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Indistinguishability operators and generalized distances: The transformation problem revisited

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Abstract

In this paper we characterize those functions that induce a fuzzy preorder from a quasi-pseudo-metric even when the considered t-norm is not continuous. On the one hand, we prove that they must be decreasing and fulfill a special property of dominance with respect to the ordinary addition and the t-norm T under consideration. On the other hand, we have shown that they must transform asymmetric triangular triplets into asymmetric T-triangular triplets. Moreover, we also study the case in which fuzzy preorders are exactly indistinguishability operators. Concretely, we show that the monotonicity of the function and the previously mentioned dominance are sufficient but not necessary conditions. In addition, we prove that such functions must transform triangular triplets into T-triangular triplets. The developed theory is illustrated by means of appropriate examples. Furthermore, we prove that the well-known technique based on the use of the pseudo-inverses of the additive generators of continuous and Archimedean t-norms is recovered as a particular case of the new one presented here.

Keywords: Fuzzy preorder, indistinguishability operator, quasi-pseudo-metric, t-norm, asymmetric T-triangular triplet, monotonicity, dominance, additive generator, pseudo-inverse.

1 Introduction

The concept of distance has become an essential tool in many branches of science such as Optimisation, Pattern Recognition, Image Analysis, Computational Biology, Computer Science, Decision and Social Choice Theory, Psychology, etc. (see, for example, [7, 20, 28]).

In general terms the resolution of problems, which arise naturally in the areas mentioned above, requires the interpretation of distance as a measure of dissimilarity (metric) or similarity (indistinguishability operator) between two objects of some nature.

Despite the usefulness and extensive use of the notion of metric, the rigidity of some of its axioms makes it impossible to obtain certain applications of interest. For this reason, different notions of distance that are more general than the classical one (i.e. notions of measurement that generalize the classical notion of metric) are currently studied in depth.

Among the aforementioned generalizations we can find the so-called quasi-pseudo-metrics [7, 11] which are defined below.

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Definition 1.1. *Let X be a non-empty set. A quasi-pseudo-metric is a function $d : X \times X \to [0, +\infty]$ which satisfies for all $x, y, z \in X$ the following:*

(M1) $d(x, x) = 0$.

(M2) $d(x, z) \leq d(x, y) + d(y, z)$.

It must be stressed that some authors call extended quasi-pseudo-metric to the notion of quasi-pseudo-metric introduced above due to the fact that they can take the value $+\infty$ (see, for instance, [7, 8, 18]). Notice, in addition, that we have denoted by $+$ the usual addition on $[0, \infty[$ extended to $[0, \infty]$, i.e., $a + \infty = \infty$ for all $a \in [0, \infty]$, and by $\leq$ the usual order on $[0, \infty[$ extended to $[0, \infty]$, where $x \leq \infty$ for all $x \in [0, \infty]$. As usual, we will write $x < y \Leftrightarrow x \leq y$ and $x \neq y$.

A special type of quasi-pseudo-metrics are the so-called quasi-metrics. Next we introduce such a notion, since it will play a distinguished role throughout this paper.

Definition 1.2. *A quasi-pseudo-metric is said to be a quasi-metric when the axiom (M1) is replaced by the next one:*

(M1') $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$.

Moreover, a pseudo-metric (metric) is a quasi-pseudo-metric (quasi-metric) satisfying the axiom of symmetry:

(M3) $d(x, y) = d(y, x)$ *for all* $x, y \in X$.

In those cases in which the method with which the distance is measured involves some kind of uncertainty, it is necessary to handle the notion of a fuzzy set in the sense of Zadeh [26]. In fact, on many occasions, when comparing two objects, it is not possible to determine categorically whether they are similar or not. This type of situation arises when there is, on the one hand, a limitation in the precision of the measurement between the objects being compared and, on the other hand, when only a certain degree of similarity can be determined between the objects being compared. The latter case is due to the fact that the comparison is carried out by means of subjective or imprecise criteria.

In this direction, the generalization of the concept of distance is given by the so-called fuzzy preorders and the indistinguishability operators, which rest strongly on the notion t-norm (see [12, 21, 23, 27]). Let us recall this concept.

Definition 1.3. *A t-norm is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ which is commutative, associative and fulfills the following conditions:*

(T1) $T(x, 1) = x$ *for all* $x \in [0, 1]$,

(T2) $T(x, y) \leq T(x, z)$ *for all* $x, y, z \in [0, 1]$ *with* $y \leq z$.

Our main reference for the theory of t-norms is [12].

Following [24] (see also [17]), fuzzy preorders and fuzzy partial orders are defined as follows.

Definition 1.4. *Given a non-empty set X, a fuzzy set $E : X \times X \to [0, 1]$ is a fuzzy preorder on X with respect to a t-norm T provided that it satisfies, for all $x, y, z \in X$, the following axioms:*

(E1) $E(x, x) = 1$.

(E2) $T(E(x, y), E(y, z)) \leq E(x, z)$.

A fuzzy preorder with respect to T is said to be a fuzzy partial order with respect to T when the axiom (E1) is replaced by the next one:

(E1') $E(x, y) = E(y, x) = 1 \Leftrightarrow x = y$.

Note that fuzzy partial orders are called T-pre-orders in [24] and fuzzy partial orders are called T-quasiorders in [15].

It is clear that the numerical value $E(x, y)$ can be interpreted as a measure of how indistinguishable x and y are. Thus, the higher the value $E(x, y)$ the greater the similarity between x and y. In particular, when $x = y$ the numerical value provides the highest degree of indistinguishability, i.e., $E(x, y) = 1$.

The symmetric fuzzy preorders are known in the literature as indistinguishability operators as the below definition states (see [17]).

Definition 1.5. *A fuzzy preorder E with respect to a t-norm T is said to be an indistinguishability operator on a non-empty set X provided that*

(E3) $E(x,y) = E(y,x)$ *for all* $x, y \in X$.

Observe that some authors call T-fuzzy equivalence (T-fuzzy equalities) what we have called indistinguishability operator (indistinguishability operator that separates points) with respect to a t-norm T (see, for example, [2, 12, 17]). Moreover, following [1, 3, 12, 17], when the condition $E(x,y) = 1$ implies $x = y$, the indistinguishability operator is said to separate points or to be separable.

Since both concepts, quasi-pseudo-metrics and fuzzy preorders, were introduced, a few works have addressed their study to explore formally the duality relationship existing between them. Among the many works studying this aspect, it is worth mentioning the references [2, 3, 5, 8, 9, 10, 22, 24, 25]. In these references a way of inducing a fuzzy preorder (indistinguishability operator) through a quasi-pseudo-metric (pseudo-metric) is developed in detail by using t-norms and the pseudo-inverse of their additive generators. Let us recall what is an additive generator and its associated pseudo-inverse, since they will be crucial in our subsequent discussion.

According to [12], a t-norm is said to be continuous provided that it is continuous in each component with respect to the Euclidean topology restricted to $[0,1]\times[0,1]$ and $[0,1]$, respectively. Moreover, Archimedean t-norms are those that for each $x, y \in]0,1[$ there exists $n \in \mathbb{N}$ such that $x_T^n < y$, where $x_T^n = T(\overbrace{x,\ldots,x}^{n})$ ($\mathbb{N}$ stands for the set of nonnegative integer numbers). Continuous and Archimedean t-norms can be represented through special functions called additive generators. Recall that a t-norm T admits an additive generator provided that there exists a strictly decreasing function $f_T : [0,1] \to [0,\infty]$ which is right-continuous at 0 with $f_T(1) = 0$ such that $f_T(x) + f_T(y) \in Range(f_T) \cup [f_T(0), \infty]$ and $T(x,y) = f_T^{(-1)}(f_T(x) + f_T(y))$ for all $x, y \in [0,1]$. Moreover, the function $f_T^{(-1)} : [0,\infty] \to [0,1]$, defined by $f_T^{(-1)}(y) = \sup\{x \in [0,1] : f_T(x) > y\}$ for all $y \in [0,\infty]$, is known as the pseudo-inverse of f_T.

In particular, the following result states an outstanding equivalence:

Theorem 1.6. *Let T be a t-norm. Then the following assertions are equivalent:*

(1) *T is continuous and Archimedean.*

(2) *T admits a continuous additive generator.*

Recall that the pseudo-inverse $f_T^{(-1)}$ of an additive generator of a continuous and Archimedean t-norm T is given by $f_T^{(-1)}(y) = (\min(f_T(0), y))$ for all $y \in [0,\infty]$.

On account of [2, 3, 8], here is a reminder of what the aforesaid way consists of.

Theorem 1.7. *Let X be a non-empty set and let T be a continuous and Archimedean t-norm which admits an additive generator f_T. If $f_T^{(-1)}$ is the pseudo-inverse of f_T, then the fuzzy set $E_{d,f_T^{(-1)}} : X \times X \to [0,1]$ defined, for all $x, y \in X$, by*

$$E_{d,f_T^{(-1)}}(x,y) = f_T^{(-1)}(d(x,y)), \tag{1}$$

is:

(1) *a fuzzy preorder on X with respect to T provided that d is a quasi-pseudo-metric on X,*

(2) *a fuzzy partial order on X with respect to T provided that d is a quasi-metric on X,*

(3) *an indistinguishability operator on X with respect to T provided that d is a pseudo-metric on X,*

(4) *an indistinguishability operator on X with respect to T that separates points provided that d is a metric on X.*

To the best of our knowledge, the technique provided by Theorem 1.7 is the only one existing in the literature for generating fuzzy preorders (indistinguishability operators) from quasi-pseudo-metrics (pseudo-metrics). Since the aforementioned technique strongly depends on the additive generators of the considered t-norms and their pseudo-inverses, the way in which fuzzy preorders can be generated from quasi-pseudo-metrics is limited. Inspired by this fact, in this paper we focus our efforts on discerning whether a new general method for inducing fuzzy preorders from quasi-pseudo-metrics can be developed. Thus, we ask if in the expression (1), the function $f_T^{(-1)}$ can be replaced by another function $g : [0,\infty] \to [0,1]$ and, at the same time, which properties must be satisfied by g in order to guarantee that $E_{d,g}$ is a fuzzy preorder (indistinguishability operator), where

$$E_{d,g}(x,y) = g(d(x,y)),$$

for all $x, y \in X$.

In the remainder of this paper we answer to the posed questions. Concretely, in Section 2 we present an extension of Theorem 1.7 in which we characterize those functions that induce a fuzzy preorder from a quasi-pseudo-metric. The aforesaid characterization shows that the functions generating fuzzy preorders from quasi-pseudo-metrics must be decreasing and fulfill a special property of dominance with respect to the ordinary addition and the t-norm. A description of such functions, in the spirit of [6, 13], in terms of asymmetric triangular triplets and asymmetric T-triangular triplets is also given. In particular, we prove that the functions under study are exactly those that transform an asymmetric triangular triplet into an asymmetric T-triangular triplet. A special attention is also paid to fuzzy partial orders. The case where fuzzy preorders are exactly indistinguishability operators is addressed in Section 3, where we show that the monotonicity of the function and the previously mentioned dominance are sufficient but not necessary conditions. Moreover, a characterization of such functions generating indistinguishability operators, and indistinguishability operators that separates points, provided by means of triangular triplets and T-triangular triplets is given. The developed theory is illustrated by means of appropriate examples. Furthermore, the technique based on the use of the pseudo-inverses of the additive generators is recovered as a particular case of the new one presented here when the t-norms involved are continuous and Archimedean. Finally, Section 4 presents the conclusions.

2 A general method for generating fuzzy preorders from quasi-pseudo-metrics

This section is devoted to get a general method for generating fuzzy preorders from quasi-pseudo-metrics in the spirit of Theorem 1.7 (statements 1) and 2)). In particular, given a non-empty set X and a quasi-pseudo-metric d on X, we explore the properties that must be verified by a function $g : [0, \infty] \to [0, 1]$ for generating a fuzzy preorder $E_{d,g}$ on X such that $E_{d,g}(x, y) = g(d(x, y))$ for all $x, y \in X$. The following concept will play a crucial role for this purpose.

Definition 2.1. *Given a function $g : [0, \infty] \to [0, 1]$ and a t-norm T, we will say that g +-dominates T if $T(g(a), g(b)) \leq g(a + b)$ for all $a, b \in [0, \infty]$.*

Taking into account the notions introduced, the first result that provides the desired characterization can be stated as follows.

Theorem 2.2. *Let T be a t-norm. If $g : [0, \infty] \to [0, 1]$ is a function, then the following assertions are equivalent:*

(1) *$E_{d,g}$ is a fuzzy preorder with respect to the t-norm T for every non-empty set X and for all quasi-pseudo-metric d on X, where $E_{d,g} : X \times X \to [0, 1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x, y) = g(d(x, y))$.*

(2) *The function g satisfies the conditions below:*

(2.1) *$g(0) = 1$,*

(2.2) *g is decreasing ($g(b) \leq g(a)$ when $a \leq b$),*

(2.3) *g +-dominates T.*

Proof. For the purpose of proving (1) $\Rightarrow$ (2), suppose that $E_{d,g}$ is a fuzzy preorder with respect to the t-norm T for every non-empty set X and for all quasi-pseudo-metrics d on X, where $E_{d,g} : X \times X \to [0, 1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x, y) = g(d(x, y))$. Consider the quasi-(pseudo)-metric d_∞ on $[0, \infty]$ defined by

$$d_\infty(x, y) = \begin{cases} \infty & \text{if } x < y \text{ and } y = \infty \\ \max\{y - x, 0\} & \text{if } x, y \in [0, \infty[\\ 0 & \text{if } x = \infty \text{ and } y \in [0, \infty] \end{cases}.$$

Therefore we have that $E_{d_\infty,g} : [0, \infty] \times [0, \infty] \to [0, 1]$ is a fuzzy preorder. Then, it is clear that $g(0) = 1$, since $g(0) = g(d_\infty(x, x)) = E_{d_\infty,g}(x, x) = 1$. Next we show that g is decreasing. With this purpose consider $a, b \in [0, \infty]$ such that $a \leq b$. The fact that $E_{d_\infty,g}$ is a fuzzy preorder gives that

$$T(E_{d_\infty,g}(0, b), E_{d_\infty,g}(b, a)) \leq E_{d_\infty,g}(0, a).$$

Whence we have that

$$T(g(d_\infty(0, b)), g(d_\infty(b, a))) \leq g(d_\infty(0, a)).$$

It follows that

$$g(b) = T(g(b), 1) = T(g(b), g(0)) \leq g(a).$$

It remains to prove that g $+$-dominates T. Let $a, b \in [0, \infty]$. We distinguish two cases:

Case 1. $a = \infty$. Then $a + b = \infty$ for all $b \in [0, \infty]$. Thus we get that

$$T(g(a), g(b)) = T(g(\infty), g(b)) \leq g(\infty) = g(a + b).$$

Case 2. $a \in [0, \infty[$. We know that

$$T(E_{d_\infty, g}(0, a), E_{d_\infty, g}(a, a + b)) \leq E_{d_\infty, g}(0, a + b).$$

Since $E_{d_\infty, g}(0, a) = g(a)$, $E_{d_\infty, g}(a, a + b) = g(b)$ and $E_{d_\infty, g}(0, a + b) = g(a + b)$ we deduce that

$$T(g(a), g(b)) = T(E_{d_\infty, g}(0, a), E_{d_\infty, g}(a, a + b)) \leq E_{d_\infty, g}(0, a + b) = g(a + b),$$

as claimed.

We now prove (2) $\Rightarrow$ (1). So, assume that g satisfies conditions (2.1), (2.2) and (2.3). Let X be a non-empty set and consider a quasi-pseudo-metric d on X. We will show that the fuzzy set $E_{d,g} : X \times X \to [0, 1]$ is a fuzzy preorder.

Firstly, $E_{d,g}(x, x) = g(d(x, x)) = g(0) = 1$, so property (E1) is fulfilled. Moreover, let $x, y, z \in X$. Since d is a quasi-pseudo-metric we get $d(x, y) \leq d(x, z) + d(z, y)$. On the one hand, the fact that g $+$-dominates T gives

$$T(g(d(x, z)), g(d(z, y))) \leq g(d(x, z) + d(z, y)).$$

On the other hand, the fact that g is decreasing implies

$$g(d(x, z) + d(z, y)) \leq g(d(x, y)),$$

and, thus, that

$$T(g(d(x, z)), g(d(z, y))) \leq g(d(x, y)).$$

Therefore,

$$T(E_{d,g}(x, z), E_{d,g}(z, y)) \leq E_{d,g}(x, y).$$

Hence property (E2) is satisfied, and $E_{d,g}$ is a fuzzy preorder on X.

□

The next example illustrates the preceding result.

Example 2.3. *Consider the function $g : [0, \infty] \to [0, 1]$ defined by*

$$g(a) = \begin{cases} e^{-a} & \text{if } a \in [0, 1[\\ e^{-2a} & \text{if } a \in [1, \infty[\\ 0 & \text{if } a = \infty \end{cases}.$$

An easy verification gives that g is decreasing. It is obvious that $g(0) = 1$. Moreover, g $+$-dominates the drastic t-norm T_D, since $g(a) < 1$ for all $a \in]0, \infty]$ and so

- *$T_D(g(a), g(b)) = \min\{g(a), g(b)\} \leq g(a + b)$ when either $a = 0$ or $b = 0$.*
- *$T_D(g(a), g(b)) = 0 \leq g(a + b)$ when $a \neq 0$ and $b \neq 0$.*

It is worth mentioning that the function g is not continuous at 1. So g cannot be the pseudo-inverse of any additive generator f of some t-norm, since the pseudo-inverse $f^{(-1)}$ of a decreasing function $f : [0, 1] \to [0, \infty]$ is continuous if and only if f is strictly decreasing on $f^{(-1)}([0, \infty[)$ (compare assertion (ii) in Remark 3.4 in [12]). Observe that an additive generator f of a t-norm is always strictly decreasing on $f^{(-1)}([0, \infty[)$ because it is strictly decreasing on $[0, 1]$.

Next consider a quasi-pseudo-metric d defined on a non-empty set X. By Theorem 2.2 the fuzzy set $E_{d,g} : X \times X \to [0, 1]$ is a fuzzy preorder with respect to T_D, where

$$E_{d,g}(x, y) = g(d(x, y)) = \begin{cases} e^{-d(x,y)} & \text{if } d(x, y) \in [0, 1[\\ e^{-2d(x,y)} & \text{if } d(x, y) \in [1, \infty[\\ 0 & \text{if } d(x, y) = \infty \end{cases}.$$

Observe that $E_{d,g}$ is a fuzzy preorder with respect to T_D on X although the t-norm T_D is not continuous (compare with Theorem 1.7).

Other result can be obtained for the special situation when the axiom (E1) of fuzzy preorder is replaced by (E1').

Theorem 2.4. *Let T be a t-norm. If $g : [0,\infty] \to [0,1]$ is a function, then the following assertions are equivalent:*

(1) *$E_{d,g}$ is a fuzzy partial order with respect to the t-norm T for every non-empty set X and for all quasi-metric d on X, where $E_{d,g} : X \times X \to [0,1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x,y) = g(d(x,y))$.*

(2) *The function g satisfies the conditions below:*

(2.1) *$g(a) = 1 \Leftrightarrow a = 0$,*

(2.2) *g is decreasing,*

(2.3) *g $+$-dominates T.*

Proof. With the aim of showing (1) $\Rightarrow$ (2), suppose that $E_{d,g}$ is a fuzzy partial order with respect to the t-norm T for every non-empty set X and for all quasi-metrics d on X, where $E_{d,g} : X \times X \to [0,1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x,y) = g(d(x,y))$. The same arguments to those given in the proof of the implication (1) $\Rightarrow$ (2) in Theorem 2.2 remain valid to show that $g(0) = 1$, g is decreasing and that g $+$-dominates T. Observe that d_∞ is a quasi-metric on X. All that remains to prove is that $g(a) = 1 \Rightarrow a = 0$. Assume that there exists $a \in [0,\infty]$ such that $g(a) = 1$. Then $d_\infty(0,a) = a$ and $g(d_\infty(0,a)) = g(a) = 1$. Thus $E_{d_\infty,g}(0,a) = 1$. Moreover, $d_\infty(a,0) = 0$ and, thus, $g(d_\infty(a,0)) = g(0) = 1$. So $E_{d_\infty,g}(a,0) = 1$. Consequently, $E_{d_\infty,g}(0,a) = E_{d_\infty,g}(a,0) = 1$ and, hence, $a = 0$ because $E_{d_\infty,g}$ is a fuzzy partial order on $[0,\infty]$.

With the goal to prove (2) $\Rightarrow$ (1), suppose that g satisfies conditions 2.1), 2.2) and 2.3). Let X be a non-empty set and consider a quasi-metric d on X. By Theorem 2.2 we have that the function $E_{d,g}$ is a fuzzy preorder with respect to T on X. Next assume that there exists $x, y \in X$ such that $E_{d,g}(x,y) = E_{d,g}(y,x) = 1$. Then $g(d(x,y)) = g(d(y,x)) = 1$. So $d(x,y) = d(y,x) = 0$. Since d is a quasi-metric we conclude that $x = y$.

□

The next result shows that the technique introduced in Theorem 1.7 is retrieved as a particular case from Theorems 2.2 and 2.4. Recall that a function $g : [0,\infty] \to [0,1]$ is strictly decreasing if $g(b) < g(a)$ when $a < b$.

Corollary 2.5. *Let X be a non-empty set and let T be a continuous and Archimedean t-norm which admits an additive generator f_T. If $f_T^{(-1)}$ is the pseudo-inverse of f_T, then the fuzzy set $E_{d,f_T^{(-1)}} : X \times X \to [0,1]$ defined, for all $x, y \in X$, by*

$$E_{d,f_T^{(-1)}}(x,y) = f_T^{(-1)}(d(x,y)),$$

is a fuzzy preorder (partial order) on X for T provided that d is a quasi-(pseudo-)metric on X.

Proof. Clearly the pseudo-inverse $f_T^{(-1)}$ is decreasing and $f_T^{(-1)}(0) = f_T^{-1}(\min\{f(0),0\}) = f_T^{-1}(0) = 1$ (see [12]). In fact

$$f_T^{(-1)}(a) = f_T^{-1}(\min\{f_T(0), a\}) = 1 \Leftrightarrow a = 0,$$

since f_T is strictly decreasing. Now we show that $f_T^{(-1)}$ $+$-dominates T, i.e., that given $a, b \in [0,\infty]$ the following inequality holds:

$$T(f_T^{(-1)}(a), f_T^{(-1)}(b)) \leq f_T^{(-1)}(a+b).$$

We know that

$$T(f_T^{(-1)}(a), f_T^{(-1)}(b)) = f_T^{-1}\left(\min\{f_T(0), f_T(f_T^{(-1)}(a)) + f_T(f_T^{(-1)}(b))\}\right).$$

Next we consider two possible cases.

Case 1. $f_T(0) \leq f_T(f_T^{(-1)}(a)) + f_T(f_T^{(-1)}(b))$. Then we obtain that

$$f_T^{-1}\left(\min\{f_T(0), f_T(f_T^{(-1)}(a)) + f_T(f_T^{(-1)}(b))\}\right) = f_T^{-1}(f_T(0)) = 0.$$

It follows that

$$T(f_T^{(-1)}(a), f_T^{(-1)}(b)) = 0 \leq f_T^{(-1)}(a+b).$$

Case 2. $f_T(f_T^{(-1)}(a)) + f_T(f_T^{(-1)}(b)) < f_T(0)$. Then we obtain that

$$f_T^{-1}\left(\min\{f_T(0), f_T(f_T^{(-1)}(a)) + f_T(f_T^{(-1)}(b))\}\right) = f_T^{-1}(f_T(f_T^{(-1)}(a)) + f_T(f_T^{(-1)}(b))).$$

The facts that $a \leq f_T(f_T^{(-1)}(a))$ and $b \leq f_T(f_T^{(-1)}(b))$ yield immediately that

$$f_T^{-1}(f_T(f_T^{(-1)}(a)) + f_T(f_T^{(-1)}(b))) \leq f_T^{(-1)}(a+b).$$

In consequence we conclude that

$$T(f_T^{(-1)}(a), f_T^{(-1)}(b)) \leq f_T^{(-1)}(a+b).$$

□

Theorem 2.2 clarifies why Corollary 2.5 fails when we eliminate the continuity of the t-norm as hypothesis. We illustrate such a fact through the example below, which has been essentially taken from [2]. Before proceeding with that, we recall the concepts of triangular triplet and asymmetric triangular triplet, which were introduced in [4] and in [16] (see also [6]), respectively, and will play a key role in the upcoming study.

Definition 2.6. *A triplet $(a,b,c) \in [0,\infty]^3$ is said to be:*

- *an asymmetric triangular triplet provided that $a \leq b + c$.*
- *a triangular triplet when $a \leq b + c$, $c \leq a + b$ and $b \leq a + c$.*

Example 2.7. *Set $X = \{x, y, z\}$. Define a quasi-pseudo-metric d on X as follows: $d(x,y) = d(y,x) = d(y,z) = d(z,y) = 1$, $d(x,z) = d(z,x) = \frac{3}{2}$ and $d(x,x) = d(y,y) = d(z,z) = 0$. Notice that actually d is a metric on X. Consider the drastic t-norm T_D. According to* [12]*, the function $f_{T_D} : [0,1] \to [0,\infty]$ is an additive generator of T_D, where*

$$f_{T_D}(a) = \begin{cases} 2 - a & \text{if } a \in [0,1[\\ 0 & \text{if } a = 1 \end{cases}.$$

It is not hard to check that a pseudo-inverse $f_{T_D}^{(-1)} : [0,\infty] \to [0,1]$ of T_D is given by

$$f_{T_D}^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in [2,\infty] \\ 2 - a & \text{if } a \in]1,2] \\ 1 & \text{if } a \in [0,1] \end{cases}.$$

It is clear that the fuzzy set $E_{d,f_{T_D}^{(-1)}} : X \times X \to [0,1]$ is not a fuzzy preorder. Certainly, we have that $E_{d,f_{T_D}^{(-1)}}(x,y) = 1$, $E_{d,f_{T_D}^{(-1)}}(y,z) = 1$ and $E_{d,f_{T_D}^{(-1)}}(x,z) = f_{T_D}^{(-1)}(\frac{3}{2}) = \frac{1}{2}$ and, thus,

$$T_D(E_{d,f_{T_D}^{(-1)}}(x,y), E_{d,f_{T_D}^{(-1)}}(y,z)) = 1 > E_{d,f_{T_D}^{(-1)}}(x,z) = \frac{1}{2}.$$

Observe that $f_{T_D}^{(-1)}$ does not $+$-dominates T_D, since

$$T_D(f_{T_D}^{(-1)}(1), f_{T_D}^{(-1)}(1)) = 1 > 0 = f_{T_D}^{(-1)}(2).$$

Notice that, in addition, the triplet $(1,1,2)$ is an asymmetric triangular triplet but the triplet $(f_{T_D}^{(-1)}(1), f_{T_D}^{(-1)}(1), f_{T_D}^{(-1)}(2))$ is not an asymmetric T_D-triangular triplet.

In the light of the above example it must be pointed out that Theorems 2.2 and 2.4 improve the technique given by Theorem 1.7 in two aspects. On the one hand, they give a complete description of those functions that are useful to generate fuzzy preorders (fuzzy partial orders) from quasi-(pseudo-)metrics, which will allow the introduction of new examples in the literature that better fit the various problems under study. On the other hand, they provide a general method to generate fuzzy preorders (fuzzy partial orders) from quasi-(pseudo-)metrics without imposing continuity on the t-norm involved.

The other notion required for our further discussion is that of asymmetric triangular triplet with respect to a t-norm. According to [16], a triplet $(a,b,c) \in [0,1]^3$ is said to be an asymmetric triangular triplet with respect to a t-norm T (or asymmetric T-triangular triplet) provided that $T(a,b) \leq c$.

As a particular case of asymmetric T-triangular triplets we find the so-called T-triangular triplets in the sense of [13]. Hence, a triplet $(a, b, c) \in [0, 1]^3$ is a triangular triplet with respect to a t-norm T (o T-triangular triplet) when $T(a, b) \leq c$, $T(a, c) \leq b$ and $T(b, c) \leq a$.

In an attempt to provide a second characterization of those functions that induce a fuzzy preorder from a quasi-pseudo-metric we introduce the following concept.

In the following, given a t-norm T, we will say that a function $g : [0, \infty] \to [0, 1]$ transforms (asymmetric) triangular triplets into (asymmetric) T-triangular triplets if $(g(b), g(c), g(a)) \in [0, 1]^3$ is a (n asymmetric) T-triangular triplet provided that $(a, b, c) \in [0, \infty]^3$ is a (n asymmetric) triangular triplet.

In view of the exposed notions we are able to prove the next proposition. It establishes the relationship between functions that transforms asymmetric triangular triplets into asymmetric T-triangular triplets and those functions which $+$-dominates T and, in addition, are decreasing.

Proposition 2.8. *Let T be a t-norm and let $g : [0, \infty] \to [0, 1]$ be a function such that $g(0) = 1$, then the following assertions are equivalent:*

(1) *g $+$-dominates T and is decreasing.*

(2) *g transforms asymmetric triangular triplets into asymmetric T-triangular triplets.*

Proof. (1) $\Rightarrow$ (2). Let $(a, b, c) \in [0, \infty]^3$ be an asymmetric triangular triplet. On the one hand, we have that $T(g(b), g(c)) \leq g(b + c)$ due to the fact that g $+$-dominates T. On the other hand, $g(b + c) \leq g(a)$ since g is decreasing and $a \leq b + c$. Then, $T(g(b), g(c)) \leq g(b + c) \leq g(a)$ and so $(g(b), g(c), g(a))$ is an asymmetric T-triangular triplet.

(2) $\Rightarrow$ (1). First we show that g $+$-dominates T. With this aim, let $a, b \in [0, \infty]$. Observe that $(a + b, a, b)$ is an asymmetric triangular triplet. So $T(g(a), g(b)) \leq g(a + b)$, since g transforms asymmetric triangular triplets into asymmetric T-triangular triplets. This proves that g $+$-dominates T. Next, we show that g is decreasing. Now, let $a, b \in [0, \infty]$ with $a \leq b$. Then, $(a, 0, b)$ is an asymmetric triangular triplet and so $T(g(0), g(b)) \leq g(a)$. Taking into account that $g(0) = 1$ we conclude that $g(b) \leq g(a)$ as we claimed. □

As immediate consequence of the previous proposition and Theorem 2.2 we obtain the next theorem, which establishes the second announced characterization.

Theorem 2.9. *Let T be a t-norm. If $g : [0, \infty] \to [0, 1]$ is a function, then the following assertions are equivalent:*

(1) *$E_{d,g}$ is a fuzzy preorder with respect to the t-norm T for every non-empty set X and for all quasi-pseudo-metric d on X, where $E_{d,g} : X \times X \to [0, 1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x, y) = g(d(x, y))$.*

(2) *The function g satisfies the conditions below:*

(2.1) *$g(0) = 1$,*

(2.2) *g transforms asymmetric triangular triplets into asymmetric T-triangular triplets.*

In the particular case of fuzzy partial orders we have the following characterization. Now we introduce an immediate consequence of Proposition 2.8 and Theorem 2.4.

Theorem 2.10. *Let T be a t-norm. If $g : [0, \infty] \to [0, 1]$ is a function, then the following assertions are equivalent:*

(1) *$E_{d,g}$ is a fuzzy partial order with respect to the t-norm T for every non-empty set X and for all quasi-metric d on X, where $E_{d,g} : X \times X \to [0, 1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x, y) = g(d(x, y))$.*

(2) *The function g satisfies the conditions below:*

(2.1) *$g(a) = 1 \Leftrightarrow a = 0$,*

(2.2) *g transforms asymmetric triangular triplets into asymmetric T-triangular triplets.*

3 A general method for generating indistinguishability operators from pseudo-metrics

In this section we continue the study made in the previous section. Thus, we address the problem of how to induce indistinguishability operators form pseudo-metrics extending the technique introduced in Theorem 1.7 (statements 2) and 3)). In particular, given a non-empty set X and a pseudo-metric d on X, we explore the properties that must be satisfied by a function $g : [0, \infty] \to [0, 1]$ for generating an indistinguishability operator $E_{d,g}$ on X such that $E_{d,g}(x, y) = g(d(x, y))$ for all $x, y \in X$.

In this context the following result can be stated.

Proposition 3.1. *Let T be a t-norm. If $g : [0, \infty] \to [0, 1]$ is a function, then from statements below (1) $\Rightarrow$ (2):*

(1) *The function g satisfies the conditions below:*

(1.1) $g(0) = 1$,

(1.2) *g is decreasing,*

(1.3) *g $+$-dominates T.*

(2) *$E_{d,g}$ is an indistinguishability operator with respect to the t-norm T for every non-empty set X and for all pseudo-metric d on X, where $E_{d,g} : X \times X \to [0, 1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x, y) = g(d(x, y))$.*

Proof. We omit the proof because it runs as the proof of implication (2) $\Rightarrow$ (1) in Theorem 2.2. Notice that the unique thing to prove is that $E_{d,g}$ fulfills (E3), but this fact is trivial due to the symmetry of the pseudo-metric. In fact,

$$E_{d,g}(x, y) = g(d(x, y)) = g(d(y, x)) = E_{d,g}(y, x),$$

for all $x, y \in X$.

□

In contrast to the asymmetric case the reciprocal of Proposition 3.1 is not true.

Example 3.2. *Define the function $g : [0, \infty] \to [0, 1]$ as follows:*

$$g(a) = \begin{cases} 1 & \text{if } a = 0 \\ 1/3 & \text{if } a \in]0, 1] \\ 1/2 & \text{if } a \in]1, \infty] \end{cases}.$$

Obviously g is not decreasing, since $g(1) = 1/3 < 1/2 = g(2)$. Now, let X be a non-empty set and consider the product t-norm T_P. Suppose that d is a pseudo-metric on X, then $E_{d,g}(x, x) = g(d(x, x)) = g(0) = 1$ for all $x \in X$. Moreover, $E_{d,g}(x, y) = g(d(x, y)) = g(d(y, x)) = E_{d,g}(y, x)$ for all $x, y \in X$. Now, we are going to see that

$$T_P(E_{d,g}(x, z), E_{d,g}(z, y)) \leq E_{d,g}(x, y),$$

for all $x, y, z \in X$.

Observe that if $d(x, y) = 0$ the previous inequality is fulfilled. Moreover, if $d(x, y) \neq 0$ and $d(x, z) = 0$ we conclude from the triangle inequality that $d(x, y) = d(z, y)$, which makes the inequality verified. The same occurs when $d(x, y) \neq 0$ and $d(z, y) = 0$, i.e., $d(x, y) = d(x, z)$ and, thus, the inequality again holds. It remains to consider that $d(x, y), d(x, z), d(z, y) \in]0, \infty]$. Then, by definition of g and $E_{d,g}$, we have that $1/3 \leq E_{d,g}(x, y)$, $E_{d,g}(x, z) \leq 1/2$ and $E_{d,g}(z, y) \leq 1/2$. Therefore,

$$T_P(E_{d,g}(x, z), E_{d,g}(z, y)) \leq \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} < \frac{1}{3} \leq E_{d,g}(x, y).$$

In view of the exposed facts we conclude that g $+$-dominates T_P.

In spite of the preceding handicap, for indisintinguishability operators we can get the following characterization.

Theorem 3.3. *Let T be a t-norm. If $g : [0, \infty] \to [0, 1]$ is a function, then the following assertions are equivalent:*

(1) *$E_{d,g}$ is an indistinguishability operator with respect to the t-norm T for every non-empty set X and for all pseudo-metric d on X, where $E_{d,g} : X \times X \to [0, 1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x, y) = g(d(x, y))$.*

(2) *The function g satisfies the conditions below:*

(2.1) $g(0) = 1$,

(2.2) *g transforms triangular triplets into T-triangular triplets.*

Proof. With the aim of proving (1) $\Rightarrow$ (2), assume that $E_{d,g}$ is an indistinguishability operator with respect to the t-norm T for every non-empty set X and for all pseudo-metrics d on X, where $E_{d,g} : X \times X \to [0,1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x,y) = g(d(x,y))$. We will see that g satisfies conditions 2.1) and 2.2)

First of all we show that $g(0) = 1$. To this end, consider the (pseudo-)metric $d_{|\cdot|}$ defined on $[0,\infty]$ by $d_{|\cdot|}(x,y) = |y-x|$ for all $x, y \in [0,\infty]$. By our assumption we conclude that $E_{d_{|\cdot|},g}$ is an indistinguishability operator on $[0,\infty]$ with respect to the t-norm T. So $E_{d_{|\cdot|},g}(x,x) = 1$ for all $x \in [0,\infty]$. Whence we deduce that

$$g(0) = g(d_{|\cdot|}(x,x)) = E_{d_{|\cdot|},g}(x,x) = 1.$$

Now suppose that $(a,b,c) \in [0,\infty]^3$ is a triangular triplet. Similar to what was done in the proof of Theorem 2.9 we consider the set $X = \{x,y,z\}$ and define on X the pseudo-metric d_X as follows:

$$d_X(x,x) = d_X(y,y) = d_X(z,z) = 0,\ d_x(x,y) = d_X(y,x) = a,$$

$$d_X(x,z) = d_X(z,x) = b \text{ and } d_X(y,z) = d_X(z,y) = c.$$

Again, by our assumption, we have that $E_{d_X,g}$ is an indistinguishability operator with respect to T on X. Hence

$$T(E_{d_X,g}(u,v), E_{d_X,g}(v,w)) \leq E_{d_X,g}(u,w),$$

for all $u, v, w \in X$. Since $E_{d_X,g}(y,x) = g(d_X(y,x)) = g(a)$, $E_{d_X,g}(x,z) = g(d_X(x,z)) = g(b)$ and $E_{d_X,g}(y,z) = g(d_X(y,z)) = g(c)$ we conclude that

$$T(g(r), g(s)) \leq g(t),$$

for all $r, s, t \in \{a,b,c\}$. So $(g(a), g(b), g(c))$ is a T-triangular triplet.

In order to prove (2) $\Rightarrow$ (1), suppose that g satisfies 2.1) and 2.2). Let X be a non-empty set and let d be a pseudo-metric on X. Consider the function $E_{d,g} : X \times X \to [0,1]$ defined by $E_{d,g}(x,y) = g(d(x,y))$ for all $x, y \in X$. Next we show that $E_{d,g}$ is an indistinguishability operator. Certainly, $E_{d,g}(x,x) = g(d(x,x)) = g(0) = 1$ for all $x \in X$. Consider $x, y, z \in X$. It is obvious that $(d(x,y), d(y,z), d(x,z)) \in [0,\infty]^3$ is a triangular triplet. Since g transforms triangular triplets into T-triangular triplets we obtain that $(g(d(x,y)), g(d(y,z)), g(d(x,z))) \in [0,1]^3$ is a T-triangular triplet. Therefore we get for all $u, v, w \in \{x,y,z\}$ that

$$T(E_{d,g}(u,v), E_{d,g}(v,w)) = T(g(d(u,v)), g(d(v,w))) \leq g(d(u,w)) = E_{d,g}(u,w).$$

Finally, $E_{d,g}(x,y) = g(d(x,y)) = g(d(y,x)) = E_{d,g}(y,x)$ for all $x, y \in X$. □

As a consequence of Theorems 2.9 and 3.3 we get the following result.

Corollary 3.4. *Let T be a t-norm. If $g : [0,\infty] \to [0,1]$ is a function, then among the assertions below* (1) $\Rightarrow$ (2)*:*

(1) *$E_{d,g}$ is a fuzzy preorder with respect to the t-norm T for every non-empty set X and for all quasi-pseudo-metric d on X, where $E_{d,g} : X \times X \to [0,1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x,y) = g(d(x,y))$.*

(2) *$E_{d,g}$ is an indistinguishability operator with respect to the t-norm T for every non-empty set X and for all pseudo-metric d on X, where $E_{d,g} : X \times X \to [0,1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x,y) = g(d(x,y))$.*

Proof. By Theorems 2.9 g transforms asymmetric triangular triplets into asymmetric T-triangular triplets. It follows that g transforms triangular triplets into T-triangular triplets. Theorem 3.3 warranties the conclusion.

□

It must be pointed out that Example 3.2 shows that statement (2) $\Rightarrow$ (1) in Corollary 3.4 is not true in general.

In the particular case of indistinguishability operators that separates points, the following characterization can be stated.

Theorem 3.5. *Let T be a t-norm. If $g : [0,\infty] \to [0,1]$ is a function, then the following assertions are equivalent:*

(1) *$E_{d,g}$ is an indistinguishability operator with respect to the t-norm T that separates points for every non-empty set X and for all metric d on X, where $E_{d,g} : X \times X \to [0,1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x,y) = g(d(x,y))$.*

(2) *The function g satisfies the conditions below:*

(2.1) *$g(a) = 1 \Leftrightarrow a = 0$,*

(2.2) *g transforms triangular triplets into T-triangular triplets.*

Proof. In an effort to prove (1) $\Rightarrow$ (2), assume that $E_{d,g}$ is an indistinguishability operator with respect to the t-norm T that separates points for every non-empty set X and for all metrics d on X, where $E_{d,g} : X \times X \to [0,1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x,y) = g(d(x,y))$. We will see that g satisfies conditions (2.1) and (2.2)

The same arguments to those given in the proof of the implication (1) $\Rightarrow$ (2) in Theorem 3.3 apply to show that $g(0) = 1$ and that g transforms triangular triplets into T-triangular triplets. Observe that we can assume that the triplet $(a,b,c) \in]0,\infty]^3$ and, thus, the same aforementioned arguments are valid because otherwise it is a simple matter to get that $(g(a), g(b), g(c))$ is a T-triangular triplet.

It remains to show that $g(a) = 1 \Rightarrow a = 0$. Assume that there exists $a \in [0,\infty]$ such that $g(a) = 1$. Consider the metric $d_{|\cdot|}$ on $[0,\infty]$ introduced in the proof of Theorem 3.3. Then $d_{|\cdot|}(0,a) = a$ and $g(d_{|\cdot|}(0,a)) = g(a) = 1$. Thus, by our assumption, we get that $E_{d_{|\cdot|},g}$ is an indistinguishability operator with respect to the t-norm T that separates points and $E_{d_{|\cdot|},g}(0,a) = 1$. So we obtain that $a = 0$.

Aiming to prove (2) $\Rightarrow$ (1), let d be a metric on a non-empty set X. By Theorem 3.3 we have that the function $E_{d,g}$ is an indistinguishability operator with respect to T on X. Next assume that there exists $x, y \in X$ such that $E_{d,g}(x,y) = 1$. Then $g(d(x,y)) = 1$. So $d(x,y) = 0$. Since d is a metric we conclude that $x = y$.

□

Theorems 2.10 and 3.5 allow to immediately obtain the following result.

Corollary 3.6. *Let T be a t-norm. If $g : [0,\infty] \to [0,1]$ is a function, then among the assertions below* (1) $\Rightarrow$ (2)*:*

(1) *$E_{d,g}$ is a fuzzy partial order with respect to the t-norm T for every non-empty set X and for all quasi-metric d on X, where $E_{d,g} : X \times X \to [0,1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x,y) = g(d(x,y))$.*

(2) *$E_{d,g}$ is an indistinguishability operator with respect to the t-norm T that separates points for every non-empty set X and for all metric d on X, where $E_{d,g} : X \times X \to [0,1]$ is the fuzzy set defined, for all $x, y \in X$, by $E_{d,g}(x,y) = g(d(x,y))$.*

The following example shows that statement (2) $\Rightarrow$ (1) in Corollary 3.6 is not true in general.

Example 3.7. *Define the function $g : [0,\infty] \to [0,1]$ by $g(0) = 1$, $g(a) = 1 - e^{-a}$ for all $a \in]0,\infty[$ and $g(\infty) = 0$. Clearly $E_{d,g}$ is an indistinguishability operator with respect to the drastic t-norm T_D that separates points for all metric d on a non-empty set X. A straightforward computation shows that g +-dominates T_D. Nevertheless, g is not decreasing. So Proposition 2.8 and Theorem 2.10 warranty that $E_{d,g}$ is not in general a fuzzy partial order with respect to the t-norm T_D for all quasi-metric d on X.*

4 Conclusions and future work

In the literature a few works have focused their efforts on exploring the duality relationship existing between quasi-pseudo-metrics and fuzzy preorders. A well-known method provides a way of inducing a fuzzy preorder (indistinguishability operator) $E_{d,g}$ through a quasi-pseudo-metric (pseudo-metric) d by using t-norms and the pseudo-inverse of their additive generators as follows:

$$E_{d,f_T^{(-1)}}(x,y) = f_T^{(-1)}(d(x,y)).$$

In this paper we have explored whether in the preceding expression the function $f_T^{(-1)}$ can be replaced by another function $g : [0,\infty] \to [0,1]$ and, at the same time, which properties must be satisfied by g in order to guarantee that $E_{d,g}$ is a fuzzy preorder (indistinguishability operator), where

$$E_{d,g}(x,y) = g(d(x,y)).$$

Hence we have characterized those functions that induce a fuzzy preorder from a quasi-pseudo-metric even when the considered t-norm is not continuous. On the one hand, we have proved that they must be decreasing and fulfill a special property of dominance with respect to the ordinary addition and the t-norm. On the other hand, we have shown that they must transform asymmetric triangular triplets into asymmetric T-triangular triplets. A special attention has been paid to fuzzy partial orders. The case where fuzzy preorders are exactly indistinguishability operators has been studied too. Concretely, we have shown that the monotonicity of the function and the previously mentioned dominance are sufficient but not necessary conditions. Moreover, we have shown that such functions must transform triangular triplets into T-triangular triplets. The developed theory has been illustrated by means of appropriate examples. Furthermore, we have proved that the technique based on the use of additive generators and their pseudo-inverses can be recovered as a particular case of the new one presented here when the t-norms involved are continuous and Archimedean.

As future work, we propose to extend the study carried out in this paper, and in the references therein, to fuzzy relations in which the transitivity axiom is modeled using conjunctors rather than t-norms, as considered in [14]. Moreover, in the same reference the minimum properties to be required on these kind of functions with the aim of preserving the correspondence between fuzzy relations and fuzzy partitions were explored. Observe that associativity of t-norms does not generally seem to play a relevant role in general in the characterizations obtained. So, inspired by the aforesaid work we could explore the developed theory involving fuzzy relations with respect to conjunctors instead t-norms clarifying the dual relationship with generalized metrics. Furthermore, we plan to explore the transformation problem when the triangle inequality of the generalized distance is modeled by means of disjunctions like t-conorms and ordinal sums of t-conorms in the spirit of [19].

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