Iranian Journal of Fuzzy Systems

Volume 21, Number 5, (2024), pp. 133-150

Original paper

University of Sistan and Baluchestan

Controllability criteria for type-2 fuzzy fractional-order dynamical system via Mittag-Leffler matrix function using granular derivative

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Abstract

This study is concerned with finding the controllability of granular type-2 fuzzy fractional-order dynamical control systems (GrT2FuFrDCS) using Mittag-Leffler matrix function. Some new concepts concerning to granular interval valued type-2 fuzzy fractional derivative is introduced in this paper. To arrive at finding the controllability of the type-2 fuzzy dynamical system, type-2 triangular fuzzy function and type-2 triangular fuzzy numbers are used to model fractional dynamical control system. Besides, the controllability Grammian matrix is defined based on Mittag-Leffler matrix function. Granular fuzzy Laplace transform and inverse Laplace transform are utilized to derive the solution of the type-2 triangular fuzzy fractional-order control systems. The derived results ensure the existence of the controllability of the proposed GrT2FuFrDCS for both linear and non-linear case. Finally, numerical examples and their simulation results are given to illustrate the merit of the obtained results.

Keywords: Horizontal membership function, type-2 triangular fuzzy number, type-2 fuzzy granular Laplace transform, type-2 fuzzy fractional differential equations, granular differentiability, granular fuzzy fractional derivative, type-2 fuzzy fractional granular controllability.

1 Introduction

In the field of science, especially engineering, in order to study the dynamical system such as mechanical, electrical, chemical, economic and biological systems, requires a mathematical model of their system's behaviour. Control problem in these fields are addressed with the help of the state space model. Because of its effectiveness and the wide range of theories and methods that have been created for state-space analysis and design, this model is frequently used as a tool to address a variety of dynamic system issues. Uncertainty in the dynamical system is tackled with the help of the fuzzy state space model which was introduced as fuzzy differential equations (FuDE) wherein the parameters, functions and the intial condition contains uncertainty [2, 10].

The concept of fractional calculus originated with Newton and Leibniz in 1695. Since then, many mathematicians have contributed to its theoretical development. Fractional calculus has gained significant importance due to its proven applications across a wide range of scientific and engineering fields. In particular, the application of fractional calculus in modelling the system dynamics allows for a more accurate representation of system behaviour, especially in scenarios involving memory effects and non-local interactions. The reason for integer order derivative being replaced by fractional order derivative is because the fractional differential operator offers a multitude of degrees of freedom enabling a more robust explanation of memory and hereditary properties across various phenomena. Incorporating fuzzy logic and fractional calculus leads to the development of fuzzy fractional differential equation (T1FuFrDE) which has gained a lot of interests among researchers [3, 32].

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Received: January 2024; Revised: May 2024; Accepted: December 2024.

https://doi.org/10.22111/ijfs.2024.47862.8422

Table 1: Abbreviations

Abbreviation	**Full Form**
FuDE	Fuzzy Differential Equations
T1FuS	Type-1 Fuzzy Set
T2FuS	Type-2 Fuzzy Set
T1TIVFuN	Type-1 Triangular Interval valued Fuzzy Number
T2TIVFuN	Type-2 Triangular Interval valued Fuzzy Number
IT2FuF	Type-2 Triangular Interval valued Fuzzy Function
IT2FuM	Type-2 Triangular Interval valued Fuzzy Matrix
T2FuN	Type-2 Fuzzy Number
GrT2D	Granular Type-2 Derivative
GrT2FuS	Granular Type-2 fuzzy Set
GrT2FuD	Granular Type-2 fuzzy Derivative
GrT2FuLT	Granular Type-2 fuzzy Laplace Transform
GrT2FuILT	Granular Type-2 fuzzy Inverse Laplace Transform
HoMF	Horizontal Membership Function
HoMFoIT2FuN	Horizontal Membership Function of Type-2 Interval valued Fuzzy Number
T2FuDE	Type-2 Fuzzy Differential Equation
T1FuFrDE	Type-1 Fuzzy Fractional Differential Equation
T2FuFrDE	Type-2 Fuzzy Fractional Differential Equation
GrT2FuFrDCS	Granular Type-2 Fuzzy Fractional Dynamical Control System
CGrT2FuFrD	Caputo Granular Type-2 Fuzzy Fractional Derivative
RDM	Relative Distance Measure
R-L	Riemann–Liouville

With a broader knowledge of type-1 fuzzy [37], Zadeh introduced a more general concept called type-2 fuzzy. This helps to model the uncertainty and imprecision in a better way. They are known as "fuzzy fuzzy" sets in which the degree of membership is a type-1 fuzzy set (T1FuS). A type-2 fuzzy set (T2FuS) comprises membership functions that reflect uncertainty in addition to the uncertainty present in the data itself. As an example, suppose we ask few people in a neighbourhood to report the temperature on a specific day. The perception of temperature as "hot" at approximately 35 degrees is generally accepted. However, variations in individual perspectives inevitably result in disparate membership functions. Consequently, each individual's definition of "hotness" at this temperature is likely to differ. As more individuals contribute their interpretations, the delineation of a specific temperature's belongingness to the "Hot" set becomes increasingly ambiguous. Each individual's characterization of belongingness transitions into a fuzzy set within a three-dimensional function, thereby giving rise to a T2FuS model. The idea is that, concepts can have diverse interpretations to different people is what drives type-2 fuzzy logic. T2FuS is a good choice for handling high degrees of uncertainty that are found in real world problems. An overview of type-2 sets was explored in [11, 12, 19], it also discusses the important concept of type-2 fuzzy which is the footprint of uncertainty. The existence and uniqueness of type-2 fuzzy solution is discussed in [23]. Authors in [17] studied interval type-2 fuzzy fractional inference system. Some of the other application of using T2FuS is elaborated in these articles [20, 35]. Incorporating the T2FuS concepts, researchers began to explore on type-2 fuzzy fractional differential equations (T2FuFDE) which make use of the concept of type-2 fuzzy number and fractional calculus. The concepts and derivatives related to type-2 fuzzy was first introduced in [12] based on the Hukuhara differentiability which has certain drawbacks which is explained in [24].

The definitions such as generalized type-2 Hukuhara Caputo fuzzy fractional derivative, strongly generalized type-2

Hukuhara Caputo fuzzy fractional derivative etc suffers from a lot of limitations and are all based on fuzzy standard interval arithmetic. Therefore granular fuzzy fractional derivative definition is defined based on horizontal membership function which was elaborated by Andrzej Piegat in [29, 34].

1.1 Advancements under granular differentiability concept:

There are a lot of achievements and applications recently in the field of granular fuzzy concept such as control systems, optimization problems, multi-agent systems, singular systems, existence and uniqueness, fuzzy stochastic systems, and Lyapunov stability. For example, article [1] deals with multi-agent systems with granular fuzzy dynamics. Authors in [6] studied on the optimal control of an epidemic disease with the help of granular fuzzy derivative. Article [27] did a detail study on the fractional descriptor dynamical system and its qualitative behaviour. Yang et. al [36] discussed the Lyapunov stability of fuzzy dynamical systems under granular differentiability concept.

Studying the dynamical system in terms of type-2 fuzzy becomes tiresome, therefore a method known as multidimensional relative distance (RDM) fuzzy arithmetic measure [29, 30, 31] is employed. Numerous articles have been written about the RDM concept [16, 22, 33]. Mazandarani et. al [15] introduced the concept of granular differentiability which overcomes all the disadvantages of Hukuhara differentiability. The review article by Mazandarani et. al [16] clearly explains the evolutions of type-1 and type-2 fuzzy fractional differential equations.

1.2 Enumerating the limitations of H, SH, GH derivatives:

1. **Catastrophe of physics law (CPLV)**.
   The significant drawback of using fuzzy standard interval arithmetic (FSIA) and related concepts in modeling uncertain system occurs when the resulting fuzzy model violates the physics laws that govern the system. While modelling a system is crucial to ensure that the mathematical model adheres to the physics laws. The devious method and direct method are the two methods for solving FuDEs. The devious approach ignores the uncertainty in the system model, considering it as a crisp system then arrive at the solution. Therefore, there is a high chance that the physics law is violated while using FSIA and related concepts like the SGH, GH and H derivative. This violation can render the resulting fuzzy model invalid and unreliable for analysis and prediction.The CPLV was first introduced in [14] and in this work by the employment of the concept of granular differentiability such a shortcoming is overcome.

2. **Unnatural behaviour in modeling (UBM)**.
   The method based on fuzzy standard interval arithmetic (FIA) fails because sometimes solution to $D_2^{\delta}(\tilde{z}(t)) = a\tilde{z}(t) + b\tilde{f}(t)$ may be different from that of the solution of $D_2^{\delta}(\tilde{z}(t)) - a\tilde{z}(t) = b\tilde{f}(t)$. The downside of derivative based of FIA is explained in detail in [15].

3. **Factorization disability.**
   Based on Hukuhara type-2 (H_2) differentiability, the following differential equation may not be equivalent [16].

$$D_2^{\delta}(z(t)) = a(\tilde{f}(\tilde{z}(t)) + \tilde{f}(\tilde{z}(t))),$$
$$D_2^{\delta}(z(t)) = a\tilde{f}(\tilde{z}(t)) + a\tilde{f}(\tilde{z}(t)).$$

4. **Unbounded solutions.**
   When we use H_2 fractional derivative definitions to solve fuzzy fractional differential equations (FuFrDEs), we might end up with solutions where the length of the closure of the support is non-decreasing. When solving fuzzy Differential Equations with fuzzy initial conditions, the resulting solution tends to become more vague over time. This means that while the solution depicts an unstable system, the system itself remains exponentially stable at all times [13, 15].

5. **Complication in computing.**
   The following type-2 fractional differential equation

$$\begin{cases} D_2^{\delta}(\tilde{z}(t)) = a\tilde{z(t)} \\ \tilde{z}(0) = \tilde{z}_0. \end{cases} \tag{1}$$

   Based on type-2 Hukuhara derivative, solving the previous equation is time consuming since the system corresponding to the first form of differentiability and the system corresponding to the second form of differentiability should be solved separately and also the solutions may not be appropriate [15].

6. **Multiplicative property.**
   DEs modeled using FSIA can result in multiple valid solutions, making it challenging to determine the precise behavior of the system [15].

7. **Solutions exhibiting monotonic ambiquity.** Since H_2 fractional derivatives rely on the H_2 derivative, it's essential to determine the monotonic behavior of the uncertainty diameter of the solution [25].

To overcome these shortcoming, the concept of granular differentiability was introduced in [15] and then Najariyan in [24] extended such a concept and introduced it based on type-2 fuzzy sets known as granular type-2 derivative (GrT2D) which employed granular type-2 fuzzy set (GrT2FuS). With the definitions defined in [24] we explore the fractional differentiability of T2FuFrDE especially using interval type-2 fuzzy numbers in the granular context where the system with type-2 fuzzy initial condition is given as ${}_{gr}{}^{\mathtt{c}}D^{\delta}\tilde{z}(t) = \tilde{\mathtt{f}}(t,\tilde{z}(t))$ and $\tilde{z}(0) \in E^2$, where E^2 is the set of all T2FuN on $\mathbb{R}$ and order $\delta \in (0,1), t \in [0,t]$ is examined.

The study of qualitative aspect of the dynamical system has become very crucial [9]. The controllability of dynamical system is an important aspect in the characteristic of the dynamical system. Controllability is the ability to move a system around in its entire space using certain permissible controls. Many researchers have concentrated on this subject since the controllability property is a significant phenomenon of both linear and nonlinear dynamical models. Articles [4, 5, 7, 8, 21, 28] talks about finding the controllability of various systems such as damped dynamical system of linear and non linear system, impulsive semilinear integrodifferential fuzzy system, stochastic system of quasi-linear stochastic evolution equations using different methods. The controllability of integer order derivative with the help of fuzzy logic control techniques has been extensively studied in [7]. Techniques in fuzzy logic control have proven to be quite helpful in providing a relatively accurate and cost-effective model that collectively captures the non linearities and uncertainties in the underlying system.

As far as we know in the literature, the research on the GrT2FuFrCDS have not been studied. Hence the topic's focus is on the controllability of type-2 fuzzy fractional differential equations in both the linear and non-linear situation. Authors in [4] examined the controllability of a mechanical system for fractional order derivative in both linear and non-linear cases, with theorems proved for each case. The article mentioned above inspires the consideration of a type-2 fuzzy dynamical control system, which makes use of the initial condition of type-2 triangular fuzzy numbers and results in

$$ {}_{gr}{}^{c}D^{\delta}_{2gr}\tilde{z}(t) = \tilde{A}_{gr}\tilde{z}(t) + \tilde{B}_{gr}\tilde{u}(t) \qquad t = [0,T] := J, \tag{2}$$

in which ${}_{gr}{}^{c}D^{\delta}_{2}$ is called CGrT2FuFrD with fractional-order $0 < \delta \leq 1$, where $\tilde{A}$ and $\tilde{B}$ are fuzzy matrices respectively. In the time intervals $[0,t]$, the input ${}_{gr}\tilde{u} \in E^2$ is fuzzy integrable. The work's primary contribution is given below:

- To analyse a novel method for determining the controllability of CGrT2FuFD, triangular type-2 fuzzy number, $\tilde{A}$, is used.
- The granular type-2 fuzzy Laplace transform (GrT2FuLT), type-2 fuzzy inverse Laplace transform (GrT2FuILT). (GrT2FuILT), and matrix Mittag-Leffler function are used to derive the solution.
- The Grammian matrix is used to determine the granular type-2 fuzzy fractional system's controllability.
- We check whether the GrT2FuFrDS system is steerable, that is, whether it can be moved from the starting point $\tilde{z}_0$ to the final stage $\tilde{z}_1$.
- The conclusion is supported by numerical simulation results from MATLAB for the granular type-2 fractional fuzzy-control problem under consideration.

The article is arranged as follows: section two deals with the definitions and theorems which are necessary to solve the system, section three explains about the Laplace transform of the granular type-2 fuzzy dynamical system, section four elucidate on the solution formulation of GrT2FuFrDS, whereas five deals with the necessary condition for proving the controllability of granular type-2 system. The final section deals with numerical simulation obtained through MATLAB code.

2 Preliminaries

The definitions of GrT2FuS and the foundational concepts of fractional calculus in the work are given in this section. The set of all real numbers is represented in this work by the letter $\mathbb{R}$, the fractional order is represented as δ, $\tilde{A}'$ and $\tilde{B}'$ denotes transpose of $\tilde{A}$ and $\tilde{B}$ matrix.

Definition 2.1. [18] *A type-2 triangular interval valued fuzzy number (T2TIVFuN) $\tilde{N}$ is represented by $\tilde{N} = (\mathfrak{a}_1, \mathfrak{a}_2, \mathrm{r}, \mathfrak{a}_3, \mathfrak{a}_4)$ where $a_{\mathcal{I}}, \mathrm{r} \in \mathbb{R}$ for $\mathcal{I} = 1, 2, 3, 4$.*

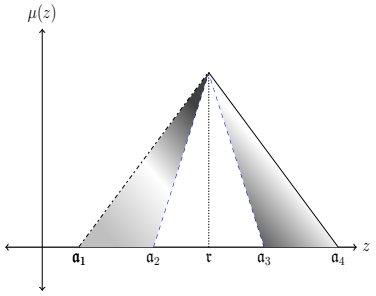

Figure 1: Representation of Type-2 Fuzzy Number

Definition 2.2. [24] *The horizontal membership function of interval type-2 fuzzy number (HoMFoIT2FuN) $\tilde{N}$ is represented as $\tilde{N}_{\mathcal{IH}}(\mu, \underline{\alpha}_{\tilde{N}}, \overline{\alpha}_{\tilde{N}}) = (\underline{f}, \overline{f})$ by $\tilde{N}_{\mathcal{IH}}$ which equals $\underline{N}_{\mathcal{IH}}(\mu, \underline{\alpha}_{\tilde{N}}) = \underline{L}^{\mu} + (\underline{R}^{\mu} - \underline{L}^{\mu})\underline{\alpha}_{\tilde{N}}$ and $\overline{N}_{\mathcal{IH}}(\mu, \overline{\alpha}_{\tilde{N}}) = \overline{L}^{\mu} + (\overline{R}^{\mu} - \overline{L}^{\mu})\overline{\alpha}_{\tilde{N}}$. Here μ is the membership degree of $\tilde{f}$ which lies between $[0, 1]$, $\underline{\alpha}, \overline{\alpha}$ are called the horizontal indices lying between $[0, 1]$.*

Remark 2.3. [24] *The HoMFoIT2FuN of fuzzy number $\tilde{N}$ can be denoted as $\mathcal{H}_2(\tilde{N})$ and are described by*

$$\tilde{N} = \mathcal{H}_2^{-1}\left(\tilde{N}_{\mathcal{IH}}\left(\mu, \underline{\alpha}_N, \bar{\alpha}_N\right)\right) \triangleq \left(\bigcup_{\mu} \mu \underline{N}^{\mu}, \bigcup_{\mu} \mu \bar{N}^{\mu}\right),$$

where

$$\underline{N}^{\mu} = \left[\inf_{\beta \geq \mu} \min_{\underline{\alpha}_N \in [0,1]} \underline{N}_{\mathcal{IH}}\left(\beta, \underline{\alpha}_N\right), \sup_{\beta \geq \mu} \max_{\underline{\alpha}_N \in [0,1]} \underline{N}_{\mathcal{IH}}\left(\beta, \underline{\alpha}_N\right)\right],$$

and

$$\overline{N}^{\mu} = \left[\inf_{\beta \geq \mu} \min_{\overline{\alpha}_N \in [0,1]} \overline{N}_{\mathcal{IH}}\left(\beta, \overline{\alpha}_N\right), \sup_{\beta \geq \mu} \max_{\overline{\alpha}_N \in [0,1]} \overline{N}_{\mathcal{IH}}\left(\beta, \overline{\alpha}_N\right)\right].$$

Definition 2.4. [24] *Consider $\tilde{C}, \tilde{D} \in E^2$. The interrelation $\tilde{C} = \tilde{D}$ and $\tilde{C} \succeq \tilde{B}$ holds good, whenever $\mathcal{H}_2(\tilde{C}) = \mathcal{H}_2(\tilde{D})$ and $\mathcal{H}_2(\tilde{C}) \geq \mathcal{H}_2(\tilde{D})$ $\forall$ $\underline{\alpha}_C = \underline{\alpha}_D \in [0, 1]$ and $\overline{\alpha}_C = \overline{\alpha}_D \in [0, 1]$.*

Remark 2.5. [24] *Let $\tilde{C}$ and $\tilde{D}$ be two T2TIVFuN. In accordance with the previous definition, relations $\tilde{C} = \tilde{D}$ and $\tilde{C} \succeq \tilde{D}$ holds good iff*

$$\begin{cases} \mathcal{H}(\underline{C}) = \mathcal{H}(\underline{D}) \\ \mathcal{H}(\overline{C}) = \mathcal{H}(\overline{D}) \end{cases},$$

and

$$\begin{cases} \mathcal{H}(\underline{C}) \geq \mathcal{H}(\underline{D}) \\ \mathcal{H}(\overline{C}) \geq \mathcal{H}(\overline{D}) \end{cases}.$$

Here $\mathcal{H}$ refers to the HoMF of type-1 interval valued fuzzy number (T1TIVFuN).

Remark 2.6. [12, 24] *The T2TIVFuN $\tilde{N}$ looks like a quintuple $(\overline{L}_N, \underline{L}_N, m_N, \underline{R}_N, \overline{R}_N)$ is the combination of two T1TIVFuN $((\underline{L}_N, m_N, \underline{R}_N), (\overline{L}_N, m, \overline{R}_N))$. The HoMF of $\tilde{N}$, i.e. $\tilde{N}_{\mathcal{IH}}(\mu, \underline{\alpha}_N, \bar{\alpha}_N) = \left(\underline{N}_{\mathcal{IH}}(\mu, \underline{\alpha}_N), \bar{N}_{\mathcal{IH}}(\mu, \bar{\alpha}_N)\right)$ where $\underline{N}_{\mathcal{IH}}(\mu, \underline{\alpha}_N) = \underline{L}_N + \mu(m_N - \underline{L}_N) + (1-\mu)(\underline{R}_N - \underline{L}_N)\underline{\alpha}_N$ & $\bar{N}_{\mathcal{IH}}(\mu, \bar{\alpha}_N) = \bar{L}_N + \mu(m_N - \bar{L}_N) + (1-\mu)(\bar{R}_N - \bar{L}_N)\bar{\alpha}_N$.*

Definition 2.7. [24] *Suppose $\tilde{C}, \tilde{D} \in E^2$ be two fuzzy numbers and let $\odot$ indicates one of the arithmetic operation such as addition, subtraction, multiplication and division, $(\tilde{C} \odot \tilde{D}) = \tilde{E} \in E^2$ iff $\mathcal{H}(E) = (\mathcal{H}_2(\tilde{C}) \odot \mathcal{H}_2(\tilde{D}))$. When $\odot$ is the division operator then 0 should not be in $H_2(\tilde{D})$.*

Remark 2.8. [24] *Based on the Definition 2.7 and the Remark 2.3, we shall prove that the following relations hold true for $\tilde{a}, \tilde{b}, \tilde{c} \in E^2$.*

1. $\tilde{a} - \tilde{b} = -(\tilde{b} - \tilde{a})$
2. $\tilde{a} - \tilde{a} = 0$
3. $\tilde{a} \div \tilde{a} = 1$
4. $(\tilde{a} + \tilde{b})\tilde{c} = \tilde{a}\tilde{b} + \tilde{a}\tilde{b}$

Proof.

$$\begin{aligned}\mathcal{H}(\tilde{a} - \tilde{b}) &= (\mathcal{H}(\underline{a}) - \mathcal{H}(\underline{b}), \mathcal{H}(\bar{a}) - \mathcal{H}(\bar{b})) \\ &= (-(\mathcal{H}(\underline{b}) - \mathcal{H}(\underline{a})), -(\mathcal{H}(\bar{b}) - \mathcal{H}(\bar{a}))) \\ &= \mathcal{H}_2(-(\tilde{\bar{b}} - \tilde{a})).\end{aligned}$$

Then using Definition 2.4, the proof is completed. □

Example 2.9. *Consider the T2TIVFuN* $\tilde{C} = (150, 160, 170, 180, 190)$. *The HoMFoIT2FuN of* $\tilde{C}$ *equals* $\mathcal{H}(\tilde{C}) \triangleq \left(\underline{C}_{\mathcal{IH}}(\mu, \underline{\alpha}_C), \overline{C}_{\mathcal{IH}}(\mu, \overline{\alpha}_C)\right) = (160 + 10\mu + 20(1-\mu)\underline{\alpha}_C, 150 + 20\mu + 40(1-\mu)\overline{\alpha}_C)$. *Using Remark 2.3, the T2TIVFuN* $\tilde{C}$ *can be described as:*

$$\left(\bigcup_{\mu} \mu[160 + 10\mu, 180 - 10\mu], \bigcup_{\mu} \mu[150 + 20\mu, 190 - 20\mu]\right),$$

depicted in Figure 2.

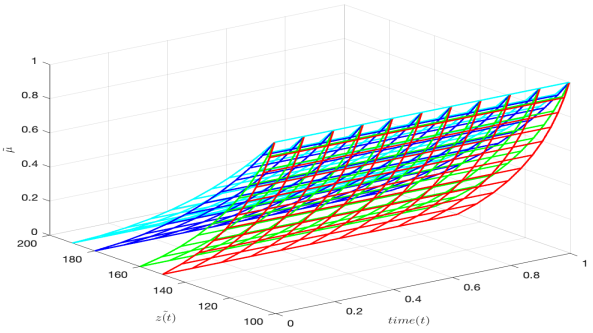

Figure 2: The horizontal membership function of $\tilde{C}$.

Definition 2.10. [24] *Let* $\tilde{p} : [a, b] \subseteq \mathbb{R} \to E^2$ *be an interval valued type-2 fuzzy function (IT2FuF) and let* $\tilde{q} : [a, b] \subseteq \mathbb{R} \to E^2$ *be an n-dimensional IT2FuF on* $[a, b]$. *Suppose* $\tilde{p} \triangleq (\underline{p}, \overline{p})$ *be an IT2FuF such that* $\underline{p} : [c, d] \subseteq \mathbb{R} \to E^1$ *and* $\overline{p} : [a, b] \subseteq \mathbb{R} \to E^1$ *be T1FuFs and* $[c, d] \subseteq [a, b]$. *Suppose that* $\tilde{p}$ *includes* $n \in \mathbb{N}$ *distinct T2TIVFNs* $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$. *Its HoMF at* $t \in [a, b]$ *is denoted by* $\mathcal{H}(\tilde{p}(t)) \triangleq \tilde{p}_{\mathcal{H}}\left(t, \mu, \underline{\alpha}_p, \overline{\alpha}_p\right)$ *and defined as* $\tilde{p}_{\mathcal{H}}\ [a, b] \times [0, 1] \times \underbrace{[0, 1] \times \ldots \times [0, 1]}_{2n} \to$ $([c, d], [a, b])$ *in which* $\underline{\alpha}_p \triangleq \left(\underline{\alpha}_{A_1}, \underline{\alpha}_{A_2}, \ldots, \underline{\alpha}_{A_n}\right)$ *and* $\overline{\alpha}_p \triangleq (\overline{\alpha}_{A_1}, \overline{\alpha}_{A_2}, \ldots, \overline{\alpha}_{A_n})$. *Moreover,*

$$\tilde{p}_{\mathcal{H}}\left(t, \mu, \underline{\alpha}_p, \overline{\alpha}_p\right) = \left(\underline{p}_{\mathcal{H}}\left(t, \mu, \underline{\alpha}_p\right), \overline{p}_{\mathcal{H}}\left(t, \mu, \overline{\alpha}_p\right)\right).$$

Definition 2.11. [24] *Suppose* $\tilde{p}(t)$ *is an IT2FuF such that* $\tilde{p}(t) = \left(\overline{L}_p(t), \underline{L}_p(t), m_p(t), \underline{R}_p(t), \overline{R}_p(t)\right)$. *The IT2FuF* $\tilde{p}(t)$ *called a triangular IT2FuF if this function maps a subset of* $\mathbb{R}$ *into the set of all triangular IT2FuNs.*

Definition 2.12. [24] *Let* $\mathcal{A} = \left[\tilde{A}_{ij}\right]_{n \times m}$ *be the matrix with* $i = 1, 2, \ldots, n$ $j = 1, 2, \ldots, m$, *supposedly called an Interval Type-2 Fuzzy Matrix (IT2FuM) if its entries are T2TIVFuN, i.e.* $\tilde{A}_{ij} \in E^2$. *On top of that, the HoMFoIT2FuN of* $\tilde{\mathcal{A}} = \left[\tilde{A}_{ij}\right]_{n \times m}$ *is equal to* $\mathcal{H}(\tilde{\mathcal{A}}) = \left[\mathcal{H}\left(\tilde{A}_{ij}\right)\right]_{n \times m}$. *The space of* $n \times m$ *IT2FuM is denoted by* $E_2^{n \times m} = E_2^n \times E_2^m$.

Definition 2.13. [24] *The T2FuN* $\tilde{\lambda}_i \in E_2, i = 1, \ldots, n$, *is called an Interval Type-2 Fuzzy eigenvalue (IT2Fu-eigenvalue) of the IT2FM* $\tilde{\mathcal{A}} \in E_2^{n\times n}$, *if*

$$\det\left(\mathcal{H}\left(\tilde{\lambda}_i\right) I_n - \mathcal{H}(\tilde{\mathcal{A}})\right) = 0,$$

holds which means $\det\left(\mathcal{H}\left(\underline{\lambda}_i\right) I_n - \mathcal{H}(\underline{\mathcal{A}})\right) = 0$ *and* $\det\left(\mathcal{H}\left(\overline{\lambda}_i\right) I_n - \mathcal{H}(\overline{\mathcal{A}})\right) = 0$, *where* $\tilde{\mathcal{A}} = (\underline{\mathcal{A}}, \overline{\mathcal{A}})$ *and* $\tilde{\lambda}_i = \left(\underline{\lambda}_i, \overline{\lambda}_i\right)$.

Definition 2.14. [24] *The distance between two T2TIVFuN* $\tilde{C}$ *and* $\tilde{D}$ *is denoted by* ${}_{gr}D_2(\tilde{C}, \tilde{D})$ *and defined as* ${}_{gr}D_2 = \frac{1}{2}(D_{gr}(\bar{C}, \bar{D}) + D_{gr}(\underline{C}, \underline{D})$ *where*

$$D_{gr}(\underline{C}, \underline{D}) = \sup_{\mu\in[0,1]} \max \|\underline{C}_{\mathcal{IH}}(\mu, \alpha_C) - \underline{D}_{\mathcal{IH}}(\mu, \alpha_D)\|,$$
$$D_{gr}(\overline{C}, \overline{D}) = \sup_{\mu\in[0,1]} \max \|\overline{C}_{\mathcal{IH}}(\mu, \alpha_C) - \overline{D}\mathcal{IH}(\mu, \alpha_D)\|.$$

Definition 2.15. [24] *Let* $\tilde{p}(t)$ *be an IT2FuF defined at all values in an open interval containing* t_0 *and* $\tilde{L} \in E^2$. *If all values of the IT2FuF* $\tilde{p}(t)$ *approach the T2FuN* $\tilde{L}$ *as the values of* $t\,(\neq t_0)$ *approach the number* t_0, *then we say that the limit of* $\tilde{p}(t)$ *as* t *approaches* t_0 *is* $\tilde{L}$. *Symbolically, this concept is denoted by* $\lim_{t\to t_0} \tilde{p}(t)$=$\tilde{L}$. *Which means* $\forall\varepsilon > 0$, $\exists$ $\delta > 0$, $\mid t - t_0 \mid < \delta \implies {}_{gr}D^2(\tilde{p}(t), \tilde{L}) < \varepsilon$.

Definition 2.16. [24] *An IT2FuF* $\tilde{p}(t)$ *is continuous at* $t = t_0$ *provided that all three of the following are true:*
(1) $\tilde{p}\,(t_0)$ *exists,*
(2) $\lim_{t\to t_0} \tilde{f}(t)$ *exists,*
(3) $\tilde{p}\,(t_0) = \lim_{t\to t_0} \tilde{f}(t)$.

Definition 2.17. [24] *The Type-2 granular derivative (GrT2D) of an IT2FF* $\tilde{f} : [a, b] \subseteq \mathbb{R} \to E^2$ *for any* $t \in [a, b]$ *is defined by*

$$\lim_{h\to 0} \frac{\tilde{p}(t+h) - \tilde{p}(t)}{h} = {}_{gr}D_2\tilde{p}(t),$$

provided that this limit exists. It should be noted that ${}_{gr}D_2\tilde{p}(t) \in E^2$ *and the limit is taken in the metric space* $\left(E^2, {}_{gr}D^2\right)$.

Definition 2.18. [24] *Let* $\tilde{p} : (a, b) \subset \mathbb{R} \to E^2$. *Then,* $\tilde{p}$ *is said to be type-2 granular differentiable (gr-differentiable for short) at a point* $t_0 \in (a, b)$ *if and only if its HoMF which is* $\mathcal{H}({}_{gr}D_2\tilde{p}(t))$ *can be differentiated wrt t. Also,*

$$\mathcal{H}({}_{gr}D_2\tilde{p}(t)) = (\mathcal{H}({}_{gr}D\underline{p}(t)), \mathcal{H}({}_{gr}D\overline{p}(t))).$$

Definition 2.19. [24] *Let* $\tilde{p} : (a, b) \subset \mathbb{R} \to E^2$. *Then,* $\tilde{p}$ *is said to be type-2 granular integrable (gr-integrable for short) at a point* $t_0 \in (a, b)$ *if and only if its HoMF which is* $\mathcal{H}(\tilde{p}(t))$ *can be integrated wrt t.*

$$\int_a^b \mathcal{H}(\tilde{p}(t)) = (\int_a^b \mathcal{H}(\underline{p}(t)), \int_a^b \mathcal{H}(\overline{p}(t))).$$

Definition 2.20. [6] *If A is a square fuzzy matrix, the matrix Mittag - Leffler function of the matrix A with two parameters* $\alpha, \beta > 0$ *is given by*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\beta(\alpha k + \beta)},$$

and the horizontal membership function of the above defintion is given by

$$E_{\alpha,\beta}({}_{gr}(z(\alpha, \mu_z)) = \sum_{k=0}^{\infty} \frac{{}_{gr}(z(\alpha, \mu_z)^k \alpha, \mu_z)}{\beta(\alpha k + \beta)} \quad \alpha \in [0, 1].$$

3 Type-2 granular fuzzy fractional derivative

In this section, the notions of granular fuzzy fractional integral, granular Riemann–Liouville, and granular Caputo fuzzy fractional derivatives are introduced. Related properties of the granular Riemann–Liouville and granular Caputo fuzzy fractional derivatives are presented. Moreover, the approximations of the granular fuzzy fractional integral and granular Caputo fuzzy fractional derivative are given.

Definition 3.1. ***Granular Fuzzy Fractional Integral*** *Consider the fuzzy function $\tilde{p} : [a,b] \subseteq \mathbb{R} \to E^2$. For the fractional order δ the granular fuzzy fractional derivative of the left and right end points of $\tilde{p}$ are defined by*

$$ {}_{gr}{}_{a}I_{2}^{\delta}{}_{t}\tilde{p}(t) = \frac{1}{\Gamma(\delta)} \oint_{a}^{t} (t-s)^{\delta-1}\tilde{f}(s)ds,$$

and

$$ {}_{gr}{}_{t}I_{2}^{\delta}{}_{b}\tilde{p}(t) = \frac{1}{\Gamma(\delta)} \oint_{t}^{b} (s-t)^{\delta-1}\tilde{f}(s)ds.$$

Definition 3.2. ***Riemann–Liouville Fuzzy Fractional Granular Derivative:*** *Suppose $\tilde{p} : [a,b] \subseteq \mathbb{R} \to E^2$ is considered, then the left and right Riemann–Liouville(R-L) fuzzy fractional granular derivative for the fractional order $\delta \in (0,1)$ is defined as follows*

1.
$$ {}_{gr}{}_{a}D_{2}^{\delta}{}_{t}\tilde{p}(t) = \frac{1}{\Gamma(1-\delta)}{}_{gr}D_{2}(\oint_{a}^{t} \frac{\tilde{p}(s)}{(t-s)^{\delta}})ds. \tag{3}$$

2.
$$ {}_{gr}{}_{t}D_{2}^{\delta}{}_{b}\tilde{p}(t) = -\frac{1}{\Gamma(1-\delta)}{}_{gr}D_{2}(\oint_{t}^{b} \frac{\tilde{p}(s)}{(s-t)^{\delta}})ds. \tag{4}$$

Definition 3.3. ***Caputo Fuzzy Fractional Granular Derivative*** *Suppose $\tilde{p} : [a,b] \subseteq \mathbb{R} \to E^2$ is considered, then the left and right Caputo fuzzy fractional granular derivative for the fractional order $\delta \in (0,1)$ is defined as follows*

1.
$$ {}_{gr}{}_{a}D_{2}^{\delta}{}_{t}\tilde{p}(t) = \frac{1}{\Gamma(1-\delta)}(\oint_{a}^{t} \frac{{}_{gr}D_{2}\tilde{p}(s)}{(t-s)^{\delta}})ds. \tag{5}$$

2.
$$ {}_{gr}{}_{t}D_{2}^{\delta}{}_{b}\tilde{p}(t) = -\frac{1}{\Gamma(1-\delta)}(\oint_{a}^{t} \frac{{}_{gr}D_{2}\tilde{p}(s)}{(s-t)^{\delta}})ds. \tag{6}$$

Remark 3.4. *As like fractional derivatives, fuzzy granular derivatives are also linear operators, which can be proved from below:*

Consider constants $c, d \in \mathbb{R}$ the fuzzy functions $\tilde{p}, \tilde{q} : [a,b] \subseteq \mathbb{R} \to E^2$ thus

1. ${}_{gr}{}_{a}D_{2}^{\delta}{}_{t}(c\tilde{p}(t) + d\tilde{q}(t)) = c({}_{gr}{}_{a}D_{2}^{\delta}{}_{t}\tilde{p}(t)) + d({}_{gr}{}_{a}D_{2}^{\delta}{}_{t}\tilde{q}(t))$.
2. ${}_{gr}{}_{t}D_{2}^{\delta}{}_{b}(c\tilde{p}(t) + d\tilde{q}(t)) = c({}_{gr}{}_{t}D_{2}^{\delta}{}_{b}\tilde{p}(t)) + d({}_{gr}{}_{t}D_{2}^{\delta}{}_{b}\tilde{q}(t))$.

4 Type-2 fuzzy granular Laplace transform

The Laplace transform is one of the most important mathematical tool in solving differential equations and transforming them to an algebraic equation for a better representation of the solution. Hence in this section we will be explaining the definitions and theorems related to type-2 fuzzy granular transform with their properties which helps us to solve the GrT2FuFrDCS.

Definition 4.1. [24] *The type-2 fuzzy function $\tilde{p} : (a,\infty) \to E^2$ is defined as type-2 fuzzy improper integrable if the type-1 fuzzy function $\overline{\tilde{p}}$ and $\underline{\tilde{p}}$ are type-2 fuzzy improper integrable on $(0,\infty)$. Moreover if the interval type-2 horizontal membership function $\mathcal{H}_2(\tilde{p}) = \mathcal{H}(\tilde{p})(t, \mathcal{H}\underline{\tilde{p}}, \mathcal{H}\overline{\tilde{p}})$, then $\oint_a^\infty \mathcal{H}_2(\tilde{p}) = (\int_a^\infty \mathcal{H}(\underline{p}), \int_a^\infty \mathcal{H}(\overline{p}))$.*

Definition 4.2. [24] *Consider a continuous IT2FuF $\tilde{p}$, if $\tilde{p}e^{-st}$ is improper fuzzy integrable on $(0,\infty)$ then $\oint_0^\infty \tilde{p}e^{-st}$ is defined as GrT2FuLT denoted by,*

$$ {}_{gr}\mathbf{L}^2[\tilde{p}(t)] = \left({}_{gr}\mathbf{L}[\underline{p}(t)], {}_{gr}\mathbf{L}[\overline{p}(t)]\right),$$

in which ${}_{gr}\mathbf{L}[\cdot]$ is the type-1 fuzzy Laplace transform, ${}_{gr}\mathbf{L}^2[\cdot]$ is the type-2 fuzzy granular Laplace transform mentioned in [26].

Theorem 4.3. [24] *Consider $\tilde{h}$ and $\tilde{g}$ to be continuous T2FuF, then we can say that the type-2 fuzzy granular Laplace transform and type-2 fuzzy granular inverse Laplace transform are linear operators, which is*

$$_{gr}\mathbf{L}^2[\tilde{a_1}\tilde{h} + \tilde{a_2}\tilde{g}] = \tilde{a_1}\,_{gr}\mathbf{L}^2\tilde{h} + \tilde{a_2}_{gr}\mathbf{L}^2\tilde{g},$$

$$_{gr}\mathbf{L}^{-2}[\tilde{a_1}\tilde{h} + \tilde{a_2}\tilde{g}] = \tilde{a_1}\,_{gr}\mathbf{L}^{-2}\tilde{h} + \tilde{a_2}_{gr}\mathbf{L}^{-2}\tilde{g},$$

where $\tilde{a_1}, \tilde{a_2} \in E^2$.

Theorem 4.4. *Consider $\tilde{h}$ to be continuous IT2FuF on $[a, +\infty)$ and $a > 0$. Suppose that $_{gr}\mathbf{L}^2[\tilde{h}(t)]$,$_{gr}\mathbf{L}^2[\oint_a^t \tilde{h}(s)ds]$ and $_{gr}\mathbf{L}^2[_{gr}D_2^{\delta}\tilde{h}(t)] = s_{gr}^{\delta}\mathbf{L}^2(\tilde{h}(t)) - s^{\delta-1}(\tilde{h}(0))$ exists.*

Proof. Based on the Definitions 2.19,4.2,2.18, which results in

$$_{gr}\mathbf{L}^2[_{gr}D_2^{\delta}\tilde{h}(t)] = (_{gr}\mathbf{L}[_{gr}D^{\delta}\underline{h}(t)],_{gr}\mathbf{L}[_{gr}D^{\delta}\overline{h}(t)]).$$

From [27] wkt,

$$_{gr}\mathbf{L}[_{gr}D^{\delta}\underline{h}(t)] = s_{gr}^{\delta}\mathbf{L}(\underline{h}(t)) - s^{\delta-1}(\underline{h}(0)),$$

and

$$_{gr}\mathbf{L}[_{gr}D^{\delta}\overline{h}(t)] = s_{gr}^{\delta}\mathbf{L}(\overline{h}(t)) - s^{\delta-1}(\underline{h}(0)).$$

Thus the first part is proved. Proof of second part is similar and is omitted. □

5 Solution representation of Cauchy type of type-2 fuzzy fractional order dynamical system (T2FuFrODS)

5.1 Problem statement

The T2FuFrODS that we have considered here for solution finding is given below

$$\begin{cases} _{gr}D_2^{\delta}\tilde{z}(t) = \tilde{A}_{gr}\tilde{z}(t) + _{gr}\tilde{f}(t,_{gr}\tilde{z}(t)), \qquad t = [0,t] := J, \\ _{gr}\tilde{z}(t_0) = _{gr}\tilde{z}(0). \end{cases} \tag{7}$$

wherein $_{gr}D_2^{\delta}$ denotes the type-2 fuzzy fractional granular derivative with the state vector $_{gr}\tilde{z}(t) \in (C[0,T], E^2)$, space of continuous function from $[0,T]$ to E^2, δ defined to be a non-integer order ranging from $0 < \delta \leq 1$ together with the type-2 fuzzy initial conditions, referred as type-2 fuzzy initial state vector $i.e, _{gr}\tilde{z}(0) \in E^2$ and $\tilde{A}$ is a $n \times n$ matrix.

If the following hypothesis holds good then the system 7 is said to be possess a solution, those hypothesis are

1. For a constant matrix N_1 the following condition satisfies $\|E_{\beta,\beta}(t^{\beta}A)\| \leq N_1$.
2. Wkt the function is from $f : [0,T] \times E^2 \to E^2$ then f satisfies,
   - the function $\tilde{f}(.,\omega) : [0,T] \to E^2$ is measurable and function $\tilde{f}(t,.)$ is continuous on $[0,T]$ where $\omega \in E^2$.
   - for $(\epsilon, \overline{\epsilon}) \in E^2$ $\tilde{f}$ satisfies the Lipschitz condition $_{gr}D_2(\tilde{f}(t,\epsilon), \tilde{f}(t,\overline{\epsilon})) \leq (N_2)_{gr}D_2(\epsilon,\overline{\epsilon})$ where N_2 is the Lipschitz constant.
   - If there exist an origin $\tilde{0} \in E^2$ then $_{gr}D_2(\tilde{f}(t,\tilde{0}),\tilde{0}) \leq N_3, N_3 > 0$.

5.2 Existence and uniqueness

Lemma 5.1. *The type-2 Cauchy system 7 is said to possess a mild solution if*

$$_{gr}\tilde{z}(t) = E_{\delta}(\tilde{A}t^{\delta})_{gr}\tilde{z}(0) + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\tilde{A}(t-s)^{\delta}{}_{gr}\tilde{f}(s)ds.$$

Proof. Rewriting the Cauchy problem 7 in parametric form of the type-2 fuzzy fractional system, we get

$$ {}_{gr}D_2^{\delta}\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu) = \tilde{A}\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu) + {}_{gr}\tilde{f}(t,\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu)), $$

for each $t \in J$. As already mentioned $\overline{\alpha},\underline{\alpha},\mu \in [0,1]$. Based on Theorem 4.4, for solving the system we will take granular Laplace transform, we will arrive at

$$\begin{aligned}
s_{gr}^{\delta}\mathbf{L}^2[{}_{gr}\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu)] - s_{gr}^{\delta-1}\mathbf{L}^2[{}_{gr}\tilde{z}(0,\underline{\alpha},\overline{\alpha},\mu)] &= \tilde{A}_{gr}\mathbf{L}^2[{}_{gr}\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu)] \\
&\quad +{}_{gr}\mathbf{L}^2[{}_{gr}\tilde{f}(t,{}_{gr}\,\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu))]. \\
[s^{\delta}I_n - \tilde{A}]_{gr}\mathbf{L}^2[{}_{gr}\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu)] &= {}_{gr}\mathbf{L}^2[{}_{gr}\tilde{z}(0,\underline{\alpha},\overline{\alpha},\mu)] \\
&\quad +{}_{gr}\mathbf{L}^2[{}_{gr}\tilde{f}(t,{}_{gr}\,\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu))]. \\
{}_{gr}\mathbf{L}^2[{}_{gr}\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu)] &= [s^{\delta}I_n - \tilde{A}]_{gr}^{-1}\mathbf{L}^2[{}_{gr}\tilde{z}(0,\underline{\alpha},\overline{\alpha},\mu)] \\
&\quad +{}_{gr}\mathbf{L}^2[{}_{gr}\tilde{f}(t,{}_{gr}\,\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu))]. \\
&= {}_{gr}\mathbf{L}^2[E_{\delta}(t^{\delta}\tilde{A}({}_{gr}\tilde{z}_0(\underline{\alpha},\overline{\alpha},\mu)))] \\
&\quad +gr\mathbf{L}^2[t^{\delta-1}E_{\delta,\delta}(\tilde{A}t^{\delta})] \\
&\quad \times_{gr}\mathbf{L}^2[{}_{gr}\tilde{f}(t,{}_{gr}\,\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu))].
\end{aligned}$$

By applying the convolution theorem,

$$\begin{aligned}
{}_{gr}\mathbf{L}^2\,[{}_{gr}\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu)] &= {}_{gr}\mathbf{L}^2[E_{\delta}(t^{\delta}\tilde{A})_{gr}\tilde{z}_0(\underline{\alpha},\overline{\alpha},\mu)] \\
&\quad +_{gr}\,\mathbf{L}^2\left[\int_0^t (t-s)^{\delta-1}E_{\delta,\delta}(t-s)^{\delta}\tilde{A})_{gr}\tilde{f}(s,{}_{gr}\,\tilde{Z}(s,\underline{\alpha},\overline{\alpha},\mu)ds\right].
\end{aligned}$$

After applying the type-2 fuzzy granular inverse Laplace transform, it forms

$$\begin{aligned}
{}_{gr}\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu) &= E_{\delta}\left(t^{\delta}\tilde{A}\right)_{gr}\tilde{z}_0(\underline{\alpha},\overline{\alpha},\mu) \\
&\quad + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left((t-s)^{\delta}\tilde{A}\right)_{gr}\tilde{f}\,(s,{}_{gr}\,\tilde{z}(s,\underline{\alpha},\overline{\alpha},\mu))\,ds.
\end{aligned}$$

which equals

$$ {}_{gr}\tilde{z}(t) = E_{\delta}\left(At^{\delta}\right)_{gr}\tilde{z}_0 + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(\tilde{A}(t-s)^{\delta}\right)_{gr}\tilde{f}(s,{}_{gr}\tilde{z}(s))ds. $$

□

6 Necessary condition for controllability of the system

$$\begin{cases} {}_{gr}D_2^{\delta}\tilde{z}(t) = A_{gr}\tilde{z}(t) + B_{gr}\tilde{u}(t), & t = [0,T] := J, \\ {}_{gr}\tilde{z}(t_0) = {}_{gr}\tilde{z}(0). \end{cases} \tag{8}$$

Definition 6.1. *A type-2 fuzzy granular fractional dynamical control is said to be granular controllable if* $\exists$ *fuzzy control* $\tilde{u}$ *such that the type-2 granular system is transferred from the initial* $z(0)$ *to final state* $z(T)$ *in the finite interval* $[0,T]$.

The solution of the above defined control system is obtained as

$$\begin{aligned}
{}_{gr}\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu) &= E_{\delta}\left(t^{\delta}A\right)_{gr}\tilde{z}_0(\underline{\alpha},\overline{\alpha},\mu) \\
&\quad + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left((t-s)^{\delta}A\right)_{gr}B\tilde{u}(s)ds.
\end{aligned} \tag{9}$$

Preposition 6.1. *Consider fuzzy fractional dynamical system 8 and let* ${}_{gr}A(\underline{\alpha},\overline{\alpha},\mu)$ *and* ${}_{gr}B(\underline{\alpha},\overline{\alpha},\mu)$ *be the horizontal membership functions of the matrices* $\tilde{A},\tilde{B}$ *, respectively. Then, type-2 fuzzy fractional dynamical system 8 is granular controllable if and only if the granular matrix*

$$\begin{aligned}&\left[{}_{gr}B(\underline{\alpha_B},\overline{\alpha_B},\mu)\quad({}_{gr}A(\underline{\alpha_A},\overline{\alpha_A},\mu))({}_{gr}B(\underline{\alpha_B},\overline{\alpha_B},\mu))\quad(({}_{gr}A\left(\underline{\alpha_A},\overline{\alpha_A},\mu\right))^2)({}_{gr}B(\underline{\alpha_B},\overline{\alpha_B},\mu))\dots\right.\\&\left.({}_{gr}A(\underline{\alpha_A},\overline{\alpha_A},\mu)^{n-1})({}_{gr}B(\underline{\alpha_B},\overline{\alpha_B},\mu))\right].\end{aligned}\tag{10}$$

has full row rank $\forall\underline{\alpha_A},\overline{\alpha_A},\mu,\underline{\alpha_B},\overline{\alpha_B}\in[0,1]$.

Proof. Writing 8 in parametric form we will obtain

$${}_{gr}D_2^{\delta}\tilde{z}(t,\underline{\alpha_z},\overline{\alpha_z},\mu)={}_{gr}\ A(\mu,\underline{\alpha_A},\overline{\alpha_A})\tilde{z}(t,\underline{\alpha_z},\overline{\alpha_z},\mu)+{}_{gr}\ B(\mu,\underline{\alpha_B},\overline{\alpha_B})_{gr}\tilde{u}(t,\underline{\alpha_u},\overline{\alpha_u},\mu),\tag{11}$$

where the initial type-2 fuzzy granular state vector if given as

$${}_{gr}z(t_0,\underline{\alpha_{z_0}},\overline{\alpha_{z_0}})={}_{gr}\ z_0(\underline{\alpha_{z_0}},\overline{\alpha_{z_0}}).$$

Here $\mu,\underline{\alpha_z},\overline{\alpha_z},\underline{\alpha_A},\overline{\alpha_A},\underline{\alpha_B},\overline{\alpha_B},\underline{\alpha_u},\overline{\alpha_u}$ are all constant values between $[0,1]$.

Then we can write the type-2 granular dynamical system as

$${}_{gr}D_2^{\delta}z^{\mu}_{(\underline{\alpha_z},\overline{\alpha_z})}(t)={}_{gr}\ A^{\mu}_{(\underline{\alpha_A},\overline{\alpha_A})}z^{\mu}_{(\underline{\alpha_z},\overline{\alpha_z})}+{}_{gr}\ B^{\mu}_{(\underline{\alpha_B},\overline{\alpha_B})}u^{\mu}_{(\underline{\alpha_u},\overline{\alpha_u})},\tag{12}$$

of which ${}_{gr}A^{\mu}_{(\underline{\alpha_A},\overline{\alpha_A})}$ and ${}_{gr}B^{\mu}_{(\underline{\alpha_B},\overline{\alpha_B})}$ are matrices which are constant. The crisp dynamical system 8 is controllable iff the below given matrix of size m has rank m of which

$$\left[{}_{gr}B^{\mu}_{(\underline{\alpha_B},\overline{\alpha_B})}\quad({}_{gr}A^{\mu}_{(\underline{\alpha_A},\overline{\alpha_A})})({}_{gr}B^{\mu}_{(\underline{\alpha_B},\overline{\alpha_B})})\quad(({}_{gr}A^{\mu}_{(\underline{\alpha_A},\overline{\alpha_A})})^2)({}_{gr}B^{\mu}_{(\underline{\alpha_B},\overline{\alpha_B})})\dots\quad(({}_{gr}A^{\mu}_{(\underline{\alpha_A},\overline{\alpha_A})})^{n-1})({}_{gr}B^{\mu}_{(\underline{\alpha_B},\overline{\alpha_B})})\right],$$

has full rank. Such that it can be concluded that 8 is controllable iff the type-2 fuzzy granular matrix 10 has rank equal to its size for all μ level set. □

Theorem 6.2. *If the type-2 fuzzy granular Grammian matrix*

$$\tilde{W}=\int_0^T(T-s)^{\delta-1}[E_{\delta,\delta}(\tilde{A}(T-s)^{\delta})\tilde{B}][E_{\delta,\delta}(\tilde{A}(T-s)^{\delta})\tilde{B}]'ds,$$

is invertible, then the linear type-2 fuzzy granular control dynamical system can be controlled.

Proof. Consider the type-2 fuzzy granular control function

$${}_{gr}\tilde{u}(t)={}_{gr}\ \tilde{B}'E_{\delta,\delta}({}_{gr}\tilde{A}'(b-t)^{\delta})_{gr}\tilde{W}^{-1}[{}_{gr}\tilde{z}_1-E_{\delta}({}_{gr}\tilde{A}_{gr}\tilde{B}^{\delta})_{gr}\tilde{z}_0].$$

Substituting $T=b$ in the solution equation 9 and expressing it in the parametric form

$${}_{gr}\tilde{z}(b,\underline{\alpha},\overline{\alpha},\mu)=E_{\delta}\left(b^{\delta}\tilde{A}\right)_{gr}\tilde{z}_0(\underline{\alpha},\overline{\alpha},\mu)+\int_0^b(b-s)^{\delta-1}E_{\delta,\delta}\left((b-s)^{\delta}\tilde{A}\right)\tilde{B}\tilde{u}(s)ds.$$

Now the type-2 fuzzy granular control function is substituted to the previous equation

$$\begin{aligned}{}_{gr}\tilde{z}(b,\underline{\alpha},\overline{\alpha},\mu)=&E_{\delta}\left(b^{\delta}\tilde{A}\right)_{gr}\tilde{z}_0(\underline{\alpha},\overline{\alpha},\mu)\\&+\int_0^b(b-s)^{\delta-1}E_{\delta,\delta}\left((b-s)^{\delta}\tilde{A}\right)\tilde{B}\tilde{B}'E_{\delta,\delta}(\tilde{A}'(b-t)^{\delta})\\&\tilde{W}^{-1}[{}_{gr}\tilde{z}_1-E_{\delta}({}_{gr}\tilde{A}\tilde{b}^{\delta})_{gr}\tilde{z}_0]ds.\\=&E_{\delta}\left(b^{\delta}A\right)_{gr}\tilde{z}_0(\underline{\alpha},\overline{\alpha},\mu)_{gr}\tilde{W}\tilde{W}^{-1}[{}_{gr}\tilde{z}_1-E_{\delta}({}_{gr}\tilde{A}b^{\delta})_{gr}\tilde{z}_0]ds.\\{}_{gr}\tilde{z}(b,\underline{\alpha},\overline{\alpha},\mu)=&{}_{gr}\tilde{z}(1,\underline{\alpha},\overline{\alpha},\mu).\end{aligned}$$

□

6.1 Controllability condition of a non-linear system

This section deals with the granular type-2 fuzzy fractional dynamical non-linear system and then prove a theorem that the considered non-linear system is controllable.

$$\begin{cases} {}_{gr}D^{\delta}_{2gr}\tilde{z}(t) = \tilde{A}_{gr}\tilde{z}(t) + \tilde{B}_{gr}\tilde{u}(t) +_{gr} \tilde{f}(t,_{gr}\tilde{z}), \qquad t \in J, \\ {}_{gr}\tilde{z}(t_0) =_{gr} \tilde{z}(0) \end{cases} \tag{13}$$

where $0 < \delta \leq 1$, ${}_{gr}\tilde{f}[0,t] \times E^2 \to E^2$ is continuous. $\tilde{A}$ is an n x n fuzzy matrix and $\tilde{B}$ is an n x m fuzzy matrix.

Theorem 6.3. *Assume that type-2 fuzzy linear granular system* (7) *is controllable, then type-2 fuzzy non-linear system represented in* (13) *on* $[0,T]$ *is controllable iff* f *satisfies the condition.*

$$\|\tilde{f}(t,\tilde{x})\| \leq b_1,$$

$$\|\tilde{f}(t,\tilde{x}_1) - f(t,\tilde{x}_2)\| \leq b_2\|\tilde{x}_1 - \tilde{x}_2\|,$$

$\tilde{x}_1, \tilde{x}_2 \in \mathbb{R}$.

Proof. To establish the theory, we define a few terms and assign them notations. They are

$$n_1 = \|z_1\|, \qquad n_2 = \|z_0\| \qquad \varepsilon = \|B\|,$$
$$\varphi = (T-t)^{\delta-1}E_{\delta,\delta}(\tilde{A}(T-t)^{\delta}), \qquad \varphi_2 = \|\varphi_1\|,$$

$$\pi = (T-t)^{\delta-1}B^{'}E_{\delta,\delta}(\tilde{A}(T-t)^{\delta})\tilde{W}^{-1}.$$

$$\begin{aligned} \|{}_{gr}\tilde{u}_n(t)\| &\leq \|{}_{gr}\tilde{B}^{'}E_{\delta,\delta}({}_{gr}\tilde{A}^{'}(b-t)^{\delta})_{gr}\tilde{W}^{-1}\|\|[{}_{gr}\tilde{z}_1 - E_{\delta}({}_{gr}\tilde{A}_{gr}\tilde{B}^{\delta})_{gr}\tilde{z}_0]\| \\ &\leq \pi[n_1 + \varphi_2 b_1 T] \\ &= \mathtt{x}. \end{aligned}$$

Applying successive approximation technique, we find first few terms

$${}_{gr}z_0(t,\underline{\alpha},\overline{\alpha},\mu) =_{gr} z_0.$$

$$\begin{aligned} {}_{gr}z_{n+1}(t,\underline{\alpha},\overline{\alpha},\mu) =& E_{\delta}\left(t^{\delta}\tilde{A}\right)_{gr}\tilde{z}_0(\underline{\alpha},\overline{\alpha},\mu) + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left((t-s)^{\delta}\tilde{A}\right) \\ & [{}_{gr}\tilde{B}\tilde{u}(s,\underline{\alpha},\overline{\alpha},\mu) +_{gr} \tilde{f}\,(s,_{gr}\tilde{z_{n+1}}(s,\underline{\alpha},\overline{\alpha},\mu))\,ds]. \end{aligned} \tag{14}$$

We need to prove that ${}_{gr}\tilde{z_n}(t)$ is Cauchy.

$$\begin{aligned} \|{}_{gr}z_{n+1}(t,\underline{\alpha},\overline{\alpha},\mu) -_{gr} z_n(t,\underline{\alpha},\overline{\alpha},\mu)\| \leq & \int_0^t \|(t-s)^{\delta-1}E_{\delta,\delta}({}_{gr}\tilde{A}(t-s)^{\delta})\|\|{}_{gr}\tilde{B}\|\|{}_{gr}u_n - \\ & {}_{gr}u_{n-1}\|ds + \int_0^t (t-s)^{\delta-1}E_{\delta,\delta}({}_{gr}\tilde{A}(t-s)^{\delta}) \\ & [{}_{gr}\tilde{f}(s,\tilde{z_n}) -_{gr} \tilde{f}(s,\tilde{z_n}-1)]ds, \\ \|{}_{gr}z_{n+1}(t,\underline{\alpha},\overline{\alpha},\mu) -_{gr} z_n(t,\underline{\alpha},\overline{\alpha},\mu)\| \leq & [\pi\varphi_2^2\varepsilon b_2 T + \varphi_2 b_2] \\ & \int_0^t \|{}_{gr}\tilde{z}_n(s,\underline{\alpha},\overline{\alpha},\mu) -_{gr} \tilde{z}_{n-1}(s,\underline{\alpha},\overline{\alpha},\mu)\|ds, \\ {}_{gr}D(z_{n+1}(t,\underline{\alpha},\overline{\alpha},\mu), z_n(t,\underline{\alpha},\overline{\alpha},\mu)) \leq & [\pi\varphi_2^2\varepsilon b_2 T + \varphi_2 b_2] \\ & \int_0^t \|{}_{gr}\tilde{z}_n(s,\underline{\alpha},\overline{\alpha},\mu) -_{gr} \tilde{z}_{n-1}(s,\underline{\alpha},\overline{\alpha},\mu)\|. \end{aligned}$$

also,

$$\begin{aligned}\|\tilde{z}_1(t,\underline{\alpha},\overline{\alpha},\mu)-\tilde{z}_0(t,\underline{\alpha},\overline{\alpha},\mu)\| &\leq n_2+[\varphi_2\epsilon\mathtt{x}+\varphi_2 b_2 t]\\ &\leq \mathtt{p}_1 T, \mathtt{p}_1>0, \quad \mathtt{p}_1=n_2+[\varphi_2\epsilon\mathtt{x}+\varphi_2 b_2]\\ {}_{gr}D(z_1(t,\underline{\alpha},\overline{\alpha},\mu),z_0(t,\underline{\alpha},\overline{\alpha},\mu)) &\leq \mathtt{p}_1 T.\end{aligned}$$

Therefore

$$\|{}_{gr}z_{n+1}(t,\underline{\alpha},\overline{\alpha},\mu)-{}_{gr}\,z_n(t,\underline{\alpha},\overline{\alpha},\mu)\|=\mathtt{p}_1[\pi\varphi_2^2\varepsilon b_2 T+\varphi_2 b_2]\frac{T^{n+1}}{(n+1)!}.$$

Since the above equation's RHS can be made very small, $\{{}_{gr}\tilde{z}_n(t)\}$ is implied as a granular Cauchy sequence in $\{{}_{gr}\tilde{Z}\}$. Since $\{{}_{gr}\tilde{Z}\}$ is complete $\{{}_{gr}\tilde{z}_n(t)\}$ converges uniformly to $\{{}_{gr}\tilde{z}\}$. Now taking limit on both sides of 14,

$\lim_{n\to\infty}({}_{gr}\tilde{z'}_{n+1}(t,\underline{\alpha},\overline{\alpha},\mu))={}_{gr}\,\tilde{z'}(t,\underline{\alpha},\overline{\alpha},\mu) \quad \forall\, t,\underline{\alpha},\overline{\alpha},\mu\in[0,1]$

Moreover,

$$\|D_2^{\delta}\tilde{z}(t)-D^{\delta}\tilde{z}_{n+1}(t)\|=\varepsilon\varphi_2 t\|u(t)-u_n(T)\|+b_2\varphi_2 T\|\tilde{z}(t)-\tilde{z}_n(t)\|.$$

This implies that, $D_2^{\delta}\tilde{z}_{n+1}(t)$ converging to $D^{\delta}\tilde{z}(t)$ as n tends to ∞ which shows that $\tilde{z}(t)=\tilde{z}_1$, that steers the non-linear type-2 fuzzy system from initial to final state, proving that the system (13) can be controlled. □

Remark 6.4. *It is important to note that the fuzzy-controllability of the system not only depends on the matrix $\tilde{A}$ and $\tilde{B}$, but also the fuzzy initial and final state. Whereas crisp-controllability of the matrix depends only on the matrix A and B.*

7 Numerical validation

The derived conditions in theorems are validated with suitable numerical examples for both linear and nonlinear granular type-2 fuzzy fractional-order system. Initially the controllability Grammian matrix is obtained, then the control and the state solution is obtained consequently. The corresponding simulation results are obtained by using MATLAB simulation.

Example 7.1. *Consider the GrT2FuD representation of the following linear GrT2FuFDCS:*

$$\begin{cases} D_2^{0.4}\tilde{z}(t)=\tilde{z}(t)+\tilde{u}(t), & t\in[0,1]\\ \tilde{z}(0)=(4,5,6,7,8),\\ \tilde{z}(1)=(15,16,17,18,19). \end{cases} \tag{15}$$

where $\tilde{A}$, $\tilde{B}=1=(1,1,1)$.
The horizontal membership function of the type-2 granular fuzzy initial state and the final state respectively is given by

$$\underline{z_0}(\mu,\alpha_{z_0})=[(5+\mu)+(1-\mu)2]\underline{\alpha}_{z_0}, \qquad \underline{z_1}(\mu,\alpha_{z_1})=[(16+\mu)+(1-\mu)2]\underline{\alpha}_{z_1},$$

$$\overline{z_0}(\mu,\alpha_{z_0})=[(4+2\mu)+(1-\mu)4]\overline{\alpha}_{z_0}, \qquad \overline{z_1}(\mu,\alpha_{z_1})=[(15+2\mu)+(1-\mu)4]\overline{\alpha}_{z_1},$$

$\tilde{z}_0$ and $\tilde{z}_1$ can be characterized as

$$[\tilde{z}(0)]^{\mu}=\bigcup_{\mu}\mu[5+\mu,7-\mu],\bigcup_{\mu}\mu[4+2\mu,8-2\mu],$$

$$[\tilde{z}(1)]^{\mu}=\bigcup_{\mu}\mu[16+\mu,18-\mu],\bigcup_{\mu}\mu[15+2\mu,19-2\mu].$$

Based on the fractional-order value $\delta=0.4$, the corresponding Mittag-Leffler matrix function is determined by the following formula,

$$E_{0.4,0.4}\left((t-s)^{0.4}\right)=\sum_{k=0}^{\infty}\frac{(t-s)^{\mathtt{k}}}{\Gamma((\mathtt{k}+1)0.4)}.$$

As discussed in Theorem (6.2) *controllability Grammian matrix is calculated as follows,*

$$\begin{aligned} \mid W(0,1) \mid &= \int_0^1 (1-s)^{0.4-1}[E_{0.4,0.4}((1-s)^{0.4})][E_{0.4,0.4}(1-s)^{(0.4)}]' ds \\ &= 17.5770 \neq 0. \end{aligned}$$

The control $\tilde{u}(t)$ *can be figured out,*

$$\begin{aligned} {}_{gr}\tilde{u}(t) =& b' E_{\delta,\delta}(\tilde{A}'(b-t)^{\delta}) W^{-1}[{}_{gr}\tilde{z}_1 - E_{\delta}(\tilde{A}b^{\delta})_{gr}\tilde{z}_0], \\ {}_{gr}\tilde{u}(t) =& E_{0.4,0.4}((1-t)^{0.4})(0.05715)[{}_{gr}\tilde{z}_1 - E_{\delta}(Ab^{\delta})_{gr}\tilde{z}_0]. \end{aligned}$$

$$\begin{aligned} {}_{gr}\tilde{u}(t) =& (\bigcup_{\mu} \mu E_{0.4,0.4}((1-t)^{0.4})(0.05715)[(16+\mu) - E_{\delta}(Ab^{\delta})(5+\mu)], \\ & E_{0.4,0.4}((1-t)^{0.4})(0.05715)[(18-\mu)) - E_{\delta}(Ab^{\delta})(4+2\mu)], \\ & \bigcup_{\mu} \mu E_{0.4,0.4}((1-t)^{0.4})(0.05715)[(15+2\mu) - E_{\delta}(Ab^{\delta})(7-\mu)], \\ & E_{0.4,0.4}((1-t)^{0.4})(0.05715)[(19-2\mu) - E_{\delta}(Ab^{\delta})(8-2\mu)]. \end{aligned}$$

Finally, the type-2 controllability solution of granular type-2 fuzzy fractional dynamical system (15) *is,*

$$\begin{aligned} {}_{gr}\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu) =& (\bigcup_{\mu} \mu \Big[E_{0.4}\left(t^{0.4}A\right)(5+\mu) \\ &+ \int_0^t (t-s)^{\delta-1} E_{0.4,0.4}\left((t-s)^{\delta}A\right)_{gr} B\tilde{u}(s)ds\Big], \Big[E_{0.4}\left(t^{0.4}A\right)(7-\mu) \\ &+ \int_0^t (t-s)^{\delta-1} E_{0.4,0.4}\left((t-s)^{\delta}A\right)_{gr} B\tilde{u}(s)ds\Big], \bigcup_{\mu} \mu \Big[E_{0.4}\left(t^{0.4}A\right)(4+2\mu) \\ &+ \int_0^t (t-s)^{\delta-1} E_{0.4,0.4}\left((t-s)^{\delta}A\right)_{gr} B\tilde{u}(s)ds\Big], \Big[E_{0.4}\left(t^{0.4}A\right)(8-2\mu) \\ &+ \int_0^t (t-s)^{\delta-1} E_{0.4,0.4}\left((t-s)^{\delta}A\right)_{gr} B\tilde{u}(s)ds\Big]). \end{aligned}$$

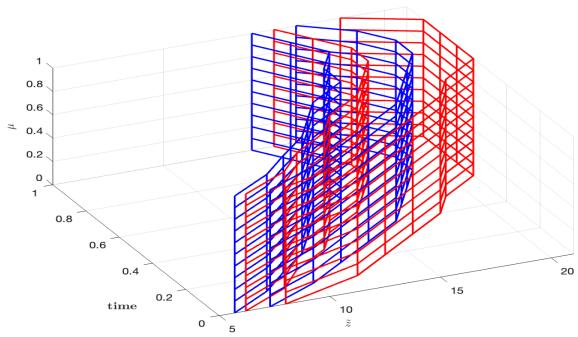

Figure 3: The state response of the type-2 granular fuzzy fractional dynamical system where $\underline{z}$ and $\overline{z}$ are represented in blue and red colours respectively.

Figure 3 obtained through MATLAB shows the steering of the system (15) *controlled trajectory from its type-2 fuzzy granular initial state to the targeted state during [0,1].*

Example 7.2. *Let us take the previous example with $\delta = 0.7$ and solve it for non-linear case.*

$$\begin{cases} D_2^{0.7}{}_{gr}\tilde{z}(t) = \tilde{z}(t) + \tilde{u}(t) + \frac{1}{\sqrt{\tilde{z}}}, & t \in [0,2] \\ \tilde{z}(0) = (0.1, 0.2, 0.3, 0.4, 0.5), \\ \tilde{z}(2) = (0.8, 0.9, 1, 2, 3) \end{cases} \tag{16}$$

with A, b = 1 and ${}_{gr}\tilde{f}(t,\tilde{z}) = \frac{1}{\sqrt{\tilde{z}}}$.
The fractional-order value $\delta = 0.7$, based on which the corresponding Mittag-Leffler matrix function is determined by the following formula:

$$E_{0.7,0.7}\left((t-s)^{0.7}\right) = \sum_{k=0}^{\infty} \frac{(t-s)^{\mathtt{k}}}{\beta((\mathtt{k}+1)0.7)}.$$

As mentioned in Theorem (6.2) *controllability Grammian matrix is given as follows,*

$$W(0,1) = \int_0^2 (2-s)^{0.7-1}[E_{0.7,0.7}((2-s)^{0.7})][E_{0.7,0.7}(2-s)^{(0.7)}]' ds \neq 0.$$

The control ${}_{gr}\tilde{u}(t)$ is given by the equation,

$${}_{gr}\tilde{u}(t) = b' E_{\delta,\delta}(\tilde{A}'(b-t)^{\delta}) W^{-1}[{}_{gr}\tilde{z}_1 - E_{\delta}(\tilde{A}b^{\delta})_{gr}\tilde{z}_0].$$

The granular type-2 fuzzy initial and final state is obtained as follows:
The type-2 controllability solution of non-linear fuzzy fractional dynamical system is,

$$\begin{aligned} {}_{gr}\tilde{z}(t,\underline{\alpha},\overline{\alpha},\mu) = & \Big(\bigcup_{\mu} \mu\Big[E_{0.4}\left(t^{0.4}\tilde{A}\right)(5+\mu) + \int_0^t (t-s)^{\delta-1} E_{0.4,0.4}\left((t-s)^{\delta}\tilde{A}\right)\tilde{B}_{gr}\tilde{u}(s)ds \\ & + \int_0^t (t-s)^{0.7-1} E_{0.7,0.7}(t-s)^{0.7} \frac{1}{\sqrt{{}_{gr}\tilde{z}}}(s)ds\Big], \\ & \Big[E_{0.4}\left(t^{0.4}\tilde{A}\right)(7-\mu) + \int_0^t (t-s)^{\delta-1} E_{0.4,0.4}\left((t-s)^{\delta}\tilde{A}\right)_{gr} B_{gr}\tilde{u}(s)ds \\ & + \int_0^t (t-s)^{0.7-1} E_{0.7,0.7}(t-s)^{0.7} \frac{1}{\sqrt{{}_{gr}\tilde{z}}}(s)ds\Big], \\ & \bigcup_{\mu} \mu\Big[E_{0.4}\left(t^{0.4}\tilde{A}\right)(4+2\mu) + \int_0^t (t-s)^{\delta-1} E_{0.4,0.4}\left((t-s)^{\delta}\tilde{A}\right)\tilde{B}_{gr}\tilde{u}(s)ds \\ & + \int_0^t (t-s)^{0.7-1} E_{0.7,0.7}(t-s)^{0.7} \frac{1}{\sqrt{{}_{gr}\tilde{z}}}(s)ds\Big], \\ & \Big[E_{0.4}\left(t^{0.4}\tilde{A}\right)(8-2\mu) + \int_0^t (t-s)^{\delta-1} E_{0.4,0.4}\left((t-s)^{\delta}\tilde{A}\right)\tilde{B}_{gr}\tilde{u}(s)ds \\ & + \int_0^t (t-s)^{0.7-1} E_{0.7,0.7}(t-s)^{0.7} \frac{1}{\sqrt{{}_{gr}\tilde{z}}}(s)ds\Big]). \end{aligned}$$

As we can see, controllability Grammian matrix, $W \neq 0$ i.e W is invertible. The non-linear function also fulfills the Lipschitz constraint of ${}_{gr}\tilde{f}(t,{}_{gr}\tilde{z})$. Meaning which the function's ${}_{gr}\tilde{f}(t,{}_{gr}\tilde{z})$ continuity and boundedness are proved. Then by Theorem 6.3, the non-linear system (16) *can be controlled.*

8 Conclusion

This paper dealt with the controllability results of GrT2FuFrDCS for both linear and non-linear system and their existence and uniqueness are proved with the respective theorems. Granular type-2 Laplace transform and Granular type-2 inverse Laplace transform are used to arrive at the solution of the granular fractional dynamical system. For validation, the state trajectory have been plotted using MATLAB. The Examples 7.1,7.2 clearly show that the type-2

granular fuzzy dynamical system can be steered which eventually fails while using any other derivatives such as the GH derivative, SGH derivative. Additionally, when we defuzzify the system, solution of ordinary dynamical system is obtained. Here, we've used granular type-2 triangular fuzzy number as the initial condition. However, in the future work, we'll choose granular type-2 trapezoidal fuzzy number to solve the controllability. In the upcoming analysis, an optimal control strategy for the dynamical system will also be investigated in order to lower the cost of the performance index with respect to type-2 fuzzy Caputo differentiability.

Declarations

Funding Statement

The authors declare that no funds, grants, or other support were received during the preparation of this manuscript.

Ethical Statement

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflict Of Interest

The authors have no relevant financial or non-financial interests to disclose.

Acknowledgment

The authors sincerely thank the reviewers for their valuable and constructive feedback during the review process. Their insightful suggestions and detailed comments have significantly improved the quality and clarity of this paper. The authors deeply appreciate their time and effort in reviewing this work.

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