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Original paper

Comment on L-convergence spaces via L-ordered co-Scott closed sets

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Abstract

Han and Pang (IJFS 2024) introduced important L-convergence structures based on L-ordered co-Scott closed sets, extending strong L-concave structures. Since L-ordered co-Scott closed sets and L-convergence structures rely on L-order, while strong L-concave structures depend on pointwise order, the relationship between them is still not clear enough. To clarify this, we provide characterizations of both using pointwise orders, revealing that the definition of L-ordered co-Scott closed sets can be simplified, as its stratified condition can be derived from the L-ordered condition. This finding will streamline proofs in Han and Pang's paper, as these conditions have been repeatedly verified. Additionally, our insights will aid in accurately establishing the relationships between various lattice-valued co-Scott closed sets and lattice-valued concave structures.

Keywords: L-convex space, L-concave space, L-ordered co-Scott closed set, L-convergence space.

1 Introduction

The concept of convex structure is an important mathematical structure [8]. As natural extensions, many kinds of fuzzy convex structures are proposed by different scholars, see Maruyama [4], Pang and Zhang [5, 12], Shi and Xiu [6, 7], Li [3], and Wu and Li [10]. To overcome the defects of fuzzy convex structures in category properties, scholars have further introduced various lattice-valued convergence structures [2, 9, 11, 13]. In particular, Han and Pang [2] recently investigated an L-convergence structure based on L-ordered co-Scott closed set, and proved that concave L-convergence space can describe strong L-concave space.

Let $L = (L, *)$ be a complete residuated lattice, that is,

(1) $(L, \leq)$ is a complete lattice with top (resp., bottom) element $\top$ (resp., $\perp$),

(2) $*$ is a commutative and associative binary operation on L s.t. $\forall a \in L, \forall \{b_j\}_{j\in J} \subseteq L$:

$$a * \bigvee_{j\in J} b_j = \bigvee_{j\in J} (a * b_j), \quad a * \top = a.$$

The operation $*$ has a residuated operation $\rightarrow: L \times L \longrightarrow L$ determined by $a \rightarrow b = \bigvee\{c \in L | a * c \leq b\}$.

A mapping $\lambda : X \rightarrow L$ is termed an L-fuzzy set in X, the set of all L-fuzzy sets in X is denoted by L^X. For each $a \in L$, we also use a to denote the constant L-fuzzy set in X valued a.

There exists a classical order $\leq$ (also called pointwise order) on L^X determined by $\forall \lambda, \mu \in L^X$, $\lambda \leq \mu$ iff $\lambda(x) \leq \mu(x)$ for each $x \in X$. The operations $\vee, \wedge, *, \rightarrow$ on L can be defined onto L^X pointwisely.

The family $\{\lambda_i\}_{i\in I} \subseteq L^X$ is referred co-directed if for every $\lambda_{i_1}, \lambda_{i_2}$, there exists λ_{i_3} s.t. $\lambda_{i_3} \leq \lambda_{i_1}$ and $\lambda_{i_3} \leq \lambda_{i_2}$. We use $\{\lambda_i\}_{i\in I} \subseteq^{cd} L^X$ to denote a co-directed subfamily of L^X.

Moreover, there exists an L-order on L^X [1], which is defined as a mapping $S : L^X \times L^X \longrightarrow L^X$ with $\forall \lambda, \mu \in L^X$, $S(\lambda, \mu) = \bigwedge\limits_{x\in X} (\lambda(x) \rightarrow \mu(x))$.

By using the L-order on L^X, Han and Pang gave the following notion.

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Definition 1.1. [2] *A mapping* $\mathbb{F}: L^X \longrightarrow L$ *is called an* L*-ordered co-Scott closed set (abbreviated as LCSC) on* L^X *provided that:*

(LCSC1) $\mathbb{F}(\top) = \top$,

(LCSC2) $S(\lambda, \mu) \leq \mathbb{F}(\lambda) \to \mathbb{F}(\mu)$ *for any* $\lambda, \mu \in L^X$ *(*L*-ordered condition),*

(LCSC3) $\bigwedge\limits_{i \in I} \mathbb{F}(\lambda_i) \leq \mathbb{F}(\bigwedge\limits_{i \in I} \lambda_i)$ *for any* $\{\lambda_i\}_{i \in I} \subseteq^{cd} L^X$,

(LCSCs) $a * \mathbb{F}(\lambda) \leq \mathbb{F}(a * \lambda)$ *for any* $\lambda \in L^X$ *and* $a \in L$ *(stratified condition),*

(LCSCcs) $a \to \mathbb{F}(\lambda) \leq \mathbb{F}(a \to \lambda)$ *for any* $\lambda \in L^X$ *and* $a \in L$ *(co-stratified condition).*

The set of all LCSC on L^X *is denoted by* $\mathbb{F}_L(X)$.

LCSC is a natural extension of the classical co-Scott closed set.

By using the L-order on $\mathbb{F}_L(X)$, Han and Pang introduced the following structure.

Definition 1.2. [2] *A mapping* $\lim: \mathbb{F}_L(X) \longrightarrow L^X$ *is termed an* L*-convergence structure on* X *provided that:*

(LCS1) $\forall x \in X$, $\lim([x])(x) = \top$, *where* $[x] \in \mathbb{F}_L(X)$ *is defined by* $\forall \lambda \in L^X$, $[x](\lambda) = \lambda(x)$,

(LCS2) $\forall \mathbb{F}, \mathbb{G} \in \mathbb{F}_L(X)$, $S(\mathbb{F}, \mathbb{G}) * \lim \mathbb{F}(x) \leq \lim \mathbb{G}(x)$ *(*L*-ordered axiom).*

The pair $(X, \lim)$ *is called an* L*-convergence space.*

Han and Pang [2] demonstrated that concave L-convergence space can describe strong L-concave space.

Definition 1.3. [12] *A subfamily* $\mathbb{C} \subseteq L^X$ *is called an* L*-concave structure on* X *provided that:*

(1) $\top, \bot \in \mathbb{C}$,

(2) $\bigvee\limits_{i \in I} \lambda_i \in \mathbb{C}$ *for any* $\{\lambda_i\}_{i \in I} \subseteq \mathbb{C}$,

(3) $\bigwedge\limits_{i \in I} \lambda_i \in \mathbb{C}$ *for any* $\{\lambda_i\}_{i \in I} \subseteq^{cd} \mathbb{C}$.

The pair $(X, \mathbb{C})$ *is called an* L*-concave space.*

An L*-concave space* $(X, \mathbb{C})$ *is termed stratified if it fulfills moreover:* $a * \lambda \in \mathbb{C}$ *for any* $\lambda \in \mathbb{C}$ *and* $a \in L$.

A stratified L*-concave space* $(X, \mathbb{C})$ *is termed strong if it fulfills moreover:* $a \to \lambda \in \mathbb{C}$ *for any* $\lambda \in \mathbb{C}$ *and* $a \in L$.

Note that Definition 1.3 is fully expressed in terms of pointwise order, while Definitions 1.1 and 1.2 additionally rely on L-order. To better establish the connection between the L-convergence structure and the L-concave structure, this paper will provide a pointwise order characterization for Definitions 1.1 and 1.2. Specifically, we will show that the condition (LCSC2) in Definition 1.1 can be divided into two conditions defined by pointwise order, one of which is the stratified condition (LCSCs), and the other is the condition (LCSC2′) stated in Proposition 2.1. This indicates that the condition (LCSCs) is redundant, allowing for a simplification of Definition 1.1. Moreover, we will show that the condition (LCS2) in Definition 1.2 can be divided into two conditions defined by pointwise order.

2 The characterization and simplification of LCSC

Proposition 2.1. *A mapping* $\mathbb{F}: L^X \longrightarrow L$ *fulfills (LCSC2) iff it fulfills (LCSCs) and (LCSC2′).*

(LCSC2′) $\lambda \leq \mu$ *implies* $\mathbb{F}(\lambda) \leq \mathbb{F}(\mu)$ *for any* $\lambda, \mu \in L^X$.

Proof. (1) (LCSC2)$\Longrightarrow$ (LCSCs). For any $\lambda \in L^X$ and $a \in L$, it follows from (LCSC2) that

$$\mathbb{F}(\lambda) \to \mathbb{F}(a * \lambda) \geq S(\lambda, a * \lambda) \geq a,$$

hence $a * \mathbb{F}(\lambda) \leq \mathbb{F}(a * \lambda)$, i.e., (LCSCs) holds.

(2) (LCSC2)$\Longrightarrow$ (LCSC2′). It follows from $S(\lambda, \mu) = \top$ iff $\lambda \leq \mu$.

(3) (LCSCs)+(LCSC2′) $\Longrightarrow$ (LCSC2). For any $\lambda, \mu \in L^X$, it follows from (LCSCs) that

$$S(\lambda, \mu) * \mathbb{F}(\lambda) \leq \mathbb{F}(S(\lambda, \mu) * \lambda) \overset{(\mathrm{LCSC2'})}{\leq} \mathbb{F}(\mu),$$

hence $S(\lambda, \mu) \leq \mathbb{F}(\lambda) \to \mathbb{F}(\mu)$, i.e., (LCSC2) holds. □

Proposition 2.1 shows that the definition of LCSC can be simplified.

Corollary 2.2. *A mapping* $\mathbb{F}: L^X \longrightarrow L$ *is an LCSC on* L^X *iff it fulfills (LCSC1), (LCSC2), (LCSC3) and (LCSCcs).*

Corollary 2.3. *A mapping* $\mathbb{F}: L^X \longrightarrow L$ *is an LCSC on* L^X *iff it fulfills (LCSC1), (LCSC2′), (LCSC3), (LCSCs) and (LCSCcs).*

Furthermore, we observe easily that (LCSC3)+(LCSC2′) $\Longleftrightarrow$ (LCSC3′).
(LCSC3′) $\bigwedge\limits_{i\in I}\mathbb{F}(\lambda_i)=\mathbb{F}(\bigwedge\limits_{i\in I}\lambda_i)$ for any $\{\lambda_i\}_{i\in I}\subseteq^{cd} L^X$.

Corollary 2.4. *A mapping $\mathbb{F}: L^X\longrightarrow L$ is an LCSC on L^X iff it fulfills (LCSC1), (LCSC3′), (LCSCs) and (LCSCcs).*

Corollaries 2.3 and 2.4 provide the characterization of LCSC that are independent of L-order.
With the aid of the characterizations mentioned above, the proofs of Propositions 3.4, 3.5, 4.3, 4.5, 4.7 and 5.2 in [2] can be simplified, as the conditions (LCSC2) and (LCSCs) have been repeatedly verified in these propositions.
Based on the definition of L-concave structures, we can present three types of lattice-valued co-Scott closed sets.

Definition 2.5. (1) *A mapping $\mathbb{F}: L^X\longrightarrow L$ is called an L-co-Scott closed set on L^X if it fulfils (LCSC1), (LCSC2′) and (LCSC3).*
(2) *An L-co-Scott closed set is called stratified if it fulfills (LCSCs).*
(3) *An L-co-Scott closed set is called L-ordered if it fulfills (LCSCs) and (LCSCcs).*

Definition 2.6. *If the relationships between (LCSC2) and (LCSCs) have not been clarified, one may define an L-co-Scott closed set on L^X as a mapping $\mathbb{F}: L^X\longrightarrow L$ satisfying (LCSC1), (LCSC2) and (LCSC3), which indeed have been stratified.*

3 The characterization of L-convergence structure

Han and Pang [2] proved that LCSC is closed to arbitrary intersection. That is, for any $\{\mathbb{F}_i\}_{i\in I}\subseteq\mathbb{F}_L(X)$, we have $\bigwedge\limits_{i\in I}\mathbb{F}_i\in\mathbb{F}_L(X)$.
Let $a\in L$ and $\mathbb{F}\in\mathbb{F}_L(X)$. We define $[a*\mathbb{F}]=\bigwedge\{\mathbb{G}\in\mathbb{F}_L(X)|a*\mathbb{F}\le\mathbb{G}\}$, then $[a*\mathbb{F}]\in\mathbb{F}_L(X)$.

Proposition 3.1. *A mapping* $\lim:\mathbb{F}_L(X)\longrightarrow L^X$ *fulfills (LCS2) iff it fulfills (LCS2′) and (LCSs).*
(LCS2′) $\mathbb{F}\le\mathbb{G}$ *implies* $\lim\mathbb{F}(x)\le\lim\mathbb{G}(x)$ *for any* $\mathbb{F},\mathbb{G}\in\mathbb{F}_L(X)$.
(LCSs) $a*\lim\mathbb{F}(x)\le\lim[a*\mathbb{F}](x)$ *for any* $\mathbb{F}\in\mathbb{F}_L(X)$ *and* $a\in L$.

Proof. (1) (LCS2)$\Longrightarrow$ (LCS2′). It follows immediately since $\mathbb{F}\le\mathbb{G}$ implies $S(\mathbb{F},\mathbb{G})=\top$.
(2) (LCS2)$\Longrightarrow$ (LCSs). For any $\mathbb{F}\in\mathbb{F}_L(X)$, $x\in X$ and $a\in L$, it follows from (LCS2) that

$$\begin{aligned}\lim[a*\mathbb{F}](x) &\ge S(\mathbb{F},[a*\mathbb{F}])*\lim\mathbb{F}(x)\\ &= \bigwedge\{S(\mathbb{F},\mathbb{G})|\mathbb{G}\in\mathbb{F}_L(X),a*\mathbb{F}\le\mathbb{G}\}*\lim\mathbb{F}(x)\\ &= \bigwedge\{S(\mathbb{F},\mathbb{G})|\mathbb{G}\in\mathbb{F}_L(X),a\le S(\mathbb{F},\mathbb{G})\}*\lim\mathbb{F}(x)\\ &\ge a*\lim\mathbb{F}(x),\end{aligned}$$

i.e., (LCSs) holds.
(3) (LCSs)+(LCS2′) $\Longrightarrow$ (LCS2). For any $\mathbb{F},\mathbb{G}\in\mathbb{F}_L(X)$, let $a=S(\mathbb{F},\mathbb{G})$. Then $a*\mathbb{F}\le\mathbb{G}$, so $[a*\mathbb{F}]\le\mathbb{G}$, it follows from (LCSs) that

$$S(\mathbb{F},\mathbb{G})*\lim\mathbb{F}(x)=a*\lim\mathbb{F}(x)\le\lim[a*\mathbb{F}](x)\overset{(\mathrm{LCS2'})}{\le}\lim\mathbb{G}(x),$$

i.e., (LCS2) holds. □

The following Corollary 3.2 provides a characterization of L-convergence structure that is independent of L-order.

Corollary 3.2. *A mapping* $\lim:\mathbb{F}_L(X)\longrightarrow L^X$ *is an L-convergence structure on X iff it fulfills (LCS1), (LCS2′) and (LCSs).*

Definition 3.3. *It is known that concave L-convergence spaces are isomorphic to strong L-concave spaces [2]. This implies that L-ordered co-Scott closed sets can characterize strong L-concave spaces. By comparing the definitions of L-co-Scott closed sets, stratified L-co-Scott closed sets, L-concave spaces, and stratified L-concave spaces, we can infer that (resp., stratified) L-co-Scott closed sets may characterize (resp., stratified) L-concave spaces. However, as noted in Remark 2.6, if the relationships between (LCSC2) and (LCSCs) are not clearly clarified, one might mistakenly conclude that stratified L-co-Scott closed sets could characterize L-concave spaces.*

4 Conclusions

In this comment, we presented several characterizations of L-ordered co-Scott closed sets and L-concave structures by fully utilizing pointwise orders. Through these characterizations, we discovered that the definition of L-ordered co-Scott closed sets can be simplified, allowing for many proofs in [2] to be streamlined. Additionally, following from that L-ordered co-Scott closed sets characterize strong L-concave spaces, we deduce that (resp., stratified) L-co-Scott closed sets may characterize (resp., stratified) L-concave spaces.

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References

[1] R. Bělohlávek, *Fuzzy relational systems: Foundations and principles*, Kluwer Academic Publishers, New York, Boston, Dordrecht, London, Moscow, (2002). https://doi.org/10.1007/978-1-4615-0633-1

[2] X. C. Han, B. Pang, *Convergence structures in L-concave spaces*, Iranian Journal of Fuzzy Systems, **21**(4) (2024), 61-80. https://doi.org/10.22111/ijfs.2024.48804.8608

[3] L. Q. Li, *On the category of enriched (L, M)-convex spaces*, Journal of Intelligent and Fuzzy Systems, **33** (2017), 3209-3216. https://doi.10.3233/JIFS-161491

[4] Y. Maruyama, *Lattice-valued fuzzy convex geometry*, RIMS Kokyuroku, **164** (2009), 22-37. http://hdl.handle.net/2433/140587

[5] B. Pang, *Fuzzy convexities via overlap functions*, IEEE Transactions on Fuzzy Systems, **31** (2023), 1071-1082. https://doi.org/10.1109/TFUZZ.2022.3194354

[6] F. G. Shi, Z. Y. Xiu, *A new approach to the fuzzification of convex structures*, Journal of Applied Mathematics, **2014** (2014), 1-12. https://doi.org/10.1155/2014/249183

[7] F. G. Shi, Z. Y. Xiu, *(L, M)-fuzzy convex structures*, Journal of Nonlinear Sciences and Applications, **10** (2017), 3655-3669. https://doi.org/10.22436/jnsa.010.07.25

[8] M. L. J. Van de Vel, *Theory of convex structures*, Elesevier Sicence Publishers, North-Holland, Amsterdam, 1993.

[9] X. Y. Wu, Y. W. Huang, *L-convex convergence relation and L-convex ideal-convergence structure*, Iranian Journal of Fuzzy Systems, **21**(5) (2024), 31-49. https://doi.org/10.22111/ijfs.2024.49227.8679

[10] X. Y. Wu, E. Q. Li, *Characterizations of L-concavities and L-convexities via derived relations*, Hacettepe Journal of Mathematics and Statistics, **52**(4) (2023), 876-895. https://doi.org/10.15672/hujms.1175332

[11] Z. Y. Xiu, *Convergence structures in L-concave spaces*, Journal of Nonlinear and Convex Analysis, **21**(12) (2020), 2693-2703.

[12] L. Zhang, B. Pang, *Strong L-concave structures and L-convergence structures*, Journal of Nonlinear and Convex Analysis, **21**(12) (2021), 2759-2769.

[13] L. Zhang, B. Pang, *Convergence structures in (L, M)-fuzzy convex spaces*, Filomat, **37** (2023), 2859-2877. https://doi.org/10.2298/FIL2309859Z