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Fuzzy time-fractional advection-dispersion and Navier-Stokes equations: A comprehensive approach

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Abstract

This paper presents a novel approach for modeling and analyzing complex systems with uncertain data by using fuzzy calculus and time-fractional differential equations. Specifically, we propose the use of the fuzzy Atangana-Baleanu time-fractional derivative with non-singular kernels for fuzzy functions as a suitable fractional derivative type for the qualitative analysis of fractional differential equations in fuzzy space. Additionally, we provide a method for numerically solving fuzzy linear time-fractional equations in fluid dynamics using the fuzzy Laplace transform iterative method. The effectiveness and practical relevance of our proposed method are demonstrated through concrete examples, including the fuzzy time-fractional Advection-Dispersion equation, the fuzzy time-fractional Navier-Stokes equation, and Couette flow. These examples showcase the potential of our method to address real-world problems in fluid dynamics and provide a clear illustration of the solution steps involved. Our findings highlight the importance of considering fuzzy calculus and time-fractional differential equations in modeling and analyzing complex systems with uncertain data.

Keywords: The fuzzy Atangana-Baleanu time-fractional derivative, the fuzzy time-fractional advection-dispersion equation, the fuzzy time-fractional Navier-Stokes equation, Couette flow.

1 Introduction

In recent years, there has been a significant surge of interest in the study of time-fractional differential equations across various scientific fields, including physics, chemistry, biology, and finance. These equations, which incorporate fractional derivatives, have become essential tools for investigating complex phenomena. By utilizing the principles of fractional calculus, they provide a nuanced approach to analyzing the temporal evolution of systems, offering insights into processes affected by non-local interactions, viscoelastic properties, and other memory-dependent behaviors [16, 18, 19, 40].

However, when applied to real-world scenarios within continuous space, time-fractional differential equations often struggle to accurately capture the complexities and uncertainties inherent in practical applications. The inherent differences between controlled laboratory conditions and real-world dynamics present challenges in achieving precise estimations. Therefore, there is a pressing need for a more adaptable modeling framework that can realistically and comprehensively represent the dynamics of complex systems [13, 14, 15, 17, 20, 28, 29, 33, 34, 43].

In response to this need, fuzzy fractional calculus has seen extensive application across numerous disciplines over the past few decades, leading to the development of a variety of non-local fuzzy fractional differential operators. These operators, available in both singular and non-singular forms, have broad applicability across diverse scientific domains. Notable examples include the fuzzy Riemann–Liouville derivative [5], the fuzzy generalized Hukuhara Caputo fractional derivative [4, 22, 42], the fuzzy Caputo-Fabrizio fractional derivative [1, 30], the fuzzy Atangana-Baleanu fractional derivative [7], the fuzzy Riemann-Liouville-Katugampola generalized Hukuhara fractional derivative [27], the fuzzy

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Caputo-Katugampola generalized Hukuhara fractional derivative [27], the fuzzy conformable fractional derivative [24, 25], the Hadamard ψ-Caputo tempered fractional derivative [8], and the weighted Caputo-type generalized Hukuhara differentiability [21]. These operators, with their multifaceted functionalities, are indispensable tools for modeling and analyzing intricate systems encountered in diverse scientific inquiries.

Nevertheless, deriving analytical solutions for most fuzzy time-fractional differential equations remains a daunting task. Thus, there is a compelling need to develop robust and efficient techniques for addressing these equations. The numerical solution of fuzzy time-fractional differential equations has gained considerable attention from researchers, leading to the exploration of various methodologies such as the fuzzy difference method [2, 3], the fuzzy Adomian decomposition method [6, 35], the fuzzy finite volume method [32, 45], the fuzzy finite difference method [44], the fuzzy generalized power series method [36], and the fuzzy variational iteration method [9, 31], among others.

This study presents a novel approach to analyzing time-fractional differential equations in the realm of fuzzy space by leveraging the fuzzy Atangana-Baleanu time-fractional derivative with a non-singular kernel. We propose this derivative as a robust tool for qualitatively assessing such equations, aiming to enhance our understanding of complex systems governed by fractional dynamics.

Furthermore, we introduce an innovative method called the fuzzy Laplace transform iterative method, specifically designed for approximating solutions to fuzzy linear time-fractional equations commonly encountered in fluid dynamics scenarios. By employing fuzzy arithmetic, fuzzy series, and the fuzzy Laplace iterative method, we directly derive fuzzy approximate solutions for fuzzy time-fractional equations. This approach eliminates the traditional requirement of transforming equations into crisp forms, offering a fresh perspective and distinct advantage.

Our research focuses on significant problems in fluid dynamics, including the fuzzy time-fractional Advection-Dispersion equation, the fuzzy time-fractional Navier-Stokes equation, and Couette flow. Through detailed examples and rigorous analysis, we demonstrate the efficacy and practical relevance of our method. By addressing real-world challenges in fluid dynamics, our approach not only proves its effectiveness but also highlights its potential utility in diverse practical applications, paving the way for new insights and advancements in the field.

The paper is organized into several sections, each focusing on a specific aspect of the proposed method. In Section 2, we revisit fundamental definitions and theorems related to the fuzzy generalized Hukuhara partial derivative and introduce the fuzzy Atangana-Baleanu time-fractional derivative. Section 3 explores the properties of the fuzzy Laplace transform and its inverse. In Section 4, we examine a fuzzy linear time-fractional equation defined using the fuzzy ABC_{gH}-time derivative and describe the fuzzy Laplace transform iterative method as a numerical approach to approximate fuzzy solutions to this problem. Section 5 demonstrates the practical relevance of the proposed method through concrete examples of fluid dynamics problems, such as the fuzzy time-fractional Advection-Dispersion equation and the fuzzy time-fractional Navier-Stokes equation, along with a special case of the fuzzy time-fractional Navier-Stokes equation, i.e., Couette flow. These examples provide a clear illustration of the solution steps involved and emphasize the applicability of the proposed method in addressing real-world problems. Finally, Section 6 presents brief conclusions.

2 Fuzzy derivatives overview

In this section, we will begin by revisiting some fundamental definitions and theorems that are relevant to the fuzzy generalized Hukuhara partial derivative. We will also introduce some notations and preliminary facts that will be used throughout the paper. Furthermore, we will delve into the fuzzy Atangana-Baleanu time-fractional derivative, which is based on the idea of partial generalized Hukuhara differentiability. This type of differentiability provides a mathematical framework for studying fuzzy functions that may not be differentiable in the Hukuhara sense. Additionally, we will highlight some of the distinctive characteristics of this form of differentiability.

Let us begin with a definition of a fuzzy number $\mathcal{A} : \mathbb{R} \longrightarrow [0, 1]$, which is a fuzzy subset of $\mathbb{R}$ satisfying the following conditions [11]:

i. $\mathcal{A}$ is normal, meaning that there exists a real number $\mathrm{t}_0 \in \mathbb{R}$ such that $\mathcal{A}(\mathrm{t}_0) = 1$.

ii. $\mathcal{A}$ is a fuzzy convex set, that is, for all $\eta, \zeta \in \mathbb{R}$ and $0 \leqslant \lambda \leq 1$, $\mathcal{A}(\lambda\eta + (1 - \lambda)\zeta) \geqslant \min(\mathcal{A}(\eta), \mathcal{A}(\zeta))$.

iii. $\mathcal{A}$ is upper semi-continuous.

iv. $\overline{\{\mathrm{t} \in \mathbb{R} : \mathcal{A}(\mathrm{t}) > 0\}}$ is compact, where $\overline{\{.\}}$ denotes the closure of $\{.\}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by $\mathcal{E}$.

A fuzzy number, denoted by $\mathcal{A}$, can be represented by a pair of ordered functions, namely $[\underline{\mathcal{A}}(\mathrm{r}), \overline{\mathcal{A}}(\mathrm{r})]$ that correspond to the lower and upper branches of the fuzzy number. This same representation can also be applied to a fuzzy function $\mathcal{G} : \mathbb{R} \times [0, \infty) \to \mathcal{E}$, where for any $\mathrm{r} \in [0, 1]$, we have $[\mathcal{G}(\mathfrak{x}, \tau)]^r = \big[\underline{\mathcal{G}}(\mathfrak{x}, \tau; \mathrm{r}), \overline{\mathcal{G}}(\mathfrak{x}, \tau; \mathrm{r})\big]$.

A triangular fuzzy number defined as a fuzzy set in $\mathcal{E}$, that is specified by an ordered triple $\mathcal{A} = (a_1, a_2, a_3)$ with $a_1 \leq a_2 \leq a_3$.

Remark 2.1. *If $\mathcal{A} = (a_1, a_2, a_3)$, $\mathcal{B} = (b_1, b_2, b_3)$ are two triangular fuzzy numbers. In this case, there are the following properties for these numbers (See* [11]*)*

1. $[\mathcal{A}]^r = [a_1 + (a_2 - a_1)r, a_3 - (a_3 - a_2)r]$ *for all* $r \in [0, 1]$.

2. *Two triangle fuzzy numbers $\mathcal{A}$ and $\mathcal{B}$ are said equal if and only if $a_1 = b_1$, $a_2 = b_2$ and $a_3 = b_3$.*

3. $\mathcal{A} \oplus \mathcal{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

4. *If there exists $\mathcal{C} \in \mathcal{E}$ such that $\mathcal{A} = \mathcal{B} \oplus \mathcal{C}$, then $\mathcal{C}$ is called Hukuhara difference(H-difference) of $\mathcal{A}$ and $\mathcal{B}$ ($\mathcal{C} = \mathcal{A} \ominus \mathcal{B}$), and $\mathcal{A} \ominus \mathcal{B} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$.*

5. $\mathcal{A} \ominus (-1)\mathcal{B} = (a_1 + b_3, a_1 + b_2, a_3 + b_1)$ *provided that $\mathcal{A} \ominus (-1)\mathcal{B}$ is a triangular fuzzy number.*

6. *If $\mathcal{A} = \mathcal{C} \ominus (-1)\mathcal{B}$, then $\mathcal{B} = -1(\mathcal{C} \ominus \mathcal{A})$.*

7. *The generalized Hukuhara difference of two fuzzy number $\mathcal{A}, \mathcal{B} \in \mathcal{E}$ is the fuzzy number $\mathcal{C}$, (if it exists), such that*

$$\mathcal{A} \ominus_{gH} \mathcal{B} = \mathcal{C} \Longleftrightarrow \begin{cases} (i).\ \mathcal{C} = (a_1 - b_1, a_2 - b_2, a_3 - b_3); \\ Or, \\ (ii).\ \mathcal{C} = (a_3 - b_3, a_2 - b_2, a_1 - b_1). \end{cases}$$

In this article, we assume that $\mathcal{A} \ominus_{gH} \mathcal{B} \in \mathcal{E}$.

8. *For all* $\lambda \in \mathbb{R}$

$$\lambda\mathcal{A} = \begin{cases} (\lambda a_1, \lambda a_2, \lambda a_3), & \text{If } \lambda \geq 0; \\ \\ (\lambda a_3, \lambda a_2, \lambda a_1), & \text{If } \lambda < 0. \end{cases}$$

Definition 2.2. [12, 41] *A fuzzy function $\mathcal{G} : \mathbb{R} \times [0, T] \to \mathcal{E}$ is said to be fuzzy generalized Hukuhara partial derivative ($[gH - p]-$derivative) with respect to $\mathfrak{x}$ at $(\mathfrak{x}, \tau) \in \mathbb{R} \times [0, T]$ if $\frac{\partial_{gH}\mathcal{G}(\mathfrak{x},\tau)}{\partial \mathfrak{x}} \in \mathcal{E}$ such that, for any sufficiently small $\hbar > 0$, the generalized Hukuhara difference $\mathcal{G}(\mathfrak{x} + \hbar, \tau) \ominus_{gH} \mathcal{G}(\mathfrak{x}, \tau)$ exists and*

$$\frac{\partial_{gH}\mathcal{G}(\mathfrak{x}, \tau)}{\partial \mathfrak{x}} = \lim_{\hbar \to 0} \frac{\mathcal{G}(\mathfrak{x} + \hbar, \tau) \ominus_{gH} \mathcal{G}(\mathfrak{x}, \tau)}{\hbar}.$$

Let $\mathcal{G}(\mathfrak{x}, \tau)$ be a $[gH - p]-$differentiable fuzzy function and $\underline{\mathcal{G}}(\mathfrak{x}, \tau; \mathrm{r}), \overline{\mathcal{G}}(\mathfrak{x}, \tau; \mathrm{r})$ both differentiable with respect to $\mathfrak{x}$ at $(\mathfrak{x}, \tau) \in \mathbb{R} \times [0, T]$. We say

- *$\mathcal{G}(\mathfrak{x}, \tau)$ is a $[i - p]-$differentiable fuzzy function with respect to $\mathfrak{x}$, if for all* $\mathrm{r} \in [0, 1]$

$$\left[\frac{\partial_{i.gH}\mathcal{G}(\mathfrak{x}, \tau)}{\partial \mathfrak{x}}\right]^r = \left[\frac{\partial}{\partial \mathfrak{x}}(\underline{\mathcal{G}}(\mathfrak{x}, \tau; \mathrm{r})), \frac{\partial}{\partial \mathfrak{x}}(\overline{\mathcal{G}}(\mathfrak{x}, \tau; \mathrm{r}))\right].$$

- *$\mathcal{G}(\mathfrak{x}, \tau)$ is a $[ii - p]-$differentiable fuzzy function with respect to $\mathfrak{x}$, if for all* $\mathrm{r} \in [0, 1]$

$$\left[\frac{\partial_{ii.gH}\mathcal{G}(\mathfrak{x}, \tau)}{\partial \mathfrak{x}}\right]^r = \left[\frac{\partial}{\partial \mathfrak{x}}(\overline{\mathcal{G}}(\mathfrak{x}, \tau; \mathrm{r})), \frac{\partial}{\partial \mathfrak{x}}(\underline{\mathcal{G}}(\mathfrak{x}, \tau; \mathrm{r}))\right].$$

Let $\mathcal{C}^{\mathcal{E}}(\mathbb{R} \times [0, T])$ represents a set of fuzzy continuous and Riemann integrable functions defined on $\mathbb{R} \times [0, T]$.

Definition 2.3. [26] *Suppose $\mathcal{G}$ and $\frac{\partial_{gH}\mathcal{G}(\mathfrak{x},\tau)}{\partial \tau} \in \mathcal{C}^{\mathcal{E}}(\mathbb{R} \times [0, T])$. In this case, the fuzzy Atangana-Baleanu time-fractional derivative ($ABC_{gH}-$time derivative) of $\mathcal{G}$ is given by:*

$${}^{ABC}_{gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x}, \tau)\} = \frac{\aleph(\alpha)}{1 - \alpha} \int_0^{\tau} \frac{\partial_{gH}\mathcal{G}(\mathfrak{x}, \xi)}{\partial \xi} \mathbb{M}_{\alpha}(\mu(\tau - \xi))\mathrm{d}\xi,$$

where $\mu = -\frac{\alpha}{1-\alpha}$ *and the normalization function* $\aleph(\alpha)$ *is selected as* $\aleph(0) = \aleph(1) = 1$ *and is composited by* $\aleph(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$. *The Mittag-Leffler function* $\mathbb{M}_\alpha(\mu\tau)$ *is given by:*

$$\mathbb{M}_\alpha(\mu\tau^\alpha) = \sum_{n=0}^{\infty} \frac{\mu^n \tau^{\alpha n}}{\Gamma(\alpha n + 1)},$$

where $0 \neq \mu \in \mathbb{R}$, $\tau \in \mathbb{C}$, *and* $\Re(\alpha) > 0$.

Let $\mathcal{G}(\mathfrak{x}, \tau)$ *be a* $ABC_{gH}-$*time differentiable fuzzy function with respect to* τ *at* $(\mathfrak{x}, \tau) \in \mathbb{R} \times [0, T]$. *We say*

- $\mathcal{G}(\mathfrak{x}, \tau)$ *is a* $[i-ABC_{gH}]-$*time differentiable over* $\mathbb{R}\times[0,T]$ *(represented by* ${}^{ABC}_{i-gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\}$*) if, for any* $\mathrm{r} \in [0,1]$, *the following holds*

$$[\,{}^{ABC}_{i-gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\}]^{\mathrm{r}} = \Big[\,{}^{ABC}D^{\alpha}_{0,\tau}\{\underline{\mathcal{G}}(\mathfrak{x},\tau;\mathrm{r})\},\ {}^{ABC}D^{\alpha}_{0,\tau}\{\overline{\mathcal{G}}(\mathfrak{x},\tau;\mathrm{r})\}\Big].$$

- $\mathcal{G}(\mathfrak{x}, \tau)$ *is a* $[ii-ABC_{gH}]-$*time differentiable over* $\mathbb{R}\times[0,T]$ *(denoted as* ${}^{ABC}_{ii-gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\}$*) if, for any* $\mathrm{r} \in [0,1]$, *we have*

$$[\,{}^{ABC}_{ii-gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\}]^{\mathrm{r}} = \Big[\,{}^{ABC}D^{\alpha}_{0,\tau}\{\overline{\mathcal{G}}(\mathfrak{x},\tau;\mathrm{r})\},\ {}^{ABC}D^{\alpha}_{0,\tau}\{\underline{\mathcal{G}}(\mathfrak{x},\tau;\mathrm{r})\}\Big].$$

Here, ${}^{ABC}D^{\alpha}_{0,\tau}\{\underline{\mathcal{G}}(\mathfrak{x},\tau;\mathrm{r})\}$ *and* ${}^{ABC}D^{\alpha}_{0,\tau}\{\overline{\mathcal{G}}(\mathfrak{x},\tau;\mathrm{r})$ *represent the Atangana-Baleanu time-fractional derivatives for the real-valued functions* $\underline{\mathcal{G}}(\mathfrak{x},\tau;\mathrm{r})$ *and* $\overline{\mathcal{G}}(\mathfrak{x},\tau;\mathrm{r})$, *respectively, as explained in* [10].

Example 2.4. *Let* $\mathcal{G}$ *be a fuzzy function defined on the interval* $[0,1]\times[0,2]$ *and taking values in* $\mathcal{E}$. *Its expression is given by:*

$$[\mathcal{G}(\mathfrak{x},\tau)]^{\mathrm{r}} = \left[3e^{-2\tau}\mathfrak{x} + 2e^{-2\tau}\mathrm{r}\mathfrak{x}, 13e^{-2\tau}\mathfrak{x} - 8e^{-2\tau}\mathrm{r}\mathfrak{x}\right].$$

For $\alpha = \frac{1}{2}$, *we have the fuzzy* ABC_{gH}*-time derivative of* $\mathcal{G}(\mathfrak{x},\tau)$ *given by:*

$$\left[\,{}^{ABC}_{ii-gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\}\right]^{\mathrm{r}} = [(-13+8r)\mathfrak{x}, -(3+2r)\mathfrak{x}]\left(\frac{e^{-2\tau}(1+\sqrt{\pi})(2e^{3\tau}Erfc[\sqrt{\tau}] + \sqrt{2}Erfi[\sqrt{2\tau}] - 2)}{3\sqrt{\pi}}\right).$$

Figures 1 (a) and (b) illustrate the fuzzy functions $\mathcal{G}(\mathfrak{x},\tau;\mathrm{r})$ *and* ${}^{ABC}_{ii-gH}\mathcal{D}^{\frac{1}{2}}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau;\mathrm{r})\}$, *respectively, for* $\mathrm{r} = 0.5$. *These figures demonstrate that the lower cut (Green) and upper cut (Orange) positions for figure (b) have shifted, indicating that* $\mathcal{G}(\mathfrak{x},\tau)$ *is differentiable* $[ii - ABC_{gH}]$ *times.*

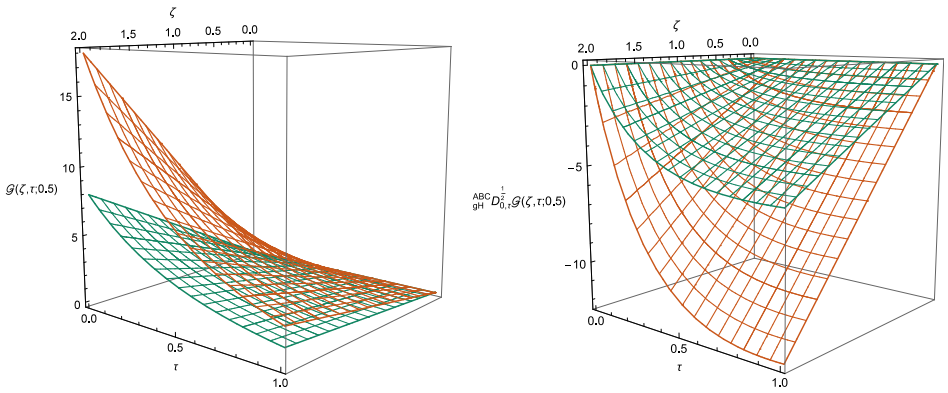

Figure 1: Plots of $\mathcal{G}(\mathfrak{x},\tau;\mathrm{r})$ (Top) and its ${}^{ABC}_{gH}\mathcal{D}^{\frac{1}{2}}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau;\mathrm{r})\}$ (Lower) for Example 2.4 in $\mathrm{r} = 0.5$.

Theorem 2.5. [26] *Let* $\mathcal{G}$ *be a function such that* $\frac{\partial_{gH}\mathcal{G}(\mathfrak{x},\tau)}{\partial\tau} \in \mathcal{C}^{\mathcal{E}}(\mathbb{R}\times[0,T])$. *The following principles apply:*

i. *If* $\mathcal{G}$ *is* $[i-p]$*-differentiable with respect to* τ, *then it is also* $[i - ABC_{gH}]$*-time differentiable.*

ii. *If* $\mathcal{G}$ *is* $[ii-p]$*-differentiable with respect to* τ, *then it is also* $[ii - ABC_{gH}]$*-time differentiable.*

3 Fuzzy Laplace transform and its key properties

In this section, our focus will be on the fuzzy Laplace transform and its key properties, including linearity, $[gH-p]-$derivative and $ABC_{gH}-$time derivative properties. We will also delve into the inverse fuzzy Laplace transform, which is utilized to convert a frequency-domain function into its corresponding time-domain function.

Let $\mathcal{G}(\mathfrak{x},\tau)$ be a piecewise continuous fuzzy function on $[0,\infty)$ of exponential order σ. We denote the fuzzy Laplace transform of $\mathcal{G}(\mathfrak{x},\tau)$ with respect to τ as $\mathcal{G}(\mathfrak{x},\mathfrak{s})$, which is defined as:

$$\mathfrak{L}_\tau\left[\mathcal{G}(\mathfrak{x},\tau)\right](\mathfrak{s})=\int_0^\infty e^{-\mathfrak{s}\tau}\mathcal{G}(\mathfrak{x},\tau)d\tau=\mathcal{G}(\mathfrak{x},\mathfrak{s}).$$

This holds provided the improper integral exists.

Additionally, the inverse fuzzy Laplace transform of $\mathcal{G}(\mathfrak{x},\mathfrak{s})$ is defined as follows:

$$\mathfrak{L}_\mathfrak{s}^{-1}\left[\mathcal{G}(\mathfrak{x},\mathfrak{s})\right](\tau)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{\mathfrak{s}\tau}\mathcal{G}(\mathfrak{x},\mathfrak{s})d\mathfrak{s}.$$

This is defined for a fixed $\sigma\in\mathbb{R}$.

Although the fuzzy Laplace transform and its inverse have similarities with the classical Laplace transform, certain assumptions must be considered to deal with the fuzzy Laplace transform's linearity. These conditions are presented below. Additionally, we can calculate the fuzzy Laplace transform for a generalized Hakuhara partial derivative by considering the type of $[gH-p]-$differentiability. We demonstrate this computation in the following remark.

Remark 3.1. *If $\mathcal{G}(\mathfrak{x},\tau)$, $\mathcal{K}(\mathfrak{x},\tau)$, and $\frac{\partial_{gH}\mathcal{G}(\mathfrak{x},\tau)}{\partial\tau}$ have fuzzy Laplace transforms, and there are no switching points within their domain, then the following hold* [23, 38]*:*

1. *For any real constants $a,b\geq 0$ (or $a,b\leq 0$), we have:*

- $\mathfrak{L}_\tau\left[a\mathcal{G}(\mathfrak{x},\tau)\oplus b\mathcal{K}(\mathfrak{x},\tau)\right](\mathfrak{s})=a\mathcal{G}(\mathfrak{x},\mathfrak{s})\oplus b\mathcal{K}(\mathfrak{x},\mathfrak{s})$.
- $\mathfrak{L}_\tau\left[a\mathcal{G}(\mathfrak{x},\tau)\ominus_{gH} b\mathcal{K}(\mathfrak{x},\tau)\right](\mathfrak{s})=b\mathcal{G}(\mathfrak{x},\mathfrak{s})\ominus_{gH} b\mathcal{K}(\mathfrak{x},\mathfrak{s})$.

2. *If $\mathcal{G}(\mathfrak{x},\tau)$ is $[i-p]-$differentiable with respect to τ, then we have:*

$$\mathfrak{L}_\tau\left[\frac{\partial_{i.gH}\mathcal{G}(\mathfrak{x},\tau)}{\partial\tau}\right](\mathfrak{s}) \quad = \quad \mathfrak{s}\mathcal{G}(\mathfrak{x},\mathfrak{s})\ominus\mathcal{G}(\mathfrak{x},0).$$

3. *If $\mathcal{G}(\mathfrak{x},\tau)$ is $[ii-p]-$differentiable with respect to τ, then we have*

$$\mathfrak{L}_\tau\left[\frac{\partial_{ii.gH}\mathcal{G}(\mathfrak{x},\tau)}{\partial\tau}\right](\mathfrak{s}) \quad = \quad (-1)\mathcal{G}(\mathfrak{x},0)\ominus(-1)\mathfrak{s}\mathcal{G}(\mathfrak{x},\mathfrak{s}).$$

Where $\mathcal{G}(\mathfrak{x},\mathfrak{s})=\mathfrak{L}_\tau\left[\mathcal{G}(\mathfrak{x},\tau)\right](\mathfrak{s})$, $\mathcal{K}(\mathfrak{x},\mathfrak{s})=\mathfrak{L}_\tau\left[\mathcal{K}(\mathfrak{x},\tau)\right](\mathfrak{s})$.

Definition 3.2. [37, 38] *Let $\hbar(\tau)$ is a piecewise continuous real-valued function, while $\mathcal{G}(\mathfrak{x},\tau)$ is a fuzzy integrable function. We use the symbol $(\hbar*\mathcal{G})(\mathfrak{x},\tau)$ to denote the convolution of $\mathcal{G}$ and $\hbar$ with respect to τ, which is defined as follows:*

$$(\hbar*\mathcal{G})(\mathfrak{x},\tau)=\int_0^\tau\hbar(\tau-\xi)\mathcal{G}(\mathfrak{x},\xi)\mathrm{d}\xi.$$

Theorem 3.3. [39] *Let $\mathcal{G}(\mathfrak{x},\tau)$ be a fuzzy piecewise continuous function that is exponentially bounded on the interval $[0,\infty)$, and let $\hbar(\tau)$ be a real-valued piecewise continuous function on the same interval. The following identity holds for the fuzzy Laplace transform:*

$$\mathfrak{L}_\tau[(\hbar*\mathcal{G})(\mathfrak{x},\tau)](\mathfrak{s})=\mathfrak{L}_\tau\left[\mathcal{G}(\mathfrak{x},\tau)\right](\mathfrak{s})\times\mathfrak{L}_\tau\left[\hbar(\tau)\right](\mathfrak{s}).$$

Where, $\Re(\mathfrak{s})>\sigma$. Here, the convolution operator $*$ is used to represent the convolution of $\hbar(\tau)$ and $\mathcal{G}(\mathfrak{x},\tau)$ with respect to the variable τ.

Theorem 3.4. [26] *Suppose $\mathcal{G}(\mathfrak{x},\tau)$ possesses fuzzy Laplace transforms, and $0<\alpha\leq 1$. The following statements hold:*

i. *If $\mathcal{G}(\mathfrak{x},\tau)$ is $[i-ABC_{gH}]-$time differentiable, then*

$$\mathfrak{L}_\tau[\ {}^{ABC}_{i-gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\}](\mathfrak{s}) = \aleph(\alpha)\Big(\frac{\mathfrak{s}^{\alpha}\mathcal{G}(\mathfrak{x},\mathfrak{s}) \ominus \mathfrak{s}^{\alpha-1}\mathcal{G}(\mathfrak{x},0)}{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}\Big). \tag{1}$$

ii. *If $\mathcal{G}(\mathfrak{x},\tau)$ is $[ii-ABC_{gH}]-$time differentiable, then*

$$\mathfrak{L}_\tau[\ {}^{ABC}_{ii-gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\}](\mathfrak{s}) = \aleph(\alpha)\Big(\frac{(-1)\mathfrak{s}^{\alpha-1}\mathcal{G}(\mathfrak{x},0) \ominus (-1)\mathfrak{s}^{\alpha}\mathcal{G}(\mathfrak{x},\mathfrak{s})}{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}\Big). \tag{2}$$

Where $\mathcal{G}(\mathfrak{x},\mathfrak{s}) = \mathfrak{L}_\tau\left[\mathcal{G}(\mathfrak{x},\tau)\right](\mathfrak{s})$.

4 An overview of fuzzy Laplace transform iterative method

In this section, we will examine a fuzzy linear time-fractional equation that is defined using the fuzzy $ABC_{gH}-$partial derivative and is subject to an initial condition over a specific interval. To approximate the fuzzy solutions to this problem, we will provide an overview of the fuzzy Laplace transform iterative method, which is a numerical method.

The following fuzzy linear time-fractional equation is considered in the sense of the fuzzy $ABC_{gH}-$time derivative:

$${}^{ABC}_{gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\}\} = \mathcal{A}\{\mathcal{G}(\mathfrak{x},\tau) \oplus \Bbbk(\mathfrak{x},\tau), \tag{3}$$

with the following fuzzy initial condition

$$\mathcal{G}(\mathfrak{x},0) = g(\mathfrak{x}). \tag{4}$$

Here, $0<\alpha\leq 1$, $(\mathfrak{x},\tau)\in\mathbb{R}^{+}\times[0,\infty)$, $\mathcal{A}\{\mathcal{G}(\mathfrak{x},\tau)\}$ is a linear operator, and $\Bbbk(\mathfrak{x},\tau)$ is a given continuous fuzzy function.

Now, we will explore the fuzzy Laplace transform iterative method as a numerical technique to approximate solutions for fuzzy linear time-fractional equations. There are two schemes that we will consider based on the type of $ABC_{gH}-$time differentiability:

- Under **Scheme 1**, we assume that $\mathcal{G}(\mathfrak{x},\tau)$ is $[i-ABC_{gH}]-$time differentiable. By using Theorem 3.4 and the initial condition (4), we apply the fuzzy Laplace transform to equation (3) with respect to τ. This results in the equation:

$$\mathfrak{L}_\tau[\ {}^{ABC}_{gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\}](\mathfrak{s}) = \mathfrak{L}_\tau[\mathcal{A}\{\mathcal{G}(\mathfrak{x},\tau)\}](\mathfrak{s}) \oplus \mathfrak{L}_\tau[\Bbbk(\mathfrak{x},\tau)](\mathfrak{s}). \tag{5}$$

  Substituting Eq. (1) into Eq. (5) leads to

$$\mathfrak{L}_\tau[\mathcal{G}(\mathfrak{x},\tau)](\mathfrak{s}) = \mathcal{Q}(\mathfrak{x},\mathfrak{s}) \oplus \Big(\frac{1}{\aleph(\alpha)}(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}})\Big)\mathfrak{L}_\tau[\mathcal{A}\{\mathcal{G}(\mathfrak{x},\tau)\}](\mathfrak{s}), \tag{6}$$

  where $\mathcal{Q}(\mathfrak{x},\mathfrak{s})$ is the following known function:

$$\mathcal{Q}(\mathfrak{x},\mathfrak{s}) = \frac{1}{\mathfrak{s}}\mathcal{G}(\mathfrak{x},0) \oplus \Big(\frac{1}{\aleph(\alpha)}(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}})\Big)\mathfrak{L}_\tau[\Bbbk(\mathfrak{x},\tau)](\mathfrak{s}).$$

  Applying the Laplace inverse transform to both sides of Eq. (6) leads to the following approximate solution:

$$\mathcal{G}(\mathfrak{x},\tau) = G(\mathfrak{x},\tau) \oplus \mathfrak{L}^{-1}_{\mathfrak{s}}\Bigg[\Big(\frac{1}{\aleph(\alpha)}(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}})\Big)\mathfrak{L}_\tau[\mathcal{A}\{\mathcal{G}(\mathfrak{x},\tau)\}](\mathfrak{s})\Bigg], \tag{7}$$

  where $G(\mathfrak{x},\tau) = \mathfrak{L}^{-1}_{\mathfrak{s}}\big[\mathcal{Q}(\mathfrak{x},\mathfrak{s})\big]$.

  Let us assume that the solution to equation (3) is a fuzzy series of the form:

$$\mathcal{G}(\mathfrak{x},\tau) = \sum_{n=0}^{\infty}\mathcal{G}_n(\mathfrak{x},\tau), \tag{8}$$

where the $\mathcal{G}_n(\mathfrak{x},\tau)$ are unknown fuzzy functions. By substituting equation (8) into equation (7), we obtain:

$$\sum_{n=0}^{\infty}\mathcal{G}_n(\mathfrak{x},\tau)=G(\mathfrak{x},\tau)\oplus\mathfrak{L}_{\mathfrak{s}}^{-1}\Big[\Big(\frac{1}{\aleph(\alpha)}(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}})\Big)\mathfrak{L}_{\tau}\Big[\mathcal{A}\Big\{\sum_{n=0}^{\infty}\mathcal{G}_n(\mathfrak{x},\tau)\Big\}\Big](\mathfrak{s})\Big]. \tag{9}$$

Using equation (9), we can define the following iterations:

$$\begin{aligned}&\mathcal{G}_0(\mathfrak{x},\tau)=G(\mathfrak{x},\tau),\\&\mathcal{G}_{n+1}(\mathfrak{x},\tau)=\mathfrak{L}_{\mathfrak{s}}^{-1}\Big[\Big(\frac{1}{\aleph(\alpha)}(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}})\Big)\mathfrak{L}_{\tau}\Big[\mathcal{A}\Big\{\mathcal{G}_n(\mathfrak{x},\tau)\Big\}\Big](\mathfrak{s})\Big].\end{aligned}$$

For $n=0,1,\cdots$.

We can thus approximate the $[i-ABC_{gH}]-$time differentiable solution $\mathcal{G}(\mathfrak{x},\tau)$ as:

$$\mathcal{G}(\mathfrak{x},\tau)=\sum_{n=0}^{k}\mathcal{G}_n(\mathfrak{x},\tau),\qquad k\to\infty. \tag{10}$$

- In **Scheme 2**, we assume that the $[ii-ABC_{gH}]-$time differentiable solution $\mathcal{G}(\mathfrak{x},\tau)$ can be expressed as

$$\mathcal{G}(\mathfrak{x},\tau)=G(\mathfrak{x},\tau)\ominus(-1)\mathfrak{L}_{\mathfrak{s}}^{-1}\Big[\Big(\frac{1}{\aleph(\alpha)}(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}})\Big)\mathfrak{L}_{\tau}[\mathcal{A}\{\mathcal{G}(\mathfrak{x},\tau)\}](\mathfrak{s})\Big], \tag{11}$$

where

$$G(\mathfrak{x},\tau)=\mathfrak{L}_{\mathfrak{s}}^{-1}\Big[\frac{1}{\mathfrak{s}}\mathcal{G}(\mathfrak{x},0)\Big]\ominus(-1)\mathfrak{L}_{\mathfrak{s}}^{-1}\Big[\Big(\frac{1}{\aleph(\alpha)}(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}})\Big)\mathfrak{L}_{\tau}[\mathbb{k}(\mathfrak{x},\tau)](\mathfrak{s})\Big].$$

We then assume that $\mathcal{G}(\mathfrak{x},\tau)$ can be expressed as an infinite sum of fuzzy functions

$$\mathcal{G}(\mathfrak{x},\tau)=\sum_{n=0}^{\infty}\mathcal{G}_n(\mathfrak{x},\tau). \tag{12}$$

Substituting this series into equation (11), we get the following equality:

$$\sum_{n=0}^{\infty}\mathcal{G}_n(\mathfrak{x},\tau)=G(\mathfrak{x},\tau)\ominus(-1)\mathfrak{L}_{\mathfrak{s}}^{-1}\Big[\Big(\frac{1}{\aleph(\alpha)}(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}})\Big)\mathfrak{L}_{\tau}\Big[\mathcal{A}\Big\{\sum_{n=0}^{\infty}\mathcal{G}_n(\mathfrak{x},\tau)\Big\}\Big](\mathfrak{s})\Big].$$

The unknown fuzzy functions can be determined through the following calculations:

$$\begin{aligned}&\mathcal{G}_0(\mathfrak{x},\tau)=G(\mathfrak{x},\tau),\\&\mathcal{G}_{n+1}(\mathfrak{x},\tau)=\ominus(-1)\mathfrak{L}_{\mathfrak{s}}^{-1}\Big[\Big(\frac{1}{\aleph(\alpha)}(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}})\Big)\mathfrak{L}_{\tau}\Big[\mathcal{A}\Big\{\mathcal{G}_n(\mathfrak{x},\tau)\Big\}\Big](\mathfrak{s})\Big],\quad n=0,1,\cdots.\end{aligned}$$

Which can be used to approximate the $[ii-ABC_{gH}]-$time differentiable solution of $\mathcal{G}(\mathfrak{x},\tau)$ for problem (3) as follow

$$\mathcal{G}(\mathfrak{x},\tau)=\sum_{n=0}^{k}\mathcal{G}_n(\mathfrak{x},\tau),\qquad k\to\infty. \tag{13}$$

5 Application of fuzzy Laplace transform iterative method in fluid dynamics

This section of the paper showcases the effectiveness of the fuzzy Laplace transform iterative method for obtaining approximate solutions to the fuzzy linear time-fractional equation in fluid dynamics. Through a series of concrete examples, including the fuzzy time-fractional Advection-Dispersion equation, the fuzzy time-fractional Navier-Stokes equation, and a special case of the fuzzy time-fractional Navier-Stokes equation, i.e., Couette flow, we demonstrate the applicability of the proposed method in addressing real-world problems. By providing a clear illustration of the solution steps involved, these examples facilitate a better understanding of the proposed approach. The practical relevance of the proposed method is emphasized through these examples, which further enhance the accuracy of the results obtained.

5.1 Fuzzy time-fractional advection-dispersion equation

The time-fractional Advection-Dispersion equation is a partial differential equation commonly used to describe the transport of solutes in groundwater flow. It is a powerful tool that has been applied to a wide range of environmental problems, including contaminant transport, sedimentation, and groundwater recharge. Despite its usefulness, accurately predicting the concentration of solutes or pollutants in real-world scenarios can be challenging due to the complex and uncertain nature of the systems under study.

To address this challenge, a fuzzy model has been proposed to account for the uncertainties and vagueness in environmental systems. This model can be used to represent the evolution of particle concentration over time and can be expressed mathematically using fuzzy logic. By incorporating fuzzy sets and fuzzy rules into the model, it becomes possible to account for the inherent uncertainties and variability in environmental systems, thereby improving the accuracy of predictions and enhancing our understanding of these complex systems. Specifically, the fuzzy mathematics model for the evolution of particle concentration is represented using the following format:

$$ {}^{ABC}_{gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\} \quad = \quad \mathcal{D}\frac{\partial^2\mathcal{G}(\mathfrak{x},\tau)}{\partial\mathfrak{x}^2} \oplus \nu\frac{\partial\mathcal{G}(\mathfrak{x},\tau)}{\partial\mathfrak{x}} \oplus \mathbb{k}(\mathfrak{x},\tau). \tag{14} $$

The initial concentration of particle is denoted by $\mathcal{G}(\mathfrak{x},0)$ and is expressed as a fuzzy function.

In this equation, $\mathcal{G}(\mathfrak{x},\tau)$ represents the concentration of the contaminant at a particular location and time, while $\mathcal{D}$ and ν are the dispersion coefficient and mean fluid velocity, respectively. The source term $\mathbb{k}(\mathfrak{x},\tau)$ represents the injection or withdrawal of contaminant at a specific location and time. The fuzzy Atangana-Baleanu time-fractional derivative of order $0<\alpha\leq 1$ is denoted by ${}^{ABC}_{gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\}$, and the fuzzy generalized Hukuhara partial derivative with respect to $\mathfrak{x}$ is represented by $\frac{\partial\mathcal{G}(\mathfrak{x},\tau)}{\partial\mathfrak{x}}$.

Now, we aim to expand the numerical method presented in Section 4 to obtain a $[i-ABC_{gH}]-$time differentiable solution of model (14). To achieve this, we begin by applying the fuzzy Laplace transform to Eq. (14) with respect to τ. Following the same approach outlined in Section 4, and using Remark 3.1, Theorem 3.4, and the initial condition $\mathcal{G}(\mathfrak{x},0)$, we obtain the following equation:

$$ \mathcal{G}(\mathfrak{x},\tau) \quad = \quad G(\mathfrak{x},\tau) \oplus \mathfrak{L}^{-1}_{\mathfrak{s}}\Big[\Big(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\aleph(\alpha)\mathfrak{s}^{\alpha}}\Big)\mathfrak{L}_{\tau}\Big[\mathcal{D}\frac{\partial^2\mathcal{G}(\mathfrak{x},\tau)}{\partial\mathfrak{x}^2} \oplus \nu\frac{\partial\mathcal{G}(\mathfrak{x},\tau)}{\partial\mathfrak{x}}\Big](\mathfrak{s})\Big], $$

where $G(\mathfrak{x},\tau)$ is defined as:

$$ G(\mathfrak{x},\tau) = \mathfrak{L}^{-1}_{\mathfrak{s}}\Big[\frac{1}{\mathfrak{s}}\mathcal{G}(\mathfrak{x},0) \oplus \Big(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\aleph(\alpha)\mathfrak{s}^{\alpha}}\Big)\mathfrak{L}_{\tau}\Big[\mathbb{k}(\mathfrak{x},\tau)\Big](\mathfrak{s})\Big]. $$

Next, we define $\mathcal{G}(\mathfrak{x},\tau)=\sum\limits_{n=0}^{\infty}\mathcal{G}_n(\mathfrak{x},\tau)$, where the unknown fuzzy functions $\mathcal{G}_n(\mathfrak{x},\tau)$ can be determined as follows:

$$ \begin{aligned} &\mathcal{G}_0(\mathfrak{x},\tau) = G(\mathfrak{x},\tau),\\ &\mathcal{G}_{n+1}(\mathfrak{x},\tau) = \mathfrak{L}^{-1}_{\mathfrak{s}}\Big[\Big(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\aleph(\alpha)\mathfrak{s}^{\alpha}}\Big)\mathfrak{L}_{\tau}\Big[\mathcal{D}\frac{\partial^2\mathcal{G}_n(\mathfrak{x},\tau)}{\partial\mathfrak{x}^2} \oplus \nu\frac{\partial\mathcal{G}_n(\mathfrak{x},\tau)}{\partial\mathfrak{x}}\Big](\mathfrak{s})\Big], \quad n=0,1,\cdots. \end{aligned} $$

To obtain a $[ii-ABC_{gH}]-$time differentiable solution of model (14), we can express it as a sum of an infinite series of functions, where each term $\mathcal{G}_n(\mathfrak{x},\tau)$ is obtained by iteratively applying a set of operators. Specifically, we have:

$$ \mathcal{G}(\mathfrak{x},\tau)=\sum_{n=0}^{\infty}\mathcal{G}_n(\mathfrak{x},\tau), $$

where the functions $\mathcal{G}_n(\mathfrak{x},\tau)$ are defined recursively as follows:

$$ \begin{aligned} &\mathcal{G}_0(\mathfrak{x},\tau) = \mathfrak{L}^{-1}_{\mathfrak{s}}\big[\frac{1}{\mathfrak{s}}\mathcal{G}(\mathfrak{x},0) \ominus (-1)\Big(\frac{1}{\aleph(\alpha)}(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}})\Big)\mathfrak{L}_{\tau}[\mathbb{k}(\mathfrak{x},\tau)](\mathfrak{s})\Big],\\ &\mathcal{G}_{n+1}(\mathfrak{x},\tau) = \ominus(-1)\mathfrak{L}^{-1}_{\mathfrak{s}}\Big[\Big(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\aleph(\alpha)\mathfrak{s}^{\alpha}}\Big)\mathfrak{L}_{\tau}\Big[\mathcal{D}\frac{\partial^2\mathcal{G}_n(\mathfrak{x},\tau)}{\partial\mathfrak{x}^2} \oplus \nu\frac{\partial\mathcal{G}_n(\mathfrak{x},\tau)}{\partial\mathfrak{x}}\Big](\mathfrak{s})\Big], \quad n=0,1,\cdots. \end{aligned} $$

Example 5.1. *Imagine a uniform aquifer with a contaminant plume that is 100 meters long and moving at an average speed of* 0.1 *meters per second. The dispersion coefficient, a measure of how much the plume spreads out as it moves,*

is 0.001 *square meters per second. At the beginning (time* $\tau = 0$*), the concentration of the contaminant at the origin of the plume is described by the function* $\mathcal{G}(\mathfrak{x}, 0) = (3\mathfrak{x} + 12, 5\mathfrak{x} + 20, 9\mathfrak{x} + 36)$*. To model this situation, we use the following fuzzy time-fractional Advection-Dispersion equation:*

$$\begin{aligned} {}^{ABC}_{gH}\mathcal{D}^{\frac{1}{2}}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\} &= 0.001\frac{\partial^2\mathcal{G}(\mathfrak{x},\tau)}{\partial\mathfrak{x}^2} \oplus 0.1\frac{\partial\mathcal{G}(\mathfrak{x},\tau)}{\partial\mathfrak{x}} \oplus \mathbb{k}(\mathfrak{x},\tau). \end{aligned}$$

Here,

$$\begin{aligned} \mathbb{k}(\mathfrak{x},\tau) = \Big(\frac{3(1+\sqrt{\pi})}{2\pi}, \frac{5(1+\sqrt{\pi})}{2\pi}, \frac{9(1+\sqrt{\pi})}{2\pi}\Big) \oplus \\ \Big((4+\mathfrak{x})(\sqrt{\pi}e^{\tau} - 2\sqrt{\tau} + \sqrt{\pi}e^{\tau}(2\tau-1))Erfc[\sqrt{\tau}]\Big) \ominus \Big(0.3e^{\tau}, 0.5e^{\tau}, 0.9e^{\tau}\Big). \end{aligned}$$

We can use the numerical methods discussed earlier to solve this equation. This can be expressed as:

$$\begin{aligned} &\mathfrak{L}_{\tau}[\mathbb{k}(\mathfrak{x},\tau)](\mathfrak{s}) = \Big(\frac{3(1+\sqrt{\pi})}{2\pi}, \frac{5(1+\sqrt{\pi})}{2\pi}, \frac{9(1+\sqrt{\pi})}{2\pi}\Big). \\ &\mathfrak{L}_{\tau}\Big[(4+\mathfrak{x})(\sqrt{\pi}e^{\tau} - 2\sqrt{\tau} + \sqrt{\pi}e^{\tau}(2\tau-1)Erfc[\sqrt{\tau}])\Big] \ominus \Big(3L_{\tau}[e^{\tau}](\mathfrak{s}), 5L_{\tau}[e^{\tau}](\mathfrak{s}), 9L_{\tau}[e^{\tau}](\mathfrak{s})\Big) \\ = \quad &\Big(\frac{18.77 - 0.3\sqrt{s} - 0.3s + 4.69\mathfrak{x}}{(\sqrt{s}-1)(\sqrt{s}+1)^2\sqrt{s}}, \frac{31.28 - 0.5\sqrt{s} - 0.5s + 7.82\mathfrak{x}}{(\sqrt{s}-1)(\sqrt{s}+1)^2\sqrt{s}}, \frac{56.31 - 0.9\sqrt{s} - 0.9s + 14.07\mathfrak{x}}{(\sqrt{s}-1)(\sqrt{s}+1)^2\sqrt{s}}\Big). \end{aligned}$$

Subsequently, we can obtain:

$$\begin{aligned} &G(\mathfrak{x},\tau) = \mathfrak{L}^{-1}_{\mathfrak{s}}\Big[\frac{1}{\mathfrak{s}}\mathcal{G}(\mathfrak{x},0) \oplus \Big(\frac{1}{\aleph(\alpha)}(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}})\Big)\mathfrak{L}_{\tau}[\mathbb{k}(\mathfrak{x},\tau)](\mathfrak{s})\Big] \\ = \quad &\Big(e^{\tau}(11.81 + 3\mathfrak{x} - 0.19Erf[\sqrt{\tau}]), e^{\tau}(19.68 + 5\mathfrak{x} - 0.31Erf[\sqrt{\tau}]), e^{\tau}(35.42 + 9\mathfrak{x} - 0.57Erf[\sqrt{\tau}])\Big). \end{aligned}$$

By setting:

$$\mathcal{G}_0(\mathfrak{x},\tau) = \Big(e^{\tau}(11.81 + 3\mathfrak{x} - 0.19Erf[\sqrt{\tau}]), e^{\tau}(19.68 + 5\mathfrak{x} - 0.31Erf[\sqrt{\tau}]), e^{\tau}(35.42 + 9\mathfrak{x} - 0.57Erf[\sqrt{\tau}])\Big),$$

and then using:

$$\begin{aligned} \mathcal{G}_1(\mathfrak{x},\tau) &= \mathfrak{L}^{-1}_{\mathfrak{s}}\Big[\Big(\frac{\sqrt{\pi}(1+\sqrt{s})}{\sqrt{s}(1+\sqrt{\pi})}\Big)\mathfrak{L}_{\tau}\Big[0.001\frac{\partial^2\mathcal{G}_0(\mathfrak{x},\tau)}{\partial\mathfrak{x}^2} \oplus 0.1\frac{\partial\mathcal{G}_0(\mathfrak{x},\tau)}{\partial\mathfrak{x}}\Big](\mathfrak{s})\Big] \\ &= \Big(0.19e^{\tau}(1 + Erf[\sqrt{\tau}]), 0.31e^{\tau}(1 + Erf[\sqrt{\tau}]), 0.57e^{\tau}(1 + Erf[\sqrt{\tau}])\Big). \end{aligned}$$

And

$$\mathcal{G}_2(\mathfrak{x},\tau) = \mathfrak{L}^{-1}_{\mathfrak{s}}\Big[\Big(\frac{\sqrt{\pi}(1+\sqrt{s})}{\sqrt{s}(1+\sqrt{\pi})}\Big)\mathfrak{L}_{\tau}\Big[0.001\frac{\partial^2\mathcal{G}_1(\mathfrak{x},\tau)}{\partial\mathfrak{x}^2} \oplus 0.1\frac{\partial\mathcal{G}_1(\mathfrak{x},\tau)}{\partial\mathfrak{x}}\Big](\mathfrak{s})\Big] = \Big(0,0,0\Big).$$

Therefore, we can express the exact $[i - ABC_{gH}]-$*time differentiable solution of* $\mathcal{G}(\mathfrak{x},\tau)$ *as:*

$$\begin{aligned} &\mathcal{G}(\mathfrak{x},\tau) = \mathcal{G}_0(\mathfrak{x},\tau) \oplus \mathcal{G}_1(\mathfrak{x},\tau) \oplus \mathcal{G}_2(\mathfrak{x},\tau) \\ = \quad &\Big(3e^{\tau}(4 + \mathfrak{x} + 6.47\times10^{-17}Erf[\sqrt{\tau}]), 4e^{\tau}(4 + \mathfrak{x} + 2.22\times10^{-17}Erf[\sqrt{\tau}]), \\ &\qquad 9e^{\tau}(4 + \mathfrak{x} - 1.48\times10^{-17}Erf[\sqrt{\tau}])\Big). \end{aligned}$$

In Figure 2, we can see the r*-cuts of* $\mathcal{G}(\mathfrak{x},\tau)$ *and* ${}^{ABC}_{gH}\mathcal{D}^{\frac{1}{2}}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\}$ *for* $\mathrm{r} = 0.1$*. Notably, the upper and lower cuts (Green and Orange) remain constant, indicating that* $\mathcal{G}(\mathfrak{x},\tau)$ *is a* $[i - ABC_{gH}]-$ *partial differentiable function with respect to* τ*.*

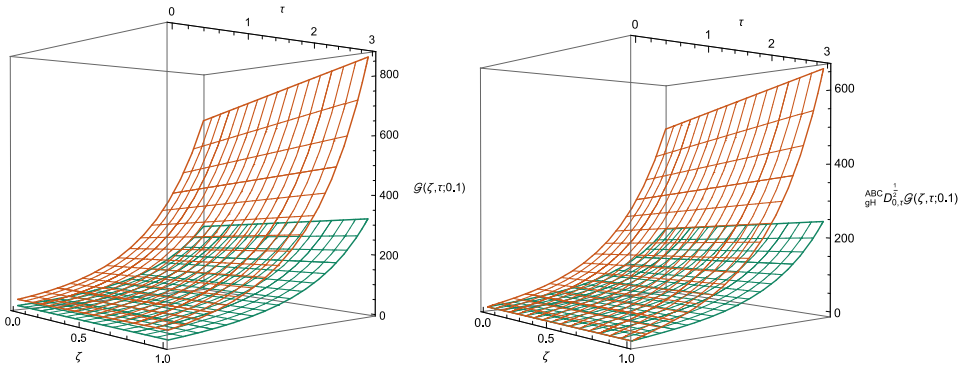

Figure 2: Plots of $\mathcal{G}(\mathfrak{x},\tau;\mathrm{r})$ and ${}^{ABC}_{gH}\mathcal{D}^{\frac{1}{2}}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau;\mathrm{r})\}$ for Example 5.1 in $\mathrm{r}=0.1$.

5.2 Fuzzy time-fractional Navier-Stokes equation

In this section, our attention will be directed towards the motion of a viscous fluid in a tube, where the flow is confined to a single dimension, usually along the length of the tube. The Navier-Stokes equations govern the behavior of this type of flow, dictating the interplay between pressure, velocity, and viscosity. External forces, such as pressure gradients that drive the flow and the tube walls' resistance, also affect the behavior of fluids moving through tubes.

To demonstrate this, let's imagine a long cylindrical pipe filled with fluid initially at rest. If a constant pressure gradient is suddenly applied along the pipe's axis, the fluid will start to move. The flow is considered to be fully developed and axially symmetric. The fluid's motion in this scenario is governed by the following equation, which takes into account the fluid's properties and the external forces acting on it:

$$\rho\ {}^{ABC}_{gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\} \quad = \quad P\oplus\mu\left(\frac{\partial^2\mathcal{G}(\mathfrak{x},\tau)}{\partial\mathfrak{x}^2}\oplus\frac{1}{\mathfrak{x}}\frac{\partial\mathcal{G}(\mathfrak{x},\tau)}{\partial\mathfrak{x}}\right)\oplus\mathbb{k}(\mathfrak{x},\tau). \tag{15}$$

Here, $\mathcal{G}(\mathfrak{x},\tau)$ denotes the fluid's velocity in the $\mathfrak{x}$ direction within the tube. The fluid's density is given by ρ, and its dynamic viscosity by μ. Additionally, $\mathbb{k}(\mathfrak{x},\tau)$ considers any external forces acting on the fluid. The fluid's pressure is denoted by $\mathfrak{p}$, while $P=-\frac{\partial\mathfrak{p}}{\partial\mathfrak{x}}$ is a fuzzy constant term.

To begin with, let's note that the fluid in question is initially at rest. Therefore, its velocity in the $\mathfrak{x}$ direction is zero, which can be expressed as $\mathcal{G}(\mathfrak{x},0)=(0,0,0)$.

Moving on to Section 4, it presents a method that can be utilized to obtain a $[i-ABC_{gH}]-$time differentiable solution for the fuzzy time-fractional Navier-Stokes equation (15). The solution can be expressed as an infinite series of functions, i.e., $\mathcal{G}(\mathfrak{x},\tau)=\sum\limits_{n=0}^{\infty}\mathcal{G}_n(\mathfrak{x},\tau)$, where

$$\begin{aligned}
\mathcal{G}_0(\mathfrak{x},\tau) &= \mathfrak{L}^{-1}_{\mathfrak{s}}\left[P\Big(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\rho\aleph(\alpha)\mathfrak{s}^{\alpha+1}}\Big)\oplus\Big(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\rho\aleph(\alpha)\mathfrak{s}^{\alpha}}\Big)\mathfrak{L}_{\tau}\Big[\mathbb{k}(\mathfrak{x},\tau)\Big](\mathfrak{s})\right],\\
\mathcal{G}_{n+1}(\mathfrak{x},\tau) &= \mathfrak{L}^{-1}_{\mathfrak{s}}\left[\Big(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\rho\aleph(\alpha)\mathfrak{s}^{\alpha}}\Big)\mathfrak{L}_{\tau}\Big[\mu\left(\frac{\partial^2\mathcal{G}_n(\mathfrak{x},\tau)}{\partial\mathfrak{x}^2}\oplus\frac{1}{\mathfrak{x}}\frac{\partial\mathcal{G}_n(\mathfrak{x},\tau)}{\partial\mathfrak{x}}\right)\Big](\mathfrak{s})\right],\quad n=0,1,\cdots.
\end{aligned}$$

Furthermore, assuming that $\mathcal{G}(\mathfrak{x},\tau)$ is a fuzzy function that is $[ii-ABC_{gH}]-$time differentiable, then the expression can be written as $\mathcal{G}(\mathfrak{x},\tau)=\mathcal{G}_0(\mathfrak{x},\tau)\oplus\sum\limits_{n=0}^{\infty}\mathcal{G}_{n+1}(\mathfrak{x},\tau)$, where $\mathcal{G}_0(\mathfrak{x},\tau)$ is a known function and can be calculated using the following formula:

$$\mathcal{G}_0(\mathfrak{x},\tau)=\ominus(-1)\mathfrak{L}^{-1}_{\mathfrak{s}}\left[P\Big(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\rho\aleph(\alpha)\mathfrak{s}^{\alpha+1}}\Big)\oplus\Big(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\rho\aleph(\alpha)\mathfrak{s}^{\alpha}}\Big)\mathfrak{L}_{\tau}\Big[\mathbb{k}(\mathfrak{x},\tau)\Big](\mathfrak{s})\right].$$

Additionally, $\mathcal{G}_{n+1}(\mathfrak{x},\tau)$ for $n=0,1,\cdots$ can be calculated recursively using the following formula:

$$\mathcal{G}_{n+1}(\mathfrak{x},\tau)=\ominus(-1)\mathfrak{L}^{-1}_{\mathfrak{s}}\left[\Big(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\rho\aleph(\alpha)\mathfrak{s}^{\alpha}}\Big)\mathfrak{L}_{\tau}\Big[\mu\left(\frac{\partial^2\mathcal{G}_n(\mathfrak{x},\tau)}{\partial\mathfrak{x}^2}\oplus\frac{1}{\mathfrak{x}}\frac{\partial\mathcal{G}_n(\mathfrak{x},\tau)}{\partial\mathfrak{x}}\right)\Big](\mathfrak{s})\right].$$

For $n = 0, 1, \cdots$.

Couette flow is a specific type of fluid flow that occurs between two parallel plates, where one plate remains stationary while the other moves with a constant velocity in the same plane as the plates. The flow is generated by the motion of the plate, which causes shear forces on the fluid. The behavior of this flow can be described mathematically using a special type of Navier-Stokes equations. In order to model the initial development of the velocity profile in Couette flow under the influence of external forces, the following equation is commonly used

$$ {}^{ABC}_{gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\} = \nu \frac{\partial^2 \mathcal{G}(\mathfrak{x},\tau)}{\partial \mathfrak{x}^2} \oplus \mathbb{k}(\mathfrak{x},\tau). $$

The diffusion coefficient, ν, controls the rate of velocity profile spread due to diffusion, and the external force term, $\mathbb{k}(\mathfrak{x},\tau)$, varies depending on the external force acting on the fluid. These external forces can be any kind of force, such as gravitational or electromagnetic, that affects the fluid flow. The right-hand side of the equation is split into two terms: the first term represents the diffusion of the fluid, while the second term represents the external force acting on the fluid.

The typical approach to setting up initial conditions for Coutte flow involves prescribing a linearly increasing velocity profile as a function of the distance from the stationary plate. Specifically, at the initial time $\tau = 0$, the velocity profile is defined as:

$$ \mathcal{G}(\mathfrak{x},0) = \frac{v_{max}}{h}\mathfrak{x}. $$

Here, v_{max} corresponds to the maximum fluid velocity, which is observed at the moving plate, h denotes the distance between the plates, and $\mathfrak{x}$ represents the distance from the stationary plate.

Example 5.2. *Consider a fluid between two parallel plates with a distance of $h = 0.1$ meters. The bottom plate is stationary, while the top plate moves at a constant velocity of $v_{max} = \Big(3.5, 4.2, 5.1\Big)$ m/s in the positive $\mathfrak{x}$ direction. The kinematic viscosity of the fluid is $\nu = 0.01$ m^2/s. We aim to model the velocity profile development in the fluid over time.*

Initially, the velocity profile of the fluid is given by $\mathcal{G}(\mathfrak{x},0) = \frac{v_{max}}{h}\mathfrak{x} = \frac{\Big(3.5,4.2,5.1\Big)}{0.1}\mathfrak{x} = (35\mathfrak{x}, 42\mathfrak{x}, 51\mathfrak{x})$.
The acceleration of the fluid caused by the moving top plate can be expressed as

$$ \begin{aligned} \mathbb{k}(\mathfrak{x},\tau) &= \Big(\frac{-51\times 2^{-\tau}}{1+\log[2]}, \frac{-42\times 2^{-\tau}}{1+\log[2]}, \frac{-35\times 2^{-\tau}}{1+\log[2]}\Big) \oplus \\ &\quad \Big(\sqrt{\frac{\log[2]\mathfrak{x}}{\pi}}(1+\sqrt{\pi})(Erfi[\sqrt{\tau}\sqrt{\log[2]}] + (-1+(2e)^{\tau}Erfc[\sqrt{\tau}])\sqrt{\log[2]}\Big) \quad m/\mathfrak{s}^2. \end{aligned} $$

To model this situation, we can use the fuzzy diffusion equation:

$$ {}^{ABC}_{gH}\mathcal{D}^{\alpha}_{0,\tau}\{\mathcal{G}(\mathfrak{x},\tau)\} = 0.01 \frac{\partial^2 \mathcal{G}(\mathfrak{x},\tau)}{\partial \mathfrak{x}^2} \oplus \mathbb{k}(\mathfrak{x},\tau). $$

We will utilize the same technique as earlier to obtain an approximation. The fuzzy Laplace transform of the acceleration function $\mathbb{k}(\mathfrak{x},\tau)$ is expressed as follows:

$$ \mathfrak{L}_{0,\tau}[\mathbb{k}(\mathfrak{x},\tau)](\mathfrak{s}) = \Big(\frac{-51\mathfrak{x}(1+\sqrt{5})\log[2]}{\sqrt{\pi}(1+\sqrt{s})\sqrt{s}(s+\log[2])}, \frac{-42\mathfrak{x}(1+\sqrt{5})\log[2]}{\sqrt{\pi}(1+\sqrt{s})\sqrt{s}(s+\log[2])}, \frac{-35\mathfrak{x}(1+\sqrt{5})\log[2]}{\sqrt{\pi}(1+\sqrt{s})\sqrt{s}(s+\log[2])}\Big). $$

Subsequently, we obtain:

$$ \begin{aligned} &\mathcal{G}_0(\mathfrak{x},\tau) = \mathfrak{L}^{-1}_{\mathfrak{s}}\Big[\frac{1}{\mathfrak{s}}\mathcal{G}(\mathfrak{x},0) \ominus (-1)\Big(\frac{1}{\aleph(\alpha)}\Big(\frac{\mathfrak{s}^{\alpha}(1-\alpha)+\alpha}{\mathfrak{s}^{\alpha}}\Big)\Big)\mathfrak{L}_{\tau}[\mathbb{k}(\mathfrak{x},\tau)](\mathfrak{s})\Big] \\ &= \Big(\frac{35\mathfrak{x}}{s}, \frac{42\mathfrak{x}}{s}, \frac{51\mathfrak{x}}{s}\Big) \\ &\quad \ominus(-1)\Big(\frac{-51\mathfrak{x}(1+\sqrt{5})\log[2]}{\sqrt{\pi}(1+\sqrt{s})\sqrt{s}(s+\log[2])}, \frac{-42\mathfrak{x}(1+\sqrt{5})\log[2]}{\sqrt{\pi}(1+\sqrt{s})\sqrt{s}(s+\log[2])}, \frac{-35\mathfrak{x}(1+\sqrt{5})\log[2]}{\sqrt{\pi}(1+\sqrt{s})\sqrt{s}(s+\log[2])}\Big) \\ &= \Big(35\mathfrak{x}2^{-\tau}, 21\mathfrak{x}2^{1-\tau}, 51\mathfrak{x}2^{-\tau}\Big), \end{aligned} $$

and

$$\mathcal{G}_1(\mathfrak{x},\tau) = \ominus(-1)\mathfrak{L}_{\mathfrak{s}}^{-1}\Big[\Big(\frac{0.01\sqrt{\pi}(1+\sqrt{s})}{\sqrt{s}(1+\sqrt{\pi})}\Big)\mathfrak{L}_{\tau}\Big[\frac{\partial^2\mathcal{G}_0(\mathfrak{x},\tau)}{\partial\mathfrak{x}^2}\Big](\mathfrak{s})\Big] = \Big(0,0,0\Big).$$

Finally

$$\begin{aligned}\mathcal{G}(\mathfrak{x},\tau) &= \mathcal{G}_0(\mathfrak{x},\tau)\oplus\mathcal{G}_1(\mathfrak{x},\tau)\oplus\mathcal{G}_1(\mathfrak{x},\tau)\oplus\mathcal{G}_3(\mathfrak{x},\tau)\oplus\cdots\\ &= \Big(35\mathfrak{x}\times 2^{-\tau}, 21\mathfrak{x}\times 2^{1-\tau}, 51\mathfrak{x}\times 2^{-\tau}\Big).\end{aligned}$$

Which is the exact fuzzy solution.

Figure 3 displays the r*-cut of the solution, along with* ${}_{gH}^{ABC}\mathcal{D}_{0,\tau}^{\frac{1}{2}}\{\mathcal{G}(\mathfrak{x},\tau)\}$*. The figure clearly illustrates that* $\mathcal{G}(\mathfrak{x},\tau)$ *is a* $[ii-ABC_{gH}]-$ *partial differentiable function with respect to* τ *as evidenced by the altered positions of the lower cut (Green) and upper cut (Orange).*

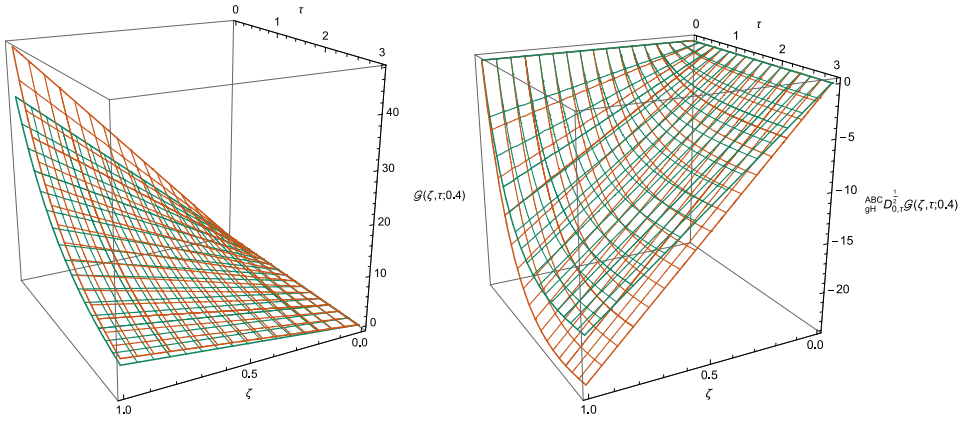

Figure 3: Plots of $\mathcal{G}(\mathfrak{x},\tau;\mathrm{r})$ and ${}_{gH}^{ABC}\mathcal{D}_{0,\tau}^{\frac{1}{2}}\{\mathcal{G}(\mathfrak{x},\tau;\mathrm{r})\}$ for Example 5.2 in $\mathrm{r}=0.4$.

6 Conclusion

In summary, our study has introduced the fuzzy Atangana-Baleanu time-fractional derivative with non-singular kernels as a promising tool for analyzing fractional differential equations within fuzzy environments. We have illustrated its effectiveness, particularly in the realm of fluid dynamics, where uncertain or imprecise data often prevail. Moreover, we have developed a numerical technique, the fuzzy Laplace transform iterative method, to approximate solutions for fuzzy linear time-fractional equations, showcasing its application through various examples such as the Advection-Dispersion equation, fuzzy time-fractional Navier-Stokes equation, and Couette flow. However, our method has limitations, especially in handling systems with extreme fuzziness or high nonlinearity. Moreover, the numerical technique may struggle with accurately representing highly nonlinear systems or those with discontinuities. Future research should focus on addressing these limitations and expanding the applicability of our approach to a wider range of fluid dynamics problems. In summary, our study contributes to the understanding and application of fuzzy calculus and time-fractional differential equations, but ongoing research is essential to refine and improve its effectiveness in tackling real-world fluid dynamics challenges.

Declarations

- Funding

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- Conflict of interest

  The authors declare that they have no conflict of interest.

- Ethics approval

  This article does not contain any studies with human participants or animals performed by any of the authors.

- Consent to participate

  It is hereby declared that all authors have equal authorship contributions to the article.

- Authors' contributions

  The idea was proposed by R. Ezzati and N. Mikaeilvand. H. Hashemi developed the theory and performed the computations. M. Nazari wrote the manuscript. R.Ezzati supervised the project. All authors discussed the results and contributed to the final manuscript.

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