin{cases} \max\limits_{i \in I_j(e)} \{\underline{x}_{ij}\} & I_j(e) \neq \emptyset \\ 0 & I_j(e) = \emptyset \end{cases} , \forall j \in J \tag{7}$$

Proof. Firstly, from Definition 2.6, we note that $j \in J_i$ iff $i \in I_j$. From Definition 2.21, $\bar{x}_j = \min\limits_{i \in I} \{\bar{x}(i)_j\}$, $\forall j \in J$. Also, according to relation (4), $\bar{x}(i)_j = \bar{x}_{ij}$ if $j \in J_i$ (i.e., $i \in I_j$), and $\bar{x}(i)_j = 1$ if $j \notin J_i$ (i.e., $i \notin I_j$). Consequently, we have

$$\bar{x}_j = \min_{i \in I} \{\bar{x}(i)_j\} = \min\{\min_{i \in I_j} \{\bar{x}(i)_j\}, \min_{i \notin I_j} \{\bar{x}(i)_j\}\} = \min\{\min_{i \in I_j} \{\bar{x}_{ij}\}, 1\}$$

which proves the relation (6). By applying similar argument, the relation (7) is resulted from Definition 2.21 and relation (5), and replacing I_j with $I_j(e)$. □

If ω is the minimum t-norm, The relation (6) is reduced to $\bar{x} = A@b = [\min_{i=1}^{m} \{(a_{ij}@b_i)\}]_{j \in J}$ (see relations (4) and (5) in [10], relation (4) in [42, 43] and relation (5) in [27]); if ω is the product t-norm, (6) is converted to $\bar{x} = A \otimes b = [\min_{i=1}^{m} \{(a_{ij} \otimes b_i)\}]_{j \in J}$ (see relations (4) and (5) in [1, 26], relation (4) in [18, 36]; and if ω is a strict t-norm, (6) gives the formulation $\bar{x} = A \diamondsuit b = [\min_{i=1}^{m} \{(a_{ij} \diamondsuit b_i)\}]_{j \in J}$ (see relation (3) in [44]) where

$$a_{ij}@b_i = \begin{cases} 1 & ,\text{if } a_{ij} \leq b_i \\ b_i & ,\text{if } a_{ij} > b_i \end{cases} , a_{ij} \otimes b_i = \begin{cases} 1 & ,\text{if } a_{ij} \leq b_i \\ b_i/a_{ij} & ,\text{if } a_{ij} > b_i \end{cases} , a_{ij} \diamondsuit b_i = \begin{cases} 1 & ,\text{if } a_{ij} \leq b_i \\ a_{ij} \varphi b_i & ,\text{if } a_{ij} > b_i \end{cases}$$

$a_{ij} \varphi b_i = \sup \{\alpha \in [0,1] \ : \ a_{ij} \circ \alpha \leq b_i\}$ and "$\circ$" denotes an arbitrary strict t-norm.

The preceding discussions can now be summarized as an algorithm shown below
In the above algorithm, computing the elements of set S_E, is a major disadvantage of the algorithm. As mentioned before, the set S_E often contains many solutions $\underline{x}(e)$ that are not minimal. Moreover, since $|S_E| = \prod_{i=1}^{m} |J_i|$ (see Remark 5), the cardinality of S_E grows exponentially with the size of the sets J_i, $\forall i \in I$. To accelerate the algorithm, we can initially remove some $e \in E$ that generate non-minimal solutions $\underline{x}(e)$. For this purpose, two basic simplification techniques will be described in Section 3 (Investigating a variety of simplification methods in the general case, where fuzzy relational equations are defined with arbitrary continuous t-norms, is the subject of our future work).

3 Two basic simplification procedures

In this section, two operations are generalized to simplify the matrix A, and a necessary and sufficient condition is derived to determine the feasibility of the main problem in the most general case, where ω is an arbitrary continuous t-norm. At first, it should be noted that if $b_i = 0$ for some $i \in I$, then the problem of solving equation $A \,\omega\, x = b$ can be always reduced to that of solving equation $A' \,\omega\, x' = b'$ in which b' has no zero component [20]. Therefore, in the rest of this paper, we assume without loss of generality that $b_i > 0$, $\forall i \in I$.

Remark 3.1. *Consider the i'th equation $\max_{j=1}^{n} \{\omega(a_{ij}, x_j)\} = b_i$ of Problem (1) and let $j_0 \in J$. Also, suppose that $\omega(a_{ij_0}, x_{j_0}) < b_i$ for each feasible solution $x \in S(A,b)$. In this case, $\max_{j=1}^{n} \{\omega(a_{ij}, x_j)\} = b_i$ can be obviously reduced to $\max_{j=1, j \neq j_0}^{n} \{\omega(a_{ij}, x_j)\} = b_i$. Hence, "resetting a_{ij_0} to zero " has no effect on the solutions of Problem (1) (since component a_{ij_0} only appears in the i'th constraint of Problem (1)). These considerations allow us to introduce the concept of an ineffective component, as stated in the following definition.*

Algorithm 1: Algorithm 1 (Resolution of the feasible region for Problem(1))

Input : Given Problem(1)
Output: The feasible region S(A,b)
Compute J_i ($\forall i \in I$) by Corollary 2.
if $J_i = \varnothing$ *for some* $i \in I$ **then**
  Stop; The problem is infeasible (Lemma 5).
end
else
  For each $j \in J_i$, Ω_{ij} is non-empty (Lemma 3 and Corollary 2).
  foreach *non-empty partial solution set* Ω_{ij} *(see Tables A2 and A3):* **do**
    Compute $\bar{x}_{ij}$ and $\underline{x}_{ij}$
  end
  Compute $\bar{x}$ from relation (6).
  if $\bar{x} \notin S(A,b)$ **then**
    Stop; The problem is infeasible (Corollary 5).
  end
  else
    Compute the elements of set $S_E = \{\underline{x}(e) \ : \ e \in E\}$ by Eq. (7).
    Remove all infeasible solutions $\underline{x}(e)$ (i.e., solutions $\underline{x}(e)$ such that $\underline{x}(e) \not\leq \bar{x}$).
    By pairwise comparison between the feasible elements of S_E, find set $\underline{S}(A,b)$ (i.e., the minimal solutions of $S(A,b)$).
    Find the feasible region, by equality $S(A,b) = \bigcup_{\underline{x} \in \underline{S}(A,b)} [\underline{x}, \bar{x}]$ (Corollary 6, part (b)).
  end
end

Definition 3.2. *Let $i_0 \in I$ and $j_0 \in J$. If $\max_{j\in J}\{\omega(a_{i_0 j}, x_j)\} = \max_{j\in J-\{j_0\}}\{\omega(a_{i_0 j}, x_j)\}$, $\forall x \in S(A,b)$, then component $a_{i_0 j_0}$ of the matrix A is referred to as an ineffective component. That is, $a_{i_0 j_0}$ can be dismissed from further consideration by resetting its value to zero.*

There are several conditions under which a component a_{ij} may be ineffective. For example, if $a_{i_0 j_0} < b_{i_0}$ (or equivalently, $j_0 \notin J_{i_0}$), then $\omega(a_{i_0 j_0}, x) < b_{i_0}$, $\forall x \in [0,1]$ (Corollary 2.9). Therefore, by Remark 3.1, the constraint $\max_{j=1}^{n}\{\omega(a_{i_0 j}, x_j)\} = b_{i_0}$ is equivalent to $\max_{j=1, j\neq j_0}^{n}\{\omega(a_{i_0 j}, x_j)\} = b_{i_0}$, that is, component $a_{i_0 j_0}$ is ineffective (Definition 3.2) and its value can be neglected in our calculations. Thus, the problem can be simplified by changing the values of these components to zero. So, we have the following lemma.

Lemma 3.3. *Suppose that matrix $A' = (a'_{ij})_{m\times n}$ is resulted from matrix A as follows:*

$$a'_{ij} = \begin{cases} a_{ij} & j \in J_i \\ 0 & j \notin J_i \end{cases} \tag{8}$$

Then, $S(A,b) = S(A',b)$.

Lemma 3.5 gives another simplification procedure which finds some ineffective components a_{ij} of A such that $a_{ij} \geq b_i$ (i.e., $j \in J_i$). Before formally presenting the lemma, some useful notations are initially introduced as follows.

Definition 3.4. *For each $i \in I$, we define $\widetilde{J}_i = \{\, j \in J_i \ : \ \underline{x}_{ij} \leq \bar{x}_j \,\}$, where $\underline{x}_{ij}$ and $\bar{x}_j$ are defined as in Corollary 2.5 and relation (6), respectively. Similarly, for each $j \in J$, define $\widetilde{I}_j = \{\, i \in I_j \ : \ \underline{x}_{ij} \leq \bar{x}_j \,\}$.*

According to Definition 3.4, we clearly have $j \in \widetilde{J}_i$ if and only if $i \in \widetilde{I}_j$.

Lemma 3.5. *Let $A' = (a'_{ij})_{m\times n}$ be obtained from A as follows:*

$$a'_{ij} = \begin{cases} a_{ij} & j \in \widetilde{J}_i \\ 0 & j \notin \widetilde{J}_i \end{cases} \tag{9}$$

Then, $S(A,b) = S(A',b)$.

Proof. Since the matrices A and A' differ only in their corresponding components (i,j) such that $j \notin \widetilde{J}_i$, in order to prove the lemma, we show that for each $j \notin \widetilde{J}_i$, a_{ij} is an ineffective component of A(see Definition 3.2). According to Remark 3.1, it is sufficient to show that if $j \notin \widetilde{J}_i$, then $\omega(a_{ij}, x_j) < b_i$, $\forall x \in S(A,b)$. But, since from Remark 2.14 $\omega(a_{ij}, x_j) \leq b_i$, $\forall x \in S(A,b)$, then the proof is reduced to show that the equality $\omega(a_{ij}, x_j) = b_i$ leads to a contradiction. Let $x \in S(A,b)$ and $j \notin \widetilde{J}_i$. If $j \notin J_i$ (i.e., $a_{ij} < b_i$), then a_{ij} is ineffective by Lemma 3.3. So, suppose that $j \in J_i - \widetilde{J}_i$(i.e., $a_{ij} \geq b_i$ and $\underline{x}_{ij} > \bar{x}_j$). Let $\omega(a_{ij}, x_j) = b_i$. Hence, $x_j \in \Omega_{ij} = [\, \underline{x}_{ij}, \bar{x}_{ij}]$ (Corollary 2.5). Therefore, $\bar{x}_j < \underline{x}_{ij} \leq x_j$ which implies $x \not\leq \bar{x}$. So, $x \notin S(A,b)$ (Theorem 2.23) that is a contradiction. □

Lemma 3.5 gives a condition to reduce matrix A. In this lemma, A' denotes the simplified matrix obtained from A after applying the above simplification process. Moreover, from the definition of $\widetilde{J}_i$, it is obvious that $\widetilde{J}_i \subseteq J_i$. Hence, if $j \notin J_i$, then $j \notin \widetilde{J}_i$. Therefore, any ineffective component a_{ij} that is reset to zero by Lemma 3.3 also vanishes by Lemma 3.5. In other words, by applying Lemma 3.5, we no longer need to use Lemma 3.3.

Many simplification processes used in the literature can be considered as special cases of Lemma 3.5 (i.g, Lemma 2 in [27]). Also, if ω is a continuous Archimedean t-norm, Theorem 14 in [38] gives another special case of Lemma 3.5. In addition, the interested reader is referred to [14] where a brief review of such processes is given.

Similar to Definition 2.1, we can define $\Omega'_{ij} = \{\, x \in [0,1] \;:\; \omega\,(a'_{ij}\,, x) = b_i \,\}$ where a'_{ij} denotes (i,j)'th components of matrix A' defined in Lemma 3.5. Based on this definition, we have $J'_i = \{\, j \in J \;:\; \Omega'_{ij} \neq \emptyset \,\}$ and $I'_j = \{\, i \in I \;:\; \Omega'_{ij} \neq \emptyset \,\}$ (see Definition 2.6), or equivalently $J'_i = \{\, j \in J \;:\; a'_{ij} \geq b_i \,\}$ and $I'_j = \{\, i \in I \;:\; a'_{ij} \geq b_i \,\}$ by Corollary 2.8. Also, by the similar way, E' is defined as the set of all functions $e : I \to \bigcup_{i\in I} J'_i$, and $S_{E'} = \{\underline{x}(e) \;:\; e \in E'\}$ (see Definitions 2.20 and 2.21). According to relation (9), $a'_{ij} > 0$ if and only if $j \in \widetilde{J}_i$. So, using our assumption that $b > 0$, for each $j \in J'_i$ (i.e., $a'_{ij} \geq b_i$) we have $a'_{ij} > 0$ (or equivalently, $j \in \widetilde{J}_i$) which implies that $J'_i \subseteq \widetilde{J}_i$. Conversely, if $j \in \widetilde{J}_i$, then $a_{ij} \geq b_i$, and also by (9), we get $a'_{ij} = a_{ij}$. Hence, $a'_{ij} = a_{ij} \geq b_i$ which implies $j \in J'_i$. So, $\widetilde{J}_i \subseteq J'_i$. Consequently, we have $J'_i = \widetilde{J}_i$, $\forall i \in I$; that is, J'_i contains only those elements $j \in J$ which belong to $\widetilde{J}_i$. These considerations lead us to introduce a new set of functions e, that helps to strengthen Theorem 2.23, as it will be stated in Corollary 3.9.

Definition 3.6. *Denote the set of all functions $e : I \to \bigcup_{i\in I} \widetilde{J}_i$ so that $e(i) \in \widetilde{J}_i$, $\forall i \in I$, by $\widetilde{E}$. Moreover, we define $S_{\widetilde{E}} = \{\, \underline{x}(e) \;:\; e \in \widetilde{E} \,\}$.*

Remark 3.7. *Based on Definitions 2.20 and 3.4 and the descriptions presented before Definition 3.6, $e \in E'$ iff $e \in \widetilde{E}$. Consequently, $S_{E'} = S_{\widetilde{E}}$. Moreover, similar to Remark 2.22, we have $\left|\, S_{\widetilde{E}} \,\right| = \left| \widetilde{E} \right| = \prod_{i=1}^{m} \left| \widetilde{J}_i \right|$. Also, since $\widetilde{J}_i \subseteq J_i$, $\forall i \in I$, we have $\widetilde{E} \subseteq E$ and $S_{\widetilde{E}} \subseteq S_E$. Therefore, $\left|\, S_{\widetilde{E}} \,\right| \leq |\, S_E \,|$ and $\left| \widetilde{E} \right| \leq |\, E \,|$.*

As previously mentioned, by deleting the infeasible intervals $[\underline{x}(e), \bar{x}]$ where $\underline{x}(e) \not\leq \bar{x}$, the feasible solution set of Problem (1) remains the same. The following theorem indicates that Lemma 3.5 can be used to remove such infeasible intervals by neglecting those vectors e which generate infeasible solutions $\underline{x}(e)$ (i.e., solutions $\underline{x}(e)$ such that $\underline{x}(e) \not\leq \bar{x}$). In other words, the infeasible solutions $\underline{x}(e)$ are not generated, if we only consider vectors $e \in E'$ generated by the components of the simplified matrix A', or equivalently vectors $e \in \widetilde{E}$ that are generated by the components of the main matrix A (see Remark 3.7).

Theorem 3.8. *Consider the main matrix A and let $e \in E$. Then, $\underline{x}(e) \in S(A,b)$ if and only if $e \in \widetilde{E}$. In other words, $S_{\widetilde{E}} \subseteq S(A,b)$.*

Proof. Let $e \in \widetilde{E}$. Based on relation (7), $\underline{x}(e)_j = 0$ if $I_j(e) = \emptyset$, and $\underline{x}(e)_j = \max_{i\in I_j(e)} \{\, \underline{x}_{ij} \,\}$ if $I_j(e) \neq \emptyset$. In the former case, clearly $\underline{x}(e)_j = 0 \leq \bar{x}_j$. In the latter case, for each $i \in I_j(e)$ we have $j = e(i) \in \widetilde{J}_i$ that implies $\underline{x}_{ij} = \underline{x}_{i\,e(i)} \leq \bar{x}_j$, $\forall i \in I_j(e)$. Therefore, $\underline{x}(e)_j = \max_{i\in I_j(e)} \{\, \underline{x}_{ij} \,\} \leq \bar{x}_j$. So, in any case, we have $\underline{x}(e)_j \leq \bar{x}_j$, $\forall j \in J$, that means $\underline{x}(e) \leq \bar{x}$. Now, by Theorem 2.23 we have $\underline{x}(e) \in S(A,b)$. The converse statement is attained by reversing the argument. □

Based on the following corollary, we can accelerate identification of the set $S(A,b)$ by reducing our search to the set $\widetilde{E}$ instead of E.

Corollary 3.9. $S(A,b) = \bigcup_{e\in\widetilde{E}} [\underline{x}(e), \bar{x}]$.

Proof. The result is obtained by Theorems 2.23 and 3.8. □

Corollary 3.10. $\underline{S}(A,b) \subseteq S_{\widetilde{E}} \subseteq S_E$.

Proof. The result is obtained by Corollary 2.25, Remark 3.7 and Theorem 3.8. □

For a special case of Corollary 3.10, if ω is the minimum t-norm, Higashi and Klir proved that $\underline{S}(A,b) \subseteq S_{\widetilde{E}}$ (Theorem 1 (iii) in [20]).
The following theorem shows that the simplified matrix A' (defined in Lemma 3.5) can be also used for deriving a necessary and sufficient condition for the feasibility of Problem (1).

Theorem 3.11. *$S(A,b) \neq \emptyset$ if and only if $\widetilde{J}_i \neq \emptyset$, $\forall i \in I$. In other words, $S(A,b) \neq \emptyset$ iff A' has no zero row.*

Proof. Let $x \in S(A,b)$. Then, from Corollary 3.9, there exists some $e' \in \widetilde{E}$ such that $x \in [\underline{x}(e'), \bar{x}]$. Since $e' \in \widetilde{E}$, then $e'(i) \in \widetilde{J}_i$, $\forall i \in I$. Hence, $\widetilde{J}_i \neq \emptyset$, $\forall i \in I$. Conversely, suppose that for each $i \in I$ there exists some $j_i \in J$ such that $j_i \in \widetilde{J}_i$. Hence, we can define the function $e' : I \to \bigcup_{i \in I} \widetilde{J}_i$ so that $e'(i) = j_i$, $\forall i \in I$. So, $e' \in \widetilde{E}$ and therefore $\underline{x}(e')$ is feasible from Theorem 3.11. Hence, $[\underline{x}(e'), \bar{x}] \neq \emptyset$ which implies that $S(A,b) \neq \emptyset$ by Corollary 3.9. □

4 Optimization of the problem

According to well-known schemes used for optimization problems such as Problem (1) [10, 12, 14, 19, 26], this optimization model can be decomposed into the following two sub-problems, one with non-positive cost coefficients and the other with positive cost coefficients, namely:

$$\begin{array}{ll} \min & z_1 = \sum_{j \in J^-} c_j\, x_j \\ & A \,\omega\, x = b \\ & x \in [0,1]^n \end{array} \tag{10}$$

and

$$\begin{array}{ll} \min & z_2 = \sum_{j \in J^+} c_j\, x_j \\ & A \,\omega\, x = b \\ & x \in [0,1]^n \end{array} \tag{11}$$

where $J^- = \{j \in J \ : \ c_j \leq 0\}$ and $J^+ = \{j \in J \ : \ c_j > 0\}$ for $j = 1, 2, ..., n$. As it will be shown in Lemma 4.1, it is easy to prove that $\bar{x}$ is an optimal solution of (10). Intuitively, when all the costs are non-positive, since x_j's are non-negative and the problem is to minimize the objective value, we should make x_j as large as possible. A similar argument will be made in Lemma 4.2 to indicate that for Sub-problem (11) one of the minimal solutions, say $\underline{x}^* \in \underline{S}(A,b)$, is an optimal solution. With the knowledge of these two special cases, the general situation can be treated by combining $\bar{x}$ and $\underline{x}^*$(see Theorem 4.3).

Lemma 4.1. *$\bar{x}$ is an optimal solution of Sub-problem (10).*

Proof. For any $x \in S(A,b)$, we know that $0 \leq x \leq \bar{x}$. Therefore, since $c_j \leq 0$, $\forall j \in J^-$, we have $\sum_{j \in J^-} c_j\, \bar{x}_j \leq \sum_{j \in J^-} c_j\, x_j$. □

Special cases of Lemma 4.1 can be found in the literature when ω is the minimum t-norm (see Lemma 4 in [10]), ω is the product t-norm (see Lemma 5 in [26]) or ω is a continuous Archimedean t-norm (see Lemma 5 in [19]).

Lemma 4.2. *Let x^* denote the optimal solution set of Sub-problem (11). Then, $x^* \bigcap \underline{S}(A,b) \neq \emptyset$. Moreover, if $\underline{x}^* \in \underline{S}(A,b)$ is an optimal solution of (11), then $\underline{x}^* = \underline{x}(e^*)$ for some $e^* \in \widetilde{E}$.*

Proof. For any $x \in S(A,b)$, from Corollary 2.25, we know that there exists a minimal solution $\underline{x} \in \underline{S}(A,b)$ such that $\underline{x} \leq x \leq \bar{x}$. Since c_j's are positive for each $j \in J^+$, we have $\sum_{j \in J^+} c_j\, \underline{x}_j \leq \sum_{j \in J^+} c_j\, x_j \leq \sum_{j \in J^+} c_j\, \bar{x}_j$. We then choose $\underline{x}^* \in \underline{S}(A,b)$ such that $\sum_{j \in J^+} c_j\, \underline{x}^*_j = \min_{\underline{x} \in \underline{S}(A,b)} \left\{ \sum_{j \in J^+} c_j\, \underline{x}_j \right\}$. Therefore, $\sum_{j \in J^+} c_j\, \underline{x}^*_j \leq \sum_{j \in J^+} c_j\, \underline{x}_j \leq \sum_{j \in J^+} c_j\, x_j$. Now, Corollary 3.10 indicates that $\underline{x}^* = \underline{x}(e^*)$ for some $e^* \in \widetilde{E}$. □

For some special cases of Lemma 4.2, we can refer to Lemma 5 in [10] (if ω is the minimum t-norm), Lemma 6 in [26] and Theorem 3.1 in [36] (if ω is the product t-norm) and Lemma 6 in [19] (if ω is a continuous Archimedean t-norm).

Theorem 4.3. *Suppose that $S(A,b) \neq \emptyset$, and $\bar{x}$ and $\underline{x}^*$ are the optimal solutions of Sub-problems (10) and (11), respectively. Define $x^* = (x^*_1, x^*_2, ..., x^*_n)$ as follows:*

$$x^*_j = \begin{cases} \bar{x}_j & c_j \leq 0 \\ \underline{x}^*_j & c_j > 0 \end{cases}, \ \forall j \in J \tag{12}$$

Then, x^* is the optimal solution of Problem (1).

Proof. Since $\underline{x}^* \in \underline{S}(A,b)$, then $\underline{x}_j^* \leq \bar{x}_j$, $\forall j \in J$. Now, according to the definition of vector x^*, we have $\underline{x}_j^* \leq x_j^* \leq \bar{x}_j$, $\forall j \in J$, which implies $x^* \in [\underline{x}^*, \bar{x}]$. Therefore, $x^* \in S(A,b)$ from Corollary 2.25, i.e., x^* is a feasible solution. Now, let $x \in S(A,b)$. From Corollary 2.25, we have $x \in [\underline{x}, \bar{x}]$ for some $\underline{x} \in \underline{S}(A,b)$. Therefore,

$$\sum_{j \in J^-} c_j \, \bar{x}_j \leq \sum_{j \in J^-} c_j \, x_j, \tag{13}$$

$$\sum_{j \in J^+} c_j \, \underline{x}_j^* \leq \sum_{j \in J^+} c_j \, \underline{x}_j \leq \sum_{j \in J^+} c_j \, x_j. \tag{14}$$

Combination of (13) and (14) results in the following:

$$\begin{array}{c} \sum_{j=1}^{n} c_j x_j^* = \sum_{j \in J^-} c_j \, x_j^* + \sum_{j \in J^+} c_j \, x_j^* = \sum_{j \in J^-} c_j \, \bar{x}_j + \sum_{j \in J^+} c_j \, \underline{x}_j^* \\ \leq \sum_{j \in J^-} c_j \, x_j + \sum_{j \in J^+} c_j \, x_j = \sum_{j=1}^{n} c_j \, x_j \end{array},$$

Hence, $\sum_{j=1}^{n} c_j x_j^* \leq \sum_{j=1}^{n} c_j x_j$, $\forall x \in S(A,b)$, which completes the proof. □

As some special cases of Theorem 4.3, see Theorem 2 in [10] (if ω is the minimum t-norm), Theorem 2 in [26] (if ω is the product t-norm) and page 290 in [19] (if ω is a continuous Archimedean t-norm).
According to relation (6), generation of the maximum solution $\bar{x}$ is not a problem. If we know how to find a minimal solution of $S(A,b)$ which solves Problem (11), then Problem (1) can be solved via Theorem 4.3 and relation (12). The following algorithm summarizes the preceding discussion.

Algorithm 2: (Solution of Problem (1))

Input : Given Problem (1)
Output: x^* (the optimal solution for the Problem (1))

Compute $\widetilde{J}_i$,$\forall i \in I$.
if $\widetilde{J}_i \neq \emptyset$, $\forall i \in I$, **then**
  continue;
end
else
  stop, the problem is infeasible (Theorem 3.11).
end
foreach $i \in I$ *and* $j \in \widetilde{J}_i$, **do**
  Compute $\bar{x}_{ij}$ and $\underline{x}_{ij}$ for partial solution set Ω_{ij}.
  Generate $\bar{x}$ from relation (6).
  Compute the elements of the set $S_{\widetilde{E}}$ from Definition 3.6 and relation (7).
  Find $\underline{S}(A,b)$, by pairwise comparison between the elements of $S_{\widetilde{E}}$.
  Find the optimal solution $\underline{x}(e^*)$ for Sub-problem (11) as the solution $\underline{x}(e) \in \underline{S}(A,b) \bigcap S_{\widetilde{E}}$ with the smallest objective value $z_2 = \sum_{j \in J^+} c_j \, \underline{x}(e)_j$.
  Find the optimal solution x^* for the Problem (1) by relation (12) (Theorem 4.3).
end

Most of the previous works in the field of fuzzy relational equations are related to FREs defined by minimum, product and Lukasiewicz t-norms. Furthermore, a few articles have dealt with wider groups of t-norms such as strict, nilpotent and Archimedean t-norms. In this article, an algorithm was presented for solving a linear optimization model constrained with the most general FREs defined by arbitrary continuous t-norms. In the next section, the algorithm was used to solve the problem with FREs defined by Mayor-Torrens t-norm, which is not strict, nilpotent or Archimedean. As it is clear from the discussion under Remark 2.14; Theorems 2.17, 2.23, 2.26 and 4.3; Corollaries 2.24, 2.25 and 3.10; and Lemma 3.5, the concept of PSS enables us to unify many propositions that have been proved separately and in different ways for particular t-norms and to generalize these propositions to all continuous t-norms.

5 Numerical example

Consider the following optimization Problem (1) defined by Mayor-Torrens t-norm:

$$\begin{aligned}\min\ z = &-7.8669\,x_1 + 9.2380\,x_2 - 9.9073\,x_3 + 5.4982\,x_4 + 6.3461\,x_5 + 7.3739\,x_6 - 8.3113\,x_7\\ &-2.0043\,x_8 - 4.8026\,x_9 + 6.0014\,x_{10}\end{aligned}$$

$$\begin{bmatrix} 0.4314 & 0.8530 & 0.4173 & 0.7803 & 0.2348 & 0.5470 & 0.9516 & 0.6443 & 0.2077 & 0.3111 \\ 0.9203 & 0.9203 & 0.9197 & 0.3897 & 0.3532 & 0.2963 & 0.7757 & 0.3786 & 0.3012 & 0.9134 \\ 0.1818 & 0.3510 & 0.0527 & 0.2417 & 0.8212 & 0.3447 & 0.4868 & 0.4116 & 0.4709 & 0.4302 \\ 0.2638 & 0.5132 & 0.4448 & 0.7379 & 0.0154 & 0.1890 & 0.7379 & 0.5328 & 0.2305 & 0.1848 \\ 0.2691 & 0.4018 & 0.0909 & 0.0965 & 0.0430 & 0.2691 & 0.1468 & 0.0507 & 0.1443 & 0.0049 \\ 0.1361 & 0.0760 & 0.3893 & 0.1320 & 0.1690 & 0.1835 & 0.4228 & 0.1390 & 0.4228 & 0.2797 \\ 0.4693 & 0.2399 & 0.3377 & 0.5479 & 0.2491 & 0.3685 & 0.5085 & 0.2759 & 0.2259 & 0.4389 \\ 0.5797 & 0.1233 & 0.9001 & 0.9427 & 0.7317 & 0.6256 & 0.5108 & 0.4502 & 0.9427 & 0.1111 \\ 0.4177 & 0.1839 & 0.3692 & 0.1752 & 0.4177 & 0.2802 & 0.2176 & 0.3225 & 0.2277 & 0.2581 \\ 0.1450 & 0.2400 & 0.1112 & 0.5000 & 0.4509 & 0.5000 & 0.3948 & 0.5070 & 0.4357 & 0.5000 \end{bmatrix} \varphi x = \begin{bmatrix} 0.9516 \\ 0.9203 \\ 0.5527 \\ 0.7379 \\ 0.2691 \\ 0.4228 \\ 0.5479 \\ 0.9427 \\ 0.4177 \\ 0.5000 \end{bmatrix}$$

where

$$\varphi^{\lambda}_{MT}(x,y) = \begin{cases} \max\{x+y-\lambda, 0\} & (x,y) \in [0,\lambda]^2 \\ \min\{x,y\} & otherwise \end{cases}$$

and $\lambda \in (0,1]$. In this example, parameter λ is set to 0.5. Also, $z_1 = -7.8669\,x_1 - 9.9073\,x_3 - 8.3113\,x_7 - 2.0043\,x_8 - 4.8026\,x_9$ is the objective function of Sub-problem (10) and $z_2 = 9.2380\,x_2 + 5.4982\,x_4 + 6.3461\,x_5 + 7.3739\,x_6 + 6.0014\,x_{10}$ is that of Sub-problem (11).

By Corollary 2.8, $J_1 = \{7\}$,$J_2 = \{1\,,\,2\}$, $J_3 = \{5\}$, $J_4 = \{4\,,\,7\}$, $J_5 = \{1\,,\,2\,,6\}$, $J_6 = \{7\,,\,9\}$, $J_7 = \{4\}$, $J_8 = \{4\,,\,9\}$, $J_9 = \{1\,,\,5\}$ and $J_{10} = \{4\,,\,6\,,\,8\,,10\}$. Hence, the only non-empty Ω_{ij} are as follows: Ω_{17}; Ω_{21}, Ω_{22}; Ω_{35}; Ω_{44}, Ω_{47}; Ω_{51}, Ω_{52}, Ω_{56}; Ω_{67}, Ω_{69}; Ω_{74}; Ω_{84}, Ω_{89}; Ω_{91}, Ω_{95}; $\Omega_{10,4}$, $\Omega_{10,6}$, $\Omega_{10,8}$, $\Omega_{10,10}$. For each equation $\omega^{0.5}_{MT}(a_{ij}, x_j) = b_i$ $(i \in I, j \in J)$, the partial solution set is given as $\Omega_{ij} = \left\{x \in [0,1] \ : \ \varphi^{\lambda}_{MT}(a_{ij}, x) = b_i\right\}$(Definition 2.1). In this case, for each Ω_{ij}, the maximum solution $\bar{x}_{ij}$ and the minimum solution $\underline{x}_{ij}$ are attained as follows:

$$\bar{x}_{ij} = \begin{cases} 1 & a_{ij} = b_i \\ b_i + \lambda - a_{ij} & \lambda \geq a_{ij} > b_i \\ b_i & a_{ij} > \lambda\,,\ a_{ij} > b_i \end{cases} \quad , \quad \underline{x}_{ij} = \begin{cases} 0 & a_{ij} \geq b_i = 0 \\ \max\{\lambda, b_i\} & a_{ij} = b_i \neq 0 \\ b_i + \lambda - a_{ij} & \lambda \geq a_{ij} > b_i \neq 0 \\ b_i & a_{ij} > \lambda\,,\ a_{ij} > b_i \neq 0 \end{cases}$$

In the following, matrices $\bar{x} = (\bar{x}_{ij})_{m\times n}$ and $\underline{x} = (\underline{x}_{ij})_{m\times n}$ present solutions $\bar{x}_{ij}$ and $\underline{x}_{ij}$ for each partial solution set Ω_{ij} $(i \in I, j \in J)$. In both matrices, the notation "*" indicates that $\Omega_{ij} = \emptyset$.

$$\bar{x} = \begin{bmatrix} * & * & * & * & * & * & 1 & * & * & * \\ 1 & 1 & * & * & * & * & * & * & * & * \\ * & * & * & * & 0.5527 & * & * & * & * & * \\ * & * & * & 1 & * & * & 1 & * & * & * \\ 1 & 0.3673 & * & * & * & 1 & * & * & * & * \\ * & * & * & * & * & * & 1 & * & 1 & * \\ * & * & * & 1 & * & * & * & * & * & * \\ * & * & * & 1 & * & * & * & * & 1 & * \\ 1 & * & * & * & 1 & * & * & * & * & * \\ * & * & * & 1 & * & 1 & * & 0.5 & * & 1 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} * & * & * & * & * & * & 0.9516 & * & * & * \\ 0.9203 & 0.9203 & * & * & * & * & * & * & * & * \\ * & * & * & * & 0.5527 & * & * & * & * & * \\ * & * & * & 0.7379 & * & * & 0.7379 & * & * & * \\ 0.5 & 0.3673 & * & * & * & 0.5 & * & * & * & * \\ * & * & * & * & * & * & 0.5 & * & 0.5 & * \\ * & * & * & 0.5479 & * & * & * & * & * & * \\ * & * & * & 0.9427 & * & * & * & * & 0.9427 & * \\ 0.5 & * & * & * & 0.5 & * & * & * & * & * \\ * & * & * & 0.5 & * & 0.5 & * & 0.5 & * & 0.5 \end{bmatrix}$$

Therefore, by Corollary 2.5 we have for example:

$\Omega_{22} = [0.9203, 1]$, $\Omega_{35} = \{0.5527\}$, $\Omega_{74} = [0.5479, 1]$, $\Omega_{10,8} = \{0.5\}$ and $\Omega_{10,10} = [0.5, 1]$.
Since $J_i \neq \emptyset$, $\forall i \in I$, then $S(a_i, b_i) \neq \emptyset$ (Lemma 5). By Definition 2.16, the unique maximum solution for each $S(a_i, b_i)$ $(i \in I)$ is obtained as follows:

$$\bar{x}(1) = \bar{x}(2) = \bar{x}(4) = \bar{x}(6) = \bar{x}(7) = \bar{x}(8) = \bar{x}(9) = (1,1,1,1,1,1,1,1,1,1), \bar{x}(3) = (1,1,1,1,0.5527,1,1,1,1,1),$$

$$\bar{x}(5) = (1,0.3673,1,1,1,1,1,1,1,1), \bar{x}(10) = (1,1,1,1,1,1,1,0.5,1,1).$$

Also, the minimal solutions of $S(a_i, b_i)$ $(i \in I)$ are:

$$\underline{x}(1,7) = (0,0,0,0,0,0,0.9516,0,0,0).$$

$$\underline{x}(2,1) = (0.9203,0,0,0,0,0,0,0,0,0), \underline{x}(2,2) = (0,0.9203,0,0,0,0,0,0,0,0).$$

$$\underline{x}(3,5) = (0,0,0,0,0.5527,0,0,0,0,0).$$

$$\underline{x}(4,4) = (0,0,0,0.7379,0,0,0,0,0,0), \underline{x}(4,7) = (0,0,0,0,0,0,0.7379,0,0,0).$$

$$\underline{x}(5,1) = (0.5,0,0,0,0,0,0,0,0,0), \underline{x}(5,2) = (0,0.3673,0,0,0,0,0,0,0,0), \underline{x}(5,6) = (0,0,0,0,0,0.5,0,0,0,0).$$

$$\underline{x}(6,7) = (0,0,0,0,0,0,0.5,0,0,0), \underline{x}(6,9) = (0,0,0,0,0,0,0,0,0.5,0).$$

$$\underline{x}(7,4) = (0,0,0,0.5479,0,0,0,0,0,0).$$

$$\underline{x}(8,4) = (0,0,0,0.9427,0,0,0,0,0,0), \underline{x}(8,9) = (0,0,0,0,0,0,0,0,0.9427,0).$$

$$\underline{x}(9,1) = (0.5,0,0,0,0,0,0,0,0,0), \underline{x}(9,5) = (0,0,0,0,0.5,0,0,0,0,0).$$

$$\underline{x}(10,4) = (0,0,0,0.5,0,0,0,0,0,0), \underline{x}(10,6) = (0,0,0,0,0,0.5,0,0,0,0), \underline{x}(10,8) = (0,0,0,0,0,0,0,0.5,0,0),$$

$$\underline{x}(10,10) = (0,0,0,0,0,0,0,0,0,0.5).$$

Therefore, by using Theorem 2.17, we have for example $S(a_1, b_1) = [\underline{x}(1,7), \bar{x}(1)]$, $S(a_4, b_4) = [\underline{x}(4,4), \bar{x}(4)] \bigcup [\underline{x}(4,7), \bar{x}(4)]$ and $S(a_5, b_5) = [\underline{x}(5,1), \bar{x}(5)] \bigcup [\underline{x}(5,2), \bar{x}(5)] \bigcup [\underline{x}(5,6), \bar{x}(5)]$. By Definition 2.21 or relation (6), $\bar{x} = (1, 0.3673, 1, 1, 0.5527, 1, 1, 0.5, 1, 1)$. So, it can be seen that the necessary and sufficient condition holds for the feasibility of the problem (Corollary 2.24). More precisely, it is easy to verify that $\bar{x} \in S(A, b)$. On the other hand, from Remark 2.22, we have $|E| = \prod_{i=1}^{10} |J_i| = 1 \times 2 \times 1 \times 2 \times 3 \times 2 \times 1 \times 2 \times 2 \times 4 = 384$, therefore the number for all vectors $e \in E$ is equal to 384. So, we have 384 solutions $\underline{x}(e)$ associated to 384 vectors e, i.e., $|S_E| = |E| = 384$ (Remark 2.22). However, each of the solutions $\underline{x}(e)$ generated by vectors $e \in E$ is not necessarily a feasible solution. As an example, for $e' = [7, 2, 5, 7, 6, 9, 4, 9, 1, 6]$, from Definition 2.21 (or relation (7)) we attain

$$\begin{aligned} \underline{x}(e') &= \max_{i \in I} \{\underline{x}(i, e'(i))\} \\ &= \max \{\underline{x}(1,7), \underline{x}(2,2), \underline{x}(3,5), \underline{x}(4,7), \underline{x}(5,6), \underline{x}(6,9), \underline{x}(7,4), \underline{x}(8,9), \underline{x}(9,1), \underline{x}(10,6)\} \ , \\ &= (0.5, 0.9203, 0, 0.5479, 0.5527, 0.5, 0.9516, 0, 0.9427, 0) \end{aligned}$$

It is obvious that $\underline{x}(e') \not\leq \bar{x}$ (actually, $\underline{x}(e')_2 > \bar{x}_2$) which means $\underline{x}(e') \notin S(A, b)$ by Theorem 2.23.

From the first simplification process (Lemma 3.3), the components a_{ij} of matrix A which meet the condition of $j \in J_i$ are as follows: a_{17}, a_{21}, a_{22}, a_{35}, a_{44}, a_{47}, a_{51}, a_{52}, a_{56}, a_{67}, a_{69}, a_{74}, a_{84}, a_{89}, a_{91}, a_{95}, $a_{10,4}$, $a_{10,6}$, $a_{10,8}$, $a_{10.10}$. So, matrix A' is obtained as follows:

$$A' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.9516 & 0 & 0 & 0 \\ 0.9203 & 0.9203 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8212 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7379 & 0 & 0 & 0.7379 & 0 & 0 & 0 \\ 0.2691 & 0.4018 & 0 & 0 & 0 & 0.2691 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4228 & 0 & 0.4228 & 0 \\ 0 & 0 & 0 & 0.5479 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9427 & 0 & 0 & 0 & 0 & 0.9427 & 0 \\ 0.4177 & 0 & 0 & 0 & 0.4177 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0.5 & 0 & 0.5 \end{bmatrix}$$

Also, by Lemma 3.5, we have $\widetilde{J_2} = \{1\} = J_2 - \{2\}$ and $\widetilde{J}_i = J_i$, $\forall i \in I - \{2\}$. For example, since $2 \in J_2$ and $\underline{x}_{22} = 0.9203 > 0.3673 = \bar{x}_2$, then $a''_{22} = 0$. Therefore, by the fact that $\widetilde{J}_i \neq \emptyset$, $\forall i \in I$, the necessary and sufficient feasibility condition stated in Theorem 3.11 will be satisfied. Also, the simplified matrix A'' is obtained as follows:

$$A'' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.9516 & 0 & 0 & 0 \\ 0.9203 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8212 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7379 & 0 & 0 & 0.7379 & 0 & 0 & 0 \\ 0.2691 & 0.4018 & 0 & 0 & 0 & 0.2691 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4228 & 0 & 0.4228 & 0 \\ 0 & 0 & 0 & 0.5479 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9427 & 0 & 0 & 0 & 0 & 0.9427 & 0 \\ 0.4177 & 0 & 0 & 0 & 0.4177 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0.5 & 0 & 0.5 \end{bmatrix}$$

Based on Remark 3.7, we have $\left| S_{\widetilde{E}} \right| = \left| \widetilde{E} \right| = \prod_{i=1}^{10} \left| \widetilde{J}_i \right| = 1\times1\times1\times2\times3\times2\times1\times2\times2\times4 = 192$. Therefore, it can be said that the above simplification process has reduced the number of minimal candidate solutions from 384 to 192, by removing 192 infeasible points $\underline{x}(e)$(Theorem 3.8). Consequently, the feasible region now has 192 minimal candidate solutions, which are the feasible points. In other words, for each $e \in \widetilde{E}$, we have $\underline{x}(e) \in S(A, b)$. However, each feasible solution $\underline{x}(e)$ ($e \in \widetilde{E}$) may not necessarily be a minimal solution for the problem. For example, by selecting $e' = [7, 1, 5, 4, 6, 9, 4, 9, 5, 8]$, the corresponding solution is obtained as $\underline{x}(e') = (0.9203, 0, 0, 0.7379, 0.5527, 0.5, 0.9516, 0.5, 0.9427, 0)$. Although $\underline{x}(e')$ is feasible (because of the inequality $\underline{x}(e') \leq \bar{x}$) but in fact it is not a minimal solution. To expand this further, we can let $e'' = [7, 1, 5, 4, 1, 9, 4, 9, 5, 8]$, then $\underline{x}(e'') = (0.9203, 0, 0, 0.7379, 0.5527, 0, 0.9516, 0.5, 0.9427, 0)$, and so obviously $\underline{x}(e'') \leq \underline{x}(e')$ which shows that $\underline{x}(e')$ is not a minimal solution.

Finally, from Lemma 4.1, the vector $\bar{x}$ is an optimal solution of Sub-problem (10). For this solution, $z_1 = -7.8669\,\bar{x}_1 - 9.9073\,\bar{x}_3 - 8.3113\,\bar{x}_7 - 2.0043\,\bar{x}_8 - 4.8026\,\bar{x}_9 = -31.89025$ and $z = c^T\bar{x} = -6.1161$. In order to find the optimal solution $\underline{x}(e^*)$ ($e^* \in \widetilde{E}$) of Sub-problem (11), initially all the minimal solutions are computed by making pairwise comparisons between all the elements of $S_{\widetilde{E}}$ (i.e., solutions $\underline{x}(e)$, $\forall e \in \widetilde{E}$), and then $\underline{x}(e^*)$ can be found amongst the obtained minimal solutions. It is worth noting that 190 feasible candidate solutions $\underline{x}(e)$ ($e \in \widetilde{E}$) are deleted by pairwise comparison. In fact, the feasible region has only two minimal solutions as follows:

$$\begin{array}{l} \underline{x}(e_1) = (0.9203, 0, 0, 0.5479, 0.5527, 0, 0.9516, 0, 0.9427, 0), \\ \underline{x}(e_2) = (0.9203, 0, 0, 0.9427, 0.5527, 0, 0.9516, 0, 0, 0) \end{array} .$$

that are generated by $e_1 = [7, 1, 5, 7, 1, 7, 4, 9, 1, 4]$ and $e_2 = [7, 1, 5, 4, 1, 7, 4, 4, 1, 4]$, respectively. So, $\underline{S}(A, b) = \{\underline{x}(e_1), \underline{x}(e_2)\}$ and $\underline{S}(A, b) \subseteq S_{\widetilde{E}} \subseteq S_E$ as stated in Corollary 3.10. Meanwhile, from Corollary 2.25, we have $S(A, b) = [\underline{x}(e_1), \bar{x}] \bigcup [\underline{x}(e_2), \bar{x}]$. By comparison of the values of the objective function for the minimal solutions, $\underline{x}(e_1)$ is an optimal solution in (11) (i.e., $e^* = e_1$). For this vector, we have $z_2 = 9.2380\,\underline{x}(e_1)_2 + 5.4982\,\underline{x}(e_1)_4 + 6.3461\,\underline{x}(e_1)_5 + 7.3739\,\underline{x}(e_1)_6 + 6.0014\,\underline{x}(e_1)_{10} = 6.52$ and $z = c^T\underline{x}(e_1) = -13.1564$. Thus, from Theorem 4.3, $x^* = (1, 0, 1, 0.5479, 0.5527, 0, 1, 0.5, 1, 0)$ and then $z^* = c^T x^* = -25.3703$.

6 Conclusions

In this paper, a class of generalized optimization models was analytically investigated in which the objective function is linear and the constraints are formed as a system of fuzzy relational equations defined by arbitrary continuous t-norms. The concept of partial solution sets and their structural properties were studied. By taking advantage of the mathematical properties of the partial solution sets, the solution sets of each single equation was determined, and then an algorithm was proposed to find the global optimal solution of the main problem in a general case. Moreover, some necessary and sufficient conditions were presented to determine the feasibility of the problem. Depending on the continuous t-norms, two basic simplification operations were proposed to speed up solution finding. It was shown that the general results obtained for continuous t-norms automatically reduce to the (apparently different) results that hold for special classes such as strict, nilpotent or Archimedean continuous t-norms. In addition, a test problem defined by Mayor-Torrens t-norm (that is continuous and non Archimedean) was then solved using the proposed algorithm. In future work, we investigate the importance and application of the concept of PSS in the expression and generalization of simplification techniques to reduce continuous FREs.

7 Appendix A

Table A.1: Summary of the most widely used continuous t-norms.

t-norm	Function	Parameter
Minimum	$\omega_{\min}(x,y)=\min\{x,y\}$	–
Product	$\omega_{prod}(x,y)=xy$	–
Lukasiewicz	$\omega_L(x,y)=\max\{x+y-1,0\}$	–
Einstein product	$\omega_E(x,y)=\frac{xy}{1+(1-x)(1-y)}$	–
Frank	$\omega_F^s(x,y)=\begin{cases}\omega_{\min}(x,y)\,if\ \ s=0\\ \omega_{prod}(x,y)\,if\ \ s=1\\ \log_s[1+((s^x-1)\,(s^y-1)/(s-1))]\ \ if\ \ s>0,s\neq 1\end{cases}$	$s\geq 0$
Yager	$\omega_Y^p(x,y)=\max\{1-[(1-x)^p+(1-y)^p]^{1/p}\,,0\}$	$p>0$
Hamacher	$\omega_H^\alpha(x,y)=\begin{cases}0 & ,\ if\ \alpha=x=y=0\\ \frac{xy}{\alpha+(1-\alpha)\,(x+y-xy)} & ,\ otherwise\end{cases}$	$\alpha\geq 0$
Dombi	$\omega_D^\lambda(x,y)=\begin{cases}0 & ,\ if\ x=0\ or\ y=0\\ \frac{1}{1+[((1-x)/x)^\lambda+((1-y)/y)^\lambda]^{1/\lambda}} & ,\ otherwise\end{cases}$	$\lambda>0$
Schweizer - Sklar	$\omega_{SS}^p(x,y)=\begin{cases}(x^p+y^p-1)^{1/p} & ,\ if\ -\infty<p<0\\ \omega_{prod}(x,y) & ,\ if\ p=0\\ (\max\{0,x^p+y^p-1\})^{1/p} & ,\ if\ 0<p<\infty\end{cases}$	$-\infty<p<\infty$
Sugeno - Weber	$\omega_{SW}^\lambda(x,y)=\max\{(x+y-1+\lambda xy)/(1+\lambda)\,,0\}$	$\lambda>-1$
Aczel-Alsina	$\omega_{AA}^p(x,y)=\begin{cases}0 & ,\ if\ x=0\ or\ y=0\\ Exp(-(\mid Ln(x)\mid^p+\mid Ln(y)\mid^p)^{1/p}) & ,\ otherwise\end{cases}$	$0<p<\infty$
Dubois - Prade	$\omega_{DP}^\gamma(x,y)=\begin{cases}\omega_{min}(x,y) & ,\ if\ \gamma=0\\ \omega_{prod}(x,y) & ,\ if\ \gamma=1\\ (xy)/\max\{x,y,\gamma\} & ,\ otherwise\end{cases}$	$\gamma\in[0,1]$
Mayor-Torrens	$\omega_{MT}^\lambda(x,y)=\begin{cases}\max\{x+y-\lambda,0\} & ,\ if\ \lambda\in(0,1]\,,(x,y)\in[0,\lambda]^2\\ \omega_{\min}(x,y) & ,\ otherwise\end{cases}$	$\lambda\in[0,1]$

Table A.2: The value of $\bar{x}_{ij}$ $(i\in I$ and $j\in J_i)$ for the t-norms stated in Table A1.

t-norm	$\bar{x}_{ij}$ if $a_{ij}>b_i>0$	$\bar{x}_{ij}$ if $a_{ij}>b_i=0$	$\bar{x}_{ij}$ if $a_{ij}=b_i$
Minimum	b_i	0	1
Product	b_i/a_{ij}	0	1
Lukasiewicz	$1+b_i-a_{ij}$	$1-a_{ij}$	1
Einstein product	$(b_i(2-a_{ij}))/(b_i+a_{ij}-b_ia_{ij})$	0	1
Frank	$\log_s[1+((s^{b_i}-1)\,(s-1))/(s^{a_{ij}}-1)]$	0	1
Yager	$1-[(1-b_i)^p-(1-a_{ij})^p]^{1/p}$	$1-[1-(1-a_{ij})^p]^{1/p}$	1
Hamacher	$[(\alpha+(1-\alpha)\,a_{ij})\,b_i]/[a_{ij}-(1-\alpha)\,(1-a_{ij})\,b_i]$	0	1
Dombi	$\frac{1}{1+[(1-b_i)/b_i)^\lambda-((1-a_{ij})/a_{ij})^\lambda]^{1/\lambda}}$	0	1
Schweizer - Sklar	$(1+b_i^p-{a_{ij}}^p)^{1/p}$	$(1-{a_{ij}}^p)^{1/p}$	1
Sugeno - Weber	$[(1+\lambda)\,b_i+(1-a_{ij})]/(1+\lambda\,a_{ij})$	$(1-a_{ij})/(1+\lambda a_{ij})$	1
Aczel-Alsina	$\exp\left(((Ln\ (b_i))^p-(Ln\ (a_{ij}))^p)^{1/p}\right)$	0	1
Dubois - Prade	$\begin{cases}b_i & ,\ if\ \gamma\leq a_{ij}\\ (\gamma\,b_i)/a_{ij} & ,\ if\ a_{ij}<\gamma\end{cases}$	0	1
Mayor-Torrens	$\begin{cases}b_i+\lambda-a_{ij} & ,\ if\ 0<a_{ij}\leq\lambda\\ b_i & ,\ if\ a_{ij}>\lambda\end{cases}$	$\begin{cases}\lambda-a_{ij} & ,\ if\ 0<a_{ij}\leq\lambda\\ 0 & ,\ if\ a_{ij}>\lambda\end{cases}$	1

Table A.3: The value of $\underline{x}_{ij}$ ($i \in I$ and $j \in J_i$) for the t-norms stated in Table A1.

t-norm	$\underline{x}_{ij}$ if $a_{ij} > b_i > 0$	$\underline{x}_{ij}$ if $a_{ij} > b_i = 0$	$\underline{x}_{ij}$ if $a_{ij} = b_i > 0$	$\underline{x}_{ij}$ if $a_{ij} = b_i = 0$
Minimum	b_i	0	b_i	0
Product	b_i/a_{ij}	0	1	0
Lukasiewicz	$1 + b_i - a_{ij}$	0	1	0
Einstein product	$(b_i(2 - a_{ij}))/(b_i + a_{ij} - b_i a_{ij})$	0	1	0
Frank	$\log_s[1 + ((s^{b_i} - 1)(s - 1))/(s^{a_{ij}} - 1)]$	0	1	0
Yager	$1 - [(1 - b_i)^p - (1 - a_{ij})^p]^{1/p}$	0	1	0
Hamacher	$[(\alpha + (1-\alpha)\, a_{ij})\, b_i]/[a_{ij} - (1-\alpha)\,(1 - a_{ij})\, b_i]$	0	1	0
Dombi	$\frac{1}{1+[(1-b_i)/b_i)^\lambda - ((1-a_{ij})/a_{ij})^\lambda]^{1/\lambda}}$	0	1	0
Schweizer - Sklar	$(1 + b_i^p - a_{ij}{}^p)^{1/p}$	0	1	0
Sugeno - Weber	$[(1 + \lambda)\, b_i + (1 - a_{ij})]/(1 + \lambda\, a_{ij})$	0	1	0
Aczel-Alsina	$\exp\left(((Ln\ (b_i))^p - (Ln\ (a_{ij}))^p)^{1/p}\right)$	0	1	0
Dubois - Prade	$\begin{cases} b_i & , \ if\ \ \gamma \le a_{ij} \\ (\gamma\, b_i)/a_{ij} & , \ if\ \ a_{ij} < \gamma \end{cases}$	0	$\max\{b_i, \gamma\}$	0
Mayor-Torrens	$\begin{cases} b_i + \lambda - a_{ij} & , \ if\ \ 0 < a_{ij} \le \lambda \\ b_i & , \ if\ \ a_{ij} > \lambda \end{cases}$	0	$\max\{b_i, \lambda\}$	0

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