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Constructions of absolutely continuous copulas with given curvilinear or opposite-curvilinear sections

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Abstract

In this paper, we address the constructions of absolutely continuous copulas characterized by given curvilinear or opposite-curvilinear sections. For a given function generated by a suitable curve within the unit square, we develop a family of absolutely continuous copulas having this function as a curvilinear section, achieved by averaging over a series of copulas sharing the common curvilinear section. Subsequently, we establish theoretical relationships between absolutely continuous copulas with given opposite-curvilinear sections and those with corresponding curvilinear sections. Using these relationships, we propose two approaches for constructing of absolutely continuous copulas with given opposite-curvilinear sections.

Keywords: Absolutely continuous copula, curvilinear section, opposite-curvilinear section, curvilinear copula.

1 Introduction

A copula is a multi-dimensional distribution function of a random vector with all uniform [0, 1] marginal distributions [14]. By Sklar's Theorem [19], it is known that for a joint distribution function H with univariate marginal distributions H_i, $i = 1, \ldots, n$, there exists a copula C so that

$$H(x_1, \ldots, x_n) = C\big(H_1(x_1), \ldots, H_n(x_n)\big),\ x_i \in \mathbb{R},\ i = 1, \ldots, n.$$

The copula C is uniquely determined when the marginal distributions H_i, $i = 1, \ldots, n$ are continuous. In this sense, copulas allow us to capture the dependence structure among random variables apart from its marginal behaviors. As a useful and important tool for modeling dependence structure, copulas have been widely applied in various domains, including risk management [23], multi-attribute utility theory [1] and actuarial science [21]. Additionally, a copula is a 1-Lipschitz aggregation operator with neutral element 1, thus establishing its applicability in the fuzzy set theory [9].

In the theoretical study of (quasi-)copulas, substantial research has focused on the constructions of copulas with predescribed values, including specific points within the unit square [8], horizontal or vertical sections [12], and diagonal or opposite-diagonal sections [15, 18]. It is noteworthy that the tail dependence coefficients of a copula are fully determined by its diagonal section. Therefore, the constructions of bivariate copulas with given diagonal or opposite-diagonal sections have been extensively investigated [2, 3, 5, 16, 20].

Recently, considerable attention has been devoted to the more general case, that is, bivariate (quasi-)copulas with given curvilinear sections (also referred to as track sections) or opposite-curvilinear sections along a curve within the unit square. For example, Fernández-Sánchez and Úbeda-Flores [7] established the conditions necessary for the existence

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of copulas with given curvilinear or opposite-curvilinear sections. Zou et al. [24] presented the best-possible bounds for families of (quasi-)copulas defined by a specific curvilinear section, while Ouyang et al. [17] further determined the conditions under which the best-possible upper bound for the family of quasi-copulas sharing a common curvilinear section is itself a copula. Additionally, Jwaid et al. [11] and Xie et al. [22] introduced the curvilinear patchwork operation, employing it to construct more (quasi-) copulas that share a common curvilinear section from the existing ones. Utilizing this curvilinear patchwork operation, Lou et al. [13] explored the construction of (quasi-)copulas with a given opposite-curvilinear section.

The aforementioned constructions generate copulas with given curvilinear or opposite-curvilinear sections that generally have a singular component [7, 11, 13, 22]. This limitation restricts the application of these constructed copulas in statistical modeling, as certain goodness-of-fit tests and estimation procedures rely on the assumption of absolute continuity for copulas [10]. However, to the best of our knowledge, there is currently no established method that constructs absolutely continuous copulas whose curvilinear or opposite-curvilinear sections coincide with a suitable function generated by a curve within the unit square.

This paper aims to address this gap by proposing methods for constructing absolutely continuous copulas with a given curvilinear or opposite-curvilinear section. For a suitable function $\Gamma_\phi : [0,1] \to [0,1]$ generated by a strictly increasing curve $\phi : [0,1] \to [0,1]$, we construct a family of absolutely continuous copulas whose curvilinear section coincides with Γ_ϕ by averaging over a series of curvilinear copulas introduced by Xie et al. [22]. Subsequently, for a given opposite-curvilinear section $\Gamma_\psi : [0,1] \to [0,1]$ generated by a strictly decreasing curve $\psi : [0,1] \to [0,1]$, we establish two significant relationships between the curvilinear section Γ_ϕ and the opposite-curvilinear section Γ_ψ. Utilizing these relationships, we construct two families of absolutely continuous copulas with the given opposite-curvilinear section Γ_ψ.

The remainder of this paper is structured as follows. Section 2 introduces essential notations concerning copulas, curvilinear sections and opposite-curvilinear sections, which will be useful later. In Section 3, we construct a family of absolutely continuous copulas whose curvilinear section coincides with a suitable function. Section 4 establishes the relationships between the opposite-curvilinear and curvilinear sections, and constructs two families of absolutely continuous copulas with a given opposite-curvilinear section. Finally, conclusions are drawn in Section 5.

2 Preliminaries

In this section, we present basic definitions and properties related to bivariate copulas, curvilinear sections and opposite-curvilinear sections that will be useful in constructing absolutely continuous copulas with given curvilinear or opposite-curvilinear sections.

2.1 Copula

In this subsection, the definition of bivariate copula and some of its theoretical properties are presented.

Definition 2.1. *A two-dimensional function C: $[0,1]^2 \to [0,1]$ is called a bivariate copula (for simplicity, a copula) if it satisfies:*

(C1) *(boundary conditions) $C(0,t) = C(t,0) = 0$ and $C(1,t) = C(t,1) = t$ for all $t \in [0,1]$;*

(C2) *(2-increasing property) for all $u_1, u_2, v_1, v_2 \in [0,1]$ with $u_1 \le u_2$ and $v_1 \le v_2$,*

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \ge 0.$$

Classical examples of bivariate copulas include the product copula $\Pi(u,v) = uv, u, v \in [0,1]$, the Fréchet upper bound $M(u,v) = \min\{u,v\}, u, v \in [0,1]$ and the Fréchet lower bound $W(u,v) = \max\{u+v-1, 0\}, u, v \in [0,1]$, which respectively represent the independence, comonotonicity and counter-monotonicity between two random variables. For any copula C, the following inequality holds:

$$W(u,v) \le C(u,v) \le M(u,v), \quad (u,v) \in [0,1]^2.$$

For more properties and examples of copulas, refer to [6] and [14].

From Definition 2.1, we know that a copula is essentially a joint distribution function of a random vector with all uniform $[0,1]$ marginal distributions. A copula C admits first partial derivatives $\frac{\partial C(u,v)}{\partial u}$ and $\frac{\partial C(u,v)}{\partial v}$ almost everywhere (a.e., for short) on $[0,1]^2$, along with the mixed second derivative $\frac{\partial^2 C(u,v)}{\partial u \partial v}$ a.e. on $[0,1]^2$ [6]. The copula C is said to be *absolutely continuous* if it has a joint density given by $\frac{\partial^2 C(u,v)}{\partial u \partial v}$, whereas if $\frac{\partial^2 C(u,v)}{\partial u \partial v} = 0$ a.e. on $[0,1]^2$, then C is *singular*. Notably, the Fréchet upper bound M and the Fréchet lower bound W are singular bivariate copulas, whereas the product copula Π is absolutely continuous.

2.2 Curvilinear sections and opposite-curvilinear sections

In this subsection, we introduce the curves under consideration in this paper, followed by a discussion of curvilinear and opposite-curvilinear sections along these curves. Additionally, we provide examples of known copulas with given curvilinear sections.

Let ϕ: $[0,1] \to [0,1]$ be a strictly increasing and differentiable function from $[0,1]$ into $[0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$. Similarly, let ψ be a strictly decreasing and differentiable function from $[0,1]$ into $[0,1]$ with $\psi(0) = 1$ and $\psi(1) = 0$. It is worth mentioning that if $\phi(u) = u$ for all $u \in [0,1]$, then the curve $\{(t, \phi(t))|t \in [0,1]\} = \{(t,t)|t \in [0,1]\}$ is the diagonal of the unit square. Similarly, if $\psi(u) = 1-u$ for all $u \in [0,1]$, then the curve $\{(t, \psi(t))|t \in [0,1]\} = \{(t, 1-t)|t \in [0,1]\}$ is the opposite-diagonal of the unit square. For simplicity, we refer to $\{(t, \phi(t))|t \in [0,1]\}$ as curve ϕ, and $\{(t, \psi(t))|t \in [0,1]\}$ as curve ψ throughout the remainder of this paper. Denote

$$\Phi := \{\phi : [0,1] \to [0,1] \mid \phi \text{ is a strictly increasing and differentiable function with } \phi(0) = 0 \text{ and } \phi(1) = 1, \text{ and } \phi'(t) \text{ is bounded on } [0,1]\},$$

and

$$\Psi := \{\psi : [0,1] \to [0,1] \mid \psi \text{ is a strictly decreasing and differentiable function with } \psi(0) = 1 \text{ and } \psi(1) = 0, \text{ and } \psi'(t) \text{ is bounded on } [0,1]\}.$$

Definition 2.2. *For any curve $\phi \in \Phi$, the curvilinear section of a copula C along this given curve is $\Gamma_{C,\phi}(t) = C(t, \phi(t))$, $t \in [0,1]$. And for any curve $\psi \in \Psi$, the opposite-curvilinear section of the copula C along this given curve is the function $\Gamma_{C,\psi} : [0,1] \to [0,1]$ defined by $\Gamma_{C,\psi}(t) = C(t, \psi(t))$, $t \in [0,1]$.*

If $\phi(u) = u$ for all $u \in [0,1]$, then $\Gamma_{C,\phi}(t) = \delta_C(t)$ is the *diagonal section* of the copula C, and if $\psi(u) = 1-u$ for all $u \in [0,1]$, then $\Gamma_{C,\psi}(t)$ is the *opposite-diagonal section* of the copula C. On the other hand, given $\phi \in \Phi$, following the notation in [4] and [24], the definition of ϕ-function is presented.

Definition 2.3. *A ϕ-function is a function $g : [0,1] \to [0,1]$ which satisfies the following conditions:*

$$\max\{0, t + \phi(t) - 1\} \le g(t) \le \min\{t, \phi(t)\} \quad \text{for any} \quad t \in [0,1], \tag{1}$$

and

$$0 \le g(t_2) - g(t_1) \le t_2 - t_1 + \phi(t_2) - \phi(t_1) \quad \text{for any} \quad 0 \le t_1 \le t_2 \le 1. \tag{2}$$

For any $\phi \in \Phi$, since $\phi'(t)$ is bounded on $[0,1]$, then from (2), we know that each ϕ-function g satisfies the Lipschitz condition.

Similarly, given $\psi \in \Psi$, following the notation in [13], a ψ-function is defined.

Definition 2.4. *A ψ-function is a function $h : [0,1] \to [0,1]$ which satisfies the following conditions:*

$$\max\{0, t + \psi(t) - 1\} \le h(t) \le \min\{t, \psi(t)\} \quad \text{for any} \quad t \in [0,1], \tag{3}$$

and

$$\psi(t_2) - \psi(t_1) \le h(t_2) - h(t_1) \le t_2 - t_1 \quad \text{for any} \quad 0 \le t_1 \le t_2 \le 1. \tag{4}$$

Remark 2.5. *Any curvilinear section $\Gamma_{C,\phi}$ of a bivariate copula C is a ϕ-function. Conversely, for every ϕ-function g, there exists at least one copula C whose curvilinear section $\Gamma_{C,\phi}$ along the curve ϕ coincides with g [4]. For instance, given any curve $\phi \in \Phi$, the functions defined by $\Pi(t, \phi(t)) = t\phi(t)$, $M(t, \phi(t)) = \min\{t, \phi(t)\}$ and $W(t, \phi(t)) = \max\{t + \phi(t) - 1, 0\}$ are all ϕ-functions. Analogously, as a direct consequence of the result established in [13], any opposite-curvilinear section $\Gamma_{C,\psi}$ of a bivariate copula C is a ψ-function, and for any ψ-function h, there exists at least one copula C whose opposite-curvilinear section $\Gamma_{C,\psi}$ along the curve ψ is h.*

Throughout this paper, it is assumed that the curve ϕ belongs to the set Φ, ψ belongs to Ψ, and Γ_ϕ, Γ_ψ denote a ϕ-function and a ψ-function, respectively.

As noted, considerable research has been devoted to the construction of bivariate copulas with given curvilinear or opposite-curvilinear sections. Specifically, the families of copulas characterized by a given ϕ-function $\Gamma_\phi : [0,1] \to [0,1]$ or a given ψ-function $\Gamma_\psi : [0,1] \to [0,1]$ have been extensively studied (see, for example, [4, 7, 11, 13, 22, 24]).

It has been established that the set of copulas with a given curvilinear section Γ_ϕ is non-empty, as is the set of copulas with a given opposite-curvilinear section Γ_ψ [4, 7, 13]. A notable example is the generalization of Γ_ϕ-*curvilinear copula* K_{Γ_ϕ}, defined as

$$K_{\Gamma_\phi}(u,v) = \min\Big\{u, v, \theta\Gamma_\phi(u) + (1-\theta)\Gamma_\phi\big(\phi^{-1}(v)\big)\Big\}, \tag{5}$$

where $0 \le \theta \le 1$ is a constant. Under the conditions that both $2t - \Gamma_\phi(t)$ and $2t - \Gamma_\phi\big(\phi^{-1}(t)\big)$ are non-decreasing on $[0,1]$, Xie et al. [22] verified that K_{Γ_ϕ} is indeed a copula with the given curvilinear section Γ_ϕ. It is noteworthy that when $\phi(u) = u$ for all $u \in [0,1]$, the copula K_{Γ_ϕ} simplifies to the diagonal copula

$$K_\delta(u,v) = \min\Big\{u, v, \frac{\delta(u)+\delta(v)}{2}\Big\},$$

which has been discussed by Fredricks and Nelsen [8], Nelsen et al. [15] and Sancetta and Satchell [18].

Due to the presence of singular components, the copula K_{Γ_ϕ} is not absolutely continuous. Furthermore, existing constructions generate copulas with given curvilinear or opposite-curvilinear sections that generally contain singular components. Therefore, the construction of absolutely continuous copulas with given curvilinear or opposite-curvilinear sections is both significant and challenging, which will be addressed in the subsequent sections.

3 The construction of absolutely continuous copulas with given curvilinear sections

In this section, we construct a family of absolutely continuous copulas with a given curvilinear section by means of averaging over a series of copulas of type (5).

To facilitate this construction, we first introduce some notations. For each ϕ-function Γ_ϕ, we define

$$t^l = \sup\{t \in [0,1] | \Gamma_\phi(t) = 0\} \quad \text{and} \quad t^r = \inf\{t \in [0,1] | \Gamma_\phi(t) = t + \phi(t) - 1\}.$$

Remark 3.1. *Note that $t^l \le t^r$. In fact, if $t^r < t^l$, then for all $t \in [t^r, t^l]$, $\Gamma_\phi(t) = t + \phi(t) - 1 = 0$, which implies that $t + \phi(t)$ is constant in $[t^r, t^l] \subset [0,1]$. This leads to a contradiction with the assumption that ϕ is strictly increasing on $[0,1]$. Furthermore, if $t^l = t^r$, then either t^l or t^r is the unique root of $t + \phi(t) = 1$.*

Define the set

$$\mathcal{S}_\phi = \Big\{\Gamma_\phi |\ \Gamma_\phi \text{ is a } \phi\text{-function with } \Gamma_\phi(t) = \frac{\Gamma_\phi^+(t) + \Gamma_\phi^-(t)}{2} < \min\{t, \phi(t)\} \text{ on } (0,1)\Big\}, \tag{6}$$

of ϕ-functions, where Γ_ϕ^+ and Γ_ϕ^- are two ϕ-functions satisfying $\Gamma_\phi^-(t) < \Gamma_\phi(t) < \Gamma_\phi^+(t)$ for all $t \in (t^l, t^r)$.

Remark 3.2. *From the definitions of t^l and t^r, it follows that $\Gamma_\phi(t) = \Gamma_\phi^+(t) = \Gamma_\phi^-(t)$ for all $t \in [0, t^l] \cup [t^r, 1]$.*

For any curve $\phi \in \Phi$, the set $\mathcal{S}_\phi$ is non-empty. In fact, consider $\Gamma_\phi^-(t) = \Pi\big(t, \phi(t)\big) = t\phi(t)$ and $\Gamma_\phi^+(t) = M\big(t, \phi(t)\big) = \min\big\{t, \phi(t)\big\}$, both of which are ϕ-functions. Since $t\phi(t) < \min\{t, \phi(t)\}$ for all $t \in (0,1)$, it follows that $\Gamma_\phi(t) = \frac{t\phi(t) + \min\{t,\phi(t)\}}{2} < \min\{t, \phi(t)\}$ on $(0,1)$. Moreover, in this case, $t^l = 0$ and $t^r = 1$, ensuring that $\Gamma_\phi^-(t) < \Gamma_\phi(t) < \Gamma_\phi^+(t)$ for all $t \in (0,1)$. Therefore, $\Gamma_\phi(t) = \frac{t\phi(t) + \min\{t,\phi(t)\}}{2}$ is an element of $\mathcal{S}_\phi$, confirming that $\mathcal{S}_\phi$ is indeed non-empty. Furthermore, the elements of $\mathcal{S}_\phi$ are rich. For example, for any two copulas C_1 and C_2 with $C_1(u,v) < C_2(u,v)$ for all $(u,v) \in (0,1)^2$, the ϕ-function $\frac{C_1\big(t,\phi(t)\big) + C_2\big(t,\phi(t)\big)}{2}$ is also an element of $\mathcal{S}_\phi$.

For a fixed $\lambda \in [0,1]$, define $\xi_\lambda(t) = \lambda\Gamma_\phi^+(t) + (1-\lambda)\Gamma_\phi^-(t)$, $t \in [0,1]$. Since both Γ_ϕ^+ and Γ_ϕ^- are ϕ-functions, their convex combination ξ_λ is also a ϕ-function.

Now, for a fixed $\lambda \in [0,1]$, we define

$$\begin{aligned} C_\lambda(u,v) &= \min\Big\{u, v, \lambda\Big(\rho\Gamma_\phi^+(u) + (1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)\Big) + (1-\lambda)\Big(\rho\Gamma_\phi^-(u) + (1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)\Big\} \\ &= \min\Big\{u, v, \rho\xi_\lambda(u) + (1-\rho)\xi_\lambda\big(\phi^{-1}(v)\big)\Big\},\ (u,v) \in [0,1]^2. \end{aligned} \tag{7}$$

Remark 3.3. *It is important to note that for each constant ρ satisfying that both $t-\rho\xi_\lambda(t)$ and $t-(1-\rho)\xi_\lambda\big(\phi^{-1}(t)\big)$ are non-decreasing on $[0,1]$, the function C_λ is a copula of type (5) generated by ξ_λ. From the proof of Theorem 36 in Xie et al. [22], this conclusion is based on verifying that the function $\rho\xi_\lambda(u)+(1-\rho)\xi_\lambda\big(\phi^{-1}(v)\big)$ is 1-Lipschitz in each argument. Indeed, given the assumptions that both $t-\rho\xi_\lambda(t)$ and $t-(1-\rho)\xi_\lambda\big(\phi^{-1}(t)\big)$ are non-decreasing on $[0,1]$, we know that for any $0\le u_1,u_2\le 1$ and $0\le v_1,v_2\le 1$,*

$$\rho|\xi_\lambda(u_2)-\xi_\lambda(u_1)|\le|u_2-u_1| \quad\text{and}\quad (1-\rho)\big|\xi_\lambda\big(\phi^{-1}(v_2)\big)-\xi_\lambda\big(\phi^{-1}(v_1)\big)\big|\le|v_2-v_1|,$$

so that

$$\begin{aligned}&\Big|\rho\xi_\lambda(u_2)+(1-\rho)\xi_\lambda\big(\phi^{-1}(v_2)\big)-\Big(\rho\xi_\lambda(u_1)+(1-\rho)\xi_\lambda\big(\phi^{-1}(v_1)\big)\Big)\Big|\\ \le{}&\rho|\xi_\lambda(u_2)-\xi_\lambda(u_1)|+(1-\rho)\big|\xi_\lambda\big(\phi^{-1}(v_2)\big)-\xi_\lambda\big(\phi^{-1}(v_1)\big)\big|\le|u_2-u_1|+|v_2-v_1|.\end{aligned}$$

Thus, $\rho\xi_\lambda(u)+(1-\rho)\xi_\lambda\big(\phi^{-1}(v)\big)$ is 1-Lipschitz in each argument. Consequently, under the assumption that both $t-\rho\xi_\lambda(t)$ and $t-(1-\rho)\xi_\lambda\big(\phi^{-1}(t)\big)$ are non-decreasing on $[0,1]$, for each $\lambda\in[0,1]$, the function C_λ of type (7) is indeed a copula.

Next, we consider a concise expression of C_λ. Observe that the equality $u=\rho\xi_\lambda(u)+(1-\rho)\xi_\lambda\big(\phi^{-1}(v)\big)$ holds along the curve $v=\phi\Big(\xi_\lambda^{-1}\big(\frac{u-\rho\xi_\lambda(u)}{1-\rho}\big)\Big)$, $u\in[0,1]$. Then, for all $v\ge\phi\Big(\xi_\lambda^{-1}\big(\frac{u-\rho\xi_\lambda(u)}{1-\rho}\big)\Big)$, $\rho\xi_\lambda(u)+(1-\rho)\xi_\lambda\big(\phi^{-1}(v)\big)\ge u$, and consequently $C_\lambda(u,v)=\min\{u,v\}$. From (1), it also follows that $\rho\xi_\lambda(u)+(1-\rho)\xi_\lambda\big(\phi^{-1}(u)\big)\le u$ for all $u\in[0,1]$ so that $\xi_\lambda^{-1}\big(\frac{u-\rho\xi_\lambda(u)}{1-\rho}\big)\ge\phi^{-1}(u)$. Then, for all $v\ge\phi\Big(\xi_\lambda^{-1}\big(\frac{u-\rho\xi_\lambda(u)}{1-\rho}\big)\Big)$, $v\ge u$ and consequently, $C_\lambda(u,v)=u$. By similar arguments, we know that $v=\rho\xi_\lambda(u)+(1-\rho)\xi_\lambda\big(\phi^{-1}(v)\big)$ on the curve $u=\xi_\lambda^{-1}\Big(\frac{v-(1-\rho)\xi_\lambda\big(\phi^{-1}(v)\big)}{\rho}\Big)$, $v\in[0,1]$, and for all $u\ge\xi_\lambda^{-1}\big(\frac{v-(1-\rho)\xi_\lambda(\phi^{-1}(v))}{\rho}\big)$, it holds that $C_\lambda(u,v)=v$. Summarizing the above results, it yields that

$$C_\lambda(u,v)=\begin{cases}u, & v\ge\phi\big(\xi_\lambda^{-1}\big(\frac{u-\rho\xi_\lambda(u)}{1-\rho}\big)\big),\\ \rho\xi_\lambda(u)+(1-\rho)\xi_\lambda\big(\phi^{-1}(v)\big), & v\le\phi\big(\xi_\lambda^{-1}\big(\frac{u-\rho\xi_\lambda(u)}{1-\rho}\big)\big)\ \text{and}\ u\le\xi_\lambda^{-1}\big(\frac{v-(1-\rho)\xi_\lambda(\phi^{-1}(v))}{\rho}\big),\\ v, & u\ge\xi_\lambda^{-1}\big(\frac{v-(1-\rho)\xi_\lambda(\phi^{-1}(v))}{\rho}\big).\end{cases}\tag{8}$$

In subsequent discussions, the copula C_λ will be used to construct absolutely continuous copulas with a given curvilinear section Γ_ϕ.

To accomplish this task, the following two auxiliary functions Λ_U and Λ_L are needed, which play an important role in the construction of absolutely continuous copulas with the given curvilinear section $\Gamma_\phi\in\mathcal{S}_\phi$. Let

$$H_U:=\{(u,v)\in[0,1]^2|v\ge\phi(u)\}\quad\text{and}\quad H_L:=\{(u,v)\in[0,1]^2|v\le\phi(u)\},$$

be two subareas of unit square. The function $\Lambda_U:H_U\to[-\infty,+\infty]$ is defined by

$$\Lambda_U(u,v)=\begin{cases}\dfrac{u-\rho\Gamma_\phi^-(u)-(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)}{\rho\Big(\Gamma_\phi^+(u)-\Gamma_\phi^-(u)\Big)+(1-\rho)\Big(\Gamma_\phi^+\big(\phi^{-1}(v)\big)-\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)}, & \text{if}\ \ \rho\Gamma_\phi^+(u)+(1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)\\ & \qquad >\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big),\\ +\infty, & \text{if}\ \ (u,v)\in\big([0,t^l]\times[0,\phi(t^l)]\big)\\ & \qquad\cup\big([t^r,1]\times[\phi(t^r),1]\big),\\ -\infty, & \text{if}\ \ (u,v)\in[0,t^l]\times[\phi(t^r),1]\end{cases}$$

and the function $\Lambda_L:H_L\to[-\infty,+\infty]$ is defined by

$$\Lambda_L(u,v)=\begin{cases}\dfrac{v-\rho\Gamma_\phi^-(u)-(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)}{\rho\Big(\Gamma_\phi^+(u)-\Gamma_\phi^-(u)\Big)+(1-\rho)\Big(\Gamma_\phi^+\big(\phi^{-1}(v)\big)-\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)}, & \text{if}\ \ \rho\Gamma_\phi^+(u)+(1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)\\ & \qquad >\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big),\\ +\infty, & \text{if}\ \ (u,v)\in\big([0,t^l]\times[0,\phi(t^l)]\big)\\ & \qquad\cup\big([t^r,1]\times[\phi(t^r),1]\big),\\ -\infty, & \text{if}\ \ (u,v)\in[t^r,1]\times[0,\phi(t^l)].\end{cases}$$

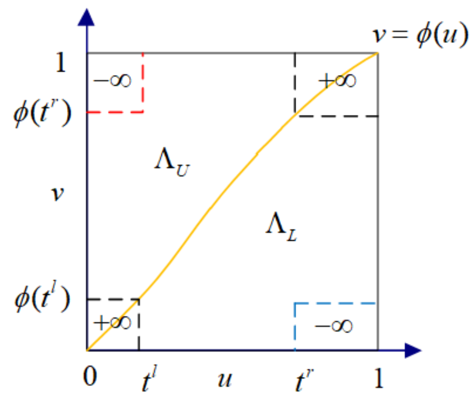


Figure 1: The supports and values of functions Λ_U and Λ_L.

Note that the values of auxiliary functions Λ_U and Λ_L on some areas are set as $+\infty$ or $-\infty$. This is done for guaranteeing that Λ_U and Λ_L are continuous except possibly at very few points. The supports and values of functions Λ_U and Λ_L are illustrated in Figure 1.

In the following, we will construct absolutely continuous copulas with a given curvilinear section $\Gamma_\phi \in \mathcal{S}_\phi$ by averaging over a series of bivariate functions of type (7). To carry out this construction, we need some conditions on the curve ϕ and Γ_ϕ, which are formalized in the following assumption.

Assumption I. There exists at least one constant $0<\rho<1$ satisfying the following conditions:

(a).

$$t^r+\phi(t^r)-1>\max\{t^l/(1-\rho), \phi(t^l)/\rho\}; \tag{9}$$

(b). for all $\lambda \in [0,1]$, both $t-\rho\xi_\lambda(t)$ and $t-(1-\rho)\xi_\lambda\big(\phi^{-1}(t)\big)$ are non-decreasing on $[0,1]$.

Here, Assumption I(b) plays an important role in ensuring that for every $\lambda \in [0,1]$, the function C_λ given by (7) is a copula; hence a convex combination of a series of functions of the form (7) is itself a copula with the given curvilinear section Γ_ϕ, as established in Theorem 3.8. Assumption I(a) is indispensable for the continuity of Λ_U and Λ_L along the specific lines (illustrated by red and blue lines in Figure 1, respectively), which underpins the properties of Λ_U and Λ_L stated in Lemma 3.7. These properties of Λ_U and Λ_L are then instrumental in verifying the absolute continuity of the copulas constructed in Theorem 3.8.

Remark 3.4. *Under Assumption I(a), it follows that $t^r+\phi(t^r)-1>t^l/(1-\rho)$ and $t^r+\phi(t^r)-1>\phi(t^l)/\rho$. From the inequality $t^r+\phi(t^r)-1>t^l/(1-\rho)$, we know that the numerator of $\Lambda_U(u,v)$ is negative when $u\downarrow t^l$ and $v\in[\phi(t^r),1]$, as well as when $v\uparrow\phi(t^r)$ and $u\in[0,t^l]$. Consequently, $\Lambda_U(u,v)$ is continuous at points (t^l,v) for all $v\in[\phi(t^r),1]$ and at points $\big(u,\phi(t^r)\big)$ for all $u\in[0,t^l]$. Similarly, the inequality $t^r+\phi(t^r)-1>\phi(t^l)/\rho$ implies that the numerator of $\Lambda_L(u,v)$ is negative when $u\uparrow t^r$ and $v\in[0,\phi(t^l)]$, as well as when $v\downarrow\phi(t^l)$ and $u\in[t^r,1]$. Thus, $\Lambda_L(u,v)$ is continuous at points (t^r,v) for all $v\in[0,\phi(t^l)]$ and at points $\big(u,\phi(t^l)\big)$ for all $u\in[t^r,1]$. Recall that under Assumption I(b), the function C_λ of type (7) is a copula for every $\lambda\in[0,1]$.*

Remark 3.5. *Assumption I is both useful and not overly restrictive. Notice that for a strictly increasing curvilinear section Γ_ϕ, $t^l=0$, and then the inequality (9) holds for all $\rho\in(0,1)$. Suppose that the derivative of $\xi_\lambda(t)$ exists on $[0,1]$, and neither $\xi'_\lambda(t)$ nor $\phi'(t)$ vanish for $t\in(0,1)$. If, in addition, the inequality*

$$\sup_{\lambda,t\in[0,1]}\left(1-\frac{\phi'(t)}{\xi'_\lambda(t)}\right)\le\inf_{\lambda,t\in[0,1]}\frac{1}{\xi'_\lambda(t)}, \tag{10}$$

holds, then for any $\sup_{\lambda,t\in[0,1]}\left(1-\frac{\phi'(t)}{\xi'_\lambda(t)}\right)\le\rho\le\inf_{\lambda,t\in[0,1]}\frac{1}{\xi'_\lambda(t)}$, Assumption I(b) is satisfied. In fact, the inequality $\rho\le\inf_{\lambda,t\in[0,1]}\frac{1}{\xi'_\lambda(t)}$ implies that for all $\lambda\in[0,1]$, $1-\rho\xi'_\lambda(t)\ge0$ for all $t\in[0,1]$. Then, by the mean value theorem, we know *that the function $t-\rho\xi_\lambda(t)$ is non-decreasing on $[0,1]$. Likewise, the inequality $\sup\limits_{\lambda,t\in[0,1]}\left(1-\frac{\phi'(t)}{\xi'_\lambda(t)}\right)\le\rho$ implies that for all $\lambda\in[0,1]$, $1-(1-\rho)\xi'_\lambda(t)/\phi'(t)\ge 0$, $t\in[0,1]$. Recalling that $\big(\xi_\lambda\circ\phi^{-1}\big)'(t)=\frac{\mathrm{d}\xi_\lambda\left(\phi^{-1}(t)\right)}{\mathrm{d}t}=\xi'_\lambda\big(\phi^{-1}(t)\big)/\phi'\big(\phi^{-1}(t)\big)$, and since $\phi^{-1}(t)$ is a strictly increasing function with $\phi^{-1}(0)=0$ and $\phi^{-1}(1)=1$, it follows that under the assumption $\sup\limits_{\lambda,t\in[0,1]}\left(1-\frac{\phi'(t)}{\xi'_\lambda(t)}\right)\le\rho$, the function $t-(1-\rho)\xi_\lambda\big(\phi^{-1}(t)\big)$ is non-decreasing on $[0,1]$ for all $\lambda\in[0,1]$.*

The following example provides the curve ϕ and $\Gamma_\phi\in\mathcal{S}_\phi$ satisfying Assumption I.

Example 3.6. *Let the curve $\phi(t)=t^\alpha$, $\alpha\ge 1$, and define $\Gamma_\phi(t)=\big(\Gamma^+_\phi(t)+\Gamma^-_\phi(t)\big)/2=(t^\alpha+t^{1+\alpha})/2$, which is a ϕ-function in $\mathcal{S}_\phi$. Observing that Γ_ϕ is strictly increasing in $[0,1]$, it follows that $t^l=0$ and $t^r=1$ so that the inequality (9) holds for all $0<\rho<1$.*

For a fixed $\lambda\in[0,1]$, straightforward calculations yield that $\xi'_\lambda(t)=\lambda\alpha t^{\alpha-1}+(1-\lambda)(1+\alpha)t^\alpha$ and $\phi'(t)=\alpha t^{\alpha-1}$. Then, $\sup\limits_{\lambda,t\in[0,1]}\left(1-\frac{\phi'(t)}{\xi'_\lambda(t)}\right)=\frac{1}{1+\alpha}$ and $\inf\limits_{\lambda,t\in[0,1]}\frac{1}{\xi'_\lambda(t)}=\frac{1}{1+\alpha}$. Thus, when $\rho=1/(1+\alpha)$, for all $\lambda\in[0,1]$, both $t-\rho\xi_\lambda(t)$ and $t-(1-\rho)\xi_\lambda\big(\phi^{-1}(t)\big)$ are non-decreasing on $[0,1]$.

Under Assumption I, the properties of Λ_U and Λ_L are presented in the following lemma.

Lemma 3.7. *Under Assumption I, the auxiliary functions Λ_U and Λ_L satisfy the following properties.*

(a). *When $1+(1-\rho)t^r-\rho\phi(t^r)-2(1-\rho)>0$, the function $\Lambda_U(u,v)$ is continuous except possibly at the point $\big(0,\phi(t^l)\big)$, and when $1+(1-\rho)t^r-\rho\phi(t^r)-2(1-\rho)\le 0$, the function $\Lambda_U(u,v)$ is continuous except possibly at points $\big(0,\phi(t^l)\big)$ and (t^r,v), $v\in[v_0,1]$, where $v_0=\inf\big\{t\in[\phi(t^r),1]\,|\,1+(1-\rho)t^r-\rho\phi(t^r)-(1-\rho)\big(t+\phi^{-1}(t)\big)\le 0\big\}$; when $1-(1-\rho)t^r+\rho\phi(t^r)-2\rho>0$, the function $\Lambda_L(u,v)$ is continuous except possibly at the point $(t^l,0)$, and when $1-(1-\rho)t^r+\rho\phi(t^r)-2\rho\le 0$, the function $\Lambda_U(u,v)$ is continuous except possibly at points $(t^l,0)$ and $\big(u,\phi(t^r)\big)$, $u\in[u_0,1]$, where $u_0=\inf\big\{t\in[t^r,1]\,|\,1-(1-\rho)t^r+\rho\phi(t^r)-\rho\big(t+\phi(t)\big)\le 0\big\}$.*

(b). *For any fixed $u\in(0,t^r)$ and $\lambda\in[0,1]$, the set $\{v\in[\phi(u),1]\,|\,\Lambda_U(u,v)=\lambda\}$ is non-empty and connected, and for any fixed $v\in\big(0,\phi(t^r)\big)$ and $\lambda\in[0,1]$, the set $\{u\in[\phi^{-1}(v),1]\,|\,\Lambda_L(u,v)=\lambda\}$ is non-empty and connected.*

(c). *For any fixed $u\in(0,t^r)$, there exist v_-, $v_+\in\big(\phi(u),1\big)$ so that*

$$\Lambda_U(u,v)\begin{cases}\in[1,+\infty], & \text{if}\quad v\in[\phi(u),v_-],\\ \in(0,1), & \text{if}\quad v\in(v_-,v_+),\\ \in[-\infty,0], & \text{if}\quad v\in[v_+,1],\end{cases}$$

and for any fixed $v\in\big(0,\phi(t^r)\big)$, there exist u_-, $u_+\in\big(\phi^{-1}(v),1\big)$ so that

$$\Lambda_L(u,v)\begin{cases}\in[1,+\infty], & \text{if}\quad u\in[\phi^{-1}(v),u_-],\\ \in(0,1), & \text{if}\quad u\in(u_-,u_+),\\ \in[-\infty,0], & \text{if}\quad u\in[u_+,1].\end{cases}$$

Proof. We only verify the properties of the function Λ_U, since these of the function Λ_L can be proved similarly.
(a). We analyze the limits of the numerator of Λ_U as the denominator tends to 0, considering four different cases.

$(a1)$. If $u\in(0,t^l]$ and $v\downarrow\phi(t^l)$, then $u-\rho\Gamma^-_\phi(u)-(1-\rho)\Gamma^-_\phi\big(\phi^{-1}(v)\big)\to u>0$.

$(a2)$. If $u\uparrow t^r<1$ and $v\in[\phi(t^r),1]$, then $u-\rho\Gamma^-_\phi(u)-(1-\rho)\Gamma^-_\phi\big(\phi^{-1}(v)\big)\to 1+(1-\rho)t^r-\rho\phi(t^r)-(1-\rho)\big(v+\phi^{-1}(v)\big)$. Thus, when $1+(1-\rho)t^r-\rho\phi(t^r)-2(1-\rho)>0$, $1+(1-\rho)t^r-\rho\phi(t^r)-(1-\rho)\big(v+\phi^{-1}(v)\big)>0$ for all $v\in[\phi(t^r),1]$, and when $1+(1-\rho)t^r-\rho\phi(t^r)-2(1-\rho)\le 0$, $1+(1-\rho)t^r-\rho\phi(t^r)-(1-\rho)\big(v+\phi^{-1}(v)\big)>0$ for all $v\in[\phi(t^r),v_0)$, where the inequality follows from the definition of v_0.

$(a3)$. If $u\in[0,t^l]$ and $v\uparrow\phi(t^r)$, then $u-\rho\Gamma^-_\phi(u)-(1-\rho)\Gamma^-_\phi(\phi^{-1}(v))\to u-(1-\rho)\big(t^r+\phi(t^r)-1\big)\le t^l-(1-\rho)\big(t^r+\phi(t^r)-1\big)<0$, where the second inequality follows from Assumption I(a) that $t^r+\phi(t^r)-1>\max\{t^l/(1-\rho),\phi(t^l)/\rho\}$.

$(a4)$. If $u\downarrow t^l$ and $v\in[\phi(t^r),1]$, then $u-\rho\Gamma^-_\phi(u)-(1-\rho)\Gamma^-_\phi(\phi^{-1}(v))\to t^l-(1-\rho)\big(v+\phi^{-1}(v)-1\big)\le t^l-(1-\rho)\big(t^r+\phi(t^r)-1\big)<0$, where the second inequality follows from Assumption I(a) that $t^r+\phi(t^r)-1>\max\{t^l/(1-\rho),\phi(t^l)/\rho\}$.

Since the denominator of Λ_U is always non-negative, then in Cases $(a1)$ and $(a2)$, Λ_U tends to $+\infty$, whereas in Cases $(a3)$ and $(a4)$, Λ_U tends to $-\infty$. Combining the above results, we obtain the property (a) of the function Λ_U.
(b). For any fixed $\lambda \in [0,1]$ and $u \in (0, t^r)$, since $\Lambda_U(u,v)$ is continuous with respect to v on $[\phi(u),1]$ and $\Lambda_U\big(u,\phi(u)\big) \geq 1$ while $\Lambda_U(u,1) \leq 0$ (due to that $u \geq \Gamma_\phi^+(u)$ for all $u \in [0,1]$ and $u - \rho\Gamma_\phi^-(u)$ is non-decreasing on [0,1] so that $u - \rho\Gamma_\phi^-(u) \leq 1 - \rho\Gamma_\phi^-(1) = 1-\rho$ for all $u \in [0,1]$), then the set $\{v \in [\phi(u),1] | \Lambda_U(u,v) = \lambda\}$ is non-empty. Furthermore, for any fixed $u \in (0,t^r)$ and $\lambda \in [0,1]$, since $\{v \in [\phi(u),1] | \Lambda_U(u,v) = \lambda\}$ is a zero set of the function

$$F_\phi(v) = u - \lambda\Big(\rho\Gamma_\phi^+(u) + (1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)\Big) - (1-\lambda)\Big(\rho\Gamma_\phi^-(u) + (1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big),$$

which is non-increasing on $[\phi(u),1]$, then it is connected.
(c). For any fixed $u \in (0,t^r)$, define

$$v_- = \sup\{v \in [\phi(u),1] | \Lambda_U(u,v) = 1\} \quad \text{and} \quad v_+ = \inf\{v \in [\phi(u),1] | \Lambda_U(u,v) = 0\}. \tag{11}$$

Since Λ_U is continuous and $\Lambda_U(u,\phi(u)) \geq 1$ while $\Lambda_U(u,1) \leq 0$, then from the definitions of v_+ and v_-, we know that $v_+ > v_-$ and obtain the desired results. By similar arguments, we can obtain the value of Λ_U. □

Under Assumption I, using the copula C_λ of type (7) and auxiliary functions Λ_U as well as Λ_L, we propose an approach for constructing absolutely continuous copulas with a given curvilinear section $\Gamma_\phi \in \mathcal{S}_\phi$, as formalized in the following theorem.

Theorem 3.8. *For each $\Gamma_\phi \in \mathcal{S}_\phi$ under Assumption I, the function*

$$C_I(u,v) = \int_0^1 C_\lambda(u,v)\mathrm{d}\lambda, \tag{12}$$

is an absolutely continuous bivariate copula with the given curvilinear section Γ_ϕ.

Proof. Recall that under Assumption I(b), for each $\lambda \in [0,1]$, C_λ is a copula. Then, the function C_I of type (12) is a convex sum of bivariate copulas, and thus from Section 3.2.4 of Nelsen [14], we know that it is a bivariate copula. Furthermore, for any $t \in [0,1]$, it holds that

$$\begin{aligned} C_I\big(t,\phi(t)\big) &= \int_0^1 C_\lambda\big(t,\phi(t)\big)\mathrm{d}\lambda = \int_0^1 \min\big\{t, \phi(t), \lambda\Gamma_\phi^+(t) + (1-\lambda)\Gamma_\phi^-(t)\big\}\mathrm{d}\lambda \\ &= \int_0^1 \big(\lambda\Gamma_\phi^+(t) + (1-\lambda)\Gamma_\phi^-(t)\big)\mathrm{d}\lambda = \frac{\Gamma_\phi^+(t) + \Gamma_\phi^-(t)}{2} = \Gamma_\phi(t), \end{aligned}$$

where the third equality follows from the property (1) of the ϕ-function, and then C_I is a copula with the given curvilinear section Γ_ϕ.

Next, we need to verify that the copula C_I of type (12) is absolutely continuous. It suffices to prove that

$$\int_{[0,1]^2} \frac{\partial^2 C_I(u,v)}{\partial u \partial v}\mathrm{d}u\mathrm{d}v = 1.$$

Noting that $H_U \cup H_L = [0,1]^2$, we calculate $\int_{H_U} \frac{\partial^2 C_I(u,v)}{\partial u \partial v}\mathrm{d}u\mathrm{d}v$ and $\int_{H_L} \frac{\partial^2 C_I(u,v)}{\partial u \partial v}\mathrm{d}u\mathrm{d}v$ respectively.
(a). We first calculate $\int_{H_U} \frac{\partial^2 C_I(u,v)}{\partial u \partial v}\mathrm{d}u\mathrm{d}v$. Since $\xi_\lambda(v) \leq \phi(v)$ for all $0 \leq v \leq 1$ by (1), it follows that

$$\xi_\lambda^{-1}\Big(\frac{v - (1-\rho)\xi_\lambda\big(\phi^{-1}(v)\big)}{\rho}\Big) \geq \phi^{-1}(v),$$

and consequently for any $(u,v) \in H_U$, $u \leq \phi^{-1}(v) \leq \xi_\lambda^{-1}\Big(\frac{v-(1-\rho)\xi_\lambda\left(\phi^{-1}(v)\right)}{\rho}\Big)$. Then, from the concise expression (8) of the generalization of Γ_ϕ-curvilinear copula, we know that for any $(u,v) \in H_U$,

$$C_\lambda(u,v) = \min\big\{u, \rho\xi_\lambda(u) + (1-\rho)\xi_\lambda\big(\phi^{-1}(v)\big)\big\}.$$

And then, using the expression of $\Lambda_U(u,v)$, the copula C_λ in H_U can be expressed as

$$C_\lambda(u,v)=\begin{cases}\lambda\Big(\rho\Gamma_\phi^+(u)+(1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)\Big) & \\ \quad+(1-\lambda)\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big), & \text{if } \Lambda_U(u,v)>\lambda,\\ u, & \text{if } \Lambda_U(u,v)\le\lambda.\end{cases}$$

Hence, for each $(u,v)\in H_U$, the form of C_λ is related to the value of $\Lambda_U(u,v)$. Due to this reason, the following three cases are discussed.

$(a1)$. If $\Lambda_U(u,v)\le 0$, then for each $\lambda\in[0,1]$, $\Lambda_U(u,v)\le\lambda$ so that

$$C_I(u,v)=\int_0^1 u\mathrm{d}\lambda=u.$$

$(a2)$. If $\Lambda_U(u,v)>1$, then for each $\lambda\in[0,1]$, $\Lambda_U(u,v)>\lambda$ so that

$$\begin{aligned}C_I(u,v)&=\int_0^1\lambda\Big(\rho\Gamma_\phi^+(u)+(1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)\Big)+(1-\lambda)\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)\mathrm{d}\lambda\\&=\rho\frac{\Gamma_\phi^+(u)+\Gamma_\phi^-(u)}{2}+(1-\rho)\frac{\Gamma_\phi^+\big(\phi^{-1}(v)\big)+\Gamma_\phi^-\big(\phi^{-1}(v)\big)}{2}\\&=\rho\Gamma_\phi(u)+(1-\rho)\Gamma_\phi\big(\phi^{-1}(v)\big).\end{aligned}$$

$(a3)$. If $0<\Lambda_U(u,v)\le 1$, then

$$\begin{aligned}C_I(u,v)&=\int_0^{\Lambda_U(u,v)}\lambda\Big(\rho\Gamma_\phi^+(u)+(1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)\Big)+(1-\lambda)\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)\mathrm{d}\lambda\\&\quad+\int_{\Lambda_U(u,v)}^1 u\mathrm{d}\lambda\\&=\frac{\big(\Lambda_U(u,v)\big)^2\Big(\rho\Gamma_\phi^+(u)+(1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)-\rho\Gamma_\phi^-(u)-(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)}{2}\\&\quad+\Lambda_U(u,v)\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)+u\big(1-\Lambda_U(u,v)\big)\\&=\frac{\Lambda_U(u,v)\Big(u-\rho\Gamma_\phi^-(u)-(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)}{2}\\&\quad+\Lambda_U(u,v)\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)+u\big(1-\Lambda_U(u,v)\big)\\&=u+\frac{\Lambda_U(u,v)}{2}\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)-u\Big).\end{aligned}\tag{13}$$

Since C_I is a bivariate copula, its mixed second derivative exists a.e. on $[0,1]^2$ [6]. In particular, in Cases $(a1)$ as well as $(a2)$, $\frac{\partial^2 C_I(u,v)}{\partial u\partial v}=0$, whereas in Case $(a3)$, $\frac{\partial^2 C_I(u,v)}{\partial u\partial v}\neq 0$. Thus, when calculating $\int_{H_U}\frac{\partial^2 C_I(u,v)}{\partial u\partial v}\mathrm{d}u\mathrm{d}v$, we only need to consider the integration on the region corresponding to Case (a3). For any $u\in[0,1]$ so that both $\Gamma_\phi^{+\prime}(u)$ and $\Gamma_\phi^{-\prime}(u)$ exist, differentiating the expression (13) of $C_I(u,v)$ in Case $(a3)$ with respect to u, it yields that

$$\frac{\partial C_I(u,v)}{\partial u}=1+\frac{\Lambda_U(u,v)}{2}\big(\rho\Gamma_\phi^{-\prime}(u)-1\big)+\frac{\partial\Lambda_U(u,v)}{\partial u}\frac{\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)-u\Big)}{2}.$$

Notice that

$$\frac{\partial\Lambda_U(u,v)}{\partial u}=\frac{1-\rho\Gamma_\phi^{-\prime}(u)-\Lambda_U(u,v)\rho\big(\Gamma_\phi^{+\prime}(u)-\Gamma_\phi^{-\prime}(u)\big)}{\rho\big(\Gamma_\phi^+(u)-\Gamma_\phi^-(u)\big)+(1-\rho)\Big(\Gamma_\phi^+\big(\phi^{-1}(v)\big)-\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)}.\tag{14}$$

Plugging (14) into the expression of $\frac{\partial C_I(u,v)}{\partial u}$, we have that

$$\frac{\partial C_I(u,v)}{\partial u}=1+\Lambda_U(u,v)\big(\rho\Gamma_\phi^{-\prime}(u)-1\big)+\frac{\rho\big(\Lambda_U(u,v)\big)^2}{2}\big(\Gamma_\phi^{+\prime}(u)-\Gamma_\phi^{-\prime}(u)\big).\tag{15}$$

Recalling that both Γ_ϕ^+ and Γ_ϕ^- satisfy the Lipschitz condition, it follows from the expression of $\Lambda_U(u,v)$ that for each fixed u, $\Lambda_U(u,\cdot)$ is locally Lipschitz whenever $\rho\Gamma_\phi^+(u)+(1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)\neq\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)$, and thus

$$\begin{aligned}\int_{H_U}\frac{\partial^2 C_I(u,v)}{\partial u\partial v}\mathrm{d}u\mathrm{d}v&=\int_0^1\int_{\phi(u)}^1\frac{\partial^2 C_I(u,v)}{\partial u\partial v}\mathrm{d}v\mathrm{d}u\\ =\int_0^{t^r}\int_{v_-}^{v_+}\frac{\partial^2 C_I(u,v)}{\partial u\partial v}\mathrm{d}v\mathrm{d}u&=\int_0^{t^r}\frac{\partial C_I(u,v)}{\partial u}\Big|_{v_-}^{v_+}\mathrm{d}u.\end{aligned}\tag{16}$$

Substituting (15) into (16) and noting that $\Lambda_U(u,v_+)=0$ as well as $\Lambda_U(u,v_-)=1$, it yields that

$$\begin{aligned}\int_{H_U}\frac{\partial^2 C_I(u,v)}{\partial u\partial v}\mathrm{d}u\mathrm{d}v&=\int_0^{t^r}\Big(1+\Lambda_U(u,v)\big(\rho\Gamma_\phi^{-\prime}(u)-1\big)+\frac{\rho\big(\Lambda_U(u,v)\big)^2}{2}\big(\Gamma_\phi^{+\prime}(u)-\Gamma_\phi^{-\prime}(u)\big)\Big)\Big|_{v_-}^{v_+}\mathrm{d}u\\ &=\int_0^{t^r}(1-\frac{\rho\Gamma_\phi^{+\prime}(u)+\rho\Gamma_\phi^{-\prime}(u)}{2})\mathrm{d}u=\int_0^{t^r}\big(1-\rho\Gamma'_\phi(u)\big)\mathrm{d}u\\ &=t^r-\rho\Gamma_\phi(t^r)=(1-\rho)t^r-\rho\phi(t^r)+\rho.\end{aligned}\tag{17}$$

(b). Next, we determine $\int_{H_L}\frac{\partial^2 C_I(u,v)}{\partial u\partial v}\mathrm{d}u\mathrm{d}v$. Since $\xi_\lambda(u)\le u$ for all $0\le u\le 1$ by (1), it follows that $\xi_\lambda^{-1}\big(\frac{u-\rho\xi_\lambda(u)}{1-\rho}\big)\ge u$, and consequently for any $(u,v)\in H_L$, $v\le\phi(u)\le\phi\Big(\xi_\lambda^{-1}\big(\frac{u-\rho\xi_\lambda(u)}{1-\rho}\big)\Big)$. Then, from the concise expression (8) of the generalization of Γ_ϕ-curvilinear copula, we have that for any $(u,v)\in H_L$,

$$C_\lambda(u,v)=\min\big\{v,\rho\xi_\lambda(u)+(1-\rho)\xi_\lambda\big(\phi^{-1}(v)\big)\big\}.$$

And then, using the expression of $\Lambda_L(u,v)$, the copula C_λ in H_L can be expressed as

$$C_\lambda(u,v)=\begin{cases}\lambda\Big(\rho\Gamma_\phi^+(u)+(1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)\Big)\\ \quad+(1-\lambda)\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big), & \text{if }\ \Lambda_L(u,v)>\lambda,\\ v, & \text{if }\ \Lambda_L(u,v)\le\lambda.\end{cases}$$

Hence, for each $(u,v)\in H_L$, the form of C_λ is related to the value of $\Lambda_L(u,v)$. Due to this reason, the following three cases are discussed.
$(b1)$. If $\Lambda_L(u,v)\le 0$, then for each $\lambda\in[0,1]$, $\Lambda_L(u,v)\le\lambda$ so that

$$C_I(u,v)=\int_0^1 v\mathrm{d}\lambda=v.$$

$(b2)$. If $\Lambda_L(u,v)>1$, then for each $\lambda\in[0,1]$, $\Lambda_L(u,v)>\lambda$ so that

$$\begin{aligned}C_I(u,v)&=\int_0^1\lambda\Big(\rho\Gamma_\phi^+(u)+(1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)\Big)+(1-\lambda)\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)\mathrm{d}\lambda\\ &=\rho\Gamma_\phi(u)+(1-\rho)\Gamma_\phi\big(\phi^{-1}(v)\big).\end{aligned}$$

$(b3)$. If $0<\Lambda_L(u,v)\le 1$, then

$$\begin{aligned}C_I(u,v)&=\int_0^{\Lambda_L(u,v)}\lambda\Big(\rho\Gamma_\phi^+(u)+(1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)\Big)+(1-\lambda)\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)\mathrm{d}\lambda\\ &\quad+\int_{\Lambda_L(u,v)}^1 v\mathrm{d}\lambda\\ &=\frac{\big(\Lambda_L(u,v)\big)^2\Big(\rho\Gamma_\phi^+(u)+(1-\rho)\Gamma_\phi^+\big(\phi^{-1}(v)\big)-\rho\Gamma_\phi^-(u)-(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)}{2}\\ &\quad+\Lambda_L(u,v)\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)\Big)+v\big(1-\Lambda_L(u,v)\big)\\ &=v+\frac{\Lambda_L(u,v)}{2}\Big(\rho\Gamma_\phi^-(u)+(1-\rho)\Gamma_\phi^-\big(\phi^{-1}(v)\big)-v\Big).\end{aligned}$$

We know that in Cases $(b1)$ as well as $(b2)$, $\frac{\partial^2 C_I(u,v)}{\partial u \partial v} = 0$, and in Case $(b3)$, $\frac{\partial^2 C_I(u,v)}{\partial u \partial v} \neq 0$. Thus, when calculating $\int_{H_L} \frac{\partial^2 C_I(u,v)}{\partial u \partial v} \mathrm{d}u\mathrm{d}v$, we only need to consider the integration on the region related with Case $(b3)$. Denote $\big(\Gamma_\phi^- \circ \phi^{-1}\big)(t) = \Gamma_\phi^-\big(\phi^{-1}(t)\big)$ and $\big(\Gamma_\phi^+ \circ \phi^{-1}\big)(t) = \Gamma_\phi^+\big(\phi^{-1}(t)\big)$. Noting that for any $v \in [0,1]$ so that both $\big(\Gamma_\phi^+ \circ \phi^{-1}\big)'(v)$ and $\big(\Gamma_\phi^- \circ \phi^{-1}\big)'(v)$ exist,

$$\frac{\partial C_I(u,v)}{\partial v} = 1 + \Lambda_L(u,v)\Big((1-\rho)\big(\Gamma_\phi^- \circ \phi^{-1}\big)'(v) - 1\Big) + \frac{(1-\rho)\big(\Lambda_L(u,v)\big)^2}{2}\Big(\big(\Gamma_\phi^+ \circ \phi^{-1}\big)'(v) - \big(\Gamma_\phi^- \circ \phi^{-1}\big)'(v)\Big).$$

Since the mixed second derivative of C_I exists a.e. on $[0,1]^2$, then using the similar arguments as that of Case (a), we have that

$$\int_{H_L} \frac{\partial^2 C_I(u,v)}{\partial u \partial v} \mathrm{d}u\mathrm{d}v = \int_0^{\phi(t^r)} \Big(1 - (1-\rho)\big(\Gamma_\phi \circ \phi^{-1}\big)'(v)\Big)\mathrm{d}v = \rho\phi(t^r) - (1-\rho)t^r + 1 - \rho. \tag{18}$$

Combining (17) and (18), it yields that

$$\int_{[0,1]^2} \frac{\partial^2 C_I(u,v)}{\partial u \partial v} \mathrm{d}u\mathrm{d}v = \int_{H_L} \frac{\partial^2 C_I(u,v)}{\partial u \partial v} \mathrm{d}u\mathrm{d}v + \int_{H_U} \frac{\partial^2 C_I(u,v)}{\partial u \partial v} \mathrm{d}u\mathrm{d}v = 1,$$

which is the desired result. □

For each $\Gamma_\phi \in \mathcal{S}_\phi$ under Assumption I, Theorem 3.8 not only verifies that the set of absolutely continuous copulas with the given curvilinear section Γ_ϕ is non-empty, but also proposes an approach for constructing a family of absolutely continuous copulas with the given curvilinear section Γ_ϕ.

Example 3.9. *Consider the curve $\phi(t) = t^\alpha$, $\alpha \geq 1$ and the ϕ-function $\Gamma_\phi(t) = \frac{t^\alpha + t^{\alpha+1}}{2}$ in $\mathcal{S}_\phi$. As discussed in Example 3.6, Assumption I is satisfied when $\rho = 1/(1+\alpha)$. Consequently, by Theorem 3.8, the function C_I of type* (12) *with $\rho = 1/(1+\alpha)$ is an absolutely continuous copula with the given curvilinear section Γ_ϕ.*

Now, an explicit expression of C_I for $\rho = 1/(1+\alpha)$ is provided. Consider the function $\Lambda_U : H_U \to [-\infty, +\infty]$ defined by

$$\Lambda_U(u,v) = \begin{cases} \frac{(1+\alpha)u - u^{1+\alpha} - \alpha v^{1+1/\alpha}}{u^\alpha - u^{1+\alpha} + \alpha\left(v - v^{1+1/\alpha}\right)}, & \text{if } \; u^\alpha + \alpha v > u^{1+\alpha} + \alpha v^{1+1/\alpha}, \\ +\infty, & \text{if } \; (u,v) \in \{(0,0),(1,1)\}, \\ -\infty, & \text{if } \; (u,v) = (0,1), \end{cases}$$

and $\Lambda_L : H_L \to [-\infty, +\infty]$ defined by

$$\Lambda_L(u,v) = \begin{cases} \frac{(1+\alpha)v - u^{1+\alpha} - \alpha v^{1+1/\alpha}}{u^\alpha - u^{1+\alpha} + \alpha(v - v^{1+1/\alpha})}, & \text{if } \; u^\alpha + \alpha v > u^{1+\alpha} + \alpha v^{1+1/\alpha}, \\ +\infty, & \text{if } \; (u,v) \in \{(0,0),(1,1)\}, \\ -\infty, & \text{if } \; (u,v) = (1,0). \end{cases}$$

In the case $\alpha > 1$, the curves

$$(1+\alpha)u - u^{1+\alpha} = \alpha v^{1+1/\alpha}\big(\text{i.e., } \Lambda_U(u,v) = 0\big), \; (1+\alpha)u - u^\alpha = \alpha v\big(\text{i.e., } \Lambda_U(u,v) = 1\big),$$

$$(1+\alpha)v - \alpha v^{1+1/\alpha} = u^{1+\alpha}\big(\text{i.e., } \Lambda_L(u,v) = 0\big) \quad \text{and} \quad v = u^\alpha\big(\text{i.e., } \Lambda_L(u,v) = 1\big),$$

divide the unit square into five regions, as illustrated in Figure 2(a) for $\alpha = 4$.

The copula C_I with $\rho = 1/(1+\alpha)$ is given by the following expressions:

A$_1$. $(1+\alpha)u - u^{1+\alpha} \leq \alpha v^{1+1/\alpha}$,

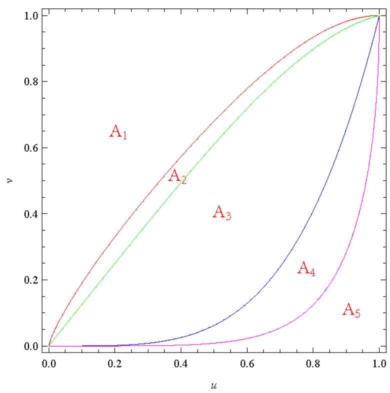
$$C_I(u,v) = u;$$

A$_2$. $(1+\alpha)u - u^{1+\alpha} > \alpha v^{1+1/\alpha}$ *and* $(1+\alpha)u - u^\alpha \leq \alpha v$,

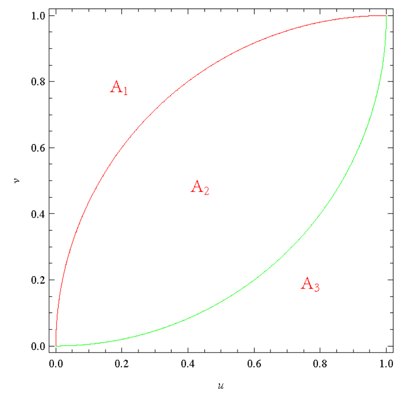
$$C_I(u,v) = u - \frac{\big((1+\alpha)u - u^{1+\alpha} - \alpha v^{1+1/\alpha}\big)^2}{2(1+\alpha)\big(u^\alpha - u^{1+\alpha} + \alpha(v - v^{1+1/\alpha})\big)};$$

A$_3$. $(1+\alpha)u - u^\alpha > \alpha v$ *and* $v > u^\alpha$,

$$C_I(u,v) = \frac{u^\alpha + u^{1+\alpha}}{2(1+\alpha)} + \frac{\alpha(v + v^{1+1/\alpha})}{2(1+\alpha)};$$



(a) The case of $\alpha = 4$



(b) The case of $\alpha = 1$

Figure 2: The divided regions of unit square in Example 3.9.

A_4. *$v \leq u^\alpha$ and $(1+\alpha)v - \alpha v^{1+1/\alpha} > u^{1+\alpha}$,*

$$C_I(u,v) = v - \frac{\big((1+\alpha)v - u^{1+\alpha} - \alpha v^{1+1/\alpha}\big)^2}{2(1+\alpha)\big(u^\alpha - u^{1+\alpha} + \alpha(v - v^{1+1/\alpha})\big)};$$

A_5. *$(1+\alpha)v - \alpha v^{1+1/\alpha} \leq u^{1+\alpha}$,*

$$C_I(u,v) = v.$$

In particular, when $\alpha = 1$, Theorem 3.8 yields an absolutely continuous copula whose diagonal section is given by $\delta(t) = (t+t^2)/2$, which has the following expressions:

A_1. *$2u - u^2 \leq v^2$,*

$$C_I(u,v) = u;$$

A_2. *$2u - u^2 > v^2$ and $2v - v^2 > u^2$,*

$$C_I(u,v) = \frac{u^4 - 4uv + 2u^2v^2 + v^4}{4(u^2 - u + v^2 - v)};$$

A_3. *$2v - v^2 \leq u^2$,*

$$C_I(u,v) = v.$$

In this scenario, the unit square is divided into three regions, as illustrated in Figure 2(b).

4 Constructions of absolutely continuous copulas with given opposite-curvilinear sections

In Section 3, we have proposed an approach for constructing absolutely continuous copulas C_I with a given curvilinear section. This section discusses the construction of absolutely continuous copulas with a given opposite-curvilinear section, achieved through the application of transformations to the copula C_I.

Note easily that if the function ψ is strictly decreasing and differentiable on $[0,1]$ with $\psi(0) = 1$ and $\psi(1) = 0$, then $1-\psi$ is a strictly increasing and differentiable function on $[0,1]$, and satisfies $1-\psi(0) = 0$ and $1-\psi(1) = 1$. Hence, the mapping $\psi \to 1-\psi$ defines a bijection between the sets Ψ and Φ, in particular, $\psi \in \Psi$ if and only if $1-\psi \in \Phi$. Also, for any ψ-function Γ_ψ, the conditions (3) and (4) imply that for any $t \in [0,1]$,

$$\max\{0, t + \big(1-\psi(t)\big) - 1\} = t - \min\{t, \psi(t)\} \leq t - \Gamma_\psi(t) \leq t - \max\{0, t + \psi(t) - 1\} = \min\{t, 1-\psi(t)\},$$

and for any $0 \leq t_1 \leq t_2 \leq 1$,

$$0 = t_1 - t_2 + (t_2 - t_1) \leq t_2 - \Gamma_\psi(t_2) - \big(t_1 - \Gamma_\psi(t_1)\big) \leq t_2 - t_1 - \psi(t_2) + \psi(t_1) = t_2 - t_1 + 1 - \psi(t_2) - \big(1 - \psi(t_1)\big).$$

Hence, the function $t - \Gamma_\psi(t)$ satisfies the conditions (1) and (2) along the curve $1-\psi$. Consequently, by Definition 2.3, it follows that for any ψ-function Γ_ψ, the function $t - \Gamma_\psi(t)$ is a ϕ-function for $\phi(t) = 1 - \psi(t)$, $t \in [0,1]$. The converse statement holds by an analogous argument.

Since for each ψ-function (respectively, ϕ-function), there exists at least one copula whose opposite-curvilinear section (respectively, curvilinear section) coincides with it, the above obtained relationship between the ψ-function Γ_ψ and the ϕ-function $t-\Gamma_\psi(t)$ enables us to establish a correspondence between an absolutely continuous copula C with the opposite-curvilinear section Γ_ψ and the absolutely continuous copula $u-C(u,1-v)$, $(u,v)\in[0,1]^2$ whose curvilinear section is $t-\Gamma_\psi(t)$ along the curve $1-\psi$. This correspondence is formalized in the following theorem, which is essential in the construction of absolutely continuous copulas with a given opposite-curvilinear section Γ_ψ.

Theorem 4.1. *For any $\psi\in\Psi$, C is an absolutely continuous copula with a given opposite-curvilinear section Γ_ψ if and only if $u-C(u,1-v)$, $(u,v)\in[0,1]^2$ is an absolutely continuous copula with the curvilinear section $\Gamma_\phi(t)=t-\Gamma_\psi(t)$ along the curve $\phi(t)=1-\psi(t)$, $t\in[0,1]$.*

Proof. It is easy to verify that C is a copula if and only if $u-C(u,1-v)$, $(u,v)\in[0,1]^2$ is a copula. Also, notice that the copula C is absolutely continuous if and only if $u-C(u,1-v)$, $(u,v)\in[0,1]^2$ is absolutely continuous.

Finally, if C is a copula with a given opposite-curvilinear section Γ_ψ, that is, $C\big(t,\psi(t)\big)=\Gamma_\psi(t)$, then the curvilinear section of the copula $u-C(u,1-v)$ along the curve $\phi=1-\psi$ is

$$\Gamma_\phi(t)=\Gamma_{1-\psi}(t)=t-C\Big(t,1-\big(1-\psi(t)\big)\Big)=t-\Gamma_\psi(t),\ t\in[0,1],$$

i.e., $u-C(u,1-v)$ is a copula with the curvilinear section $\Gamma_\phi(t)=t-\Gamma_\psi(t)$ along the curve $\phi(t)=1-\psi(t)$. And the converse is also true. □

Theorem 4.1 therefore indicates that constructing absolutely continuous copulas with a given opposite-curvilinear section Γ_ψ can be accomplished by constructing absolutely continuous copulas with the corresponding curvilinear section $\Gamma_\phi(t)=t-\Gamma_\psi(t)$ along the curve $\phi=1-\psi$, which is shown in the following.

To facilitate this construction, we first introduce some notations. For each ψ-function Γ_ψ, we define

$$s^l=\sup\{t\in[0,1]|\Gamma_\psi(t)=t\}\quad\text{and}\quad s^r=\inf\{t\in[0,1]|\Gamma_\psi(t)=\psi(t)\}.$$

Remark 4.2. *Since ψ is strictly decreasing on $[0,1]$, we can verify easily that $s^r\ge s^l$ by contradiction. And if $s^l=s^r$, then either s^l or s^r is the unique root of $\psi(t)=t$.*

Define the set

$$\mathcal{S}_\psi=\{\Gamma_\psi|\ \Gamma_\psi\text{ is a }\psi\text{-function with }\Gamma_\psi(t)=\frac{\Gamma^+_\psi(t)+\Gamma^-_\psi(t)}{2}>\max\{0,t+\psi(t)-1\}\text{ on }(0,1)\},\tag{19}$$

of ψ-functions, where Γ^+_ψ and Γ^-_ψ are two ψ-functions satisfying that $\Gamma^-_\psi(t)<\Gamma_\psi(t)<\Gamma^+_\psi(t)$ for all $t\in(s^l,s^r)$. The set $\mathcal{S}_\psi$ is non-empty and the elements in $\mathcal{S}_\psi$ are rich. For example, for any two copulas C_1 and C_2 with $C_1(u,v)<C_2(u,v)$ for all $(u,v)\in(0,1)^2$, the ψ-function $\Big(C_1\big(t,\psi(t)\big)+C_2\big(t,\psi(t)\big)\Big)/2$ is an element in $\mathcal{S}_\psi$.

As mentioned above, the construction of absolutely continuous copulas with a given opposite-curvilinear section $\Gamma_\psi\in\mathcal{S}_\psi$ can be achievable by first constructing absolutely continuous copulas with the corresponding curvilinear section $\Gamma_\phi(t)=t-\Gamma_\psi(t)$ along the curve $\phi=1-\psi$. To implement this construction, the following assumption on the curve ψ and Γ_ψ is needed.

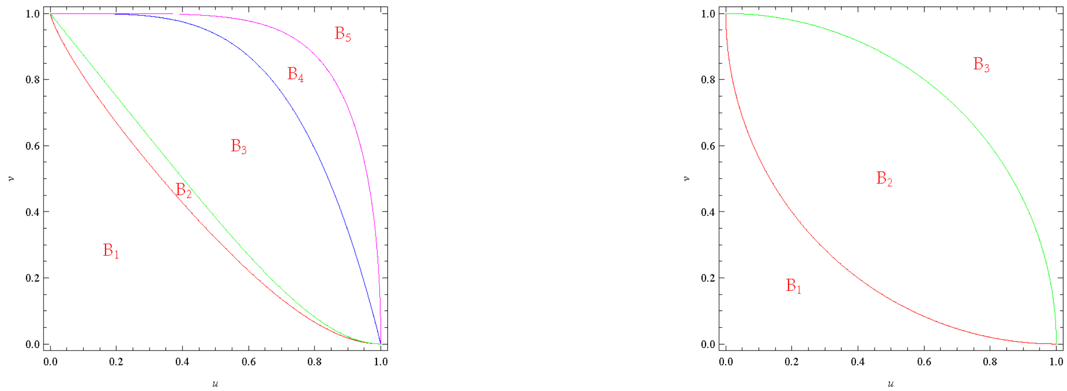
Assumption II. There exists at least one constant $0<\rho<1$ satisfying the following conditions:

(a).
$$s^r-\psi(s^r)>\max\{s^l/(1-\rho),\big(1-\psi(s^l)\big)/\rho\};\tag{20}$$

(b). for all $\lambda\in[0,1]$, both $t-\rho\big(t-\lambda\Gamma^+_\psi(t)-(1-\lambda)\Gamma^-_\psi(t)\big)$ and $t-(1-\rho)\Big(\psi^{-1}(1-t)-\lambda\Gamma^+_\psi\big(\psi^{-1}(1-t)\big)-(1-\lambda)\Gamma^-_\psi\big(\psi^{-1}(1-t)\big)\Big)$ are non-decreasing on $[0,1]$.

Here, Assumption II(b) is indispensable for obtaining copulas with the curvilinear section $\Gamma_\phi(t)=t-\Gamma_\psi(t)$ via a convex sum of a series of functions of the form (7), and thereby for constructing copulas with the original opposite-curvilinear section Γ_ψ. Moreover, Assumption II(a) serves to ensure that the resulting copulas are absolutely continuous, since it implies that Assumption I(a) holds for $\Gamma_\phi(t)=t-\Gamma_\psi(t)$ along the curve $\phi(t)=1-\psi(t)$, $t\in[0,1]$.

Notice easily that for each ψ-function $\Gamma_\psi\in\mathcal{S}_\psi$ satisfying Assumption II, the associated function $\Gamma_\phi(t)=t-\Gamma_\psi(t)$ belongs to $\mathcal{S}_\phi$ with $\phi(t)=1-\psi(t)$, $t\in[0,1]$ and satisfies Assumption I. Then, from Theorems 3.8 and 4.1, we obtain directly the following theorem.



(a) The case of $\alpha = 4$ (b) The case of $\alpha = 1$

Figure 3: The divided regions of unit square in Example 4.4

Theorem 4.3. *For any $\psi \in \Psi$ and $\Gamma_\psi \in \mathcal{S}_\psi$ under Assumption II, let C_I be an absolutely continuous copula of type* (12) *with the curvilinear section $\Gamma_\phi(t) = t - \Gamma_\psi(t)$ along the curve $\phi(t) = 1 - \psi(t)$. Then, $u - C_I(u, 1-v)$, $(u, v) \in [0, 1]^2$ is an absolutely continuous copula with the given opposite-curvilinear section Γ_ψ.*

For each $\Gamma_\psi \in \mathcal{S}_\psi$ satisfying Assumption II, Theorem 4.3 guarantees that the set of absolutely continuous copulas with the given opposite-curvilinear section Γ_ψ is non-empty, and provides an approach for constructing a family of absolutely continuous copulas with the given opposite-curvilinear section Γ_ψ.

Example 4.4. *Let the curve $\psi(t) = 1 - t^\alpha \in \Psi$, $\alpha \geq 1$, and define $\Gamma_\psi(t) = \frac{(t-t^\alpha)+(t-t^{\alpha+1})}{2}$, which is a ψ-function $\in \mathcal{S}_\psi$. Observing that $s^l = 0$ and $s^r = 1$, it follows that for every $0 < \rho < 1$, the inequality* (20) *holds.*

Straightforward calculations yield that for all $\lambda \in [0, 1]$, when $\rho \leq 1/(1+\alpha)$, the function $t - \rho\big(t - \lambda\Gamma_\psi^+(t) - (1-\lambda)\Gamma_\psi^-(t)\big)$ is non-decreasing on $[0, 1]$ and when $\rho \geq 1/(1+\alpha)$, the function $t - (1-\rho)\Big(\psi^{-1}(1-t) - \lambda\Gamma_\psi^+\big(\psi^{-1}(1-t)\big) - (1-\lambda)\Gamma_\psi^-\big(\psi^{-1}(1-t)\big)\Big)$ is non-decreasing on $[0, 1]$. Therefore, for $\rho = 1/(1+\alpha)$, the condition (b) of Assumption II is satisfied.

In this setting, the associated curvilinear section is $\Gamma_\phi(t) = t - \Gamma_\psi(t) = \frac{t^\alpha + t^{\alpha+1}}{2}$, which is an element of $\mathcal{S}_\phi$ with $\phi(t) = 1 - \psi(t) = t^\alpha$, $t \in [0, 1]$. Let C_I be an absolutely continuous copula of type (12) *constructed by the curve $\phi(t) = 1 - \psi(t)$, $t \in [0, 1]$ in Example 3.9. Then, by Theorem 4.3, the function $C_{\Gamma_\psi}(u, v) = u - C_I(u, 1-v)$, $(u, v) \in [0, 1]^2$ defines an absolutely continuous copula with the given opposite-curvilinear section Γ_ψ along the curve ψ, which is given by the following expressions for $\alpha > 1$:*

B_1. $(1+\alpha)u - u^{1+\alpha} \leq \alpha(1-v)^{1+1/\alpha}$,

$$C_{\Gamma_\psi}(u, v) = 0;$$

B_2. $(1+\alpha)u - u^{1+\alpha} > \alpha(1-v)^{1+1/\alpha}$ *and* $(1+\alpha)u - u^\alpha \leq \alpha(1-v)$,

$$C_{\Gamma_\psi}(u, v) = \frac{\big((1+\alpha)u - u^{1+\alpha} - \alpha(1-v)^{1+1/\alpha}\big)^2}{2(1+\alpha)\Big(u^\alpha - u^{1+\alpha} + \alpha\big(1 - v - (1-v)^{1+1/\alpha}\big)\Big)};$$

B_3. $(1+\alpha)u - u^\alpha > \alpha(1-v)$ *and* $1 - v > u^\alpha$,

$$C_{\Gamma_\psi}(u, v) = u - \frac{u^\alpha + u^{1+\alpha}}{2(1+\alpha)} - \frac{\alpha\big(1 - v + (1-v)^{1+1/\alpha}\big)}{2(1+\alpha)};$$

B_4. $1 - v \leq u^\alpha$ *and* $(1+\alpha)(1-v) - \alpha(1-v)^{1+1/\alpha} > u^{1+\alpha}$,

$$C_{\Gamma_\psi}(u, v) = u + v - 1 + \frac{\big((1+\alpha)(1-v) - u^{1+\alpha} - \alpha(1-v)^{1+1/\alpha}\big)^2}{2(1+\alpha)\Big(u^\alpha - u^{1+\alpha} + \alpha\big(1 - v - (1-v)^{1+1/\alpha}\big)\Big)};$$

B_5. $(1+\alpha)(1-v) - \alpha(1-v)^{1+1/\alpha} \leq u^{1+\alpha}$,

$$C_{\Gamma_\psi}(u, v) = u + v - 1.$$

The unit square is partitioned into five regions, as illustrated in Figure 3(a) for $\alpha = 4$.

In particular, when $\alpha = 1$, Theorem 4.3 yields an absolutely continuous copula with the given opposite-diagonal section $\delta(t) = (t - t^2)/2$, which has the following expressions:

B$_1$. $2u - u^2 \leq (1-v)^2$,

$$C_{\Gamma_\psi}(u, v) = 0;$$

B$_2$. $2u - u^2 > (1-v)^2$ *and* $1 - v^2 > u^2$,

$$C_{\Gamma_\psi}(u, v) = \frac{\left(u^2 - 2u + (1-v)^2\right)^2}{4(u - u^2 + v - v^2)};$$

B$_3$. $1 - v^2 \leq u^2$,

$$C_{\Gamma_\psi}(u, v) = u + v - 1.$$

In this scenario, the unit square is divided into three regions, as illustrated in Figure 3(b).

It is further to note that $\phi(t) = \psi(1-t)$ is a strictly increasing and differentiable function on $[0,1]$ with $\phi(0) = 0$ and $\phi(1) = 1$ if ψ is strictly decreasing and differentiable on $[0,1]$ with $\psi(0) = 1$ and $\psi(1) = 0$. That is, $\psi \in \Psi$ implies $\phi(t) = \psi(1-t) \in \Phi$, and the converse implication is analogous. Also, for any ψ-function Γ_ψ, the conditions (3) and (4) imply that for any $t \in [0,1]$,

$$\begin{aligned}\max\{0, t + \psi(1-t) - 1\} &= \psi(1-t) - \min\{1-t, \psi(1-t)\} \leq \psi(1-t) - \Gamma_\psi(1-t) \\ &\leq \psi(1-t) - \max\{0, 1 - t + \psi(1-t) - 1\} = \min\{t, \psi(1-t)\},\end{aligned}$$

and for any $0 \leq t_1 \leq t_2 \leq 1$,

$$\begin{aligned}0 &\leq \psi(1-t_2) - \Gamma_\psi(1-t_2) - \big(\psi(1-t_1) - \Gamma_\psi(1-t_1)\big) \\ &\leq (1-t_1) - (1-t_2) + \psi(1-t_2) - \psi(1-t_1) = t_2 - t_1 + \psi(1-t_2) - \psi(1-t_1).\end{aligned}$$

Thus, the function $\psi(1-t) - \Gamma_\psi(1-t)$ satisfies the conditions (1) and (2) along the curve $\psi(1-t)$. By Definition 2.3, it follows that for any ψ-function Γ_ψ, $\psi(1-t) - \Gamma_\psi(1-t)$ is a ϕ-function with $\phi(t) = \psi(1-t)$, $t \in [0,1]$. The converse statement follows by an analogous argument.

Since for each ψ-function (respectively, ϕ-function), there exists at least one copula whose opposite-curvilinear section (respectively, curvilinear section) coincides with it, then by exploiting the above obtained relationship between the ψ-function Γ_ψ and the ϕ-function $\psi(1-t) - \Gamma_\psi(1-t)$ and employing the similar argument as that of Theorem 4.1, we directly establish a correspondence between an absolutely continuous copula C whose opposite-curvilinear section is Γ_ψ and the absolutely continuous copula $v - C(1-u, v)$, $(u,v) \in [0,1]^2$ whose curvilinear section is $\psi(1-t) - \Gamma_\psi(1-t)$ along the curve $\psi(1-t)$. This correspondence also plays an important role in the construction of absolutely continuous copulas with a given opposite-curvilinear section Γ_ψ.

Theorem 4.5. *For any $\psi \in \Psi$, C is an absolutely continuous copula with a given opposite-curvilinear section Γ_ψ if and only if $v - C(1-u, v)$, $(u,v) \in [0,1]^2$ is an absolutely continuous copula with the curvilinear section $\Gamma_\phi(t) = \psi(1-t) - \Gamma_\psi(1-t)$ along the curve $\phi(t) = \psi(1-t)$, $t \in [0,1]$.*

We next propose an alternative approach for constructing absolutely continuous copulas with a given opposite-curvilinear section. The set $\mathcal{S}_\psi$ of ψ-functions defined in (19) remains under consideration. In this case, s^l and s^r are defined as

$$s^l = \sup\{t \in [0,1] | \Gamma_\psi(1-t) = \psi(1-t)\} \quad \text{and} \quad s^r = \inf\{t \in [0,1] | \Gamma_\psi(1-t) = 1-t\}.$$

Theorem 4.5 shows that the construction of absolutely continuous copulas with a given opposite-curvilinear section $\Gamma_\psi \in \mathcal{S}_\psi$ can alternatively be effected by constructing absolutely continuous copulas whose curvilinear section along the curve $\phi(t) = \psi(1-t)$ is $\Gamma_\phi(t) = \psi(1-t) - \Gamma_\psi(1-t)$, $t \in [0,1]$. To facilitate this construction, we impose conditions on the curve ψ and Γ_ψ, which is formalized in the following assumption.

Assumption III. There exists at least one constant $0 < \rho < 1$ satisfying the following conditions:

(a).

$$s^r + \psi(1-s^r) - 1 > \max\{s^l/(1-\rho), \psi(1-s^l)/\rho\}; \tag{21}$$

(b). for all $\lambda \in [0,1]$, both $t - \rho\big(\psi(1-t) - \lambda\Gamma_\psi^+(1-t) - (1-\lambda)\Gamma_\psi^-(1-t)\big)$ and $t - (1-\rho)\Big(t - \lambda\Gamma_\psi^+\big(\psi^{-1}(t)\big) - (1-\lambda)\Gamma_\psi^-\big(\psi^{-1}(t)\big)\Big)$ are non-decreasing on $[0,1]$.

In this context, Assumption III(b) is instrumental for obtaining copulas with the curvilinear section $\Gamma_\phi(t) = \psi(1-t) - \Gamma_\psi(1-t)$ along the curve $\phi(t) = \psi(1-t)$ via a convex combination of a series of functions of the form (7), and thus for constructing copulas with the original opposite-curvilinear section Γ_ψ. Additionally, Assumption III(a) ensures that the resulting copulas are absolutely continuous, since it implies that Assumption I(a) holds for $\Gamma_\phi(t) = \psi(1-t) - \Gamma_\psi(1-t)$ along the curve $\phi(t) = \psi(1-t)$, thereby providing the continuity properties required in the analysis.

Observe that for each ψ-function $\Gamma_\psi \in \mathcal{S}_\psi$ satisfying Assumption III, the function $\Gamma_\phi(t) = \psi(1-t) - \Gamma_\psi(1-t)$ is an element of $\mathcal{S}_\phi$ along the curve $\phi(t) = \psi(1-t)$ satisfying Assumption I. Theorems 3.8 and 4.5, thus yield the following theorem, which constructs another family of absolutely continuous copulas with the given opposite-curvilinear section $\Gamma_\psi \in \mathcal{S}_\psi$.

Theorem 4.6. *For any $\psi \in \Psi$ and $\Gamma_\psi \in \mathcal{S}_\psi$ under Assumption III, let C_I be an absolutely continuous copula of type* (12) *with the curvilinear section $\Gamma_\phi(t) = \psi(1-t) - \Gamma_\psi(1-t)$ along the curve $\phi(t) = \psi(1-t)$, $t \in [0,1]$. Then, $v - C_I(1-u, v)$, $(u,v) \in [0,1]^2$ is an absolutely continuous copula with the given opposite-curvilinear section Γ_ψ.*

For each $\Gamma_\psi \in \mathcal{S}_\psi$ under Assumption III, Theorem 4.1 provides a complementary approach for constructing absolutely continuous copulas with the given opposite curvilinear section Γ_ψ.

Example 4.7. *Let the curve $\psi(t) = (1-t)^\alpha \in \Psi$, $\alpha \geq 1$, and define $\Gamma_\psi(t) = \frac{(1-t)^\alpha - (1-t)^{\alpha+1}}{2}$, which is a ψ-function in $\mathcal{S}_\psi$. In this setting, observing that $s^l = 0$ and $s^r = 1$, it follows that for all $0 < \rho < 1$, the inequality* (21) *holds. Straightforward calculations yield that for all $\lambda \in [0,1]$, when $\rho = 1/(1+\alpha)$, the condition (b) of Assumption III holds.*

In this setting, $\Gamma_\phi(t) = \psi(1-t) - \Gamma_\psi(1-t) = \frac{t^\alpha + t^{\alpha+1}}{2}$ is a ϕ-function in $\mathcal{S}_\phi$ with $\phi(t) = \psi(1-t) = t^\alpha$, $t \in [0,1]$. Let C_I be an absolutely continuous copula of type (12) *constructed by $\phi(t) = \psi(1-t)$, $t \in [0,1]$ in Example 3.9. Then, using the expression of C_I and from Theorem 4.6, we know that $C_{\Gamma_\psi}(u,v) = v - C_I(1-u,v)$, $(u,v) \in [0,1]^2$ with $\rho = 1/(1+\alpha)$ is an absolutely continuous copula with the given curvilinear section Γ_ψ, which is given by the following expressions for $\alpha > 1$:*

C$_1$. $(1+\alpha)(1-u) - (1-u)^{1+\alpha} \leq \alpha v^{1+1/\alpha}$,

$$C_{\Gamma_\psi}(u,v) = u + v - 1;$$

C$_2$. $(1+\alpha)(1-u) - (1-u)^{1+\alpha} > \alpha v^{1+1/\alpha}$ *and* $(1+\alpha)(1-u) - (1-u)^\alpha \leq \alpha v$,

$$C_{\Gamma_\psi}(u,v) = u + v - 1 + \frac{\big((1+\alpha)(1-u) - (1-u)^{1+\alpha} - \alpha v^{1+1/\alpha}\big)^2}{2(1+\alpha)\Big((1-u)^\alpha - (1-u)^{1+\alpha} + \alpha\big(v - v^{1+1/\alpha}\big)\Big)};$$

C$_3$. $(1+\alpha)(1-u) - (1-u)^\alpha > \alpha v$ *and* $v > (1-u)^\alpha$,

$$C_{\Gamma_\psi}(u,v) = v - \frac{(1-u)^\alpha + (1-u)^{1+\alpha}}{2(1+\alpha)} - \frac{\alpha\big(v + v^{1+1/\alpha}\big)}{2(1+\alpha)};$$

C$_4$. $v \leq (1-u)^\alpha$ *and* $(1+\alpha)v - \alpha v^{1+1/\alpha} > (1-u)^{1+\alpha}$,

$$C_{\Gamma_\psi}(u,v) = \frac{\big((1+\alpha)v - (1-u)^{1+\alpha} - \alpha v^{1+1/\alpha}\big)^2}{2(1+\alpha)\Big((1-u)^\alpha - (1-u)^{1+\alpha} + \alpha\big(v - v^{1+1/\alpha}\big)\Big)};$$

C$_5$. $(1+\alpha)v - \alpha v^{1+1/\alpha} \leq (1-u)^{1+\alpha}$,

$$C_{\Gamma_\psi}(u,v) = 0.$$

The five divided regions of unit square is illustrated in Figure 4.

In particular, when $\alpha = 1$, from Theorem 4.6, we also obtain the absolutely continuous copula with the given opposite-diagonal section $\delta(t) = (t - t^2)/2$, which is presented in Example 4.4.

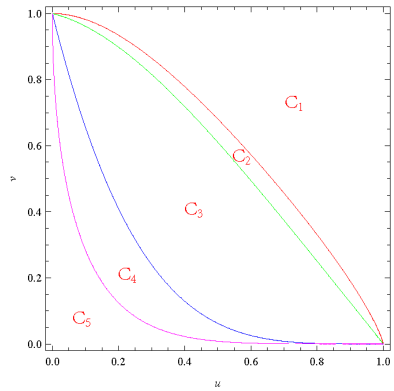


Figure 4: The divided regions of unit square in Example 4.7

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5 Conclusions

This paper establishes a framework for constructing absolutely continuous copulas with a given ϕ-function Γ_ϕ along a curve ϕ, achieved by averaging over a series of generalization of Γ_ϕ-curvilinear copulas, subject to appropriate assumptions. We further established two relationships: (i) between strictly increasing curves ϕ and strictly decreasing curves ψ on $[0, 1]$, and (ii) between opposite-curvilinear sections Γ_ψ along the curve ψ and curvilinear sections Γ_ϕ along the curve ϕ. Utilizing these relationships and the constructed absolutely continuous copula with given curvilinear sections, we construct two distinct families of absolutely continuous copulas having Γ_ψ as the opposite-curvilinear section. These contributions provide valuable insights into the constructions of absolutely continuous copulas whose curvilinear or opposite-curvilinear sections are defined by suitable curves within the unit square, thereby providing approaches for generating absolutely continuous copulas with given curvilinear or opposite-curvilinear sections.

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