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Bornological convergence and separation in (L, M)-fuzzy bornological vector spaces

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Abstract

In this paper, the notions of (L, M)-fuzzy bornological convergence and separation in (L, M)-fuzzy bornological vector spaces are introduced. Some properties of (L, M)-fuzzy bornological convergence and separation are discussed. The relationships between (L, M)-fuzzy bornologial convergence and separation in (L, M)-fuzzy bornological vector spaces are proposed. Moreover, the relationships of bornological convergence and separation between the framework of (L, M)-fuzzy bornological vector spaces and L-bornological vector spaces are discussed.

Keywords: (L, M)-fuzzy bornological vector space, (L, M)-fuzzy bornological convergence, separation, L-bornological vector space.

1 Introduction

S.T. Hu [11, 12] first introduced the axiomatic definition of bornology to define boundedness in a general topological space. Many researchers further studied bornology from different aspects, such as different hyperspace topologies [2, 5, 15, 17], optimization theory [4], topologies in function spaces [3, 18], and so on [6, 8, 10, 21, 27]. With the development of fuzzy set theory, three types of fuzzy bornologies have been proposed, including fuzzifying bornology [24], L-bornology [1, 19], and (L, M)-fuzzy bornology [16, 26].

In the framework of L-bornology, Paseka et al. [19] introduced L-bornological vector spaces and L-bornological vector systems, and studied the categorical properties of them. Moreover, it is proved that the category of L-bornological vector spaces is isomorphic to a full reflective subcategory of the category of L-bornological vector systems. In [28], Zhang and Zhang discussed the induced I-bornological vector spaces by general bornological vector spaces and I-bornological linear mappings. Since bornological convergence and separation constitute the fundamental facets of the theory of bornological vector spaces, Jin and Yan [13] proposed L-Mackey convergence and separation in L-bornological vector spaces, and discussed the relationships between them.

In an L-bornology, the bounded sets are fuzzy, but the bornology comprising those bounded sets is a crisp subset. Šostak and Uļjane [24] introduced $(L, *)$-valued bornology in another way, which is considered as an L-subset of 2^X. In the setting of $(L, *)$-valued bornology, they proposed induced L-valued bornologies by fuzzy metrics and relative compactness-type L-valued bornologies in Chang-Goguen L-topological spaces. Adopting the terminology of fuzzy topology, we call this bornology an M-fuzzifying bornology. In the setting of M-fuzzifying bornology, Jin and Yan [14] dicussed bornological convergence and separation properties. Later, Shen and Yan [22] discussed fuzzifying bornologies induced by fuzzy pseudo-norms. They proved that the degree of bornological convergence is equivalent to the degree of topological convergence.

In 2017, Šostak and Uļjane [26] introduced M-valued bornology on the L-powerset of an L-valued set (X, E) (where $E : X \times X \to L$ is an L-valued equality on X), which is called LM-fuzzy bornology for short. Recently, Liang et al. [16] introduced a new kind of M-valued L-fuzzy bornological vector spaces, namely (L, M)-fuzzy bornological vector

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spaces. The categorical properties were studied, and its relationships with M-fuzzifying bornology from the perspective of category were discussed. As a continuation of [16], we will focus on bornological convergence and separation in (L, M)-fuzzy bornological vector spaces, and disscuss the relationships between them.

The structure of this paper is organized as follows. In Section 2, we review some preliminaries that are needed in the subsequent sections. In Section 3, we introduce separation of (L, M)-fuzzy bornological vector spaces, and study the relationships of separation between the categories of (L, M)-fuzzy bornological vector spaces and L-bornological vector spaces. In Section 4, we introduce (L, M)-fuzzy bornological convergence in (L, M)-fuzzy bornological vector spaces, and discuss the relationships between (L, M)-fuzzy bornologial convergence and separation in (L, M)-fuzzy bornological vector spaces.

2 Preliminaries

Throught this paper, L (resp. M) is a frame with order-reversing involution. The smallest element and the largest element in L (resp. M) are denoted by $\perp_L$ and $\top_L$ (resp. $\perp_M$ and $\top_M$). An element a in L is called a prime element [9] if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. An element a in L is called co-prime element if $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$.

For a nonempty set X, 2^X denotes the powerset of X, and L^X denotes the set of all L-subsets on X. For all $a \in L$ and $U \in L^X$, $U_{[a]} = \{x \in X \mid U(x) \geqslant a\}$, $U^{(a)} = \{x \in X \mid U(x) \nleqslant a\}$ [23].

Let $f : X \longrightarrow Y$ be a mapping between two nonempty sets X, Y. The forward L-power operator $f^{\rightarrow} : L^X \longrightarrow L^Y$ and the backward L-powerset operator $f^{\leftarrow} : L^Y \longrightarrow L^X$ induced by f are defined as follows [20]:

(1) $\forall\ A \in L^X, y \in Y, f^{\rightarrow}(A)(y) = \bigvee\limits_{f(x)=y} A(x)$;

(2) $\forall\ B \in L^Y, f^{\leftarrow}(B) = B \circ f$.

Let X be a vector space over the field of real or complex numbers $\mathbb{K}$. Then for any $A, B \in L^X$, $x \in X$, and $k \in \mathbb{K}$, the scalar multiplication and addition operator in L^X are defined as follows:

(1) $(A + B)(x) = \bigvee\limits_{y+z=x} (A(y) \wedge B(z))$;

(2) $(kA)(x) = A\left(\frac{x}{k}\right), k \neq 0$;

(3) $(0A)(x) = \begin{cases} \bigvee\limits_{y \in X} A(y), & x = \theta; \\ \perp_L, & x \neq \theta. \end{cases}$

Definition 2.1. [19] *Let $\{X_i\}_{i \in I}$ be a family of sets. For all $A_i \in L^{X_i}$, define a mapping $\prod\limits_{i \in I} A_i : \prod\limits_{i \in I} X_i \longrightarrow L$ as follows $\prod\limits_{i \in I} A_i(x) = \bigwedge\limits_{i \in I} A_i(P_i(x))$, where $P_i : \prod\limits_{i \in I} X_i \longrightarrow X_i$ is the projection.*

Definition 2.2. [1, 19] *An L-bornology on a set X is a subfamily $\mathcal{B} \subseteq L^X$ satisfying:*

(LB1) $\bigvee\limits_{B \in \mathcal{B}} B(x) = \top_L$, $\forall\ x \in X$;

(LB2) $\forall B \in \mathcal{B}$, $A \in L^X$ *with* $A \leqslant B \Rightarrow A \in \mathcal{B}$;

(LB3) $A, B \in \mathcal{B} \Rightarrow A \vee B \in \mathcal{B}$.

The pair $(X, \mathcal{B})$ is called an L-bornological space.

Proposition 2.3. [13] *Let $(X, \mathcal{B})$ be an L-bornological space. Then $(X, \mathbb{K}, \mathcal{B})$ is an L-bornological vector space if and only if $\mathcal{B}$ satisfies the following conditions:*

(LB4) $U, V \in \mathcal{B} \Rightarrow U + V \in \mathcal{B}$;

(LB5) $\forall t \in \mathbb{K}, U \in \mathcal{B} \Rightarrow tU \in \mathcal{B}$;

(LB6) $U \in \mathcal{B} \Rightarrow \bigvee\limits_{|t| \leqslant 1} tU \in \mathcal{B}$.

Definition 2.4. [13] *Let $(X, \mathbb{K}, \mathcal{B})$ be an L-bornological vector space. Then $(X, \mathbb{K}, \mathcal{B})$ is L-separated if and only if $suppM = \{\theta\}$ for all fuzzy vector subspace $M \in \mathcal{B}$, where $suppM = \{x \in X \mid M(x) \neq \perp_L\}$ is the support set of M.*

Definition 2.5. [16, 25] *An M-valued L-fuzzy bornology, or an (L, M)-fuzzy bornology for short on a set X is a mapping $\mathscr{B} : L^X \longrightarrow M$ that satisfies:*

(LMB1)*]$\mathscr{B}(x_{\top_L}) = \top_M$;*

(LMB2) *for each $A, B \in L^X$, $A \leqslant B \Rightarrow \mathscr{B}(A) \geqslant \mathscr{B}(B)$;*

(LMB3) *$\mathscr{B}(A \vee B) \geqslant \mathscr{B}(A) \wedge \mathscr{B}(B)$, $\forall A, B \in L^X$.*

The pair $(X, \mathscr{B})$ is called an (L, M)-fuzzy bornological space. Given (L, M)-fuzzy bornological spaces $(X, \mathscr{B}_X)$ and $(Y, \mathscr{B}_Y)$, a mapping $f : X \longrightarrow Y$ is called (L, M)-fuzzy bounded provided that $\mathscr{B}_X(A) \leqslant \mathscr{B}_Y(f^{\rightarrow}(A))$ for any $A \in L^X$.

Proposition 2.6. [16] *Let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological space. Then*
(1) *$\forall\ r \in P(M)$, $\mathscr{B}^{(r)} = \{A \in L^X : \mathscr{B}(A) \nleqslant r\}$ is an L-bornology on X.*
(2) *$\forall\ r \in M$, $\mathscr{B}_{[r]} = \{A \in L^X : \mathscr{B}(A) \geqslant r\}$ is an L-bornology on X.*

Proposition 2.7. [16] *Supposed that $\{(X_i, \mathscr{B}_i)\}_{i \in I}$ is a family of (L, M)-fuzzy bornological spaces and $X = \prod\limits_{i \in I} X_i$. Define $\prod\limits_{i \in I} \mathscr{B}_i : L^X \longrightarrow M$ by*

$$\prod_{i \in I} \mathscr{B}_i(A) = \bigvee_{A \leqslant \prod_{i \in I} A_i} \bigwedge_{i \in I} \mathscr{B}_i(A_i), \forall A \in L^X.$$

Then $(X, \prod\limits_{i \in I} \mathscr{B}_i)$ is an (L, M)-fuzzy bornological space, which is called the product space of $\{(X_i, \mathscr{B}_i)\}_{i \in I}$.

Proposition 2.8. [16] *Let X be a vector space over $\mathbb{K}$, and let $(X, \mathscr{B})$ be an (L, M)-fuzzy bornological space. Then $(X, \mathbb{K}, \mathscr{B})$ is an (L, M)-fuzzy bornological vector space if and only if $\mathscr{B}$ satisfies the following conditions:*
(BV3) *$\mathscr{B}(A) \wedge \mathscr{B}(B) \leqslant \mathscr{B}(A + B)$, $\forall A, B \in L^X$;*
(BV4) *$\mathscr{B}(A) \leqslant \mathscr{B}(\lambda A)$, $\forall A \in L^X, \lambda \in \mathbb{K}$;*
(BV5) *$\mathscr{B}(A) \leqslant \mathscr{B}\left(\bigvee\limits_{|\lambda| \leqslant 1} \lambda A\right)$, $\forall A \in L^X$.*

3 Separation of (L, M)-fuzzy bornological vector spaces

In this section, we introduce separation of (L, M)-fuzzy bornological vector spaces, and discuss the relationships of separation between the categories of (L, M)-fuzzy bornological vector spaces and L-bornological vector spaces.

Definition 3.1. *Let $(X, \mathbb{K}, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space. Define the degree $Sep(X, \mathbb{K}, \mathscr{B})$ to which $(X, \mathbb{K}, \mathscr{B})$ is separated as follows:*

$$Sep(X, \mathbb{K}, \mathscr{B}) = \bigwedge_{A \in Vecs(X)} \left\{ \mathscr{B}(A)' : \exists \lambda \in L, A^{(\lambda)} \neq \{\theta\} \right\},$$

where $Vecs(X) = \left\{A \in L^X : \forall\ r \in L, A^{(r)} \text{ is a vector subspace of } X\right\}$.

Remark 3.2. *Let $L = M = \{0, 1\}$, and $Sep(X, \mathbb{K}, \mathscr{B}) = 1$. Then for all $A \in 2^X$, if A is the vector subspace of X and $A \neq \{\theta\}$, we have $\mathscr{B}(A)' = 1$, i.e., $\mathscr{B}(A) = 0$. This is exactly the definition of separation in classical bornological vector spaces.*

By combining Propositions 2.3 and 2.6, we have the following conclusion.

Lemma 3.3. *Let $(X, \mathbb{K}, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space. Then*
(1) *For any $r \in P(M)$, $\left(X, \mathbb{K}, \mathcal{B}^{(r)}\right)$ is an L-bornological vector space.*
(2) *For any $r \in M$, $\left(X, \mathbb{K}, \mathcal{B}_{[r]}\right)$ is an L-bornological vector space.*

Proposition 3.4. *Let $(X, \mathbb{K}, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space. Then for any $r \in P(M)$, $Sep(X, \mathbb{K}, \mathscr{B}) \geqslant r'$ if and only if $\left(X, \mathbb{K}, \mathscr{B}^{(r)}\right)$ is L-separated.*

Proof. Necessity. Take any $r \in P(M)$. If $\left(X, \mathbb{K}, \mathscr{B}^{(r)}\right)$ is not L-separated, then there exists a fuzzy vector subspace $A \in L^X$ such that $A \in \mathscr{B}^{(r)}$ and $\text{supp}A \neq \{\theta\}$. Hence there exists $\lambda \in L$ such that $A^{(\lambda)} \neq \{\theta\}$ and $A \in Vecs(X)$. This implies

$$\bigvee_{A \in Vecs(X)} \left\{\mathscr{B}(A) : \exists \lambda \in L, A^{(\lambda)} \neq \{\theta\}\right\} \nleqslant r.$$

It follows immediately that

$$Sep(X, \mathbb{K}, \mathscr{B}) = \bigwedge_{A \in Vecs(X)} \left\{\mathscr{B}(A)^{'} : \exists \lambda \in L, A^{(\lambda)} \neq \{\theta\}\right\} \ngeqslant r^{'},$$

which is a contradiction.

Sufficiency. Take any $r \in P(M)$. If $Sep(X, \mathbb{K}, \mathscr{B}) \ngeqslant r^{'}$, then there exist $A \in Vecs(X)$ and $\lambda \in L$ such that $A^{(\lambda)} \neq \{\theta\}$ and $\mathscr{B}(A)^{'} \ngeqslant r^{'}$. This implies that A is a fuzzy vector subsapce of X and $A \in \mathscr{B}^{(r)}$, but $\text{supp}A \neq \{\theta\}$. Thus $\left(X, \mathbb{K}, \mathscr{B}^{(r)}\right)$ is not L-separated, which is a contradiction. □

Proposition 3.5. *Suppose that $(X, \mathbb{K}, \mathscr{B}_X)$ and $(Y, \mathbb{K}, \mathscr{B}_Y)$ are two (L, M)-fuzzy bornological vector spaces, and that $f : X \longrightarrow Y$ is an (L, M)-fuzzy bounded linear mapping. Then $Sep(Y, \mathbb{K}, \mathcal{B}_Y) \leqslant Sep(X, \mathbb{K}, \mathcal{B}_X)$.*

Proof. Since f is a linear mapping, it follows that for all $A \in Vecs(X)$, $f^{\rightarrow}(A) \in Vecs(Y)$. This implies

$$\begin{aligned} Sep(Y, \mathbb{K}, \mathscr{B}_Y) &= \bigwedge_{B \in Vecs(Y)} \left\{\mathscr{B}_Y(B)^{'} : \exists \mu \in L, B^{(\mu)} \neq \{\theta_Y\}\right\} \\ &\leqslant \bigwedge_{A \in Vecs(X)} \left\{\mathscr{B}_Y(f^{\rightarrow}(A))^{'} : \exists \mu \in L, (f^{\rightarrow}(A))^{(\mu)} \neq \{\theta_Y\}\right\} \\ &\leqslant \bigwedge_{A \in Vecs(X)} \left\{\mathscr{B}_X(A)^{'} : \exists \mu \in L, A^{(\mu)} \neq \{\theta_X\}\right\} \\ &= Sep(X, \mathbb{K}, \mathscr{B}_X). \end{aligned}$$

This completes the proof. □

Proposition 3.6. *Let $(X, \mathbb{K}, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space and let Y be a vector subspace of X. Define $\mathscr{B}|_Y : L^Y \longrightarrow M$ by*

$$\mathscr{B}|_Y(U) = \bigvee_{V \in L^X, V|_Y = U} \mathscr{B}(V), \ \forall\, U \in L^Y.$$

Then $(Y, \mathbb{K}, \mathscr{B}|_Y)$ is an (L, M)-fuzzy bornological vector space, which is called the subspace of $(X, \mathbb{K}, \mathscr{B})$.

Proof. It suffices to show that $\mathscr{B}|_Y$ satisfies (LMB1)-(LMB3) and (BV3)-(BV5).

(LMB1) Take any $x \in Y \subseteq X$. Since $\mathscr{B}(x_{\top_L}) = \top_M$, we have $\mathscr{B}|_Y(x_{\top_L}) = \top_M$.

(LMB2) Take any $A, B \in L^Y$ with $A \leqslant B$. For any $U \in L^X$ with $U|_Y = B$, let

$$V(x) = \begin{cases} A(x), & x \in Y; \\ U(x), & x \notin Y. \end{cases}$$

For any $x \in Y$, $V(x) = A(x) \leqslant B(x) = U|_Y(x) = U(x)$, it follows that $V \leqslant U$. This implies $\mathscr{B}(U) \leqslant \mathscr{B}(V)$. Since $V|_Y = A$, it holds that

$$\mathscr{B}|_Y(B) = \bigvee_{U \in L^X, U|_Y = B} \mathscr{B}(U) \ \leqslant \bigvee_{V \in L^X, V|_Y = A} \mathscr{B}(V) = \mathscr{B}|_Y(A).$$

(LMB3) For any $A, B \in L^Y$, we can obtain that

$$\begin{aligned}
\mathscr{B}|_Y(A) \wedge \mathscr{B}|_Y(B) &= \bigvee_{V \in L^X, V|_Y = A} \mathscr{B}(V) \wedge \bigvee_{U \in L^X, U|_Y = B} \mathscr{B}(U) \\
&= \bigvee_{V \in L^X, V|_Y = A} \ \bigvee_{U \in L^X, U|_Y = B} (\mathscr{B}(V) \wedge \mathscr{B}(U)) \\
&\leqslant \bigvee_{V \in L^X, V|_Y = A} \ \bigvee_{U \in L^X, U|_Y = B} \mathscr{B}(V \vee U) \\
&\leqslant \bigvee_{W \in L^X, W|_Y = A \vee B} \mathscr{B}(W) \\
&= \mathscr{B}|_Y(A \vee B).
\end{aligned}$$

In a similar way, we can check (BV3).
(BV4) For any $U \in L^X$ and $\lambda \in \mathbb{K}$, we have $\lambda(U|_Y) = (\lambda U)|_Y$. Then for any $A \in L^Y$,

$$\begin{aligned}
\mathscr{B}|_Y(A) &= \bigvee_{U \in L^X, U|_Y = A} \mathscr{B}(U) \\
&\leqslant \bigvee_{U \in L^X, (\lambda U)|_Y = \lambda A} \mathscr{B}(\lambda U) \\
&= \bigvee_{W \in L^X, W|_Y = \lambda A} \mathscr{B}(W) \\
&= \mathscr{B}|_Y(\lambda A).
\end{aligned}$$

In a similar way, we can check (BV5). Therefore, $(Y, \mathbb{K}, \mathscr{B}|_Y)$ is an (L, M)-fuzzy bornological vector space. □

Proposition 3.7. *Let $(X, \mathbb{K}, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space and $(Y, \mathbb{K}, \mathscr{B}|_Y)$ be the subspace of $(X, \mathbb{K}, \mathscr{B})$. Then $Sep(X, \mathbb{K}, \mathscr{B}) \leqslant Sep(Y, \mathbb{K}, \mathscr{B}|_Y)$.*

Proof. For any $B \in L^Y$, let

$$A(x) = \begin{cases} B(x), & x \in Y; \\ \perp_L, & x \notin Y. \end{cases}$$

First, we show that $\mathscr{B}|_Y(B) = \mathscr{B}(A)$. On one hand, since $A|_Y = B$, it follows that $\mathscr{B}|_Y(B) \geqslant \mathscr{B}(A)$. On the other hand, for any $C \in L^X$ with $C|_Y = B$, we have $C \geqslant A$, consequently, $\mathscr{B}(C) \leqslant \mathscr{B}(A)$. This implies

$$\mathscr{B}|_Y(B) = \bigvee_{C \in L^X, C|_Y = B,} \mathscr{B}(C) \leqslant \mathscr{B}(A).$$

Therefore, $\mathscr{B}|_Y(B) = \mathscr{B}(A)$. Since for any $r \in L$, $A^{(r)} = B^{(r)}$, it holds that

$$\begin{aligned}
Sep(X, \mathbb{K}, \mathscr{B}) &= \bigwedge_{A \in Vecs(X)} \left\{ \mathscr{B}(A)' : \exists \lambda \in L, A^{(\lambda)} \neq \{\theta\} \right\} \\
&\leqslant \bigwedge_{B \in Vecs(Y)} \left\{ \mathscr{B}|_Y(B)' : \exists \lambda \in L, B^{(\lambda)} \neq \{\theta\} \right\} \\
&= Sep(Y, \mathbb{K}, \mathscr{B}|_Y).
\end{aligned}$$

This completes the proof. □

By combining Definition 2.5 and Proposition 2.8, we have the following result.

Proposition 3.8. *Let $\{(X, \mathbb{K}, \mathscr{B}_i)\}_{i \in I}$ be a family of (L, M)-fuzzy bornological vector spaces. Define $\bigwedge_{i \in I} \mathscr{B}_i : L^X \longrightarrow M$ by $\left(\bigwedge_{i \in I} \mathscr{B}_i\right)(A) = \bigwedge_{i \in I} \mathscr{B}_i(A), \forall A \in L^X$. Then $\left(X, \mathbb{K}, \bigwedge_{i \in I} \mathscr{B}_i\right)$ is an (L, M)-fuzzy bornological vector space.*

Proposition 3.9. *Let $\{(X,\mathbb{K},\mathscr{B}_i)\}_{i\in I}$ be a family of (L,M)-fuzzy bornological vector spaces. Then $\bigwedge\limits_{i\in I} Sep(X,\mathbb{K},\mathscr{B}_i)\leqslant Sep\left(X,\mathbb{K},\bigwedge\limits_{i\in I}\mathscr{B}_i\right)$.*

Proof. By Definition 3.1, it holds that

$$\begin{aligned}\bigwedge_{i\in I} Sep(X,\mathbb{K},\mathscr{B}_i) &= \bigwedge_{i\in I}\bigwedge_{A\in Vecs(X)}\left\{\mathscr{B}_i(A)^{'}:\exists\lambda\in L, A^{(\lambda)}\neq\{\theta\}\right\}\\ &\leqslant \bigwedge_{A\in Vecs(X)}\left\{\left(\bigwedge_{i\in I}\mathscr{B}_i(A)\right)^{'}:\exists\lambda\in L, A^{(\lambda)}\neq\{\theta\}\right\}\\ &= Sep\left(X,\mathbb{K},\bigwedge_{i\in I}\mathscr{B}_i\right).\end{aligned}$$

This completes the proof. □

Proposition 3.10. *Let $\{(X_i,\mathbb{K},\mathscr{B}_i)\}_{i\in I}$ be a family of (L,M)-fuzzy bornological vector spaces and $X=\prod\limits_{i\in I}X_i$. Then $(X,\mathbb{K},\prod\limits_{i\in I}\mathscr{B}_i)$ is an (L,M)-fuzzy bornological vector space, which is called the product space of $\{(X_i,\mathbb{K},\mathscr{B}_i)\}_{i\in I}$. Moreover, the projection $P_i:X\longrightarrow X_i$ is (L,M)-fuzzy bounded.*

Proof. By Proposition 2.7, it suffices to show that $(X,\mathbb{K},\prod\limits_{i\in I}\mathscr{B}_i)$ satisfies (BV3)-(BV5).

(BV3) Take any $A,B\in L^X$ and $A_i,B_i\in L^{X_i}$ such that $A\leqslant\prod\limits_{i\in I}A_i$ and $B\leqslant\prod\limits_{i\in I}B_i$. Since for any $x\in X$,

$$\begin{aligned}(A+B)(x) &\leqslant \left(\prod_{i\in I}A_i+\prod_{i\in I}B_i\right)(x)\\ &= \bigvee_{y+z=x}\left((\prod_{i\in I}A_i)(y)\wedge(\prod_{i\in I}B_i)(z)\right)\\ &= \bigvee_{y+z=x}\left(\bigwedge_{i\in I}A_i(P_i(y))\right)\wedge\left(\bigwedge_{i\in I}B_i(P_i(z))\right)\\ &\leqslant \bigwedge_{i\in I}\bigvee_{y+z=x}(A_i(P_i(y))\wedge B_i(P_i(z)))\\ &\leqslant \prod_{i\in I}(A_i+B_i)(x),\end{aligned}$$

it follows that $A+B\leqslant\prod\limits_{i\in I}(A_i+B_i)$. Then

$$\begin{aligned}\prod_{i\in I}\mathscr{B}_i(A)\wedge\prod_{i\in I}\mathscr{B}_i(B) &= \left(\bigvee_{A\leqslant\prod\limits_{i\in I}A_i}\bigwedge_{i\in I}\mathscr{B}_i(A_i)\right)\wedge\left(\bigvee_{B\leqslant\prod\limits_{i\in I}B_i}\bigwedge_{i\in I}\mathscr{B}_i(B_i)\right)\\ &= \bigvee_{A\leqslant\prod\limits_{i\in I}A_i}\bigvee_{B\leqslant\prod\limits_{i\in I}B_i}\bigwedge_{i\in I}(\mathscr{B}_i(A_i)\wedge\mathscr{B}_i(B_i))\\ &\leqslant \bigvee_{A+B\leqslant\prod\limits_{i\in I}(A_i+B_i)}\bigwedge_{i\in I}\mathscr{B}_i(A_i+B_i)\\ &= \prod_{i\in I}\mathscr{B}_i(A+B).\end{aligned}$$

In a similar way, we can check (BV4) and (BV5). Therefore, $(X, \mathbb{K}, \prod_{i\in I} \mathscr{B}_i)$ is an (L, M)-fuzzy bornological vector space. Next, we show that P_i is (L, M)-fuzzy bounded. For any $\prod_{i\in I} A_i \in L^X, i \in I$ and $x_i \in X_i$,

$$\begin{aligned}
P_i^{\rightarrow}\left(\prod_{i\in I} A_i\right)(x_i) &= \bigvee_{P_i(x)=x_i} \left(\prod_{i\in I} A_i\right)(x) \\
&= \bigvee_{P_i(x)=x_i} \bigwedge_{i\in I} A_i(P_i(x)) \\
&= \bigwedge_{i\in I} A_i(x_i) \\
&\leqslant A_i(x_i),
\end{aligned}$$

it follows that $P_i^{\rightarrow}\left(\prod_{i\in I} A_i\right) \leqslant A_i$. Then for any $A \in L^X$ and $i \in I$,

$$\begin{aligned}
\prod_{i\in I} \mathscr{B}_i(A) &= \bigvee_{A\leqslant \prod_{i\in I} A_i} \bigwedge_{i\in I} \mathscr{B}_i(A_i) \\
&\leqslant \bigvee_{A\leqslant \prod_{i\in I} A_i} \mathscr{B}_i(A_i) \\
&\leqslant \bigvee_{P_i^{\rightarrow}(A)\leqslant A_i} \mathscr{B}_i(A_i) \\
&\leqslant \mathscr{B}_i(P_i^{\rightarrow}(A)).
\end{aligned}$$

This completes the proof. □

Theorem 3.11. *Let $\{(X_i, \mathbb{K}, \mathscr{B}_i)\}_{i\in I}$ be a family of (L, M)-fuzzy bornological vector spaces and $X = \prod_{i\in I} X_i$. Then $\bigwedge_{i\in I} Sep(X_i, \mathbb{K}, \mathscr{B}_i) \leqslant Sep(X, \mathbb{K}, \prod_{i\in I} \mathscr{B}_i)$.*

Proof. Take any $\alpha \in M$ with $\alpha \nleqslant Sep(X, \mathbb{K}, \prod_{i\in I} \mathscr{B}_i)$. Then there exist $A \in Vecs(X)$ and $\lambda \in L$ such that $A^{(\lambda)} \neq \{\theta\}$ and $\alpha \nleqslant \prod_{i\in I} \mathscr{B}_i(A)'$. Since $(P_i^{\rightarrow}(A))^{(\lambda)} = P_i(A^{(\lambda)})$, we know that there exists $i_0 \in I$ such that $(P_{i_0}^{\rightarrow}(A))^{(\lambda)} \neq \{\theta_{i_0}\}$ and $P_{i_0}^{\rightarrow}(A) \in Vecs(X_{i_0})$. It follows from Proposition 3.10 that $\prod_{i\in I} \mathscr{B}_i(A) \leqslant \mathscr{B}_{i_0}(P_{i_0}^{\rightarrow}(A))$, hence $\alpha \nleqslant \mathscr{B}_{i_0}(P_{i_0}^{\rightarrow}(A))'$. This implies

$$\begin{aligned}
\alpha &\nleqslant \bigwedge_{A_{i_0}\in Vecs(X_{i_0})} \left\{\mathscr{B}_{i_0}(A_{i_0})' : \exists\lambda \in L, A_{i_0}^{(\lambda)} \neq \{\theta_{i_0}\}\right\} \\
&= Sep(X_{i_0}, \mathbb{K}, \mathscr{B}_{i_0}),
\end{aligned}$$

consequently, $\alpha \nleqslant \bigwedge_{i\in I} Sep(X_i, \mathbb{K}, \mathscr{B}_i)$. Therefore, $\bigwedge_{i\in I} Sep(X_i, \mathbb{K}, \mathscr{B}_i) \leqslant Sep(X, \mathbb{K}, \mathscr{B})$ as desired. □

4 (L, M)-fuzzy bornological convergence

In this section, we introduce (L, M)-fuzzy bornological convergence in (L, M)-fuzzy bornological vector spaces, and discuss the relationships between (L, M)-fuzzy bornologial convergence and separation in (L, M)-fuzzy bornological vector spaces. In [10], the author introduced circled sets as follows.

Definition 4.1. [7] *Let X be a vector space over $\mathbb{K}$ and $A \in L^X$. We say that A is circled if $tA \leqslant A$ for any $t \in \mathbb{K}$ with $|t| \leqslant 1$. The set of all circled L-subsets is denoted by $Cir(X, \mathbb{K})$.*

Definition 4.2. *Let $(X,\mathbb{K},\mathscr{B})$ be an (L,M)-fuzzy bornological vector space. For any $\lambda\in L$, $x^n\in X$, and $\{x^n_{\lambda_n}\}_{n\in N}\subseteq J(L^X)$ with $\bigvee_{n\in N}\bigwedge_{m\geqslant n}\lambda_m\geqslant\lambda$, define the degree $Con(x^n_{\lambda_n},\theta_\lambda)$ to which $\{x^n_{\lambda_n}\}_{n\in N}$ is (L,M)-fuzzy bornologically convergent to θ_λ as follows:*

$$Con(x^n_{\lambda_n},\theta_\lambda)=\bigvee_{A\in Cir(X,\mathbb{K}),\mu_n\to 0}\left\{\mathscr{B}(A): x^n_{\lambda_n}\nleqslant\mu_nA^{'},\forall n\in N\right\}.$$

Moreover, for any $x\in X$, the degree $Con(x^n_{\lambda_n},x_\lambda)$ to which $\{x^n_{\lambda_n}\}_{n\in N}$ is (L,M)-fuzzy bornologically convergent to x_λ is defined as follows:

$$Con(x^n_{\lambda_n},x_\lambda)=Con((x^n-x)_{\lambda_n},\theta_\lambda).$$

Remark 4.3. *Let $L=M=\{0,1\}$, and $Con(x^n_{\lambda_n},\theta_\lambda)=1$. Then there exist $A\in Cir(X,\mathbb{K})$ and $\{\mu_n\}_{n\in N}\subseteq\mathbb{K}$ such that $\mathscr{B}(A)=1$, $\lim_{n\to\infty}\mu_n=0$, and for all $n\in N, x^n_{\lambda_n}\nleqslant\mu_nA^{'}$. This is exactly the definition of bornological convergence in classical bornological vector space.*

Proposition 4.4. *Let $(X,\mathbb{K},\mathscr{B})$ be an (L,M)-fuzzy bornological vector space. Then for all $\{x^n_{\lambda_n}\}_{n\in N},\{y^n_{\lambda_n}\}_{n\in N}\subseteq J(L^X)$, and $x_\lambda,y_\lambda\in J(L^X)$ with $\bigvee_{n\in N}\bigwedge_{m\geqslant n}\lambda_m\geqslant\lambda$,*

$$Con(x^n_{\lambda_n},x_\lambda)\wedge Con(y^n_{\lambda_n},y_\lambda)\leqslant Con(x^n_{\lambda_n}+y^n_{\lambda_n},x_\lambda+y_\lambda).$$

Proof. Take any $A,B\in Cir(X,\mathbb{K})$, and $\{\alpha_n\}_{n\in N},\{\beta_n\}_{n\in N}\subseteq\mathbb{K}$ such that $\lim_{n\to\infty}\alpha_n=0$, $\lim_{n\to\infty}\beta_n=0$, and for all $n\in N,(x^n-x)_{\lambda_n}\nleqslant\alpha_nA^{'}$, $(y^n-y)_{\lambda_n}\nleqslant\beta_nB^{'}$. Then for all $n\in N$, $\lambda_n\nleqslant\alpha_nA^{'}(x^n-x)\vee\beta_nB^{'}(y^n-y)$. Since $A,B\in Cir(X,\mathbb{K})$, it follows that $A+B\in Cir(X,\mathbb{K})$ and $\lambda_n\nleqslant(\alpha_n\vee\beta_n)(A+B)^{'}(x^n-x+y^n-y)$, i.e., $(x^n-x+y^n-y)_{\lambda_n}\nleqslant(\alpha_n\vee\beta_n)(A+B)^{'}$. Therefore,

$$\begin{aligned}
Con(x^n_{\lambda_n},x_\lambda)\wedge Con(y^n_{\lambda_n},y_\lambda) &= Con((x^n-x)_{\lambda_n},\theta_\lambda)\wedge Con((y^n-y)_{\lambda_n},\theta_\lambda)\\
&= \bigvee_{\substack{A\in Cir(X,\mathbb{K})\\ \alpha_n\to 0}}\left\{\mathscr{B}(A):(x^n-x)_{\lambda_n}\nleqslant\alpha_nA^{'},\forall n\in N\right\}\wedge\\
&\quad\ \bigvee_{\substack{B\in Cir(X,\mathbb{K})\\ \beta_n\to 0}}\left\{\mathscr{B}(B):(y^n-y)_{\lambda_n}\nleqslant\beta_nB^{'},\forall n\in N\right\}\\
&= \bigvee_{\substack{A,B\in Cir(X,\mathbb{K})\\ \alpha_n\to 0,\beta_n\to 0}}\left\{\mathscr{B}(A)\wedge\mathscr{B}(B):(x^n-x)_{\lambda_n}\nleqslant\alpha_nA^{'},(y^n-y)_{\lambda_n}\nleqslant\beta_nB^{'},\forall n\in N\right\}\\
&\leqslant \bigvee_{\substack{A,B\in Cir(X,\mathbb{K})\\ \alpha_n\to 0,\beta_n\to 0}}\left\{\mathscr{B}(A+B):(x^n-x+y^n-y)_{\lambda_n}\nleqslant(\alpha_n\vee\beta_n)(A+B)^{'},\forall n\in N\right\}\\
&\leqslant \bigvee_{\substack{C\in Cir(X,\mathbb{K})\\ \gamma_n\to 0}}\left\{\mathscr{B}(C):(x^n-x+y^n-y)_{\lambda_n}\nleqslant\gamma_nC^{'},\forall n\in N\right\}\\
&= Con((x^n-x+y^n-y)_{\lambda_n},\theta_\lambda)\\
&= Con(x^n_{\lambda_n}+y^n_{\lambda_n},x_\lambda+y_\lambda).
\end{aligned}$$

This completes the proof. □

Proposition 4.5. *Let $(X,\mathbb{K},\mathscr{B})$ be an (L,M)-fuzzy bornological vector space. Then for any $\{x^n_{\lambda_n}\}_{n\in N}\subseteq J(L^X)$, $x_\lambda\in J(L^X)$, $\{t_n\}_{n\in N}\subseteq\mathbb{K}$ and $t\in\mathbb{K}$ with $\bigvee_{n\in N}\bigwedge_{m\geqslant n}\lambda_m\geqslant\lambda$ and $\lim_{n\to\infty}t_n=t$,*

$$Con(x^n_{\lambda_n},x_\lambda)\leqslant Con(t_nx^n_{\lambda_n},tx_\lambda).$$

Proof. Take any $A\in Cir(X,\mathbb{K})$ and $\{\alpha_n\}_{n\in N}\subseteq\mathbb{K}$ such that $\lim_{n\to\infty}\alpha_n=0$ and for all $n\in N,(x^n-x)_{\lambda_n}\nleqslant\alpha_nA^{'}$. Let

$\beta_n = t_n\alpha_n \vee (t_n - t), \forall n \in N$. Then $\lim_{n\to\infty}\beta_n = 0$, and

$$\begin{aligned}\beta_n\left(A+\bigvee_{|\mu|\leqslant 1}\mu x_{\top_L}\right)(t_nx^n - tx) &= \beta_n\left(A+\bigvee_{|\mu|\leqslant 1}\mu x_{\top_L}\right)(t_nx^n - t_nx + t_nx - tx)\\ &\geqslant \beta_n A\left(t_n(x^n - x)\right) \wedge \beta_n\left(\bigvee_{|\mu|\leqslant 1}\mu x_{\top_L}\right)((t_n - t)x)\\ &= \beta_n A\left(t_n(x^n - x)\right)\\ &\geqslant \alpha_n A(x^n - x).\end{aligned}$$

This implies $(t_nx^n - tx)_{\lambda_n} \not\leqslant \beta_n\left(A+\bigvee_{|\mu|\leqslant 1}\mu x_{\top_L}\right)^{'}, \forall n \in N$. Since $A, \bigvee_{|\mu|\leqslant 1}\mu x_{\top_L} \in Cir(X,\mathbb{K})$, we have $A + \bigvee_{|\mu|\leqslant 1}\mu x_{\top_L} \in Cir(X,\mathbb{K})$. By combining (LMB1), (BV3) and (BV5), it holds that $\mathscr{B}(A) \leqslant \mathscr{B}\left(A+\bigvee_{|\mu|\leqslant 1}\mu x_{\top_L}\right)$. Therefore,

$$\begin{aligned}Con(x^n_{\lambda_n}, x_\lambda) &= \bigvee_{\substack{A\in Cir(X,\mathbb{K})\\ \alpha_n\to 0}}\left\{\mathscr{B}(A) : (x^n - x)_{\lambda_n} \not\leqslant \alpha_n A^{'}, \forall n \in N\right\}\\ &\leqslant \bigvee_{\substack{A\in Cir(X,\mathbb{K})\\ \beta_n\to 0}}\left\{\mathscr{B}(A) : (t_nx^n - tx)_{\lambda_n} \not\leqslant \beta_n\left(A+\bigvee_{|\mu|\leqslant 1}\mu x_{\top_L}\right)^{'}, \forall n \in N\right\}\\ &\leqslant \bigvee_{\substack{B\in Cir(X,\mathbb{K})\\ \beta_n\to 0}}\left\{\mathscr{B}(B) : (t_nx^n - tx)_{\lambda_n} \not\leqslant \beta_n B^{'}, \forall n \in N\right\}\\ &= Con(t_nx^n_{\lambda_n}, tx_\lambda).\end{aligned}$$

This completes the proof. □

Proposition 4.6. *Let $(X,\mathbb{K},\mathscr{B}_X)$ and $(Y,\mathbb{K},\mathscr{B}_Y)$ be two (L,M)-fuzzy bornological vector spaces, and let $f : X \longrightarrow Y$ be an (L,M)-fuzzy bounded linear mapping. Then for any $\{x^n_{\lambda_n}\} \subseteq J(L^X)$ and $x_\lambda \in J(L^X)$ with $\bigvee_{n\in N}\bigwedge_{m\geqslant n}\lambda_m \geqslant \lambda$,*

$$Con_X(x^n_{\lambda_n}, x_\lambda) \leqslant Con_Y(f(x^n)_{\lambda_n}, f(x)_\lambda).$$

Proof. Take any $A \in Cir(X,\mathbb{K})$ and $\{\alpha_n\}_{n\in N} \subseteq \mathbb{K}$ such that $\lim_{n\to\infty}\alpha_n = 0$ and for all $n \in N, (x^n - x)_{\lambda_n} \not\leqslant \alpha_n A^{'}$. Since f is an (L,M)-fuzzy bounded linear mapping, it follows that $f^{\to}(A) \in Cir(Y,\mathbb{K})$, $\mathscr{B}_X(A) \leqslant \mathscr{B}_Y(f^{\to}(A))$, and $\alpha_n A(x^n - x) \leqslant \alpha_n f^{\to}(A)(f(x^n - x))$. This implies $(f(x^n) - f(x))_{\lambda_n} \not\leqslant \alpha_n f^{\to}(A)^{'}$. Therefore,

$$\begin{aligned}Con_X(x^n_{\lambda_n}, x_\lambda) &= \bigvee_{\substack{A\in Cir(X,\mathbb{K})\\ \alpha_n\to 0}}\left\{\mathscr{B}_X(A) : (x^n - x)_{\lambda_n} \not\leqslant \alpha_n A^{'}, \forall\, n \in N\right\}\\ &\leqslant \bigvee_{\substack{f^{\to}(A)\in Cir(Y,\mathbb{K})\\ \alpha_n\to 0}}\left\{\mathscr{B}_Y(f^{\to}(A)) : (f(x^n) - f(x))_{\lambda_n} \not\leqslant \alpha_n f^{\to}(A)^{'}, \forall\, n \in N\right\}\\ &\leqslant \bigvee_{\substack{B\in Cir(Y,\mathbb{K})\\ \alpha_n\to 0}}\left\{\mathscr{B}_Y(B) : (f(x^n) - f(x))_{\lambda_n} \not\leqslant \alpha_n B^{'}, \forall\, n \in N\right\}\\ &= Con_Y(f(x^n)_{\lambda_n}, f(x)_\lambda),\end{aligned}$$

as desired. □

By Proposition 3.6, we have the following result.

Proposition 4.7. *Let $(X,\mathbb{K},\mathscr{B})$ be an (L,M)-fuzzy bornological vector space, and let $(Y,\mathbb{K},\mathscr{B}|_Y)$ be the subspace of $(X,\mathbb{K},\mathscr{B})$. Then for any $\{x^n_{\lambda_n}\}_{n\in N} \subseteq J(L^Y)$, and $x_\lambda \in J(L^Y)$ with $\bigvee_{n\in N}\bigwedge_{m\geqslant n}\lambda_m \geqslant \lambda$,*

$$Con_X(x^n_{\lambda_n}, x_\lambda) \leqslant Con_Y(x^n_{\lambda_n}, x_\lambda).$$

In classical bornological vector spaces, a bornological vector space is separated if and only if every bornologically convergent sequence has a unique limit. Next we disscuss the many-valued logical counterpart of this classical result. In [13], the authors introduced the relationship between bornologial convergence and separation in the setting of L-bornological vector space as follows:

Definition 4.8. [13] *Let $(X, \mathbb{K}, \mathcal{B})$ be an L-bornological vector space and $\lambda \in J(L)$, $S = \{x^n_{\lambda_n}\}_{n\in N} \subseteq J(L^X)$, where $\bigvee\limits_{n\in N} \bigwedge\limits_{m\geqslant n} \lambda_m \geqslant \lambda$. Then S is said to converge bornologically to θ_λ if there exists a circled L-subset $B \in \mathcal{B}$ and a sequence $\{s_n\}$ of scalars tending to 0 such that $x^n_{\lambda_n} \nleqslant s_n B'$ for all $n \in N$. Moreover, $S = \{x^n_{\lambda_n}\}_{n\in N}$ is said to convergent to x_λ if $(x^n - x)_{\lambda_n}$ converges to θ_λ, and it is denoted by $x^n_{\lambda_n} \xrightarrow{M} x_\lambda$.*

Proposition 4.9. [13] *Let $(X, \mathbb{K}, \mathcal{B})$ be an L-bornological vector space and $\lambda \in J(L)$. If $x^n_{\lambda_n} \xrightarrow{M} x_\lambda$ and $x^n_{\lambda_n} \xrightarrow{M} y_\lambda$, then $(X, \mathbb{K}, \mathcal{B})$ is L-separated if and only if $x = y$.*

By combining Lemma 3.3, Definitions 4.2 and 4.8, we have the following lemma.

Lemma 4.10. *Let $(X, \mathbb{K}, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space, $\lambda \in L$ and $\{x^n_{\lambda_n}\}_{n\in N} \subseteq J(L^X)$ with $\bigvee\limits_{n\in N} \bigwedge\limits_{m\geqslant n} \lambda_m \geqslant \lambda$. Then for any $r \in P(M)$, $Con(x^n_{\lambda_n}, x_\lambda) \nleqslant r$ if and only if $\{x^n_{\lambda_n}\}_{n\in N}$ is convergent to x_λ in L-bornological vector space $\left(X, \mathbb{K}, \mathscr{B}^{(r)}\right)$.*

Theorem 4.11. *Let $(X, \mathbb{K}, \mathscr{B})$ be an (L, M)-fuzzy bornological vector space. If there exists $r \in P(M)$ such that $Con(x^n_{\lambda_n}, x_\lambda) \nleqslant r$ and $Con(x^n_{\lambda_n}, y_\lambda) \nleqslant r$, then $Sep(X, \mathbb{K}, \mathscr{B}) \geqslant r'$ if and only if $x = y$.*

Proof. By Lemma 4.10, $Con(x^n_{\lambda_n}, x_\lambda) \nleqslant r$ and $Con(x^n_{\lambda_n}, y_\lambda) \nleqslant r$ if and only if $x^n_{\lambda_n} \xrightarrow{M} x_\lambda$ and $x^n_{\lambda_n} \xrightarrow{M} y_\lambda$. By combining Propositions 3.4 and 4.9, it follows that $Sep(X, \mathbb{K}, \mathscr{B}) \geqslant r'$ if and only if $x = y$. □

5 Conclusions

In this paper, we endowed the concepts of bornological convergence and separation in (L, M)-fuzzy bornological vector spaces with some degrees. We presented the lattice-valued forms of elementary conclusions related to these concepts. Moreover, we discussed the relationships between (L, M)-fuzzy bornologial convergence and separation in (L, M)-fuzzy bornological vector spaces. As we all know, pseudo-norms play an important role in the theory of bornological vector spaces. In the future, we will consider how to induce (L, M)-fuzzy bornological vector spaces by fuzzy pseudo-norms, and discuss the relationships between bornological convergence and topological convergence in the framework of (L, M)-fuzzy bornological vector spaces.

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