

ON FUZZY CONVEX LATTICE-ORDERED SUBGROUPS

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ABSTRACT. In this paper, the concept of fuzzy convex subgroup (resp. fuzzy convex lattice-ordered subgroup) of an ordered group (resp. lattice-ordered group) is introduced and some properties, characterizations and related results are given. Also, the fuzzy convex subgroup (resp. fuzzy convex lattice-ordered subgroup) generated by a fuzzy subgroup (resp. fuzzy subsemigroup) is characterized. Furthermore, the Fundamental Homomorphism Theorem is established. Finally, it is proved that the class of all fuzzy convex lattice-ordered subgroups of a lattice-ordered group G forms a complete Heyting sublattice of the lattice of fuzzy subgroups of G .

1. Introduction

After Zadeh [13] introduced the concept of fuzzy set, Rosenfeld [9] applied it to group theory and developed the theory of fuzzy subgroups. Next, many authors worked on fuzzy group theory (see [2, 3, 6, 7]).

Also, many authors have worked on fuzzy lattice theory. They introduced the concept of fuzzy sublattice, fuzzy ideal, fuzzy prime ideal, in a lattice and gave some interesting results (see [1, 5, 11]).

The partially ordered algebraic systems play an important role in algebra. Some important concepts in partially ordered systems are lattice-ordered groups, lattice-ordered rings and vector lattices. These concepts play a major role in the study of rings of continuous functions, and functional analysis, besides being a major branch in algebra. So, it seems that the study of the fuzzy set theory of lattice-ordered algebraic structures, in order to get a better insight in fuzzy algebra, is necessary. To this end, Saibaba [10] considered L -fuzzy lattice-ordered groups, where L is a complete lattice. He introduced the notions of L -fuzzy lattice-ordered subgroup and L -fuzzy lattice-ordered ideal of a lattice-ordered group and constructed the quotient lattice-ordered groups using L -fuzzy lattice-ordered ideals. Particularly, he obtained the homomorphism theorem for lattice-ordered groups induced by L -fuzzy lattice-ordered ideals.

Convex subgroups play an important role in the study of compatible orders and positive cones of an ordered group, especially in characterizing the lattice-ordered subgroups generated by a subsemigroup. So, it seems that the study of this notion from fuzzy point of view would be a useful tool to extend their properties in the case of fuzzy setting. Thus, in this paper, the concept of fuzzy convex subgroup of an ordered group is defined, and some important results are given. In particular,

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an equivalent condition for a subset of a group G/λ , where λ is a fuzzy normal subgroup of G , to be the positive cone of a compatible order on G/λ is given, and using this the order on G/λ is described. It is prove that any fuzzy convex subsemigroup λ of the positive cone of lattice-ordered group G generates a fuzzy convex lattice-ordered subgroup of G which is, of course, the least fuzzy convex lattice-ordered subgroup of G that contains λ . Then, using this result, it is shown that the set of all fuzzy convex subsemigroups of the positive cone of G and the set of all fuzzy convex lattice-ordered subgroups of G are order isomorphism. Finally, the lattice of all fuzzy convex lattice-ordered subgroups of a lattice-ordered group is studied and some related results are obtained.

2. Preliminaries

In this section we give some definitions and results from the literature. For more details, we refer the reader to the references [1, 4, 8, 10].

Definition 2.1. Let $(E; \leq)$ be an ordered set and $x, y \in E$ with $x \leq y$.

- An *interval* in E is a subset $\{z \in E : x \leq z \leq y\}$, denoted by $[x, y]$.
- A nonempty subset A of E is called *convex* if $[a, b] \subseteq A$, for all $a, b \in A$ with $a \leq b$.

Definition 2.2. A subset D of an ordered set $(E; \leq)$ is called a *down-set* if $y \leq x$ and $x \in D$ imply $y \in D$. Down-set D is called a *principal down-set* if there exists $x \in D$ such that $D = \{y \in E : y \leq x\}$ which is denoted by x^\downarrow . Similarly, a principal upper-set is defined as a subset $x^\uparrow = \{y \in E : y \geq x\}$.

Example 2.3. Obviously \mathbb{R} under the natural ordering is an ordered set in which every interval $[a, b]$ is a convex subset. Also, every $(-\infty, a]$ and $[a, \infty)$ (with $a \in \mathbb{R}$) is a principal down-set and a principal upper set in \mathbb{R} , respectively.

Definition 2.4. Let S be a set endowed with a law of composition that is written multiplicatively. By a *compatible order* on S , we mean an order \leq with respect to which all translations $y \mapsto xy$ and $y \mapsto yx$ are isotone. That is $x \leq y$ implies that $xz \leq yz$ and $zx \leq zy$.

Definition 2.5. By an *ordered group* we mean a group on which a compatible order is defined, that is, a structure $(G; \cdot, e, \leq)$ such that $(G; \cdot, e)$ is a group and \leq is a compatible order on G .

Example 2.6. Under addition and the natural ordering, each of \mathbb{Z} , \mathbb{Q} , and \mathbb{R} is an ordered group.

Definition 2.7. Let $(G; \cdot, e, \leq)$ be an ordered group and $x \in G$. x is called *positive* if $x \geq e$, and is called *negative* if $x \leq e$. The set of all positive elements of G is called the *positive cone* and the set of all negative elements of G is called the *negative cone* of G , denoted by $P(G)$ and $N(G)$, respectively.

Example 2.8. In ordered additive group \mathbb{Z} , every positive integer is positive and every negative integer is negative.

Obviously, in an ordered group G , $N(G) = e^\downarrow$ and $P(G) = e^\uparrow$. Moreover, it is easy to see that for subgroup H of G , $P(H) = H \cap P(G)$.

Definition 2.9. A *lattice-ordered group* is an ordered group (G, \cdot, e, \leq) such that (G, \leq) is a lattice.

Example 2.10. One can see that \mathbb{Z} , \mathbb{Q} and \mathbb{R} are lattice-ordered group under the natural ordering .

In any lattice-ordered group we have $(a \vee b)x = ax \vee bx$ and $a \wedge b = b(a \vee b)^{-1}a$.

Definition 2.11. For every $x \in G$, the *positive part* of x and dually, the *negative part* of x is defined by $x_+ = x \vee e \in P(G)$ and $x_- = x \wedge e \in N(G)$, respectively.

Theorem 2.12. In any lattice-ordered group G , the following hold:

- (1) $(x_+)^{-1} = (x^{-1})_-$ and $(x_-)^{-1} = (x^{-1})_+$,
- (2) $x \vee y = (yx^{-1})_+x$ and $x \wedge y = x(x^{-1}y)_-$,
- (3) $x = x_+x_- = x_-x_+$,
- (4) $x \leq y$ if and only if $x_+ \leq y_+$ and $x_- \leq y_-$.

Definition 2.13. Let G be a group. A fuzzy subset λ of G is said to be a *fuzzy subgroup* if

- (1) $\lambda(xy) \geq \lambda(x) \wedge \lambda(y)$,
- (2) $\lambda(x^{-1}) \geq \lambda(x)$.

Any fuzzy subset of a group which satisfies the condition (1) of Definition 2.13 is called a *fuzzy subsemigroup*.

Proposition 2.14. Let G be a group. Every fuzzy subgroup λ of G satisfies the following:

- (1) $(\forall x \in G) \lambda(e) \geq \lambda(x)$,
- (2) $(\forall x \in G) (\forall n \in \mathbb{N}) \lambda(x^n) \geq \lambda(x)$,

Proposition 2.15. Let G be a group. A fuzzy subset λ of G is a fuzzy subgroup if and only if every non-empty level subset λ_t (with $t \in [0, 1]$) is a subgroup of G .

Definition 2.16. Let G be a group. A fuzzy subgroup λ of G is called a *fuzzy normal subgroup* if it satisfies the following equivalent conditions:

- (1) $\lambda(xy) = \lambda(yx)$,
- (2) $\lambda(xyx^{-1}) = \lambda(y)$,
- (3) $\lambda(xyx^{-1}) \geq \lambda(y)$,
- (4) $\lambda(xyx^{-1}) \leq \lambda(y)$,

Example 2.17. Consider the ordered additive group \mathbb{Z} , and define a fuzzy subset μ of \mathbb{Z} by

$$\mu(n) = \begin{cases} 1/2 & ; n = 2k, k \in \mathbb{Z} \setminus \{0\}, \\ 1 & ; n = 0, \\ 0 & ; \text{otherwise.} \end{cases}$$

Then μ is a fuzzy normal subgroup of \mathbb{Z} .

We recall that for fuzzy subgroup λ of G , the fuzzy subset $x\lambda$ of G defined by $(x\lambda)(y) = \lambda(x^{-1}y)$ is called a fuzzy coset of λ . Denote the set of all fuzzy cosets of λ by G/λ . If λ is a fuzzy normal subgroup of G , G/λ together with the operations induced by those of G form a group.

Theorem 2.18. *Let G be a group and λ a fuzzy normal subgroup of G .*

- (1) $\lambda_{\lambda(e)}$ is a normal subgroup of G .
- (2) $x\lambda = y\lambda$ if and only if $\lambda(x) = \lambda(y)$.

Definition 2.19. A fuzzy subset ν of a lattice $(L; \vee, \wedge)$ is said to be a *fuzzy sublattice* if

- (1) $\nu(x \vee y) \geq \nu(x) \wedge \nu(y)$,
- (2) $\nu(x \wedge y) \geq \nu(x) \wedge \nu(y)$.

Example 2.20. Let $L = \{0, a, b, c, d, e\}$ be a lattice with the following Hasse diagram (Figure 1). Define a fuzzy subset μ of L by

$$\mu(0) = 1, \mu(a) = \mu(b) = \mu(c) = \mu(d) = 1/2, \mu(e) = 0.$$

It is not difficult to see that μ is a fuzzy sublattice of L .

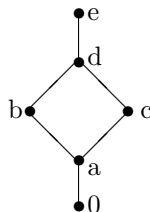


FIGURE 1. The Hasse Diagram of L

Proposition 2.21. *A fuzzy subset μ of a lattice $(L; \vee, \wedge)$ is a fuzzy sublattice if and only if every nonempty level subset μ_t (with $t \in [0, 1]$) is a sublattice of L .*

Definition 2.22. A fuzzy subset λ of a lattice-ordered group G is said to be a *fuzzy lattice-ordered subgroup* or briefly, *fuzzy ℓ -subgroup* if

- (1) $\lambda(xy^{-1}) \geq \lambda(x) \wedge \lambda(y)$,
- (2) $\lambda(x \vee y) \wedge \lambda(x \wedge y) \geq \lambda(x) \wedge \lambda(y)$.

One can easily check that the fuzzy subset defined in Example 2.17 is a fuzzy ℓ -subgroup of \mathbb{Z} .

As usual, in this paper, we write xy instead of $x \cdot y$.

3. Fuzzy Convex Subgroups

We start this section by an important result about fuzzy subgroups of an ordered group. This result tell us which subsets of a group G/λ , where λ is a fuzzy normal subgroup of G , would be the positive cone with respect to a compatible order on G/λ .

In this section, G will denote an ordered group.

For subset X of G , let $X^{-1} = \{x^{-1} : x \in X\}$.

Theorem 3.1. *Let λ be a fuzzy normal subgroup of G . A subset P of G/λ is the positive cone of a compatible order on G/λ if and only if*

- (1) $P \cap P^{-1} = \{\lambda\}$,
- (2) $P^2 = P$,
- (3) $(\forall x \in G) (x\lambda)P(x^{-1}\lambda) = P$.

Moreover, $P \cup P^{-1} = G/\lambda$ if and only if this order is total.

Proof. (\Rightarrow) Let \leq be a compatible order on G/λ and $P(G/\lambda)$ be the associated positive cone.

(1) If $x\lambda \in P(G/\lambda) \cap P(G/\lambda)^{-1}$, then $x\lambda \geq \lambda$ and $x\lambda = y^{-1}\lambda$, where $y\lambda \geq \lambda$. Then, $x\lambda \leq \lambda$ and $x\lambda \geq \lambda$ whence $x\lambda = \lambda$, i.e., $P(G/\lambda) \cap P(G/\lambda)^{-1} = \{\lambda\}$ and so (1) holds.

(2) Let $x\lambda, y\lambda \in P(G/\lambda)$. Then, $x\lambda \geq \lambda$ and $y\lambda \geq \lambda$ and so $xy\lambda \geq \lambda$. This implies that $P(G/\lambda)^2 \subseteq P(G/\lambda)$. The converse inclusion is obviously hold.

(3) Let $x \in G$ and $y\lambda \in P(G/\lambda)$. Then, $y\lambda \geq \lambda$ and so

$$(x\lambda)(y\lambda)(x\lambda)^{-1} \geq (x\lambda)\lambda(x\lambda)^{-1} = \lambda$$

whence $(x\lambda)P(G/\lambda)(x\lambda)^{-1} \subseteq P(G/\lambda)$. Replacing x by x^{-1} , the converse inclusion obtained.

(\Leftarrow) Suppose that P is a subset of G/λ that satisfies the conditions (1)-(3). Define a relation \leq_λ on G/λ by

$$x\lambda \leq_\lambda y\lambda \Leftrightarrow yx^{-1}\lambda \in P.$$

We show that \leq_λ is a compatible order on G/λ . Obviously, \leq_λ is reflexive. Let $x\lambda \leq_\lambda y\lambda$ and $y\lambda \leq_\lambda x\lambda$. Then, $yx^{-1}\lambda \in P$ and $xy^{-1}\lambda \in P$ whence $(yx^{-1}\lambda)^{-1} = xy^{-1}\lambda \in P$. Hence, $yx^{-1}\lambda \in P \cap P^{-1} = \{\lambda\}$ and so $x\lambda = y\lambda$, which shows that \leq_λ is antisymmetric. Now, suppose that $x\lambda \leq_\lambda y\lambda$ and $y\lambda \leq_\lambda z\lambda$. Then, $yx^{-1}\lambda \in P$ and $zy^{-1}\lambda \in P$. Now, from

$$zx^{-1}\lambda = zy^{-1}yx^{-1}\lambda = (zy^{-1}\lambda)(yx^{-1}\lambda) \in P^2 = P$$

it follows that $x\lambda \leq_\lambda z\lambda$, proving that \leq_λ is transitive.

For compatibility, let $x\lambda \leq_\lambda y\lambda$. Then, $yx^{-1}\lambda \in P$ and so by (3), for all $a, b \in G$,

$$\begin{aligned} ayb(axb)^{-1}\lambda &= aybb^{-1}x^{-1}a^{-1}\lambda = ayx^{-1}a^{-1}\lambda \\ &= (a\lambda)(yx^{-1}\lambda)(a\lambda)^{-1} \in (a\lambda)P(a\lambda)^{-1} = P \end{aligned}$$

whence $(a\lambda)(x\lambda)(b\lambda) \leq_\lambda (a\lambda)(y\lambda)(b\lambda)$.

Now, $\lambda \leq_\lambda x\lambda$ if and only if $x\lambda \in P$, proving that P is the positive cone of \leq_λ .

Assume that $P \cup P^{-1} = G/\lambda$ and $x\lambda, y\lambda \in G/\lambda$. Then, $xy^{-1}\lambda \in P$ or $xy^{-1}\lambda \in P^{-1}$ so that $xy^{-1}\lambda \geq \lambda$ or $xy^{-1}\lambda \leq \lambda$ whence $x\lambda \geq y\lambda$ or $x\lambda \leq y\lambda$ proving G/λ is totally ordered. Conversely, if G/λ is totally ordered, for all $x \in G$, $x\lambda \geq \lambda$ or $x\lambda \leq \lambda$ which shows that $x\lambda \in P$ or $x\lambda \in P^{-1}$, i.e., $G/\lambda = P \cup P^{-1}$. \square

Note 3.2. The conditions (2) and (3) of Theorem 3.1 tell us that the positive cone of an ordered group is a normal subsemigroup.

Definition 3.3. Let G be an ordered group. A fuzzy subgroup λ of G is said to be a *fuzzy convex subgroup* if for all $a, b, c \in G$ with $a \leq c \leq b$ we have $\lambda(c) \geq \lambda(a) \wedge \lambda(b)$.

Proposition 3.4. *A fuzzy subset λ of G is a fuzzy convex subgroup if and only if every nonempty level subset λ_t (with $t \in [0, 1]$) is a convex subgroup of G .*

Proof. It is easy. □

Example 3.5. Consider the additive abelian group $\mathbb{R} \times \mathbb{R}$ on which the relation \leq is defined as

$$(x, y) \leq (z, w) \Leftrightarrow x = z, y = w \text{ or } x \leq z, y < w$$

is a compatible order and the set $A = \{(x, 0) : x \in \mathbb{R}\}$ is a convex subgroup of $\mathbb{R} \times \mathbb{R}$ (see [4, Theorem 9.1 and Exercise 9.13]). Now, define the fuzzy subset μ in $\mathbb{R} \times \mathbb{R}$ by $\mu(x, y) = b$ if $y = 0$, and $\mu(x, y) = 0$ otherwise, $b \in [0, 1]$. It is easy to see that the only nonempty level subsets of μ are A and $\mathbb{R} \times \mathbb{R}$ which are convex subgroups of $\mathbb{R} \times \mathbb{R}$. Thus, μ is a fuzzy convex subgroup of $\mathbb{R} \times \mathbb{R}$, by Proposition 3.4.

As the first result about fuzzy convex subgroups, we shall prove that fuzzy convex subgroups of an ordered group G , is order-reversing on $P(G)$ and is order-preserving on $N(G)$.

Proposition 3.6. *Any fuzzy convex subgroup of G satisfies*

- (1) $(\forall x, y \in G)$ if $e \leq x \leq y$ then $\lambda(x) \geq \lambda(y)$,
- (2) $(\forall x, y \in G)$ if $y \leq x \leq e$ then $\lambda(x) \geq \lambda(y)$.

Definition 3.7. A fuzzy subset λ of G is called a *fuzzy down-set* if every nonempty level subset λ_t (with $t \in [0, 1]$) is a down-set of G .

Example 3.8. It is easy to check that the fuzzy sublattice given in Example 2.20 is a fuzzy down-set of L .

As a characterization of fuzzy down-sets we get the following proposition.

Proposition 3.9. *A fuzzy subset λ of G is a fuzzy down-set of G if and only if it is order-reversing.*

Proof. It is easy. □

Notation. For fuzzy subset λ of G , let λ_+ denotes $\lambda|_{P(G)}$, the restriction of λ to $P(G)$.

Theorem 3.10. *Let λ be a fuzzy subgroup of G . Then, $\lambda_+ = \lambda \wedge \chi_{P(G)}$. Moreover, the following are equivalent:*

- (1) λ is a fuzzy convex subgroup of G ,
- (2) λ_+ is a fuzzy down-set of $P(G)$.

Proof. Let $x \in P(G)$. Then, $\lambda \wedge \chi_{P(G)}(x) = \lambda(x) \wedge \chi_{P(G)}(x) = \lambda(x) = \lambda_+(x)$.

(1) \Rightarrow (2) Let $y, x \in P(G)$ and $y \leq x$. Then,

$$\lambda_+(y) = \lambda \wedge \chi_{P(G)}(y) = \lambda(y) \geq \lambda(x) = \lambda_+(x)$$

proving λ_+ is a fuzzy down-set of $P(G)$, by Proposition 3.9.

(2) \Rightarrow (1) We show that λ_t is convex, for every $t \in [0, 1]$ with $\lambda_t \neq \emptyset$. For this, let $x \leq y \leq z$ and $x, z \in \lambda_t$, for $t \in [0, 1]$. Then, $e \leq x^{-1}y \leq x^{-1}z$ and $e, x^{-1}z \in P(\lambda_t)$. Now,

$$\lambda(x^{-1}y) = \lambda_+(x^{-1}y) \geq \lambda_+(x^{-1}z) = \lambda(x^{-1}z) \geq \lambda(x) \wedge \lambda(z) \geq t$$

whence $x^{-1}y \in \lambda_t$. Hence, $y \in x\lambda_t = \lambda_t$, i.e., $\lambda(y) \geq t = \lambda(x) \wedge \lambda(z)$, Which completes the proof. \square

Theorem 3.11. *Let λ be a fuzzy normal subgroup of G which is constant on $P(G)$. Then, $\pi(P(G)) = \{x\lambda : x \in P(G)\}$ is the positive cone of a compatible order on G/λ if and only if λ is fuzzy convex.*

Proof. Let $Q = \{x\lambda : x \in P(G)\}$ be the positive cone of a compatible order on G/λ . To show that λ is fuzzy convex, we shall prove that λ_+ is order-reversing. Let $e \leq y \leq x$. We first observe that since λ_+ is constant, so $\lambda(x) = \lambda_+(x) = \lambda(e)$, for all $x \in P(G)$ whence $x\lambda = \lambda$. Now,

$$(y\lambda)^{-1} = (y\lambda)^{-1}\lambda = (y\lambda)^{-1}x\lambda = y^{-1}x\lambda \in Q$$

and so $y\lambda \in Q \cap Q^{-1} = \{\lambda\}$. This implies that

$$\lambda_+(y) = \lambda(y) = \lambda(e) \geq \lambda(x) = \lambda_+(x).$$

Hence, by Proposition 3.9 and Theorem 3.10, it follows that λ is fuzzy convex.

It is easy to verify The converse. It is enough to show that Q satisfies the conditions (1)-(3) of Theorem 3.1. \square

Remark 3.12. If λ is a fuzzy convex normal subgroup of ordered group G which is constant on $P(G)$, the order \leq_λ on G/λ can be described as in Theorem 3.1:

$$\begin{aligned} x\lambda \leq_\lambda y\lambda &\Leftrightarrow yx^{-1}\lambda \in Q \\ &\Leftrightarrow yx^{-1}\lambda = z\lambda = \lambda, \text{ for } z \in P(G) \\ &\Leftrightarrow \lambda(yx^{-1}) = \lambda(z) = \lambda(e). \end{aligned}$$

Then, $yx^{-1} \in \lambda_{\lambda(e)}$ and so $y\lambda_{\lambda(e)} = x\lambda_{\lambda(e)}$. Since, $z \in \lambda_{\lambda(e)}$, so $xz = yw$, where $\lambda(w) = \lambda(e)$ and so $y \geq xw^{-1}$, where $\lambda(w^{-1}) = \lambda(e)$. Conversely, assume that $zx \leq y$, for $z \in G$ with $\lambda(z) = \lambda(e)$. From $yx^{-1} = y(zx)^{-1}z$ it follows that $yx^{-1}\lambda = y(zx)^{-1}z\lambda = y(zx)^{-1}\lambda \in Q$. This shows that $x\lambda \leq_\lambda y\lambda$.

Therefore,

$$x\lambda \leq_\lambda y\lambda \Leftrightarrow y \geq zx, \text{ for } z \in G \text{ with } \lambda(z) = \lambda(e).$$

Theorem 3.13. *Let λ be a fuzzy subgroup of G . Then $\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}$ is the least fuzzy convex subgroup of G contains λ .*

Proof. Let $x, y \in G$. Then,

$$\bigvee \{\lambda(z) : z \leq x\} \wedge \bigvee \{\lambda(w) : w \leq y\} \leq \bigvee \{\lambda(z) : z \leq xy\} = \lambda \circ \chi_{P(G)}(xy).$$

Similarly,

$$\bigvee \{\lambda(w) : x \leq w\} \wedge \bigvee \{\lambda(z) : y \leq z\} \leq \bigvee \{\lambda(t) : xy \leq t\} = \lambda \circ \chi_{N(G)}(xy).$$

Therefore,

$$(\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)})(x) \wedge (\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)})(y) \leq (\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)})(xy).$$

For the last property it suffices to observe that

$$\lambda \circ \chi_{N(G)}(a) \leq \lambda \circ \chi_{P(G)}(a^{-1}) \quad \text{and} \quad \lambda \circ \chi_{P(G)}(a) \leq \lambda \circ \chi_{N(G)}(a^{-1}).$$

Hence, $\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}$ is a fuzzy subgroup of G .

Now, observe that

$$\begin{aligned} \lambda \circ \chi_{P(G)}(x) &= \bigvee \{ \lambda(y) \wedge \chi_{P(G)}(z) : x = yz \} = \bigvee \{ \lambda(y) : x = yz, z \geq e \} \\ &= \bigvee \{ \lambda(y) : y \leq x \} \end{aligned}$$

and similarly, $\lambda \circ \chi_{N(G)}(x) = \bigvee \{ \lambda(y) : x \leq y \}$.

From, $\lambda(x) \leq \bigvee \{ \lambda(y) : y \leq x \}$ and $\lambda(x) \leq \bigvee \{ \lambda(y) : x \leq y \}$ it follows that

$$\begin{aligned} \lambda(x) &\leq \left(\bigvee \{ \lambda(y) : x \geq y \} \right) \wedge \left(\bigvee \{ \lambda(y) : x \leq y \} \right) \\ &= (\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)})(x). \end{aligned}$$

Hence, $\lambda \leq \lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}$.

Now, we prove that $\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}$ is fuzzy convex. Let $a, b, c \in G$ be such that $a \leq c \leq b$. Then,

$$\begin{aligned} \lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}(a) &= \bigvee \{ \lambda(t) : t \leq a \} \wedge \bigvee \{ \lambda(t) : a \leq t \} \\ &\leq \bigvee \{ \lambda(t) : t \leq a \} \\ &\leq \bigvee \{ \lambda(t) : t \leq c \}. \end{aligned}$$

Similarly,

$$\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}(b) \leq \bigvee \{ \lambda(s) : b \leq s \} \leq \bigvee \{ \lambda(s) : c \leq s \}.$$

Thus,

$$\begin{aligned} &(\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}(a)) \wedge (\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}(b)) \\ &\leq \bigvee \{ \lambda(t) : t \leq c \} \wedge \bigvee \{ \lambda(s) : c \leq s \} \\ &= \lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}(c) \end{aligned}$$

proving $\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}$ is fuzzy convex.

Thus, $\lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}$ is a fuzzy convex subgroup of G containing λ .

Now, let μ be a fuzzy convex subgroup of G containing λ and $x \in G$. Then,

$$\begin{aligned} \lambda \circ \chi_{P(G)} \wedge \lambda \circ \chi_{N(G)}(x) &= \bigvee \{ \lambda(y) : y \leq x \} \wedge \bigvee \{ \lambda(z) : x \leq z \} \\ &\leq \bigvee \{ \mu(y) : y \leq x \} \wedge \bigvee \{ \mu(z) : x \leq z \} \\ &\leq \bigvee \{ \mu(y) \wedge \mu(z) : y \leq x \leq z \} \\ &\leq \mu(x) \end{aligned}$$

completes the proof. □

Now, we recall some notions and results from ordered groups. For more details we refer the reader to [4].

If H is a convex normal subgroup of G , G/H forms an ordered group whose order ' \leq_H ' can be described as

$$xH \leq_H yH \Leftrightarrow (\exists h \in H) hx \leq y.$$

Theorem 3.14. *Let G and H be ordered groups. If $f : G \rightarrow H$ is a group homomorphism, then f is isotone if and only if $f(P(G)) \subseteq P(H)$.*

Proof. See [4], the corollary just after Theorem 9.5. □

Definition 3.15. An *order isomorphism* between ordered sets E and F is an isotone bijection f whose inverse f^{-1} is isotone. When E and F are ordered isomorphic we write $E \simeq F$.

Example 3.16. Consider the ordered groups $(\mathbb{R}, +)$ and (\mathbb{R}^+, \cdot) . It is easy to check that the mapping $\phi : (\mathbb{R}, +) \rightarrow (\mathbb{R}^+, \cdot)$ defined by $\phi(x) = e^x$ is an order isomorphism.

Definition 3.17. Let G and H be ordered groups.

- (1) G and H are said to be *isomorphic* if there is an order isomorphism between G and H which is also a group isomorphism.
- (2) A mapping $f : G \rightarrow H$ is said to be *exact* if $f(P(G)) = P(H)$.

Example 3.18. Consider the ordered additive groups $(\mathbb{Z}, +)$ and $(2\mathbb{Z}, +)$, and define a mapping $f : \mathbb{Z} \rightarrow 2\mathbb{Z}$, $f(n) = 2n$. Obviously, f is an exact group homomorphism.

Theorem 3.19. *Let G and H be ordered groups. Then, the following are equivalent:*

- (1) $G \simeq H$,
- (2) *there is an exact group isomorphism $f : G \rightarrow H$.*

Theorem 3.20. *Let G and H be ordered groups. If $f : G \rightarrow H$ is an exact group homomorphism, then $Im f \simeq G/ker f$.*

Theorem 3.21. *If λ is a fuzzy convex normal subgroup of G , $G/\lambda \simeq G/\lambda_{\lambda(e)}$.*

Proof. Define a mapping $f : G/\lambda \rightarrow G/\lambda_{\lambda(e)}$ by $f(x\lambda) = x\lambda_{\lambda(e)}$. It is easy to check that f is a group isomorphism. Also, from

$$x\lambda \leq_\lambda y\lambda \Leftrightarrow y \geq xz, \text{ for } z \in \lambda_{\lambda(e)} \Leftrightarrow y\lambda_{\lambda(e)} \leq y\lambda_{\lambda(e)}$$

it follows that f is an order isomorphism. It remains to prove that f is exact. Let $x\lambda \in P(G/\lambda)$. Then, $\lambda = e\lambda \leq_\lambda x\lambda$ and so $x \geq ye = y$, for $y \in \lambda_{\lambda(e)}$. This implies that $\lambda_{\lambda(e)} \leq_{\lambda_{\lambda(e)}} x\lambda_{\lambda(e)} = f(x\lambda)$ and hence $f(x\lambda) \in P(G/\lambda_{\lambda(e)})$. Conversely, if $z\lambda_{\lambda(e)} \in P(G/\lambda_{\lambda(e)})$ then $\lambda_{\lambda(e)} \leq_{\lambda_{\lambda(e)}} z\lambda_{\lambda(e)} = f(z\lambda)$, for $z\lambda \in G/\lambda$. Since, f is an order isomorphism, $\lambda = f^{-1}(\lambda_{\lambda(e)}) \leq z\lambda$, whence $z\lambda \in P(G/\lambda)$, completes the proof. □

Corollary 3.22. *(The Fundamental Homomorphism Theorem)*

Let λ be a fuzzy convex normal subgroup of G and $f : G \rightarrow H$ be an exact epimorphism with $ker f = \lambda_{\lambda(e)}$. Then, $G/\lambda \simeq H$.

Proof. The proof follows from Theorems 3.20 and 3.21. \square

4. Fuzzy Convex ℓ -subgroups

In this section, G will denote a lattice-ordered group, unless otherwise stated. We first give a characterization of fuzzy ℓ -subgroups.

Proposition 4.1. *A fuzzy subgroup λ of G is a fuzzy ℓ -subgroup if and only if for all $x \in G$, $\lambda(x \vee e) \geq \lambda(x)$.*

Proof. The necessity is obvious.

Sufficiency: Let $x \in G$. Then

$$\lambda(x \vee y) = \lambda((xy^{-1} \vee e)y) \geq \lambda(xy^{-1} \vee e) \wedge \lambda(y) \geq \lambda(xy^{-1}) \wedge \lambda(y) \geq \lambda(x) \wedge \lambda(y).$$

Likewise, we conclude that $\lambda(x \wedge y) \geq \lambda(x) \wedge \lambda(y)$. \square

In the theory of lattice-ordered groups an important part is played by those ℓ -subgroups that are also convex subsets. Here, we study this part from fuzzy set theory point of view.

Definition 4.2. A fuzzy ℓ -subgroup of G which is convex is said to be a *fuzzy convex ℓ -subgroup*.

Example 4.3. Consider the ordered additive abelian group $\mathbb{R} \times \mathbb{R}$ defined in Example 3.5. It is easy to see that $(\mathbb{R} \times \mathbb{R}, \leq)$ is a lattice in which A is a convex ℓ -subgroup of $\mathbb{R} \times \mathbb{R}$. Hence, μ is a fuzzy convex ℓ -subgroup, by Proposition 2.21.

Denote the set of all fuzzy convex ℓ -subgroups of G by $\mathcal{FCL}(G)$.

Theorem 4.4. *Let $\{\lambda_i : i \in I\}$ be a family of fuzzy ℓ -subgroups of G , and $\lambda = \bigcap_{i \in I} \lambda_i$. If λ_i is fuzzy convex, for all $i \in I$, then λ is fuzzy convex.*

Proof. It is easy. \square

Definition 4.5. [4] Let G be a lattice-ordered group. Then $x, y \in G$ are said to be *orthogonal* or *disjoint* if $|x| \wedge |y| = e$, where $|x| = x \vee x^{-1}$.

Theorem 4.6. [4] *Let G be a lattice-ordered group and $x_1, \dots, x_n \in P(G)$ are pairwise orthogonal. Then $x_1 \vee \dots \vee x_n = x_1 \dots x_n$. Consequently, mutually orthogonal elements of $P(G)$ commute.*

Definition 4.7. A fuzzy subsemigroup λ of G is called a *quasi fuzzy subsemigroup* if

$$(FS) \quad \lambda(e) \geq \lambda(x), \text{ for all } x \in G.$$

Example 4.8. Consider the ordered group (\mathbb{R}^+, \cdot) in which \mathbb{N} is a subsemigroup of \mathbb{R} . Let $t \in [0, 1]$ be fix and define a fuzzy subset μ of \mathbb{R} by $\mu(x) = t$, if $x \in \mathbb{N}$, and $\mu(x) = 0$, otherwise. Then μ is a quasi fuzzy subsemigroup of \mathbb{R} .

Obviously, any fuzzy subgroup satisfies (FS).

Theorem 4.9. *Let λ be a quasi fuzzy subsemigroup of $P(G)$. Then,*

$$\langle \lambda \rangle(x) = \vee \{ \lambda(a) \wedge \lambda(b) : x = ab^{-1} \}$$

is the least fuzzy convex ℓ -subgroup of G containing λ .

Proof. Let $\mu(x) = \vee \{ \lambda(a) \wedge \lambda(b) : x = ab^{-1} \}$. Obviously, $\lambda \subseteq \mu$, and $\mu(x^{-1}) \geq \mu(x)$. Let $x, y \in G$ and $\epsilon > 0$. Then there exist $a, b, c, d \in P(G)$ such that $x = ab^{-1}$, $y = cd^{-1}$ and $\mu(x) - \epsilon < \lambda(a) \wedge \lambda(b)$ and $\mu(y) - \epsilon < \lambda(c) \wedge \lambda(d)$. Now, let $\alpha = (b \wedge c)^{-1}b$ and $\beta = (b \wedge c)^{-1}c$. It is easy to see that $e \leq \alpha \leq b$, $e \leq \beta \leq c$ and $\alpha \wedge \beta = e$. Then, α and β and so α^{-1} and β commute, by Theorem 4.6. Consequently,

$$\begin{aligned} xy &= (ab^{-1})(cd^{-1}) = a\alpha^{-1}(b \wedge c)^{-1}(b \wedge c)\beta d^{-1} = a\alpha^{-1}\beta d^{-1} = a\beta\alpha^{-1}d^{-1} \\ &= (a\beta)(d\alpha)^{-1}. \end{aligned}$$

Thus,

$$\begin{aligned} \mu(xy) &> \lambda(a\beta) \wedge \lambda(d\alpha) \\ &\geq \lambda(a) \wedge \lambda(\beta) \wedge \lambda(d) \wedge \lambda(\alpha) \\ &\geq \lambda(a) \wedge \lambda(b) \wedge \lambda(c) \wedge \lambda(d) \\ &> (\mu(x) - \epsilon) \wedge (\mu(y) - \epsilon) \\ &\geq \mu(x) \wedge \mu(y) - \epsilon \end{aligned}$$

Hence, $\mu(xy) \geq \mu(x) \wedge \mu(y)$ which shows that μ is a fuzzy subgroup of G .

Now, let $x \in G$ be such that $\mu(x) = t$ and n be a sufficiently large natural number. Then

$$\vee \{ \lambda(a) \wedge \lambda(b) : x = ab^{-1} \} > s = t - 1/n$$

whence $\lambda(a_0) \wedge \lambda(b_0) > s$, for some $a_0, b_0 \in P(G)$ with $x = a_0b_0^{-1}$. But, we have

$$e_G \leq x \vee e_G \leq a_0 \vee e_G = a_0$$

and so

$$\mu(x \vee e) \geq \lambda(x \vee e) \geq \lambda(a_0) > s.$$

Thus, $\mu(x \vee e) \geq t = \mu(x)$.

To prove that μ is fuzzy convex, we shall show that $P(\mu_t)$, with $t \in [0, 1]$ and $\mu_t \neq \emptyset$, is a down-set of $P(G)$. For this assume that $e_G \leq g \leq x \in P(\mu_t)$. Then,

$$\vee \{ \lambda(a) \wedge \lambda(b) : x = ab^{-1} \} = \mu(x) > s$$

where $s = t - 1/n$, n is a sufficiently large natural number, and so there exist $a_0, b_0 \in P(G)$ such that $x = a_0b_0^{-1}$ and $\lambda(a_0) \wedge \lambda(b_0) > s$. On the other hand from $e_G \leq g \leq x = a_0b_0^{-1}$ and that λ is fuzzy convex it follows that $\lambda(g) > s$ and so $\mu(g) \geq t$. This implies that $g \in P(\mu_t)$, which shows that $P(\mu_t)$ is convex.

Therefore μ is the least fuzzy convex ℓ -subgroup of G , and so $\langle \lambda \rangle = \mu$. □

Corollary 4.10. *Let λ_i with $i \in \mathbb{N}_k = \{1, 2, \dots, k\}$ be a quasi fuzzy subsemigroup of $P(G)$ and $\lambda_k = \cup_{i \in \mathbb{N}_k} \lambda_i$. Then,*

$$\langle \lambda_k \rangle(x) = \vee \{ \lambda_k(a) \wedge \lambda_k(b) : x = ab^{-1} \}.$$

Lemma 4.11. *If λ is a quasi fuzzy convex subsemigroup of $P(G)$, then $\langle \lambda \rangle_+ = \lambda$.*

Proof. Obviously, $\langle \lambda \rangle_+ \geq \lambda$. Let $\epsilon > 0$ and $x \in P(G)$. Then there exist $a_0, b_0 \in P(G)$ such that $x = a_0 b_0^{-1}$ and $\lambda(a_0) \wedge \lambda(b_0) > \langle \lambda \rangle_+(x) - \epsilon$. From $e \leq x = a_0 b_0^{-1}$ it follows that $e \leq x \leq a_0$ and so $\lambda(x) \geq \lambda(a_0) > \langle \lambda \rangle_+(x) - \epsilon$ proving $\lambda \geq \langle \lambda \rangle_+$. Hence, $\langle \lambda \rangle_+ = \lambda$. \square

Denote the set of all quasi fuzzy subsemigroups of G by $\mathcal{QFS}(G)$.

Theorem 4.12. $\mathcal{FCL}(G) \simeq \mathcal{QFS}(P(G))$.

Proof. Define the mappings

$$\Phi : \mathcal{FCL}(G) \longrightarrow \mathcal{QFS}(P(G)) \quad \text{and} \quad \Psi : \mathcal{QFS}(P(G)) \longrightarrow \mathcal{FCL}(G)$$

by $\Phi(\mu) = \mu_+$ and $\Psi(\lambda) = \langle \lambda \rangle$, respectively. Obviously, Φ and Ψ are isotone. Now, $\Psi(\Phi(\mu)) = \Psi(\mu_+) = \langle \mu_+ \rangle = \mu$ and $\Phi(\Psi(\lambda)) = \Phi(\langle \lambda \rangle) = \langle \lambda \rangle_+ = \lambda$, by Lemma 4.11. Hence, Φ and Ψ are mutually inverse order isomorphisms. \square

Notation. Let $\mathcal{F}(G)$ denote the set of all fuzzy subgroups of G , and for fuzzy subset μ of G , $\langle \mu \rangle = \{\nu : \mu \subseteq \nu, \nu \in \mathcal{F}(G)\}$ denote the fuzzy subgroup of G generated by μ . Also, for fuzzy subset μ of G , let $\mu^{-1}(x) = \mu(x^{-1})$, $\mu^1 = \mu$ and for $n \in \mathbb{N}$, let $\mu^n = \mu^{n-1} \circ \mu$, where $\mu \circ \nu(x) = \vee\{\mu(a) \wedge \nu(b) : x = ab\}$, for fuzzy subsets μ and ν of G , regarding that the supremum of an empty set is zero.

Theorem 4.13. [8] *If μ is a fuzzy subset of G , then*

$$\langle \mu \rangle = \bigvee_{n=1}^{\infty} (a_e \vee \mu \vee \mu^{-1})^n$$

where $a_e = \wedge\{\eta(e) : \mu \subseteq \eta, \eta \in \mathcal{F}(G)\}$.

Theorem 4.14. $\mathcal{FCL}(G)$ is a complete Heyting sublattice of the lattice of all fuzzy subgroups of G .

Proof. Let $\{\mu_i : i \in I\}$ be a family of fuzzy convex ℓ -subgroups of G . Obviously, $\bigwedge \mu_i$ is a fuzzy ℓ -subgroup of G . From

$$\left(\bigwedge_{i \in I} \mu_i \right)_+ = \left(\bigwedge_{i \in I} \mu_i \right) \wedge \chi_{P(G)} = \bigwedge_{i \in I} (\mu_i \wedge \chi_{P(G)}) = \bigwedge_{i \in I} (\mu_i)_+$$

and that $(\bigwedge_{i \in I} \mu_i)_+$ is a fuzzy down-set, it follows that $\bigwedge \mu_i$ is fuzzy convex.

Let $\mu = \bigvee_{i \in I} \mu_i$. We shall show that $\langle \mu \rangle$ is a fuzzy convex ℓ -subgroup of G . Let $e \leq g \leq a$. Then, $\mu_i(g) \geq \mu_i(a)$ and so $\mu(g) = \bigvee \mu_i(g) \geq \bigvee \mu_i(a) = \mu(a)$. Similarly, $\mu^{-1}(g) \geq \mu^{-1}(a)$. This implies that

$$(a_e \vee \mu \vee \mu^{-1})(g) \geq (a_e \vee \mu \vee \mu^{-1})(a).$$

By induction we can prove that

$$(a_e \vee \mu \vee \mu^{-1})^n(g) \geq (a_e \vee \mu \vee \mu^{-1})^n(a), \quad \forall n \in \mathbb{N}$$

whence $\langle \mu \rangle_+(x) \geq \langle \mu \rangle_+(a)$ which shows that $\langle \mu \rangle$ is a fuzzy convex subgroup of G , by Theorem 3.10.

Now, let $x = a_1 a_2 \cdots a_n$, for some $a_1, a_2, \dots, a_n \in G$. Then,

$$e \leq x \vee e \leq (a_1 \vee e) \cdots (a_n \vee e).$$

Since, every μ_i is a fuzzy ℓ -subgroup of G , we can see that for any $n \in \mathbb{N}$,

$$(a_e \vee \mu \wedge \mu^{-1})^n(a_i \vee e) \geq (a_e \vee \mu \wedge \mu^{-1})^n(a_i)$$

and so

$$\begin{aligned} \langle \mu \rangle(x \vee e) &\geq \langle \mu \rangle((a_1 \vee e) \cdots (a_n \vee e)) \\ &\geq \langle \mu \rangle(a_1 \vee e) \wedge \cdots \wedge \langle \mu \rangle(a_n \vee e) \\ &= \bigvee_{n=1}^{\infty} (a_e \vee \mu \vee \mu^{-1})^n(a_1 \vee e) \wedge \cdots \wedge \bigvee_{n=1}^{\infty} (a_e \vee \mu \vee \mu^{-1})^n(a_n \vee e) \\ &\geq \bigvee_{n=1}^{\infty} (a_e \vee \mu \vee \mu^{-1})^n(a_1) \wedge \cdots \wedge \bigvee_{n=1}^{\infty} (a_e \vee \mu \vee \mu^{-1})^n(a_n) \\ &= \bigvee_{n=1}^{\infty} [(a_e \vee \mu \vee \mu^{-1})^n(a_1) \wedge \cdots \wedge (a_e \vee \mu \vee \mu^{-1})^n(a_n)] \\ &= \langle \mu \rangle(x) \end{aligned}$$

proving that $\langle \mu \rangle$ is a fuzzy ℓ -subgroup.

Now, let ν be another fuzzy convex ℓ -subgroup of G . Then,

$$\begin{aligned} \nu \wedge \bigvee \mu_i(x) &= \nu(x) \wedge \bigvee (a_e \vee (\cup \mu_i) \vee (\cup \mu_i)^{-1})^n(x) \\ &= \bigvee ((a_e \wedge \nu) \vee (\nu \wedge \cup \mu_i) \vee (\nu \wedge (\cup \mu_i)^{-1}))^n \\ &\leq \bigvee (a_e \vee (\nu \wedge \cup \mu_i) \vee (\nu \wedge \cup \mu_i)^{-1})^n \\ &= \langle \cup(\nu \wedge \mu_i) \rangle = \bigvee(\nu \wedge \mu_i) \end{aligned}$$

proving that $\mathcal{FCL}(G)$ is a Heyting lattice. \square

5. Conclusions

In this paper, we studied convex subgroups and convex ℓ -subgroups from fuzzy set theory point of view. In this respect, we gave some characterizations and related results that help us to identify those subsets of ordered groups that can be the positive cone of a compatible order. Using this, we recognized the order on the quotient ordered group induced by a fuzzy convex subgroup. Furthermore, we characterized the fuzzy convex subgroup and fuzzy convex ℓ -subgroup generated by a fuzzy subgroup and fuzzy subsemigroup, respectively. Finally, we obtained an important result that says that the class of all fuzzy convex ℓ -subgroups of a lattice-ordered group is a complete Heyting sublattice of the lattice of all fuzzy subgroups of that group.

There's still many subjects to work in this area of research, say, interval-valued fuzzy ℓ -subgroups, intuitionistic fuzzy ℓ -subgroups and so on that could be the subject of further research.

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