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MFS: Dynamic decision making approach to select optimal alternative in the presence of uncertainty

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Abstract

The concept of a multi-fuzzy set (MFS) is a hybrid mathematical approach that aids reasoning and decision-making in situations characterized by imprecise information and multiple occurrences. Researchers have developed robust MFS-based frameworks that facilitate decision-making by identifying optimal alternatives. However, these frameworks often struggle to select the desired alternative in uncommon situations. To address the limitations of existing methods, we incorporated relations and operators with MFS to better measure levels of uncertainty. To enhance the effectiveness of decision-making, we proposed two approaches based on the significance of the criteria within the MFS framework. First, we introduced a relative weight-based approach, where the weight of each criterion is estimated dynamically. Second, we developed a normalized-based decision-making approach, which generates optimal solutions based on normalized score factors. We demonstrated the effectiveness of our proposed approaches using semi-realistic cases in health science and healthcare management. Furthermore, we evaluated the performance of the healthcare system across different socio-economic regions, which helped to illustrate the relative importance of the criteria.

Keywords: Multi-fuzzy set, pivot point, pivot degree, score factor, healthcare system.

1 Introduction

A fuzzy set-based decision-making approach utilizes the principles of fuzzy set theory to address the imprecision and uncertainty inherent in decision-making processes. Fuzzy sets transform the precise input data into fuzzy sets by assigning membership degrees to each element. This includes defining linguistic variables and fuzzy membership functions that represent the degree of membership for each element of the fuzzy set sets.

The real world is filled with uncertainty, imprecision, and vagueness. Most concepts in daily life are vague rather than precise. As a result, researchers have become interested in developing problem-solving models in vague environments over the last few decades. Traditional tools do not always succeed in addressing these problems. A variety of existing theories, such as probability theory [20], fuzzy set theory [38], rough set theory [40], vague set theory [5], and interval mathematics [25], are well known and often serve as useful mathematical approaches to model vagueness in inputs. Each of these theories has inherent challenges, as noted in [35]. In 1999, Molodtsov introduced soft set theory as a new mathematical tool for addressing uncertainties without the complications that existing methods face [7]. This theory has proven beneficial in many fields, including decision-making [2, 18, 19, 29], data analysis [1, 22, 39], forecasting [23], and more.

Fuzzy set is one of the most important mathematical tools introduced by Zadeh [38] to manage uncertain situations effectively. Subsequently, researchers focused on this area and modified it into several extended versions to tackle complex problems [9, 14, 26, 28, 31]. Some well-known extended terminologies include type-2 fuzzy set [13], L-fuzzy sets [24], intuitionistic fuzzy set [4], interval-value fuzzy set [25, 37], intuitionistic multi-fuzzy set [30], picture fuzzy

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set [10, 33], and Pythagorean fuzzy set [17]. The fuzzy set can also be combined with other mathematical models to create hybrid concepts for improved performance. In this context, multi-fuzzy set (another extension of the fuzzy set) was introduced by Sebastian and Ramakrishnan [3]. This type of fuzzy set employs ordered sequences of membership functions and offers an alternative method to represent problems not encompassed by other extensions of fuzzy set theory, such as pixel color. Subsequently, the notion of multi-fuzzy complex numbers and sets was introduced by Dey and Pal [15]. These authors presented multi-fuzzy complex nilpotent matrices over a distributive lattice. Recently, Yong et al. [36] proposed the concept of the multi-fuzzy soft set for application in decision-making, which represents a more general fuzzy soft set. The normalized multi-fuzzy set was reintroduced as fuzzy distributed sets (FDS) in [6]. The author [6] introduced basic operations on FDS in an alternative manner, demonstrating their utilization in real-life decision-making methods. The concepts of Type-2 multi-fuzzy set introduced by Kar et al. [21] proposed a decision-making approach based on Hamming and Euclidean distances using this framework. Sun et al. [32] integrated fuzzy concepts into clustering techniques and presented a novel clustering method that does not rely on the traditional identification of positive and negative ideal solutions. Yang et al. [36] combined the concepts of multi-fuzzy set and soft set, introducing a hybrid model called multi-fuzzy soft set. Initially, multi-fuzzy-based decision-making approaches were common, establishing a threshold value [36] to eliminate less important information. These approaches typically considered either choice value or partial score value. Challenges associated with this approach include estimating the threshold value and not accounting for all reported data [12]. In contrast, distance-based comparison techniques randomly select a saddle point and estimate the distance of each alternative from it [11]. Final decisions are made based on these estimated distances. Distance-based approaches are well-suited for clustering, whereas intensity-related issues may arise from contradictions due to positive or negative distances. In this study, we introduce relevant functionalities and alternative decision-making approaches: dynamic weight-based and normalized-based methods to address such situations. These approaches can be compared to alternatives without relying on external information. In the weight-based approach, criteria weights are estimated based on their importance and evaluated significantly, with the weight value of the criteria depending on their dispersion, intensity level, and the mean value of observation reports. Similarly, in the normalized-based approach, the score factors of the alternatives are calculated independently, without interaction with other alternatives. Essentially, the score factors represent the individual strength of the alternatives. The resultant score of the dynamic approach and the score factor of the normalized approach respectively measure the individual and relative importance of other alternatives for decision-making. Furthermore, proposed functionalities such as pivot point, pivot degree, and merge operation play a crucial role in making accurate decisions.

This article examines the non-normalization multi-fuzzy set and presents several relevant operators for semi-realistic decision-making problems. To facilitate our discussion, we first review some background on fuzzy sets, intuitionistic fuzzy sets, picture fuzzy sets, and multi-fuzzy sets in Section 2. Additionally, this section introduces novel relations and operations of the multi-fuzzy set, including the score function, pivot point, pivot degree, merge operation, and distance measurement technique. In Section 3, the multi-fuzzy set is leveraged to analyze decision-making problems, proposing two algorithms that effectively manage some unusual situations in decision-making. A real-life case study is included in Section 4 to validate our proposed methods through compassion analysis. Finally, Section 5 highlights some significant conclusions.

2 Preliminary

Fuzzy set (FS) introduced by Zadeh [38] manages the uncertainty and vagueness of the system. The fuzzy set considers the degree of membership of the elements to represent their belonging within the set and provides a well-organized method for constructing uncertain decision-making systems using uncertain information, which represents experts' opinions. Atanassov [4] modified the concept of the fuzzy set into the intuitionistic fuzzy set (IFS) by introducing the non-membership degree of the elements for improved accuracy. The IFS includes a hesitation margin as a degree of hesitation (the hesitation margin is equal to the complement of the sum of membership and non-membership degrees). The concept of IFS as a generalized fuzzy set is more attractive and practically useful for solving various real-life problems. The idea and semantic depiction of IFS are found to be more significant, imaginative, and appropriate due to the introduction of the belongingness degree, non-belongingness degree, and hesitation margin. The authors in [8] demonstrated that the IFS is beneficial in situations where explaining the problem using a linguistic variable in terms of a membership function alone seems inadequate. Due to the flexibility of IFS in handling uncertainty, it has greater applicability in human consistent reasoning under imperfectly defined facts and imprecise knowledge. Similarly, Coug [10] further modified the intuitionistic fuzzy set and extended it into a picture fuzzy set (PFS) by incorporating the neutral membership degree, which represents the special functionality of the elements. The term "picture" in the PFS indicates that this set is a direct extension of FS and IFS, and manages the imprecision, vagueness, fuzziness, and uncertainty of the problem efficiently and easily. The multi-fuzzy set is a more generalized version of the fuzzy set with multiple instances of membership degrees instead of various dependent degrees [3]. Essentially, multi-fuzzy set theory is an extension of fuzzy set theory [38], intuitionistic fuzzy set theory [4], and L-fuzzy set theory [21]. The functionalities of the different fuzzy sets and their logical significance are shown in Table 1.

The definitions and properties of fuzzy sets and their extension over the common universe U are outlined below. Let X represent a classical set of objects called the universe of discourse, where the elements of X are denoted by k, k_1, k_2, and k_3. Consider, A, B, C, and M represent the FS, IFS, PFS, and MFS, respectively. Their representation is provided below:

i. Fuzzy Set

$$A = \{(k_1, \mu_A(k_1)), k_1 \in X\}. \tag{2.1}$$

Here k_1 is the element of A and $\mu_A(k_1) \in [0, 1]$ is called the positive membership degree of k_1.

ii. Intuitionistic Fuzzy Set

$$B = \{(k_2, \mu_B(k_2), \nu_B(k_2)), k_2 \in X\}. \tag{2.2}$$

Here k_2 is the element of B and the $\mu_B(k_2) \in [0, 1]$ and $\nu_B(k_2) \in [0, 1]$ are called the positive and negative membership degree of k respectively. These two parameters $(\mu_B(k_2), \nu_B(k_2))$ of the intuitionistic fuzzy set B satisfy the following condition $\forall k_2 \in X, 0 \leq \mu_B(k_2) + \nu_B(k_2) \leq 1$.

iii. Picture Fuzzy set

$$C = (k_3, \mu_C(k_3), \eta_C(k_3), \nu_A(k_3)), k3 \in X. \tag{2.3}$$

Here k_3 is the element of C and $\mu_C(k3) \in [0, 1]$, $\eta_C(k_3) \in [0, 1]$, and $\nu_C(k_3) \in [0, 1]$ are called the positive, neutral and negative membership degree of $k3$, respectively. These three parameters of the picture fuzzy set satisfy the following condition $0 \leq \mu_C(k_3), \eta_C(k_3), \nu_C(k_3) \leq 1$. $\pi_C(k_3)$ is called the degree of refusal membership of k_3 in X and computed as $\pi_C(k_3) = 1 - \mu_C(k_3) - \eta_C(k_3) - \nu_C(k_3)$.

Table 1: Versions of fuzzy sets and their anatomy

Fuzzy Terminology	Instance of membership degree	Behavioural significant of membership degree/s	Dependency among the membership degree/s
Fuzzy Set [38]	positive	–	–
IFS [4]	positive negative	different	dependent
PFS [10]	positive neutral negative	different	dependent
MFS [3]	n times of positive, n>1	same	independent

2.1 Multi-fuzzy set

Multi-fuzzy sets allow more nuanced representation of uncertainty than traditional fuzzy sets and their extended versions. By incorporating multiple membership degrees of fuzzy set, it can capture various degrees of uncertainty or ambiguity more accurately [3, 27].

Let a multi-fuzzy set M in X be a set of ordered sequences, where l is the dimension of M, which is defined in Eq. (2.4) as,

$$M^l = \{k, \mu_1(k), \mu_2(k), ..., \mu_i(k), ..., \mu_l(k))\}|k \in X. \tag{2.4}$$

The function $\mu_{M^l} = (\mu_1, \mu_2, ..., \mu_l)$ is called the multi-membership degrees of multi-fuzzy set M^l. The set of all multi-fuzzy sets in X is denoted by $M^lFS(X)$.

Remark 2.1. *Clearly, a multi-fuzzy set of dimension 1 is a Zadeh's fuzzy set* [38]*, and a multi-fuzzy set of dimension 2 is an Atanassov's intuitionistic fuzzy set* [4]*. Similarly, a multi-fuzzy set with dimension 3 is the Coung's picture fuzzy set* [10].

Remark 2.2. *Let $M^l \in M^l FS(X)$. If $M^l = \{k, (0, 0, 0, ..., 0)\}$, then M^l is called the null multi-fuzzy set of dimension l, denoted by Φ_0 . If $M^l = \{k, (1, 1, 1, ..., 1)\}$, then M^l is called the absolute multi-fuzzy set of dimension l, denoted by Λ_1.*

2.1.1 Non-normalized multi-fuzzy set and its characteristic

In the case of the non-normalized multi-fuzzy set, the summation of all membership degrees will be more than 1. The M^l is called the non-normalized MFS if and only if $\sum_{i=1}^{l} \mu_i(k) \geq 1$. The multi-membership degree can be redefined as $\frac{1}{l}(\mu_1(k), \mu_2(k), ..., \mu_l(k))$, which leads to the non-normalized multi-fuzzy set to be a normalized multi-fuzzy set.
Let $Z_1 = \{(k, \mu_1(k), \mu_2(k), ..., \mu_i(k), ..., \mu_l(k))|k \in X\}$ and $Z_2 = \{(k, \beta_1(k), \beta_2(k), ..., \beta_i(k), ..., \beta_l(k))|k \in X\}$ are the two MFSs of dimension l in X. Then, we can define the following operators and functions:

i. The standard complement Z_1^c of the MFS Z_1 over the X is defined as,
$\mu_i^c(k) = 1 - \mu_i(k), \forall k \in X, i = 1, 2, ..., l$.

ii. The standard intersection between the two MFSs Z_1 and Z_2 ($Z_1 \cap Z_2$) over the X is defined as,
$Z_1 \cap Z_2 = \{(k, \mu_1(k) \wedge \beta_1(k), \mu_2(k) \wedge \beta_2(k), ..., \mu_l(k)) \wedge \beta_l(k)|k \in X\}$.

iii. The standard union of the two MFSs Z_1 and Z_2 ($Z_1 \cup Z_2$) over the X is defined as,
$Z_1 \cup Z_2 = \{(k, \mu_1(k) \vee \beta_1(k), \mu_2(k) \vee \beta_2(k), ..., \mu_l(k)) \vee \beta_l(k)|k \in X\}$.

iv. The two MFSs Z_1 and Z_2 are said to be equal ($Z_1 = Z_2$), if and only if $\mu_i(k) = \beta_i(k), \forall k \in X, 1 \leq i \leq l$.

v. Let Z_1 be the MFS over X the cardinality of Z_1, which is a natural number, is defined as,
$|Z| = \sum_i (Count_Z(\mu_i(k) = \mu_j(k))|i < j \leq l)$. Here, the cardinality is the dimension of the $MFS(Z_1)$.

vi. Pivot point (P_{Z_1}) of the MFS Z_1 indicates the maximum number of occurrences of the same membership degree and defined as, $P_{Z_1} = \{\mu_i(k)|max_i(Count_{Z_1}(\mu_i(k) = \mu_j(k))\forall(i < j \leq l))\}$.

vii. Pivot degree PvD_{Z_1} of pivot points (P_{Z_1}) indicate the number of instances, which is defined as
$PvD_{Z_1} = \{\mu_i(k)|\mu_i(k) = \mu_j(k) \wedge P_{Z_1} \in \mu_i(k)\}\forall i, j$.

viii. Score function of the MFS (Z_1) is defined as, $S(Z_1) = \sum_{\mu_i(k) \in Z_1}(\mu_i(k))$.

ix. Distance between two MFSs Z_1 and Z_2 can be estimated as, $dis(Z_1, Z_2) = \sum_{i=1}^{k} |\mu_i(k) - \beta_i(k)|$.
The properties of the distance between two MFSs are as follows:

  a. $dis(Z_1, Z_2) \geq 0$,
  b. $dis(Z_1, Z_2) = dis(Z_2, Z_1)$,
  c. $dis(Z_1, Z_2) = 0$, if $Z_1 = Z_2$.

x. The standard merge between two MFSs Z_1 and Z_2 can be defined as,

$Z_1 * Z_2 = \{(k, \rho_1(k), \rho_2(k), ..., \rho_l(k))|k \in X\}$, where $\rho_i(k) = \frac{\mu_i(k)+\beta_i(k)-\mu_i(k)\beta_i(k)}{2-\mu_i(k)\beta_i(k)}$.

xi. The score factor of the MFS (Z_1) is defined as,

$$SF(Z_1) = avg_i(\mu_i(k)) * \frac{min(\mu_i(k))}{max(\mu_i(k))}. \tag{2.5}$$

xii. The normalized factor for MFSs $Z_j (j = 1, 2, ..., m)$ is defined as,

$$n_i^j(k) = \frac{\mu_i^j(k) - \mu_{*i}^j(k)}{\mu_i^{*j}(k) - \mu_{*i}^j(k)}, \tag{2.6}$$

where $\mu_i^{*j}(k) = max_j(\mu_i^j(k))$ and $\mu_{*i}^j(k) = min_j(\mu_i^j(k))$.

Remark 2.3. *The merge operation is a special type operator that can normalize the rigid situation of the boundary value in the max and min operations.*

Remark 2.4. *Pivot degree PvD_{Z_1} indicates the partial strength of the MFS, whereas the score value denotes the overall strength of the MFS.*

Remark 2.5. *Pivot degree PvD_{Z_1} can be used to represent the stability of the MFS. If the pivot degree PvD_{Z_1} of the MFS Z_1 is l, then the MFS is fully stable where the MFS is unstable for the pivot degree 1.*

Remark 2.6. *The score factor of the MFS is an important characteristic of the MFS that can be used for measuring the strength of the MFS; a higher score factor indicates that the strength of the MFS is high.*

Example 2.7. *Let S be the set of all students. For each student s in S, the non-normalized membership values are denoted as $\mu_{Eng}(s)$, $\mu_{Phy}(s)$, $\mu_{Chem}(s)$, $\mu_{BS}(s)$, $\mu_{Math}(s)$, and $\mu_{CS}(s)$. These values correspond to the marks in English, Physics, Chemistry, Bio-Science, Mathematics, and Computer Science, respectively. The performance of each student can be approximated by their collection of marks using the multi-membership function $\mu_{Eng}, \mu_{Phy}, \mu_{Chem}, \mu_{BS}, \mu_{Math}, \mu_{CS}$. This performance can be represented as a multi-fuzzy set:*

$$M_S = \{(s, \mu_{Eng}(s), \mu_{Phy}(s), \mu_{Chem}(s), \mu_{BS}(s), \mu_{Math}(s), \mu_{CS}(s))|s \in S\}.$$

This provides a structured way to analyze and interpret the academic performance of students across multiple subjects. The performance of students in their end-semester examinations, which consist of six isolated papers, cannot be accurately described using a typical membership function of an ordinary fuzzy set. Instead, it can be characterized by a six-dimensional membership function, denoted as M_S. To better understand a multi-fuzzy set, it can be viewed as being composed of ordinary fuzzy sets as its building blocks.

2.1.2 Normalized multi-fuzzy set and its characteristic

A normalized multi-fuzzy set is a specific type of fuzzy set in which the aggregated membership degrees of the multi-fuzzy set (MFS) are equal to or less than 1. For an MFS Z, if the sum of the membership degrees satisfies $\sum_{i=1}^{k} \mu_i(k) \leq 1$ for all $k \in X$, then the multi-fuzzy set of dimension l is classified as a normalized multi-fuzzy set. Ildar [6] redefined the concept of normalized MFS and introduced new terminology referred to as Distributed Fuzzy Sets (DFS), along with several novel relations and operators. In this subsection, we will define some of these well-known relations and operators as follows:

i. Normalized MFS-based t-conorm is a function, F: [0,1] × [0,1] → [0,1]. For all $\mu_1(k), \mu_2(k), \mu_3(k) \in [0, 1]$, the following properties are satisfied [6]:

(a) $F(\mu_1(k), \mu_2(k)) = F(\mu_2(k), \mu_1(k))$ (commutatively),

(b) $F(\mu_1(k), F(\mu_2(k), \mu_3(k))) = F(F(\mu_1(k), \mu_2(k)), \mu_3(k)))$ (associativity),

(c) $F(\mu_1(k), \mu_2(k)) \leq F(\mu_1(k), \mu_3(k))$, whenever $\mu_2(k) \leq \mu_3(k)$ (monotonicity),

(d) $F(\mu_1(k), 0) = F(0, \mu_1(k)) = 0$ (Boundary condition).

(e) $F(\mu_1(k), 1) = F(1, \mu_1(k)) = 1$.

(f) $F_M(\mu_1(k), \mu_2(k)) = max(\mu_1(k), \mu_2(k))$ (Maximum).

(g) $F_P(\mu_1(k), \mu_2(k)) = \mu_1(k) + \mu_2(k) - \mu_1(k) * \mu_2(k)$ (Probabilistic sum).

ii. A union of two MFSs based on t-conorm F is a function D × D → D, defined for any two MFSs $Z_1 = \{(k, \mu_1(k), \mu_2(k), ..., \mu_i(k), ..., \mu_l(k))|k \in X\}$ and $Z_2 = \{(k, \beta_1(k), \beta_2(k), ..., \beta_i(k), ..., \beta_l(k))|k \in X\}$ in D as follows:

$$D(Z_1, Z_2) = P = (p_1, p_2, ..., p_k), \text{ where } p_i = \frac{F(\mu_i, \beta_i)}{\sum_{i=1}^{k} F(\mu_i, \beta_i)}.$$

Example 2.8. *Consider that the pixels of the color image are denoted by the set X and the normalized membership degrees for the red, green, and blue values of each pixel x (where $x \in X$) are represented as $\mu_r(x)$, $\mu_g(x)$, and $\mu_b(x)$, respectively. Therefore, the color image can be approximated by a collection of pixels characterized by the multi-membership function (μ_r, μ_g, μ_b). This is expressed as a multi-fuzzy set:*

$$M_{Image} = \{(x, \mu_r(x), \mu_g(x), \mu_b(x))|x \in X\}.$$

In a two-dimensional image, the colors of the pixels cannot be accurately represented by the membership function of a conventional fuzzy set. Instead, they can be characterized using a three-dimensional membership function (μ_r, μ_g, μ_b). A multi-fuzzy set can be better understood by considering ordinary fuzzy sets as its foundational building blocks.

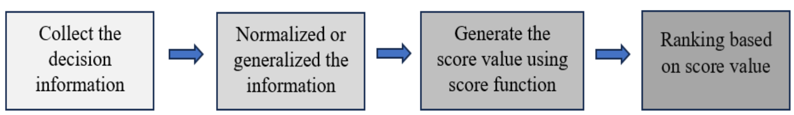

Figure 1: FMS-based Decision-Making Approach

3 Dynamic weight-based decision-making approaches

Healthcare systems play a crucial role in our communities, as they are responsible for delivering quality treatment for both common illnesses and critical surgical procedures. This is a fundamental requirement for any healthcare system, as it enables rapid patient recovery while minimizing major health risks. In this article, we present an alternative Multi-Criteria Decision Making (MCDM) approach to evaluate the overall performance of hospitals based on the services they offer and the resources available to them. First, we introduce a dynamic weight-based decision-making method that takes into account individual differences and the overall strengths of each hospital. Next, we propose a normalized decision-making concept that emphasizes the relative importance of various factors tailored to individual needs.

3.1 Dynamic weight estimation technique of the criteria

The criteria weights are crucial for analyzing decision information during the aggregation process. Typically, individual experts assign these weights, often distributing them randomly or equally. Such weight estimation techniques can diminish the acceptance of the final decision-making results. In this article, we propose a dynamic approach to estimating the weights of the criteria based on the dispersion of the decision information.

The core concept of our weight estimation technique is that the weight of each criterion is linked to its overall dispersion and cumulative importance. We utilize the range of membership degrees and the overall significance of the criteria to dynamically estimate the weights. A higher mean of the membership degrees coupled with greater dispersion plays a significant role in the decision-making process. Conversely, if the dispersion is zero, the weight of that criterion becomes zero as well, resulting in no impact on the decision-making.

Based on this heuristic, our proposed weight estimation technique is defined as follows:

$$w_j = \frac{max_j(\mu_i^j) - min_j(\mu_i^j)}{1 - avg_j(\mu_i^j)}. \tag{3.1}$$

Here, μ_i^j represents the membership degree of the j^{th} instance of the i^{th} alternative and $max_j(\mu_i^j)$, $min_j(\mu_i^j)$, and $avg_j(\mu_i^j)$ represent the maximum, minimum, and average membership degrees of the j^{th} criterion, respectively.

3.2 Intermediate factors and their importance

This article discusses Score Factor (SF) and Normalized Factor (NF). The SF is calculated using Equation (2.5), and its range is from 0 to 1. If any criterion value is zero, the SF value will also be zero. This outcome indicates that the alternative is not ideal and holds less importance for the system. Similarly, the NF is calculated using Equation (2.6). The magnitude of the NF signifies the relative importance of an alternative compared to others. The most favorable situation occurs when the NF value reaches the maximum mean number of criteria, indicating that the alternative is the most desirable and can dominate others in relation to all criteria. These factors are estimated for different purposes. The SF measures the individual strength of an alternative, while the NF assesses its relative importance in comparison to other alternatives. Ultimately, the evaluation of an alternative is conducted differently, with both factors considered from different perspectives.

3.3 Dynamic decision-making approaches

In this section, we present an approach to solving MCDM problems using a MFS framework to manage information from multiple instances. The general procedure for dynamic MFS-based decision-making under uncertainties is illustrated in Fig. 1. This dynamic MFS-based approach is significant for decision-making as it accommodates multiple instances of an alternative, which cannot be effectively captured using FS, IFS, or PFS methods. The proposed frameworks, which are based on relative weights and normalized factors using MFSs, consider l types of features for n alternatives, denoted as μ_i^j. Here, μ_i^j represents the membership degree of the j^{th} instance of the i^{th} alternative. The alternatives can thus be expressed as follows:

$$Alt_i = \{(\mu_i^1, \mu_i^2, ..., \mu_i^l)\}, \quad i = 1, 2, ..., n.$$

In this context, we introduce two MCDM approaches to identify the most desirable alternative for the system.

3.3.1 Relative weight-based decision-making approach

The relative weight-based MCDM approach works as follows:

Step 1. Construct the decision table ($D = (\mu_i^j(k)), i = 1, 2, ..., n$ and $j = 1, 2, ..., l$) according to experts' opinion in the form of MFS.

Step 2. Calculate the relative weight w_j of each parameter based on decision table (D) by the Eq.(3.1).

Step 3. Compute the final score ($FS_i, i = 1, 2, ..., n$) of the alternatives using following equation:

$$FS_i = \sum_{j=1}^{k} w_j \mu_i^j(k). \tag{3.2}$$

Step 4. Rank the alternatives according to their final score value from Eq. (3.2).

Remark 3.1. *The weight w_j is influenced by the dispersion and overall strength of the fuzzy set. If all membership degrees within the fuzzy set are equal, the weight will be zero. Conversely, a high-strength fuzzy set has a high weight because it produces a high average membership degree.*

3.3.2 Normalization based decision-making approach

The normalized weight-based MCDM approach operates as follows:

Step 1. Construct the decision table ($D = (\mu_i^j(k))$, $j = 1, 2, ..., l$ and $i = 1, 2, ..., n$) according to experts' opinion in the form of MFS.

Step 2. Calculate the score factor of each alternative as follows:

$$SF_i = avg_j(\mu_i^j(k)) * \frac{min_j(\mu_i^j(k))}{max_j(\mu_i^j(k))}. \tag{3.3}$$

Step 3. Normalize the decision table (D) and generate the normalized matrix (N) using Eq.(2.6)

Step 4. Compute the resultant score ($RS_i, i = 1, 2, ..., n$) of the alternatives using following equation:

$$RS_i = SF_i + \sum_j n_i^j(k). \tag{3.4}$$

Step 5. Rank the alternatives according to their resultant score value.

Table 2: Triangular fuzzy number

Linguistic Observation	**Fuzzy Number**
Very Good	0.75,1,1
Good	0.5,0.75,1
Moderately Good	0.25,0.5,0.75
Average	0,0.25,0.5
Poor	0,0,0.25

Remark 3.2. *The final score of the alternatives consists of two components: the normalized factor ($\sum_j n_i^j(k)$), which indicates the relative importance of each alternative compared to others, and the score factor (SF_i), which measures the individual significance of the alternative.*

Remark 3.3. *A conflict arises when multiple alternatives yield the same score, whether final or resultant. This issue is addressed by calculating the geometric aggregation factor for the conflicting alternatives ($GAF_i = \Pi_j \mu_i^j(k)$); a higher value of this factor indicates a more desirable alternative.*

Table 3: Decision table (D) according to observation report of the health experts

Hospital	e_1	e_2	e_3	e_4	e_5	e_6
$Hspt_1$	0.57	0.77	0.83	0.65	0.92	0.79
$Hspt_2$	0.73	0.82	0.85	0.83	0.93	0.67
$Hspt_3$	0.81	0.74	0.71	0.68	0.9	0.69
$Hspt_4$	0.92	0.71	0.79	0.75	0.89	0.77
$Hspt_5$	0.73	0.76	0.87	0.73	0.9	0.69

4 Numerical illustration

Fuzzy multisets allow for the inclusion of multiple membership degrees, which means that individual responses can be repeated. We primarily utilize fuzzy multisets for decision-making problems where multiple instances of observations exist. For example, we can measure the performance of hospitals after gathering sufficient information in this field. According to the World Health Organization (WHO) guidelines [34], we identified six factors for evaluating hospital performance. These factors are clinical effectiveness (e_1), production efficiency (e_2), patient-centeredness (e_3), safety (e_4), staff orientation (e_5), and responsive governance (e_6). We assessed five hospitals, denoted as $Hspt_i(i = 1, 2, ..., 5)$, across these six performance dimensions. Health experts evaluated the hospitals based on these dimensions and provided their opinions using a fuzzy information system [16]. The experts' evaluations are represented through five multi-fuzzy sets, from which we constructed the corresponding decision table (Table 3).

To determine the relative status of the hospitals, we illustrated our proposed methods based on the decision table. We generated a criteria-wise weight vector using Eq. (3.1) to evaluate the score values of the hospitals, and we normalized the matrix using Eq. (2.6), as shown in Table 4 and Table 5, respectively. The overall score according to the function defined in section 2, the final score based on the dynamic weight-based approach, and the resultant score according to the normalized-based approach are all presented in Table 6.

Table 4: Relative weight of each criterion according to decision Table 3

Criteria	e_1	e_2	e_3	e_4	e_5	e_6
Related weight	1.4	0.46	0.84	0.67	0.44	0.43

Table 5: Normalized decision matrix(N) according to decision matrix Table 3

Hospital	e_1	e_2	e_3	e_4	e_5	e_6
$Hspt_1$	0.0	0.55	0.75	0.0	0.75	1.0
$Hspt_2$	0.46	1.0	0.88	1.0	1.0	0.0
$Hspt_3$	0.68	0.27	0.0	0.17	0.25	0.17
$Hspt_4$	1.0	0.0	0.5	0.56	0.0	0.83
$Hspt_5$	0.46	0.45	1.0	0.44	0.25	0.17

The overall score values and the contributions from various criteria for the hospitals are represented in a stack-based bar plot, as shown in Fig. 2. The overall scores do not provide a clear ranking among the hospitals because of the conflicting results from Hospital, $Hspt_1$ (4.53), $Hspt_3$ (4.53), and the scores of $Hspt_2$ (4.83) and $Hspt_4$ (4.83). Additionally, Fig. 4 indicates that the observation reports from the hospitals are completely uncontrolled, showing no clear relationships among them. In this context, our two proposed methods can effectively rank the hospitals without conflicts. Using the relative weight-based decision-making approach, the final scores for the hospitals were 3.03, 3.37, 3.22, 3.5, and 3.28, respectively. Similarly, the normalized decision-making method produced scores of 3.22, 4.6, 1.72, 3.16, and 2.82 for the hospitals, respectively. The ranking sequence according to the relative weight-based approach is $Hspt_4$, $Hspt_2$, $Hspt_5$, $Hspt_3$, and $Hspt_1$. For the normalized-based approach, the ranking sequence is $Hspt_2$, $Hspt_1$, $Hspt_4$, $Hspt_5$, and $Hspt_3$. The rankings generated by these approaches are conflict-free and meaningful. Fig. 3 illustrates the statistical analysis of the hospitals, where the score values reflect overall performance factors such as range of displacement, mean value, and intensity level of the observation report. Based on the generated scores, $Hspt_2$ holds a distinct importance according to the proposed methods. The relative weight-based approach takes individual importance into account, while the normalized approach emphasizes relative importance. Although the overall scores for some hospitals are the same, the ranking order is clear and significant. The comprehensive services offered by the hospitals validate the effectiveness of the proposed methods, making the resultant ranking sequence more scientific. The geometric aggregation factor, defined as $GAF_i = \Pi_j \mu_i^j$, will be utilized in situations where more than one alternative yields the same final score.

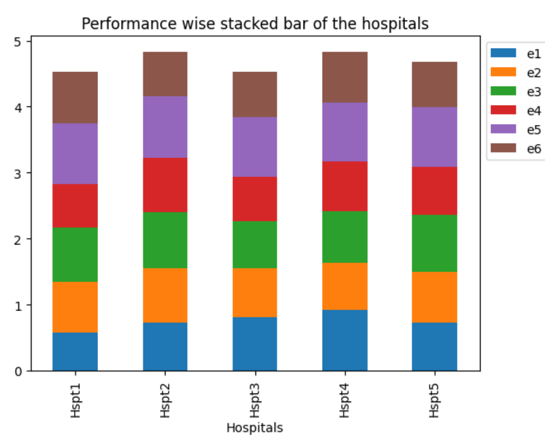

Figure 2: Overall observation report of Hospitals

Table 6: Overall score, generated final score and resultant score of the hospitals

Hospital	$Hspt_1$	$Hspt_2$	$Hspt_3$	$Hspt_4$	$Hspt_5$
Overall Score	4.53	4.83	4.53	4.83	4.68
Final Score (FS)	3.03	3.37	3.22	3.5	3.28
Resultant Score (RS)	3.22	4.6	1.72	3.16	2.82

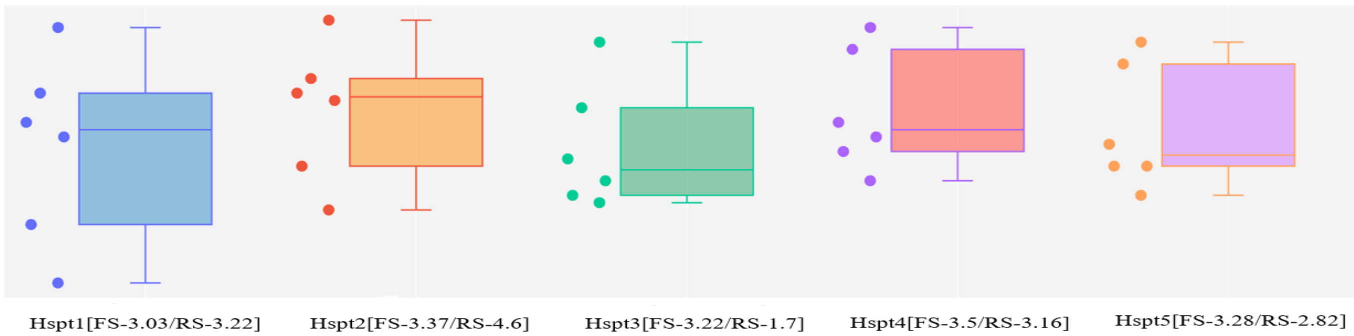

Figure 3: Statistical observation report of the hospitals.

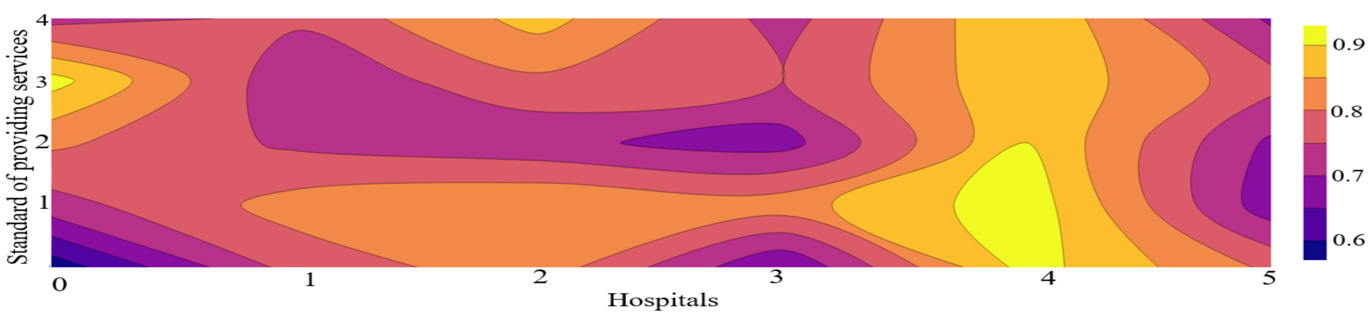

Figure 4: Comparative analysis of the hospitals

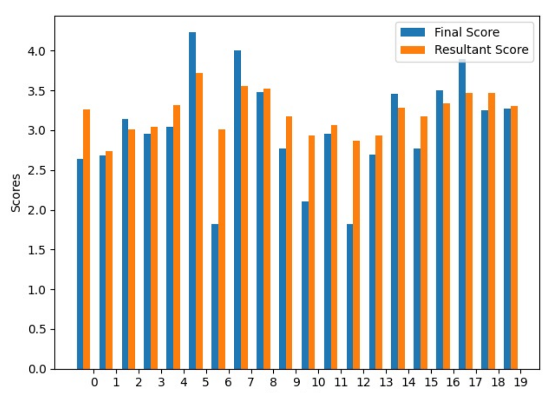

Figure 5: Status of the hospitals in rural area

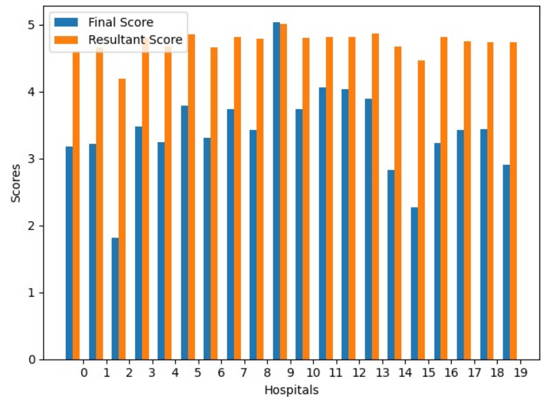

Figure 6: Status of the hospitals in semi-urban area

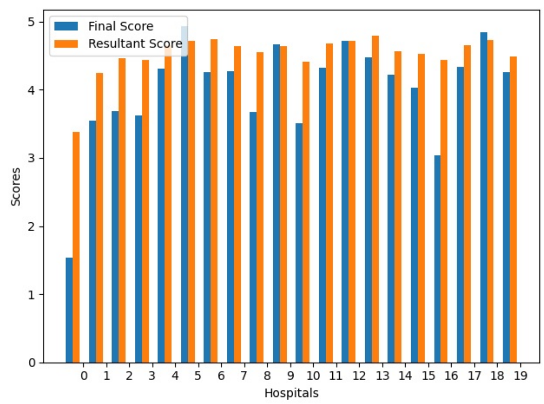

Figure 7: Status of the hospitals in urban area

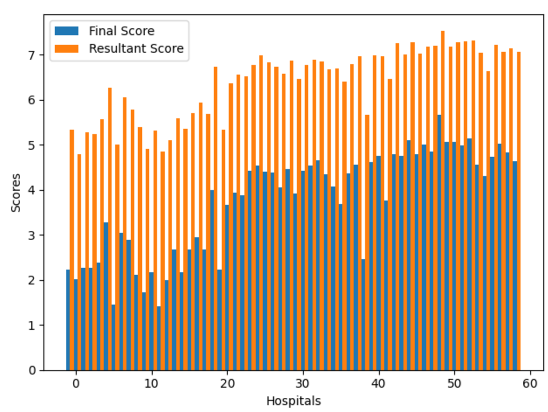

Figure 8: Status of the hospitals within the all area

Table 7: Region-wise performance of the hospitals based on normalized approach

Region	Region-wise performance		Global performance	
	Minimum Score	Maximum Score	Minimum Score	Maximum Score
Rural	1.67	4.42	1.21	4.11
Semi-Urban	1.89	4.91	2.23	4.53
Urban	1.52	4.95	4.09	5.59

Table 8: Region-wise performance of the hospitals based on relative weight approach

Region	Region wise performance		Global performance	
	Minimum Score	Maximum Score	Minimum Score	Maximum Score
Rural	2.61	3.65	4.82	6.81
Semi-Urban	4.21	4.86	5.36	6.79
Urban	3.32	4.76	6.26	7.24

Table 9: ± 20% tolerance based pivot point with maximum pivot degree for each region

Region	Relative weight-based approach		Normalize-based approach	
	Pivot Point	Pivot Degree	Pivot Point	Pivot Degree
Rural	2.62	8	3.21	13
Semi-Urban	3.13	11	4.56	17
Urban	4.12	9	4.65	10

5 Case study

An efficient healthcare system can significantly contribute to a country's economy, development, and industrialization. Healthcare plays a crucial role in promoting the overall physical and mental well-being of communities worldwide. According to the general guidelines provided by the World Health Organization (WHO) [34], several key factors have been discussed. However, focusing on a single direction of data may lead to errors due to insufficient point-to-point investigation by health experts. To minimize these errors, we gather generalized opinions from various hospital stakeholders. Consequently, we have collected information from a range of sources, including health experts, healthcare workers, governing authorities, and feedback from patients and their guardians. This feedback is provided in both qualitative terms and satisfaction levels. The use of linguistic terms can introduce uncertainty in decision-making, which we address by reprocessing the linguistic hedges represented conveniently in Table 2. We gather decision-making information from three types of regions: urban, semi-urban, and rural. Each of these regions presents unique challenges in providing quality service, infrastructure, and access to health experts and workers. For our analysis, we randomly selected sixty hospitals, with twenty from each region, out of a total of 250 hospitals from which data was collected. The opinions of these hospitals are evaluated using a multi-fuzzy information system.

Recognizing the significance of the performances of the proposed methods allows for a comparison of hospital effectiveness, both in specific areas and overall. The region-wise minimum and maximum performance scores of the hospitals are generated using the proposed approaches based on the same dataset represented in Tables 7 and 8, considering both local and global aspects. The performance scores of the hospitals shift from a local perspective to a global one. Specifically, the score for rural hospitals decreases, while the scores for urban hospitals increase, reflecting the better quality of services provided by urban hospitals compared to those in rural areas. In the global aspect, the service range of hospitals expands, with urban hospitals benefiting more than rural hospitals. The area-wise performance of hospitals is illustrated in Figures 5, 6, and 7 for rural, semi-urban, and urban hospitals, respectively. Figure 8 showcases the overall performance based on the same hospital observation reports in a sequence that includes rural, semi-urban, and urban areas. We also applied a ±20% tolerance to assess the area-wise hospital standards for both approaches, with the resulting data among the three regions displayed in Table 9. For the relative weight-based approach, the evaluated pivot points were 2.62, 3.13, and 4.12 for rural, semi-urban, and urban hospitals, respectively, with corresponding pivot degrees of 8, 11, and 9. In contrast, the normalized-based approach resulted in estimated pivot points of 3.21, 4.56, and 4.65 for the same regions, along with corresponding pivot degrees of 13, 17, and 10. The outcomes of both approaches are closely aligned in terms of pivot degrees and pivot points, although the range of pivot degrees and the magnitude of the pivot points differ from the region-wise results. The relative weight-based approach appears to be more robust than the normalized-based approach, as indicated in Table 9. In the relative weight-based approach, the pivot degrees are relatively consistent; however, the urban area's pivot point significantly surpasses those of the semi-urban and rural areas. Conversely, in the normalized-based approach, area-wise pivot degrees are more fragmented, and the pivot point for urban hospitals is notably poorer compared to those of semi-urban and rural hospitals. This information highlights that hospitals in remote areas tend to provide more conventional services, resulting in a moderate overall performance.

6 Conclusion

Robust frameworks based on MFS that incorporate a decision-making approach have been introduced in various cutting-edge literature to identify optimal alternatives. However, existing methods often fail to select the preferred alternative in unusual situations. To address this issue, we have integrated relative weight-based and normalized decision-making approaches with MFS to measure varying levels of uncertainty. To assess the actual effectiveness of the alternatives in decision-making, we dynamically estimate the weights of the criteria and utilize normalized score factors to identify optimal solutions. We demonstrated the effectiveness of our proposed approaches using semi-realistic issues in health science and healthcare management. The outcomes of these approaches are aligned concerning pivot degrees and pivot points, which are unique to different socio-economic regions. Notably, our findings indicate that the relative weight-based approach is significantly more effective than the normalized-based approach, as shown in Table 9. From our case studies, it is evident that hospitals in remote areas offer relatively standard services, and their overall performance is moderate, which correlates with expert opinions. In the future, this work can be further developed by incorporating weight estimation in either static or dynamic modes and aggregating the membership degrees to generate resultant scores. Additionally, a specialized decision-making method could be introduced where both normalized and non-normalized MFS coexist.

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