POWERSET OPERATOR FOUNDATIONS FOR CATALG FUZZY SET THEORIES

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ABSTRACT. The paper sets forth in detail categorically-algebraic or catalg foundations for the operations of taking the image and preimage of (fuzzy) sets called forward and backward powerset operators. Motivated by an open question of S. E. Rodabaugh, we construct a monad on the category of sets, the algebras of which generate the fixed-basis forward powerset operator of L. A. Zadeh. On the next step, we provide a direct lift of the backward powerset operator using the notion of categorical biproduct. The obtained framework is readily extended to the variable-basis case, justifying the powerset theories currently popular in the fuzzy community. At the end of the paper, our general variety-based setting postulates the requirements, under which a convenient variety-based powerset theory can be developed, suitable for employment in all areas of fuzzy mathematics dealing with fuzzy powersets, including fuzzy algebra, logic and topology.

1. Introduction

Every working mathematician will readily acknowledge at least one encounter with the so-called powersets and powerset operators. More precisely, given a set X, there exists the set $\mathcal{P}(X) = \{S \,|\, S \subseteq X\}$ of all subsets of X (including the empty one) called the *powerset* of X. Every map $X \xrightarrow{f} Y$ extends to the respective powersets providing the *forward powerset operator* $\mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$ and the *backward powerset operator* $\mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$ defined by

$$f^{\rightarrow}(S) = \{f(x) \,|\, x \in S\}, \quad f^{\leftarrow}(T) = \{x \,|\, f(x) \in T\}.$$

The notions, however being simple, are much exploited in almost all branches of modern mathematics, including algebra, logic and topology. Just to mention a few examples, notice that the famous representation theorems for distributive lattices of M. Stone [117] (generalizing that for Boolean algebras [116]) and H. Priestley [78] (combining the theorems of M. Stone and G. Birkhoff [13]) are based on particular

subsets of powersets (the sets of prime ideals) and backward powerset operators (the action of the respective equivalence functors on morphisms). The results provided (and continue to provide) extremely useful topological representations for various algebraic structures, giving birth to a powerful theory of what have come to be known as *natural dualities* [20]. In particular, they play a crucial role in *monadic logic*, started by P. Halmos [41] (see the collection of papers on the topic [42]) and based on *monadic Boolean algebras*, i.e., Boolean algebras equipped with an additional unary operation (called (*existential*) *quantifier*) with certain properties. The theory of such structures is an algebraic treatment of the logic of propositional functions of one argument, with Boolean operations and a single (existential) quantifier. There exists the famous representation theorem of P. Halmos [41], claiming that every monadic Boolean algebra is isomorphic to a *functional monadic Boolean algebra* (Boolean subalgebra of a powerset B^X for a Boolean algebra B and a set X), the proof based on the above-mentioned representation theorem of M. Stone [116]. Some authors tried to use a weaker structure, i.e., R. Cignoli [18] considered the case of bounded distributive lattices (arriving at the concept of *Q-distributive lattice*), establishing a duality between (appropriately defined) quantifiers on them and certain equivalence relations on their Priestley spaces. An up to date state of the field is contained in [19], where J. Cīrulis considers quantifiers on multiplicative semilattices, with the aim "to find out how weak a lattice-structure may be in order that a reasonable theory of existential quantifiers on it still could be developed". The author of this manuscript himself studied extensively the case of *monadic quantale algebras* [108] arriving at several representation theorems for the structure.

Topology is definitely the most important area of application for powersets and powerset operators, being actually generated by them. The basic notions of *topology* on a set and *continuity* of a map are defined in terms of powersets and backward powerset operators, all other topological concepts being (implicitly) dependant on the basic ones. No wonder that immediately after the concept of *fuzzy set* was brought into light by the famous paper of L. A. Zadeh [122], researchers started to ponder over the fuzzification of powersets and powerset operators. The former notion enjoyed an easy extension to the new framework, suggested by the nature of fuzziness in question, i.e., the set I^X of all maps $X \xrightarrow{\alpha} I$ from a given set X to the unit interval $[0, 1]$ (called the *basis* of the powerset) gave a substitute for $\mathcal{P}(X)$. The latter concept, however, posed some difficulties in its definition. The first approach was given by L. A. Zadeh [122], where he introduced the desired operators $I^X \xrightarrow{f_I^\rightarrow} I^Y$ and $I^Y \xrightarrow{f_I^\leftarrow} I^X$ as follows:

$$(f_I^\rightarrow(\alpha))(y) = \bigvee\{\alpha(x) \mid f(x) = y\}, \quad f_I^\leftarrow(\beta) = \beta \circ f.$$

While the first definition has been accepted right from the start, the second one caused some concerns even in L. A. Zadeh himself. The problem gained in influence with the appearance of *L-fuzzy set* of J. A. Goguen [36], where the unit interval I was replaced by a suitable complete lattice L (actually *quantale*).

It was fuzzy topology that gave the topic its decisive push. The concept of *(stratified) (L-)fuzzy topology*, introduced in the celebrated papers of C. L. Chang [17],

J. A. Goguen [37] and R. Lowen [65], resulted in the theory of *fixed-basis fuzzy topological spaces*, the idea of stratification being due to R. Lowen, but the term itself coined by P. M. Pu and Y. M. Liu [79]. Many researchers have pursued the line [8, 9, 39, 40, 43, 46, 45, 52, 61, 63, 97, 119, 121], exploiting successfully the existing approach to powersets, until all of a sudden the proposed framework appeared to be a rather restrictive one. The trouble arose from the cunning idea of B. Hutton [49], providing the first variable-basis approach, which resulted in the theory of *variable-basis singleton topological spaces*. Further developments were rapid, in 1983 S. E. Rodabaugh [83] introducing the first variable-basis category for topology, with the underlying sets of the spaces allowed to be non-singletons. The proposed theory, getting its final accomplishment in [87] where it was coined as *variable-basis lattice-valued* or *point-set lattice-theoretic (poslat) topology*, induced a bunch of results on its properties [26, 27, 84, 85, 87], the inventor himself being always in the front line. Since the underlying machinery relied on category theory, P. Eklund [28, 29, 30] started the research on its categorical properties, initiating *categorical fuzzy topology.*

The idea of variable-basis gave rise to the new challenge of introducing the operators acting on powersets with different basis, i.e., maps $L^X \to M^Y$ and $M^Y \to L^X$. The first fruitful solution was proposed by S. E. Rodabaugh [83, 84], which nowadays has become almost a *de facto* standard in the fuzzy community. Strangely enough, the situation has changed completely, the backward powerset operator having an almost ready definition, whereas the forward one, in general, left with no expression whatsoever. The issue of *ground category* (wisely suggested by P. Eklund in [28]) acquired importance in the developments and was fixed at the product category $\mathbf{Set} \times \mathbf{C}$, with $\mathbf{Set}$ standing for the category of sets and maps, and $\mathbf{C}$ being a subcategory of $\mathbf{Loc}$ (*locales*) introduced by J. Isbell [51] and extensively studied by P. T. Johnstone [53], or the dual of a category of some other algebraic structures enjoying the existence of arbitrary joins and a binary operation (with no distributivity assumed). In the new framework, every $\mathbf{Set} \times \mathbf{C}$-morphism $(X, L) \xrightarrow{(f,\varphi)} (Y, M)$ provided the backward powerset operator $M^Y \xrightarrow{(f,\varphi)^\leftarrow} L^X$ defined by ($(-)^{op}$ stands for the actual map)

$$(f, \varphi)^\leftarrow(\alpha) = \varphi^{op} \circ \alpha \circ f.$$

The respective definition of the forward powerset operator appeared to be in need of more sophistication. Two additional maps $L \xrightarrow{[\varphi]} M$ and $L^Y \xrightarrow{\langle[\varphi]\rangle} M^Y$ given by $[\varphi](a) = \bigwedge\{b \,|\, a \leqslant \varphi^{op}(b)\}$ and $\langle[\varphi]\rangle(\alpha) = [\varphi] \circ \alpha$ respectively, suggested the following definition of $L^X \xrightarrow{(f,\varphi)^\rightarrow} M^Y$ (notice the use of the fixed-basis operator)

$$(f, \varphi)^\rightarrow = \langle[\varphi]\rangle \circ f_L^\rightarrow. \tag{1}$$

It was already S. E. Rodabaugh himself, who had some doubts on whether to use $f_M^\rightarrow \circ \langle[\varphi]\rangle$ in (1). Soon it became clear that in order to develop a fruitful theory of lattice-valued sets (which successfully began to force out its fixed-basis counterpart), both operators needed a sober mathematical standpoint. As an initiator of the topic, S. E. Rodabaugh [86, 88, 90] gave the first strict foundations

for (poslat) fuzzy set theories, justifying at last not only the choice of L. A. Zadeh but also the traditional crisp case. Motivated by the ideas of E. G. Manes [67] and P. T. Johnstone [53], he proposed an approach based on two cornerstones:

(1) An adjunction between **Set** and the category **CSLat**($\bigvee$) of $\bigvee$-semilattices and $\bigvee$-preserving maps [3].
(2) *Galois connection* or *adjunction* between partially ordered sets [35].

The respective foundation scheme for the standard crisp operators is rather simple, being incorporated in the following two steps:

Step 1: Item (1) gives a covariant powerset functor **Set** $\xrightarrow{(-)^{\rightarrow}}$ **CSLat**($\bigvee$), $(X \xrightarrow{f} Y)^{\rightarrow} = \mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$.
Step 2: Since $f^{\rightarrow}$ is $\bigvee$-preserving, Item (2) provides a unique upper adjoint to $f^{\rightarrow}$ in the form of $f^{\leftarrow}$.

The classical case done with, S. E. Rodabaugh proceeds to generalized powerset operators, postulating two criteria [88]:

I. The analogue of $f^{\rightarrow}$ (resp. $f^{\leftarrow}$) should be a unique lifting of $f^{\rightarrow}$ (resp. $f^{\leftarrow}$), i.e., the analogue of $f^{\rightarrow}$ (resp. $f^{\leftarrow}$) should be a *generalization* of $f^{\rightarrow}$ (resp. $f^{\leftarrow}$) in a natural way.
II. The analogue of $f^{\rightarrow}$ should be a lower adjoint to the analogue of $f^{\leftarrow}$, i.e., the relationship $f^{\rightarrow} \dashv f^{\leftarrow}$ between $f^{\rightarrow}$ and $f^{\leftarrow}$ should hold for their analogues.

Having the criteria in hand, the author constructs the desired operators, which luckily enough satisfy both of them. A more attentive look, however, on the machinery employed casts some doubts on its relevancy. The main deficiencies seem to be a somewhat artificial nature of the requirements arising during the construction process in [88, Theorems 2.6(A,B)] and the rather unmotivated choice of the expression for $(f, \varphi)^{\rightarrow}$ in [88, Definition 3.12], that together with some minor (but very annoying) typos makes a more transparent setting highly advisable. Some improvements have already been done by S. E. Rodabaugh himself in [90], where more category theory was brought into account to provide a general definition of *powerset theory* [90, Definition 3.5]. The new setting presented still inherits the drawbacks of its ancestors, missing the point of clarifying the true relation between powerset operators. It is the purpose of this paper to propose a more convenient framework for the traditional as well as fuzzy powerset theories, which underlines their categorical aspects in the first place.

Motivated by the problem posed in [86, Open Question 6.17], we present a monad on the category **Set**, the algebras of which generate a functor lifting the above-mentioned fixed-basis forward powerset operator $f_L^{\rightarrow}$ of L. A. Zadeh and J. A. Goguen. The machinery employed is based on the category Q-**Mod** of modules over a given quantale Q [60, 72, 112] suggested by the well-known category R-**Mod** of modules over a ring R [7, 47]. On the next step, we employ the concept of categorical *biproduct* to construct another functor, lifting directly the backward powerset operator $f_L^{\leftarrow}$. The functors obtained have a ready extension to the

variable-basis framework, justifying the already mentioned approach of S. E. Rodabaugh. The advantage of our setting is the streamlining of categorically-algebraic (catalg) properties of powerset operators, void of their point-set lattice-theoretic dependencies. In particular, we show the true essence of the duality between the operators in question, which basically arises from the fact that forward powerset operators correspond to coproducts and backward ones to products in categories, the notion of categorical biproduct providing the missing link between the concepts.

Following our current trend in developing catalg aspects of the theory of fuzzy sets [101, 102, 103], at the end of the paper we switch to the setting of varieties of algebras, postulating the requirements, under which a convenient variety-based powerset theory can be developed. As a result, we arrive at a framework suitable for employment in all areas of fuzzy mathematics dealing with fuzzy powersets, including fuzzy algebra, logic and topology. The theory developed is illustrated by two examples coming from the realm of the latter field, namely, by the respective theories of *variety-based topological spaces* and *systems*, which recently has acquired some influence in the fuzzy community [26, 40, 100]. With the achievements of this paper, we would like to underline once more (following the steps of [101, 102, 103]) the fruitfulness of change of the current fuzzy setting from poslat (point-set lattice-theoretic) to catalg (categorically-algebraic), the latter one being more flexible, transparent and trustworthy.

The necessary categorical background can be found in [3, 44, 66, 67]. For the notions of universal algebra we recommend [16, 21, 67]. The reader should be aware that although we tried to make the paper as much self-contained as possible, the lack of space still compelled us to leave some details for his/her own perusal.

2. Varieties of Algebras

The cornerstone of our approach to fuzziness is the notion of *algebra*. The structure is to be thought of as a set with a family of operations defined on it, satisfying certain identities, e.g., semigroup, monoid, group and also complete lattice, frame, quantale. In case of *finitary algebras*, i.e., those induced by a set of finite operations, there are (at least) two ways of describing the resulting entities [16, 21]. The first one, being rather categorical, uses the concept of *variety*, i.e., a class of algebras closed under homomorphic images, subalgebras and direct products. The second one, being highly algebraic, is based on the notion of *equational class*, namely, providing a set of identities and taking precisely those algebras which satisfy all of them. The well-known HSP-theorem of G. Birkhoff [12] says that varieties and equational classes coincide. Motivated by the algebraic structures popular in fuzzy topology, where unions are usually represented as arbitrary joins, this paper is bound to include infinitary cases as well and therefore extends the categorical approach of varieties to cover its needs, leaving aside the infinitary algebraic machineries of *equationally-definable class* [67] and *equational category* [64, 95] (which, however, still are referred to later on).

Definition 2.1. Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a (possibly proper) class of cardinal numbers. An Ω-*algebra* is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$ (denoted by A), consisting of a set A and a

family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$, called n_λ-*ary operations* on A. An Ω-*homomorphism* $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{f} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{f} B$ making the diagram

$$
\begin{array}{ccc}
A^{n_\lambda} & \xrightarrow{\ f^{n_\lambda}\ } & B^{n_\lambda} \\
{\scriptstyle \omega_\lambda^A} \downarrow & & \downarrow {\scriptstyle \omega_\lambda^B} \\
A & \xrightarrow{\ f\ } & B
\end{array}
$$

commute for every $\lambda \in \Lambda$. $\mathbf{Alg}(\Omega)$ is the construct of Ω-algebras and Ω-homomorphisms, with the underlying functor denoted by $|-|$.

Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A *variety of Ω-algebras* (also called a *variety*) is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras) and $\mathcal{E}$-quotients (homomorphic images). The objects (resp. morphisms) of a variety are called *algebras* (resp. *homomorphisms*).

Given a variety $\mathbf{A}$ and a subclass $\Omega' \subseteq \Omega$, an Ω'-*reduct* of $\mathbf{A}$ (or just *reduct* if the context is clear) is a pair $(||-||, \mathbf{B})$, where $\mathbf{B}$ is a variety of Ω'-algebras and $\mathbf{A} \xrightarrow{||-||} \mathbf{B}$ is a concrete functor.

We illustrate the definition by two important varieties, providing the basis for all other examples of the concept (notice the use of the notations from [90]).

Definition 2.2. Given $\Xi \in \{\bigvee, \bigwedge\}$, a Ξ-*semilattice* is a partially ordered set having arbitrary Ξ. A Ξ-*semilattice homomorphism* is a map preserving arbitrary Ξ. $\mathbf{CSLat}(\Xi)$ is the construct of Ξ-semilattices and their homomorphisms.

More sophisticated examples are given by the constructs $\mathbf{SQuant}$, $\mathbf{Quant}$ and $\mathbf{Frm}$ of semi-quantales, quantales and frames, extremely popular in lattice-valued topology [90] and obtained as suitable extensions of the objects of $\mathbf{CSLat}(\bigvee)$. For convenience of the reader as well as to feel free in using them throughout the paper, we recall from [60, 90] the respective definitions.

Definition 2.3. A *semi-quantale* (*s-quantale* for short) is a $\bigvee$-semilattice equipped with a binary operation $\otimes$ (called *multiplication*). An *s-quantale homomorphism* is a $\bigvee$-semilattice homomorphism which also preserves $\otimes$. $\mathbf{SQuant}$ is the construct of s-quantales and their homomorphisms.

An s-quantale A is called *commutative* (*cs-quantale* for short) provided that $\otimes$ is commutative. $\mathbf{CSQuant}$ is the full subcategory of $\mathbf{SQuant}$ of all cs-quantales.

An s-quantale A is called *involutive* (*is-quantale* for short) provided that A has a unary operation $(-)^*$ (called *involution*) such that $a^{**} = a$, $(a \otimes b)^* = b^* \otimes a^*$ and $(\bigvee S)^* = \bigvee_{s \in S} s^*$ for every $a, b \in A$ and every $S \subseteq A$. An s-quantale homomorphism is *involutive* provided that it preserves the involution. $\mathbf{ISQuant}$ is the category of is-quantales and their homomorphisms.

An s-quantale A is called *unital* (*us-quantale* for short) provided that $\otimes$ has the unit $e \in A$. An s-quantale homomorphism is *unital* provided that it preserves the unit. $\mathbf{USQuant}$ is the construct of unital s-quantales and their homomorphisms.

An s-quantale A is called a *quantale* provided that $\otimes$ is associative and distributes over $\bigvee$ from the right and from the left. **Quant** is the full subcategory of **SQuant** of all quantales.

A uc-quantale A is called a *frame* provided that $\otimes$ is idempotent. **Frm** is the full subcategory of **UCQuant** of all frames.

It is easy to see that **CSLat**$(\bigvee)$ is a $\{\bigvee\}$-reduct of **SQuant**; **SQuant** is a $\{\bigvee, \otimes\}$-reduct of **ISQuant**, **USQuant**, **CSQuant** and **Quant**; **UCQuant** is a $\{\bigvee, \otimes, e\}$-reduct of **Frm**. The reader should also be aware of the difference between is-quantales and *DeMorgan s-quantales* of [90], the latter ones being equipped with an order-reversing involution having nothing in common with the multiplication. The objects of **SQuant** are proposed in [90] as the basic mathematical structure for doing poslat fuzzy topology upon (the statement confirmed by us in [101, 103]) since their properties constitute the minimum allowing the obtained categories for topology (and these include many well-known categories) to be topological over their ground categories. A significant drawback of the new structure is the lack of knowledge on its properties, contrasting sharply the situation with its luckier counterpart quantale [60, 68, 72, 92, 94, 112] extremely useful in the non-commutative approach to topology [15, 22, 69, 70], shown to be more general than its respective fuzzy analogue [104]. The relevance of frames is (at least) twofold: on one hand, they provide a nice fuzzification framework for different topological structures (based, essentially, on idempotency and commutativity of $\wedge$), on the other, they constitute the backbone of *pointless topology* developed in [51, 53, 71, 74, 80].

Having examined different modifications of the construct **CSLat**$(\bigvee)$, we are going now to pay attention to its counterpart **CSLat**$(\bigwedge)$. The following definition is motivated by the construct of *closure spaces* [5, 6, 31] which has been known for a long time already (see, e.g., the famous book of G. Birkhoff [14]) and is getting more and more attention in recent years.

Definition 2.4. A *closure lattice* (*c-lattice* for short) is a $\bigwedge$-semilattice, with the bottom element denoted by $\perp$. A *c-lattice homomorphism* is a map preserving arbitrary $\bigwedge$ and $\perp$. **CL** is the construct of c-lattices and their homomorphisms.

The reader should be aware that **CL** is a (non-full) subcategory of **CSLat**$(\bigwedge)$ and also that the latter category is a $\{\bigwedge\}$-reduct of the former. Moreover, there are some interrelations between s-quantales and c-lattices, i.e., every s-quantale is a c-lattice and vice versa (use the Duality Principle of ordered sets). On the other hand, easy considerations (e.g., the case of morphisms) clearly show that the two constructs are not equivalent. In fact, the concept of c-lattice is more general than the respective one of s-quantale, due to the lack of an additional binary operation.

Remark 2.5. From now on we fix a variety **A** and use the following notations [27, 87, 90].

- The dual of the category **A** is denoted by **LoA** (the "**Lo**" comes from "localic"), whose objects (resp. morphisms) are called *localic algebras* (resp. *homomorphisms*). Following the widely-accepted notation of [53], the dual of **Frm** is denoted by **Loc**, with the respective change in the naming of

objects. Given a localic algebra A, $\mathbf{S}_A$ will stand for the subcategory of $(\mathbf{Lo})\mathbf{A}$ with the only morphism 1_A.

- Given a morphism f of a category $\mathbf{C}$, the respective morphism of $\mathbf{C}^{op}$ is denoted by f^{op} and vice versa. To distinguish between set maps (or, more generally, morphisms) and homomorphisms, we will denote the former ones by f, g, h (also α, β, γ in case of fuzzy sets) reserving φ, ψ, ϕ for the latter.

3. Quantale Modules

This section provides a monad on the category $\mathbf{Set}$ of sets and maps, the algebras of which generate the fixed-basis forward powerset operator of L. A. Zadeh [122], answering the question of S. E. Rodabaugh [86, Open Question 6.17] on its existence. For convenience of the reader, we begin by recalling from [112] some algebraic preliminaries.

Definition 3.1. A *(left) module* over a given quantale Q (Q-*module* for short) is a $\bigvee$-semilattice A, equipped with a map $Q \times A \xrightarrow{*} A$ (called Q-*action on A*) such that

(1) $q * (\bigvee S) = \bigvee_{s \in S}(q * s)$ for every $q \in Q$ and every $S \subseteq A$;
(2) $(\bigvee S) * a = \bigvee_{s \in S}(s * a)$ for every $a \in A$ and every $S \subseteq Q$;
(3) $q_1 * (q_2 * a) = (q_1 \otimes q_2) * a$ for every $q_1, q_2 \in Q$ and every $a \in A$.

A Q-*module homomorphism* $A \xrightarrow{\varphi} B$ is a $\bigvee$-semilattice homomorphism such that $\varphi(q * a) = q * \varphi(a)$ for every $a \in A$ and every $q \in Q$. Q-$\mathbf{Mod}$ is the construct of Q-modules and their homomorphisms.

A Q-module A over a u-quantale Q is called *unital* (*u-Q-module* for short) provided that $e * a = a$ for every $a \in A$. Given a u-quantale Q, $\mathbf{U}Q$-$\mathbf{Mod}$ is the full subcategory of Q-$\mathbf{Mod}$ of all u-Q-modules.

Definition 3.1 was motivated by the category $(\mathbf{U})R$-$\mathbf{Mod}$ of (unital) left modules over a ring R with identity [7, 47]. The first lattice analogy of ring module appeared in [54], in connection with analysis of descent theory. Although the authors work with commutative structures, most of their results are valid in the absence of the property. Moreover, modules over a u-quantale formed the central idea in the unified treatment of process semantics developed in [2]. Nowadays, the topic provides a rich source for investigation [59, 60, 72, 73, 92, 94].

Example 3.2. Given the two-element u-quantale $\mathbf{2} = \{\bot, \top\}$, the categories $\mathbf{U2}$-$\mathbf{Mod}$ and $\mathbf{CSLat}(\bigvee)$ are isomorphic (compare with integers $\mathbb{Z}$ in case of the category $\mathbf{U}R$-$\mathbf{Mod}$). A nice challenge provides the case of $Q = \mathbf{1}$, namely, $\mathbf{1}$-$\mathbf{Mod}$ is isomorphic to the category $\mathbf{CSLat}(\bigvee)$, whereas $\mathbf{U1}$-$\mathbf{Mod}$ is equivalent (but not isomorphic) to the one-element terminal category.

Example 3.3. Every (u-)quantale is a (u-)module over each of its (u-)subquantales, with action given by multiplication.

The reader may be aware of the fact (mentioned in different forms in [3, 53, 67, 92, 94]) that the category $\mathbf{CSLat}(\bigvee)$ is monadic over the category $\mathbf{Set}$, the monad in question being the well-known powerset monad [3, Example 20.2(3)].

The result can be readily extended to the more general category Q-**Mod**. Being in need of some particular properties of the achievement, we provide a brief account of the machinery employed. The approach is based on a mixture of the results from [60, 112], modified to suite our current framework. We begin with a preliminary notion, extending the concepts of both Q-module and quantale.

Definition 3.4. An *algebra* over a given uc-quantale Q (Q-*algebra* for short) is a u-Q-module A, which is also a quantale such that $q * (a \otimes b) = (q * a) \otimes b = a \otimes (q * b)$ for every $a, b \in A$ and every $q \in Q$. A Q-*algebra homomorphism* is a Q-module homomorphism, which is also a quantale homomorphism. Q-**Alg** is the construct of Q-algebras and their homomorphisms.

A Q-algebra A is called *unital* (u-Q-*algebra* for short) provided that A is a u-quantale. **U**Q-**Alg** is the (non-full) subcategory of Q-**Alg** of all u-Q-algebras and unital Q-algebra homomorphisms.

Definition 3.4 was motivated by the category K-**Alg** of algebras over a given commutative ring K with identity [7, 47]. Some properties of the new category can be found in [98, 115, 111, 113]. In particular, the first two references consider a generalization of the notion, providing an extension of the famous sobriety-spatiality equivalence [53, 71] to the framework of partial algebras.

Example 3.5. Given the two-element uc-quantale $\mathbf{2} = \{\bot, \top\}$, the categories $\mathbf{2}$-**Alg** and **Quant** are isomorphic.

Example 3.6. Every uc-quantale is an algebra over each of its u-subquantales, with action given by multiplication.

The next simple but rather useful lemma was suggested by [47, Theorem III.1.10], providing a generalization of [60, Lemma 1.1.11].

Lemma 3.7. *Every Q-algebra can be embedded into a u-Q-algebra.*

Proof. Given a Q-algebra A, define the underlying set of a new algebra B by $|A| \times |Q|$, equipping it with the component-wise joins and thus, obtaining a $\bigvee$-semilattice. The required binary operation on B is given by $(a_1, q_1) \otimes (a_2, q_2) = ((a_1 \otimes a_2) \vee (q_2 * a_1) \vee (q_1 * a_2), q_1 \otimes q_2)$, with the unit $e_B = (\bot, e_Q)$. The Q-action is equally simple, being given by $q_1 * (a, q_2) = (q_1 * a, q_1 \otimes q_2)$. Straightforward computations show that B is a u-Q-algebra. To convince the reader, we show the most challenging part, i.e., associativity of the multiplication: $(a_1, q_1) \otimes ((a_2, q_2) \otimes (a_3, q_3)) = (a_1, q_1) \otimes ((a_2 \otimes a_3) \vee (q_3 * a_2) \vee (q_2 * a_3), q_2 \otimes q_3) = ((a_1 \otimes (a_2 \otimes a_3)) \vee (a_1 \otimes (q_3 * a_2)) \vee (a_1 \otimes (q_2 * a_3)) \vee ((q_2 \otimes q_3) * a_1) \vee (q_1 * (a_2 \otimes a_3)) \vee (q_1 * (q_3 * a_2)) \vee (q_1 * (q_2 * a_3)), (q_1 \otimes (q_2 \otimes q_3)) = ((a_1 \otimes a_2) \otimes a_3) \vee ((q_2 * a_1) \otimes a_3) \vee ((q_1 * a_2) \otimes a_3) \vee (q_3 * (a_1 \otimes a_2)) \vee (q_3 * (q_2 * a_1)) \vee (q_3 * (q_1 * a_2)) \vee ((q_1 \otimes q_2) * a_3), (q_1 \otimes q_2) \otimes q_3) = ((a_1 \otimes a_2) \vee (q_2 * a_1) \vee (q_1 * a_2), q_1 \otimes q_2) \otimes (a_3, q_3) = ((a_1, q_1) \otimes (a_2, q_2)) \otimes (a_3, q_3)$. The embedding in question $A \overset{\varphi}{\hookrightarrow} B$ is provided by $\varphi(a) = (a, \bot)$. $\square$

Corollary 3.8. *Every quantale can be embedded into a u-quantale.*

Proof. By Example 3.5, every quantale is a $\mathbf{2}$-algebra. $\square$

Following the notations of [60], we will denote the u-Q-algebra B constructed in Lemma 3.7 by $A[e]$, with e reserved for its unit. It appears that the construction gives rise to some important consequences.

Theorem 3.9. $\mathbf{U}Q\text{-}\mathbf{Alg}$ *is a monoreflective subcategory of Q-$\mathbf{Alg}$.*

Proof. Suppose $\mathbf{U}Q\text{-}\mathbf{Alg} \xrightarrow{\; E \;} Q\text{-}\mathbf{Alg}$ is the inclusion functor. Given a Q-algebra A, Lemma 3.7 provides a Q-$\mathbf{Alg}$-embedding $A \xrightarrow{\eta_A} EA[e]$ defined by $\eta_A(a) = (a, \bot)$. We show that η_A is the desired reflection arrow. Given a Q-$\mathbf{Alg}$-morphism $A \xrightarrow{\varphi} EB$, define $A[e] \xrightarrow{\overline{\varphi}} B$ by $\overline{\varphi}(a, q) = \varphi(a) \vee (q * e_B)$. In the following, we prove that $\overline{\varphi}$ is the required extension of φ. To see that $\overline{\varphi}$ is a $\mathbf{U}Q\text{-}\mathbf{Alg}$-morphism, notice that $\overline{\varphi}(\bigvee_{i \in I}(a_i, q_i)) = \overline{\varphi}(\bigvee_{i \in I} a_i, \bigvee_{i \in I} q_i) = \varphi(\bigvee_{i \in I} a_i) \vee ((\bigvee_{i \in I} q_i) * e_B) = (\bigvee_{i \in I} \varphi(a_i)) \vee (\bigvee_{i \in I}(q_i * e_B)) = \bigvee_{i \in I} \overline{\varphi}(a_i, q_i)$ and $\overline{\varphi}((a_1, q_1) \otimes (a_2, q_2)) = \overline{\varphi}((a_1 \otimes a_2) \vee (q_2 * a_1) \vee (q_1 * a_2), q_1 \otimes q_2) = (\varphi(a_1) \otimes \varphi(a_2)) \vee (q_2 * \varphi(a_1)) \vee (q_1 * \varphi(a_2)) \vee ((q_1 \otimes q_2) * e_B) = (\varphi(a_1) \otimes \varphi(a_2)) \vee (\varphi(a_1) \otimes (q_2 * e_B)) \vee ((q_1 * e_B) \otimes \varphi(a_2)) \vee ((q_1 * e_B) \otimes (q_2 * e_B)) = (\varphi(a_1) \vee (q_1 * e_B)) \otimes (\varphi(a_2) \vee (q_2 * e_B)) = \overline{\varphi}(a_1, q_1) \otimes \overline{\varphi}(a_2, q_2)$, all other properties being similarly straightforward. Moreover, it appears that $E\overline{\varphi} \circ \eta_A(a) = \overline{\varphi}(a, \bot) = \varphi(a)$. Given another $\mathbf{U}Q\text{-}\mathbf{Alg}$-morphism $A[e] \xrightarrow{\psi} B$ with the same property, $\psi(a, q) = \psi(a, \bot) \vee \psi(\bot, q) = (E\psi \circ \eta_A(a)) \vee (q * \psi(e)) = \varphi(a) \vee (q * e_B) = \overline{\varphi}(a, q)$. $\square$

Corollary 3.10. *There exists an adjoint situation $(\eta, \epsilon) : F \relbar\mapsto E : \mathbf{U}Q\text{-}\mathbf{Alg} \to Q\text{-}\mathbf{Alg}$.*

Proof. To use it constantly throughout the paper, we recall from [3] the standard scheme of obtaining an adjoint situation from the existence of universal arrows. Given a Q-$\mathbf{Alg}$-morphism $A \xrightarrow{\varphi} B$, $F(A \xrightarrow{\varphi} B) = A[e] \xrightarrow{\varphi[e]} B[e]$, with $\varphi[e]$ defined by commutativity of the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\;\eta_A\;} & EA[e] \\
{\scriptstyle \varphi}\big\downarrow & & \big\downarrow{\scriptstyle E\varphi[e]} \\
B & \xrightarrow[\;\eta_B\;]{} & EB[e]
\end{array}
$$

and therefore $(\varphi[e])(a, q) = (\varphi(a), q)$. Given a u-$Q$-algebra A, $(EA)[e] \xrightarrow{\epsilon_A} A$ is defined by commutativity of the diagram

$$
\begin{array}{ccc}
EA & \xrightarrow{\;\eta_{EA}\;} & E(EA)[e] \\
& {\scriptstyle 1_A}\searrow & \big\downarrow{\scriptstyle E\epsilon_A} \\
& & EA
\end{array}
$$

and therefore $\epsilon_A(a, q) = a \vee (q * e_A)$. $\square$

Corollary 3.11. $\mathbf{UQuant}$ *is a monoreflective subcategory of* $\mathbf{Quant}$.

The next important consequence of Lemma 3.7 requires the additional definition of a new category.

Definition 3.12. Given a Q-algebra A, A_Q-**Mod** is the category, with objects all A-modules D, which are also u-Q-modules such that $a * (q * d) = (q * a) * d = q * (a * d)$ for every $a \in A$, $q \in Q$, $d \in D$. Morphisms of the category are A-module homomorphisms, which are also Q-module homomorphisms.

Theorem 3.13. *The categories A_Q-**Mod** and $\mathbf{U}A[e]$-**Mod** are concretely isomorphic.*

Proof. We construct two functors A_Q-**Mod** $\xrightarrow{G}$ $\mathbf{U}A[e]$-**Mod** and $\mathbf{U}A[e]$-**Mod** $\xrightarrow{F}$ A_Q-**Mod** such that $FG = 1_{A_Q\text{-}\mathbf{Mod}}$ and $GF = 1_{\mathbf{U}A[e]\text{-}\mathbf{Mod}}$. To obtain the first one, suppose $D \xrightarrow{\varphi} D'$ is an A_Q-**Mod**-morphism. Define the action of $A[e]$ on D by $(a, q) *_G d = (a * d) \vee (q * d)$ and check the conditions of Definition 3.1. For distributivity of $*_G$ over $\bigvee$ on the left, notice that $(a, q) *_G (\bigvee S) = (a * (\bigvee S)) \vee (q * (\bigvee S)) = (\bigvee_{s \in S}(a * s)) \vee (\bigvee_{s \in S}(q * s)) = \bigvee_{s \in S}((a * s) \vee (q * s)) = \bigvee_{s \in S}((a, q) *_G s)$ and similarly on the right. Further, $(a_1, q_1) *_G ((a_2, q_2) *_G d) = (a_1, q_1) *_G ((a_2 * d) \vee (q_2 * d)) = (a_1 * (a_2 * d)) \vee (a_1 * (q_2 * d)) \vee (q_1 * (a_2 * d)) \vee (q_1 * (q_2 * d)) \overset{(\dagger)}{=} ((a_1 \otimes a_2) * d) \vee ((q_2 * a_1) * d) \vee ((q_1 * a_2) * d) \vee ((q_1 \otimes q_2) * d) = ((a_1 \otimes a_2) \vee (q_2 * a_1) \vee (q_1 * a_2), q_1 \otimes q_2) *_G d = ((a_1, q_1) \otimes (a_2, q_2)) *_G d$, using the assumption on the objects of A_Q-**Mod** in $(\dagger)$. The fact that $e *_G d = (\bot * d) \vee (e_Q * d) = d$ concludes the proof of $(D, *_G)$ being in $\mathbf{U}A[e]$-**Mod**. Moreover, $\varphi((a, q) *_G d) = (a * \varphi(d)) \vee (q * \varphi(d)) = (a, q) *_G \varphi(d)$ and therefore the assignment $G(D \xrightarrow{\varphi} D') = (D, *_G) \xrightarrow{\varphi} (D', *'_G)$ defines a functor.

For the second functor let $D \xrightarrow{\varphi} D'$ be in $\mathbf{U}A[e]$-**Mod**. Define the A-action on D by $a *_F^A d = (a, \bot) * d$ (cf. the embedding of Lemma 3.7) and get an A-module.

For the Q-action, notice that there is a u-quantale embedding $Q \overset{\phi}{\hookrightarrow} A[e]$, $\phi(q) = (\bot, q)$, and set $q *_F^Q d = (\bot, q) * d$, getting a u-$Q$-module. For the coherence condition, notice that $(a, \bot) \otimes (\bot, q) = (q * a, \bot) = (\bot, q) \otimes (a, \bot)$ implies $a *_F^A (q *_F^Q d) = ((a, \bot) \otimes (\bot, q)) * d = (q * a, \bot) * d = (q * a) *_F^A d$ and similarly $q *_F^Q (a *_F^A d) = (q * a) *_F^A d$, concluding the proof that $(D, *_F^A, *_F^Q)$ is in A_Q-**Mod**. Moreover, $\varphi(a *_F^A d) = \varphi((a, \bot) * d) = (a, \bot) * \varphi(d) = a *_F^A \varphi(d)$ and similarly for the Q-action, i.e., the assignment $F(D \xrightarrow{\varphi} D') = (D, *_F^A, *_F^Q) \xrightarrow{\varphi} (D', *'^A_F, *'^Q_F)$ defines a functor.

Given an A_Q-**Mod**-object D, $FG(D) = (D, *_{FG}^A, *_{FG}^Q)$ implies $a *_{FG}^A d = (a, \bot) *_G d = (a * d) \vee (\bot * d) = a * d$, the case $q *_{FG}^Q d = q * d$ being similar, and therefore $FG = 1_{A_Q\text{-}\mathbf{Mod}}$. On the other hand, given D in $\mathbf{U}A[e]$-**Mod**, $GF(D) = (D, *_{GF})$ implies $(a, q) *_{GF} d = (a *_F^A d) \vee (q *_F^Q d) = ((a, \bot) * d) \vee ((\bot, q) * d) = (a, q) * d$, providing $GF = 1_{\mathbf{U}A[e]\text{-}\mathbf{Mod}}$. $\qquad\square$

Corollary 3.14. *Given a quantale Q, the categories Q-**Mod** and $\mathbf{U}Q[e]$-**Mod** are concretely isomorphic.*

Proof. By Example 3.2, every Q-module D has the unique u-$\mathbf{2}$-module structure, which is preserved by Q-module homomorphisms. For the coherence condition, notice that given $q \in Q$, $c \in \mathbf{2}$ and $d \in D$,

$$q * (c * d) = c * (q * d) = (c * q) * d = \begin{cases} q * d, & c = \top \\ \bot, & c = \bot. \end{cases}$$

It follows that $Q_{\mathbf{2}}$-**Mod** is essentially Q-**Mod** and then Theorem 3.13 provides the promised statement.   □

Meta-mathematically restated, the category Q-**Mod** as an entity is redundant in mathematics and therefore, in future, we will restrict our attention to the case of u-modules only. On the other hand, the reader should be aware that being obsolete from the categorical point of view, modules can potentially have much importance in other perspectives. Everything depends on the long-standing meta-mathematical problem of how deep one can penetrate into the nature of mathematical objects, using methods of category theory. To make the case more exciting, we postulate the following problem.

Problem 3.15. Do we really need non-unital quantale modules in mathematics?

The next theorem, stemming from [112] (cf. also [60, Theorem 2.1.2]), shows the existence of free u-Q-modules, bringing us closer to the promised monad, generating the forward powerset operator.

Theorem 3.16. *The underlying functor of* $\mathbf{U}Q$-**Mod** *has a left adjoint.*

Proof. Given a set X, the set Q^X of all maps $X \xrightarrow{\alpha} Q$ equipped with the point-wise structure is a u-Q-module. There exists a map $X \xrightarrow{\eta_X} |Q^X|$, with $\eta_X(x) = \{x\}$ given by

$$\{x\}(y) = \begin{cases} e, & x = y \\ \bot, & x \neq y. \end{cases}$$

We show that the map provides the required universal arrow. Notice that every $\alpha \in Q^X$ has a (unique) representation in the form $\alpha = \bigvee_{x \in X}(\alpha(x) * \{x\})$. Given a map $X \xrightarrow{f} |A|$, define $Q^X \xrightarrow{\overline{f}} A$ by $\overline{f}(\alpha) = \bigvee_{x \in X}(\alpha(x) * f(x))$. Straightforward computations show that the map is the desired lift.   □

Corollary 3.17. *There exists an adjoint situation* $(\eta, \epsilon) : F \dashv\!\vdash | - | : \mathbf{U}Q\text{-}\mathbf{Mod} \to$ **Set**.

Proof. With the scheme proposed in the proof of Corollary 3.10 in mind, notice that given a map $X \xrightarrow{f} Y$, $F(X \xrightarrow{f} Y) = Q^X \xrightarrow{Ff} Q^Y$, with $Ff(\alpha) = \bigvee_{x \in X}(\alpha(x) * \{f(x)\})$, i.e., $(Ff(\alpha))(y) = \bigvee\{\alpha(x) \mid f(x) = y\}$. Given a u-$Q$-module A, $F|A| \xrightarrow{\epsilon_A} A$ is defined by $\epsilon_A(\alpha) = \bigvee_{a \in A}(\alpha(a) * a)$.   □

The importance of Corollary 3.17 is strikingly manyfold. On one hand, it lifts directly the fixed-basis forward powerset operator of L. A. Zadeh justifying its "correctness" without involving the intricate technique of [86, 88, 90]. On the other hand, the case $Q = \mathbf{2}$ provides the background for the traditional forward powerset operator already mentioned in Introduction. Moreover, with the adjoint situation available, we can obtain the desired result on monadicity of $\mathbf{U}Q$-**Mod** in (at least) two ways, the first one being purely theoretical, whereas the second one relying on the explicit construction. Starting with the easiest first one, for the moment we restrict the notion of *concrete category* to "categories whose objects are sets with

some kind of structure and whose morphisms are structure-preserving mappings, the composition law being the usual composition of functions" [53, p. 16]. To get the intuition for the concept, notice that every variety (in the sense of Definition 2.1) satisfies the just mentioned requirement. The field of topology provides another rich source of examples, e.g., the categories **Top**, **Unif** and **Prox** of topological, uniform and proximity spaces [120] fit the framework as well. For more examples and motivation, the reader is advised to consult [3, Examples 3.3(2)], where the categories in question are called *constructs*. On the other hand, it is important to underline that the more rigid approach of [3, Definition 5.1(2)] does not suit our framework, allowing such highly non-concrete (but still, concretizable) categories as, e.g., the one-element ones $\bullet\ \overset{\curvearrowleft}{}\ id$.

Remark 3.18. From now we will call the just introduced restricted concrete categories by *strictly concrete*. The term is by no means standard and is used only to make referring to the concept easier.

Clearly, the category **UQ-Mod** provides a nice example of the concept. Further, a strictly concrete category is called *equationally presentable* "if its objects can be described by (a proper class of) operations and equations" [53, p. 23]. It is straightforward that **UQ-Mod** falls into the framework. For the statement of the next proposition the reader is referred to [53, p. 23] and for its proof to perusal of [67, Chapter 1].

Proposition 3.19. *Let **C** be an equationally presentable strictly concrete category. If the forgetful functor **C** $\xrightarrow{|-|}$ **Set** has a left adjoint, then **C** is equivalent to a category monadic over **Set**.*

In view of Theorem 3.16, **UQ-Mod** satisfies all the requirements of Proposition 3.19 and therefore enjoys the stated property. To provide the technique, rather important in the subsequent developments, we prove the result explicitly, following the path, proposed in [53] (taken up from [67, Chapter 1]). As an additional advantage, we show that **UQ-Mod** is not just equivalent but, in fact, is isomorphic to a monadic category.

Definition 3.20. The adjoint situation of Corollary 3.17 gives rise to a monad $\mathsf{T} = (T, \eta, \mu)$ on the category **Set** defined by $T = |-|F$, $\mu = |-|\epsilon F$ and called the *Q-powerset monad on* **Set**. For $Q = \mathbf{2}$, T provides the standard powerset monad on **Set** [3, Example 20.2(3)].

Consider the comparison functor **UQ-Mod** $\xrightarrow{K}$ **Set**$^\mathsf{T}$ (**Set**$^\mathsf{T}$ is the respective category of T-algebras [3]) given by $K(A \xrightarrow{\varphi} B) = (|A|, |\epsilon_A|) \xrightarrow{|\varphi|} (|B|, |\epsilon_B|)$. We are going to show that K is a concrete isomorphism. For the sake of convenience, we use a modification of the usual set-theoretical notations, i.e., every subset $S \subseteq X$ is also considered as a map $X \xrightarrow{S} Q$ given by

$$S(x) = \begin{cases} e, & x \in S \\ \bot, & x \notin S \end{cases}$$

(notice that we do not exclude the trivial case of $e = \bot$, i.e., $Q = \mathbf{1}$). In particular, the map corresponding to a finite set $\{x_1, \ldots, x_n\}$ uses the same notation.

Lemma 3.21. *Given a map $TX \xrightarrow{h} X$, (X, h) is a T-algebra iff the following conditions are fulfilled:*

(1) $h(\{x\}) = x$ *for every $x \in X$;*
(2) $h(\bigvee_{\alpha \in TX}(\bar{\beta}(\alpha) * \alpha)) = h(\bigvee_{\alpha \in TX}(\bar{\beta}(\alpha) * \{h(\alpha)\}))$ *for every $\bar{\beta} \in TTX$.*

In case of the category $\mathbf{CSLat}(\bigvee)$, the second condition is equivalent to

$2'$. $h(\bigcup \mathcal{S}) = h(\{h(S) \mid S \in \mathcal{S}\})$ *for every $\mathcal{S} \in \mathcal{P}(\mathcal{P}(X))$.*

Proof. Recall from [3, Definition 20.4] that (X, h) is a T-algebra iff the following diagrams commute:

$$
\begin{array}{ccc}
TTX & \xrightarrow{\mu_X} & TX \\
{\scriptstyle Th}\downarrow & & \downarrow{\scriptstyle h} \\
TX & \xrightarrow{h} & X
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & TX \\
& {\scriptstyle 1_X}\searrow & \downarrow{\scriptstyle h} \\
& & X.
\end{array}
$$

$\square$

Theorem 3.22. *The category $\mathbf{U}Q\text{-}\mathbf{Mod}$ is a monadic construct.*

Proof. We show the existence of a concrete functor $\mathbf{Set}^{\mathsf{T}} \xrightarrow{G} \mathbf{U}Q\text{-}\mathbf{Mod}$, which is inverse to K. Given a T-algebra morphism $(X, h) \xrightarrow{f} (X', h')$, define the required structure on X as follows:

(1) for $x, y \in X$ let $x \leqslant_h y$ iff $h(\{x, y\}) = y$;
(2) for $S \subseteq X$ let $\bigvee^h S = h(S)$;
(3) for $q \in Q$ and $x \in X$ let $q *_h x = h(q * \{x\})$.

Verification that the proposed structure gives a u-Q-module relies heavily on the description of T-algebras of Lemma 3.21. To show that $\leqslant_h$ is a partial order on X, start with the observation that $h(\{x, x\}) = h(\{x\}) = x$, i.e., $x \leqslant_h x$. Moreover, if $x \leqslant_h y$ and $y \leqslant_h x$, then $x = h(\{y, x\}) = h(\{x, y\}) = y$. Lastly, $x \leqslant_h y$ and $y \leqslant_h z$ give $h(\{x, z\}) = h(\{x\} \vee \{z\}) = h(\{h(\{x\})\} \vee \{h(\{y, z\})\}) = h(\{x\} \vee \{y, z\}) = h(\{x, y, z\}) = h(\{x, y\} \vee \{z\}) = h(\{h(\{x, y\})\} \vee \{h(\{z\})\}) = h(\{y\} \vee \{z\}) = h(\{y, z\}) = z$, i.e., $x \leqslant_h z$.

To see that $\bigvee^h$ provides a $\bigvee$-semilattice structure on X, notice that given $s \in S$, $h(\{s, h(S)\}) = h(\{s\} \vee \{h(S)\}) = h(\{h(\{s\})\} \vee \{h(S)\}) = h(\{s\} \vee S) = h(S)$, i.e., $s \leqslant_h \bigvee^h S$. On the other hand, if $s \leqslant_h x$ for every $s \in S$, then $h(\{\bigvee^h S, x\}) = h(\{\bigvee^h S\} \vee \{x\}) = h(\{h(S)\} \vee \{h(\{x\})\}) = h(S \vee \{x\})$. For $S = \varnothing$, $h(S \vee \{x\}) = h(\{x\}) = x$, whereas $S \neq \varnothing$ implies $h(S \vee \{x\}) = h(\bigvee_{s \in S}(\{s\} \vee \{x\})) = h(\bigvee_{s \in S}\{s, x\}) = h(\bigvee_{s \in S}\{h(\{s, x\})\}) = h(\bigvee_{s \in S}\{x\}) = h(\{x\}) = x$, i.e., both cases give $\bigvee^h S \leqslant_h x$.

For the Q-action notice that $q *_h (\bigvee^h S) = h(q * \{h(S)\}) = h(q * S) = h(\bigvee_{s \in S}(q * \{s\})) = h(\bigvee_{s \in S}\{h(q * \{s\})\}) = \bigvee^h_{s \in S}(q *_h s)$ and $(\bigvee P) *_h x = h((\bigvee P) * \{x\}) = h(\bigvee_{p \in P}(p * \{x\})) = h(\bigvee_{p \in P}\{h(p * \{x\})\}) = \bigvee^h_{p \in P}(p *_h x)$. Moreover, $q_1 *_h (q_2 *_h x) = $

$h(q_1 * \{h(q_2 * \{x\})\}) = h(q_1 * (q_2 * \{x\})) = h((q_1 \otimes q_2) * \{x\}) = (q_1 \otimes q_2) *_h x$ and $e_Q *_h x = h(e_Q * \{x\}) = h(\{x\}) = x$, i.e., $(X, \leqslant_h, \bigvee^h, *_h)$ is a u-Q-module.

Since $f(\bigvee^h S) = f \circ h(S) = h' \circ Tf(S) = h' \circ Tf(\bigvee_{s \in S}\{s\}) = h'(\bigvee_{s \in S}\{f(s)\}) = h'(\{f(s) \mid s \in S\}) = \bigvee'^{h'}_{s \in S} f(s)$ and $f(q *_h x) = f \circ h(q * \{x\}) = h' \circ Tf(q * \{x\}) = h'(q * \{f(x)\}) = q *'_{h'} f(x)$, the assignment $G((X, h) \xrightarrow{f} (X', h')) = (X, \bigvee^h, *_h) \xrightarrow{f} (X', \bigvee'^{h'}, *'_{h'})$ defines a concrete functor.

For a u-Q-module A, $GK(A) = (|A|, \leqslant_h, \bigvee^h, *_h)$ implies $\bigvee^h S = \epsilon_A(S) = \bigvee S$ and $q *_h a = \epsilon_A(q * \{a\}) = q * a$, i.e., $GK = 1_{\mathbf{UQ\text{-}Mod}}$. Moreover, for a T-algebra (X, h), $KG(X, h) = (X, |\epsilon_{G(X,h)}|)$ with $\epsilon_{G(X,h)}(\alpha) = \bigvee^h_{x \in X}(\alpha(x) *_h x) = h(\{h(\alpha(x) * \{x\}) \mid x \in X\}) = h(\bigvee_{x \in X}\{h(\alpha(x) * \{x\})\}) = h(\bigvee_{x \in X}(\alpha(x) * \{x\})) = h(\alpha)$, providing $KG = 1_{\mathbf{Set}^\mathsf{T}}$. $\square$

Corollary 3.23. *The category* $\mathbf{CSLat}(\bigvee)$ *is a monadic construct.*

Corollary 3.24. *The category* $\mathbf{UQ\text{-}Mod}$ *is concretely complete, cocomplete, wellpowered, extremally co-wellpowered, and has regular factorizations, lifted uniquely from* $\mathbf{Set}$. *The monomorphisms in* $\mathbf{UQ\text{-}Mod}$ *are precisely the homomorphisms with injective underlying maps.*

Proof. Follows from [3, Proposition 20.34, Corollary 7.38]. $\square$

Theorem 3.22, apart from having some important consequences, provides an answer to the open question of S. E. Rodabaugh [86, Open Question 6.17] on the existence of a monad T on $\mathbf{Set}$, the algebras of which generate the fixed-basis forward powerset operator. Our result clearly shows that the desired algebras are just quantale modules, giving rise to the following meta-mathematical statement.

Meta-Theorem 3.25. *Given a unital quantale* Q, *there exists a monad on* $\mathbf{Set}$, *the algebras of which generate the fixed-basis forward powerset operator* $(-)\overrightarrow{_Q}$ *in the sense of L. A. Zadeh.*

An additional advantage is provided by the fact that the respective powerset operator is presented in the form of a functor and that makes investigating of its properties considerably easier, the point missed in [86, 88] and partly in [90]. On the other hand, a new challenge rises immediately, namely, whether it is possible to obtain a lifting of the backward powerset operator as well. It will be the purpose of the subsequent developments to show the possibility of such a lift. The cornerstone of the approach will be (rather unexpectedly) the well-known notion of *biproduct* in categories, the motivating push being again given by the above-mentioned category of modules over a given quantale.

Having the merit of concrete limits, the category $\mathbf{UQ\text{-}Mod}$ never has concrete colimits. In fact, one can prove a striking (but, alas, extremely nice) property that smashes concreteness completely. Start by recalling from [44, Definition 40.4] the notion of *biproduct* in category. The concept, however being sufficiently simple, requires some preliminary definitions and notational remarks (the reader is supposed to have some knowledge of *abelian categories*, up to the level of [44, Chapter XI] at least).

Definition 3.26. A category **C** is called *pointed* provided that every hom-set $\mathbf{C}(A, B)$ contains a (necessarily unique) zero morphism denoted by 0_{AB} or just 0_A in case of $A = B$.

Example 3.27. Every category which has a zero object is pointed.

Example 3.28. The singleton $\bigvee$-lattice **1** is a zero object in the category $\mathbf{U}Q\text{-}\mathbf{Mod}$ and therefore it is pointed, the constant maps to the bottom element $\bot$ playing the role of zero morphisms.

It is important to keep in mind that there is plenty of pointed categories having no zero object, i.e., the existence condition is sufficient but not necessary. On the other hand, a category is pointed iff it can be fully embedded in a category with a zero object [44, Exercise V.12F(b)].

Notation 3.29. Given a pointed category **C** and a family of **C**-objects $(A_i)_{i \in I}$, for every $j, k \in I$ we let

$$A_j \xrightarrow{\delta_{jk}} A_k = \begin{cases} 1_{A_j}, & j = k \\ 0_{A_j A_k}, & j \neq k. \end{cases}$$

Definition 3.30. Let $(A_i)_{i \in I}$ be a set-indexed family of objects in a pointed category **C**. Then the family $((\mu_i)_{i \in I}, B, (\pi_i)_{i \in I})$ is called a *biproduct of* $(A_i)_{i \in I}$ provided that the following conditions hold:

  (1) $(B, (\pi_i)_{i \in I})$ is a product of $(A_i)_{i \in I}$;
  (2) $((\mu_i)_{i \in I}, B)$ is a coproduct of $(A_i)_{i \in I}$;
  (3) $\pi_k \circ \mu_j = \delta_{jk}$.

The (object-part of a) biproduct will be denoted by $\bigoplus_{i \in I} A_i$.

A pointed category **C** *has (finite) biproducts* provided that every (finite) set-indexed family of **C**-objects has a biproduct.

Lemma 3.31. *The category* $\mathbf{U}Q\text{-}\mathbf{Mod}$ *has biproducts.*

Proof. Suppose $(A_i)_{i \in I}$ is a set-indexed family of u-Q-modules and, moreover, let $\mathcal{P} = (\prod_{i \in I} |A_i|, (\pi_i)_{i \in I})$ be the product of their underlying sets. The point-wise structure on $\prod_{i \in I} |A_i|$ provides a product in the category $\mathbf{U}Q\text{-}\mathbf{Mod}$. Given $j \in I$, define $A_j \xrightarrow{\mu_j} \prod_{i \in I} A_i$ by $\mu_j(a) = (a_i)_{i \in I}$, with

$$a_i = \begin{cases} a, & i = j \\ \bot, & i \neq j, \end{cases}$$

thus getting a sink $\mathcal{C} = ((\mu_i)_{i \in I}, \prod_{i \in I} A_i)$ in $\mathbf{U}Q\text{-}\mathbf{Mod}$. Straightforward computations show that $\mathcal{C}$ is a coproduct of $(A_i)_{i \in I}$ since given another sink $\mathcal{T} = (A_i \xrightarrow{\varphi_i} B)_{i \in I}$ in $\mathbf{U}Q\text{-}\mathbf{Mod}$, the map $\prod_{i \in I} A_i \xrightarrow{\varphi} B$ defined by $\varphi((a_i)_{i \in I}) = \bigvee_{i \in I} \varphi_i(a_i)$ is the unique $\mathbf{U}Q\text{-}\mathbf{Mod}$-morphism such that $\varphi \circ \mathcal{C} = \mathcal{T}$. Verification of the last condition of Definition 3.30 is left to the reader. $\square$

Corollary 3.32. *The category* $\mathbf{CSLat}(\bigvee)$ *has biproducts.*

In order to continue, we need an additional property of biproducts, which provides an extremely useful machinery for incorporating in the notion of the enrichment of a given category with an extra structure.

4. **Quantaloids**

It is a well-known fact that the category R-**Mod** of (left) R-modules over a given ring R has the merit of its hom-sets being equipped with the structure of an abelian group, with the composition of morphisms acting distributively from the left and from the right. On the other hand, in this category finite products coincide with finite coproducts, providing finite biproducts. Strikingly enough, it can be shown that these two seemingly unrelated properties are linked. More explicitly, a pointed category $\mathbf{C}$ with finite products has finite biproducts iff there is a (unique) semiadditive structure on $\mathbf{C}$ [44, Theorem 40.13]. By Lemma 3.31 of the current paper we know that the category $\mathbf{U}Q$-**Mod** has not only finite but all set-indexed biproducts. A category-minded reader can ask whether the existence of additional biproducts gives any extra property to the semiadditive structure in question. It is the main purpose of this section to clarify the matter.

We begin by introducing enriched categories, suitable in dealing with arbitrary biproducts. The concept is actually a well-known one, having taken its proper place in mathematics quite a long time ago.

Definition 4.1. A *quantaloid* is a category $\mathbf{Q}$ such that:

(1) for every $\mathbf{Q}$-objects A and B, the hom-set $\mathbf{Q}(A, B)$ is a $\bigvee$-semilattice;
(2) composition of morphisms in $\mathbf{Q}$ preserves $\bigvee$ in both variables.

In the language of enriched category theory [55], quantaloids are precisely the categories enriched in the category $\mathbf{CSLat}(\bigvee)$. To get the intuition for the new concept, consider the following examples.

Example 4.2. A quantaloid with one object is just a u-quantale. Thus, quantaloids can be thought of as quantales "with many objects".

Example 4.3. The category $\mathbf{U}Q$-**Mod**, with the point-wise structure on its hom-sets, is a (large) quantaloid. On the other hand, none of the categories **Frm**, **Quant** and **SQuant**, enriched in a similar manner, produces a quantaloid (for **Frm** consider, e.g., the empty set $\mathbf{Frm}(\mathbf{1}, \mathbf{2})$).

The next important lemma establishes a link between quantaloids and pointed categories, paving the way for considering biproducts in the new setting. The result improves that of J. Rosický [96, p. 330], which postulates the same claim under the unnecessary requirement of the existence of a zero object.

Lemma 4.4. *Every quantaloid is a pointed category, where the zero morphisms are the bottom elements of the respective* hom-*sets.*

Proof. Given a quantaloid $\mathbf{Q}$, consider the bottom element $\perp_{AB}$ of a hom-set $\mathbf{Q}(A, B)$. It will be enough to show that $\perp_{AB}$ is a zero morphism. Given $\mathbf{Q}$-morphisms $C \underset{g}{\overset{f}{\rightrightarrows}} A$, $\perp_{AB} \circ f = (\bigvee \varnothing) \circ f = \bigvee \varnothing = \perp_{CB} = \perp_{AB} \circ g$, proving that $\perp_{AB}$ is constant. Similarly one shows that $\perp_{AB}$ is a coconstant morphism. $\square$

The reader should be aware that the notion of quantaloid was introduced by K. I. Rosenthal in [93] (the concept was also studied by A. Pitts in [75] under the name of **SL**-*category*). It appeared that the concept has applications in different areas of theoretical computer science. In particular, S. Abramsky and S. Vickers [2] use quantaloids to introduce the notion of typing on processes; R. Betti and S. Kasangian [11] indicate how categories enriched in a certain quantaloid provide an appropriate categorical framework for considering tree automata; J. Rosický [96] shows that the new semantics for concurrent computation, called *interaction categories* [1], can be naturally described in the language of quantaloids (for an up to date state of the theory of quantaloids the reader is referred to the treatise of K. I. Rosenthal [94]). Moreover, the notion fits perfectly into our current framework. We begin with a couple of helpful results, the proofs of which can be easily extended to the set-indexed framework from the finite one given in [44].

Proposition 4.5. *Given a quantaloid* **Q** *and a set-indexed family* $(A_i \xrightarrow{\mu_i} B \xrightarrow{\pi_i} A_i)_{i \in I}$ *of* **Q**-*morphisms, the following conditions are equivalent:*

(1) $((\mu_i)_{i \in I}, B, (\pi_i)_{i \in I})$ *is a biproduct of* $(A_i)_{i \in I}$;

(2) $(B, (\pi_i)_{i \in I})$ *is a product of* $(A_i)_{i \in I}$ *and for every* $i, j \in I$, $\pi_j \circ \mu_i = \delta_{ij}$;

(3) $((\mu_i)_{i \in I}, B)$ *is a coproduct of* $(A_i)_{i \in I}$ *and for every* $i, j \in I$, $\pi_j \circ \mu_i = \delta_{ij}$;

(4) $\bigvee_{i \in I}(\mu_i \circ \pi_i) = 1_B$ *and for every* $i, j \in I$, $\pi_j \circ \mu_i = \delta_{ij}$.

Proof. It will be enough to show that (2) and (4) are equivalent.

Ad (2) $\Rightarrow$ (4). Since composition is $\bigvee$-distributive, $\pi_j \circ \bigvee_{i \in I}(\mu_i \circ \pi_i) = \bigvee_{i \in I}(\pi_j \circ \mu_i \circ \pi_i) = \bigvee_{i \in I}(\delta_{ij} \circ \pi_i) = \pi_j = \pi_j \circ 1_B$ for every $j \in I$, and therefore $\bigvee_{i \in I}(\mu_i \circ \pi_i) = 1_B$.

Ad (4) $\Rightarrow$ (2). Given a **Q**-source $\mathcal{S} = (C \xrightarrow{\varphi_i} A_i)_{i \in I}$, define $C \xrightarrow{\varphi} B = C \xrightarrow{\bigvee_{i \in I}(\mu_i \circ \varphi_i)} B$. Then $\pi_j \circ \varphi = \pi_j \circ \bigvee_{i \in I}(\mu_i \circ \varphi_i) = \bigvee_{i \in I}(\pi_j \circ \mu_i \circ \varphi_i) = \bigvee_{i \in I}(\delta_{ij} \circ \varphi_i) = \varphi_j$ for every $j \in I$. Given another **Q**-morphism $C \xrightarrow{\psi} B$ with the same property, $\psi = 1_B \circ \psi = (\bigvee_{i \in I}(\mu_i \circ \pi_i)) \circ \psi = \bigvee_{i \in I}(\mu_i \circ \pi_i \circ \psi) = \bigvee_{i \in I}(\mu_i \circ \varphi_i) = \varphi$. $\square$

Proposition 4.6. *Given a quantaloid* **Q** *and a* **Q**-*product* $(B, (\pi_i)_{i \in I})$ *of a set-indexed family* $(A_i)_{i \in I}$, *the pair* $(B, (\pi_i)_{i \in I})$ *can be uniquely completed to a biproduct* $((\mu_i)_{i \in I}, B, (\pi_i)_{i \in I})$ *of* $(A_i)_{i \in I}$.

Proof. Given $j \in I$, there exists a unique **Q**-morphism $A_j \xrightarrow{\mu_j} B$, defined by $\pi_i \circ \mu_j = \delta_{ji}$ for every $i \in I$. By Proposition 4.5(2), $(\mu_i)_{i \in I}$ is the unique family, making $((\mu_i)_{i \in I}, B, (\pi_i)_{i \in I})$ a biproduct of $(A_i)_{i \in I}$. $\square$

Corollary 4.7. *For every quantaloid* **Q**, *the following are equivalent:*

(1) **Q** *has products;*

(2) **Q** *has coproducts;*

(3) **Q** *has biproducts.*

The next result (Theorem 4.9), being the main achievement of the section, provides a generalization of [44, Proposition 40.12], replacing finite biproducts with the set-indexed ones. For convenience of the reader, we recall from [44] some notational conventions, namely, given a pointed category **C** with biproducts and a

subset $S \subseteq \mathbf{C}(A, B)$, there exist $\mathbf{C}$-morphisms (the dashed ones), defined by commutativity of the following diagrams:

We also require an additional lemma, which is based on the respective result of [44, Lemma 40.11].

Lemma 4.8. *Given a pointed category $\mathbf{C}$, if the triples $((\mu_i^A)_{i \in I}, A, (\pi_i^A)_{i \in I})$ and $((\mu_j^B)_{j \in J}, B, (\pi_j^B)_{j \in J})$ are biproducts of the families $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ respectively, and $(A_i \xrightarrow{f_{ij}} B_j)_{i \in I, j \in J}$ is a family of $\mathbf{C}$-morphism, then $\bigoplus_{i \in I} A_i \xrightarrow{[\langle f_{ij} \rangle_{j \in J}]_{i \in I}} \bigoplus_{j \in J} B_j = \bigoplus_{i \in I} A_i \xrightarrow{\langle [f_{ij}]_{i \in I} \rangle_{j \in J}} \bigoplus_{j \in J} B_j$.*

Proof. Notice that $\pi_k^B \circ [\langle f_{ij} \rangle_{j \in J}]_{i \in I} \circ \mu_l^A = \pi_k^B \circ \langle f_{lj} \rangle_{j \in J} = f_{lk} = [f_{ik}]_{i \in I} \circ \mu_l^A = \pi_k^B \circ \langle [f_{ij}]_{i \in I} \rangle_{j \in J} \circ \mu_l^A$ for every $l \in I$, $k \in J$. $\square$

Theorem 4.9. *Every pointed category $\mathbf{C}$ with biproducts has a unique quantaloid structure, given by each of the following:*

$$\text{if } S \subseteq \mathbf{C}(A, B), \text{ then } A \xrightarrow{\bigvee S} B = \begin{cases} A \xrightarrow{\Delta} \bigoplus_{s \in S} A \xrightarrow{[s]_{s \in S}} B = \\ A \xrightarrow{\langle s \rangle_{s \in S}} \bigoplus_{s \in S} B \xrightarrow{\nabla} B = \\ A \xrightarrow{\Delta} \bigoplus_{s \in S} A \xrightarrow{\bigoplus_{s \in S} s} \bigoplus_{s \in S} B \xrightarrow{\nabla} B. \end{cases}$$

Proof. The proof follows the path of the respective one of [44, Proposition 40.12]. First we show uniqueness, i.e., if $\mathbf{C}$ is a quantaloid, then each of the above formulas provides its quantaloid structure. From distributivity of composition over $\bigvee$ and the characterization of biproducts in Proposition 4.5(4), we have:

$$[s]_{s \in S} \circ \Delta = [s]_{s \in S} \circ 1_{\bigoplus_{s \in S} A} \circ \Delta = [s]_{s \in S} \circ \bigvee_{t \in S} (\mu_t^A \circ \pi_t^A) \circ \Delta =$$

$$\bigvee_{t \in S} ([s]_{s \in S} \circ \mu_t^A \circ \pi_t^A \circ \Delta) = \bigvee S,$$

$$\nabla \circ \langle s \rangle_{s \in S} = \nabla \circ 1_{\bigoplus_{s \in S} B} \circ \langle s \rangle_{s \in S} = \nabla \circ \bigvee_{t \in S} (\mu_t^B \circ \pi_t^B) \circ \langle s \rangle_{s \in S} =$$

$$\bigvee_{t \in S} (\nabla \circ \mu_t^B \circ \pi_t^B \circ \langle s \rangle_{s \in S}) = \bigvee S,$$

$$\nabla \circ (\bigoplus_{s \in S} s) \circ \Delta = \nabla \circ \bigvee_{t \in S} (\mu_t^B \circ \pi_t^B) \circ (\bigoplus_{s \in S} s) \circ \bigvee_{r \in S} (\mu_r^A \circ \pi_r^A) \circ \Delta =$$

$$\bigvee_{t \in S} \bigvee_{r \in S} (\nabla \circ \mu_t^B \circ \pi_t^B \circ (\bigoplus_{s \in S} s) \circ \mu_r^A \circ \pi_r^A \circ \Delta) = \bigvee_{t \in S} \bigvee_{r \in S} (\pi_t^B \circ (\bigoplus_{s \in S} s) \circ \mu_r^A) = \bigvee S.$$

Secondly, we show that each of the above-mentioned equalities does define a quantaloid structure on $\mathbf{C}$. Keeping the characterization of $\bigvee$-semilattices of Lemma 3.21 in mind, we prove that if Θ is any of the three aforesaid operations, then

(1) $\Theta(\{\varphi\}) = \varphi$ for every $\varphi \in \mathbf{C}(A, B)$;
(2) $\Theta(\bigcup \mathcal{S}) = \Theta\{\Theta(S) \,|\, S \in \mathcal{S}\}$ for every $\mathcal{S} \in \mathcal{P}(\mathcal{P}(\mathbf{C}(A, B)))$.

For the sake of convenience, denote the first two of them by $\bigvee$ and $\bigsqcup$ respectively. Item (1) is then immediate: $A \xrightarrow{\bigvee\{\varphi\}} B = A \xrightarrow{1_A} A \xrightarrow{\varphi} B = A \xrightarrow{\varphi} B = A \xrightarrow{\varphi} B \xrightarrow{1_B} B = A \xrightarrow{\bigsqcup\{\varphi\}} B$. To check Item (2), an additional property of the operations in question is required, i.e., neutrality of 0_{AB} w.r.t. both $\bigvee$ and $\bigsqcup$.

Suppose $0_{AB} \notin S \subseteq \mathbf{C}(A, B)$ and let $T = S \bigcup \{0_{AB}\}$. To show $\bigvee S = \bigvee T$, define a $\mathbf{C}$-morphism $\bigoplus_{t \in T} A \xrightarrow{\varphi} \bigoplus_{s \in S} A$ by $\pi_s^S \circ \varphi = \pi_s^T$ for every $s \in S$. Then given $r \in S$, $\pi_s^S \circ \mu_r^S = \delta_{rs} = \pi_s^T \circ \mu_r^T = \pi_s^S \circ \varphi \circ \mu_r^T$ for every $s \in S$, implies $\mu_r^S = \varphi \circ \mu_r^T$. On the other hand, $\pi_s^S \circ 0_{A \bigoplus_{s \in S} A} = 0_A = \pi_s^T \circ \mu_{0_{AB}}^T = \pi_s^S \circ \varphi \circ \mu_{0_{AB}}^T$ for every $s \in S$, gives $0_{A \bigoplus_{s \in S} A} = \varphi \circ \mu_{0_{AB}}^T$. With Δ' standing for the $\mathbf{C}$-morphism $A \xrightarrow{\Delta'} \bigoplus_{t \in T} A$, it follows that $\pi_s^S \circ \varphi \circ \Delta' = \pi_s^T \circ \Delta' = 1_A = \pi_s^S \circ \Delta$ for every $s \in S$, proving $\varphi \circ \Delta' = \Delta$. Moreover, $[s]_{s \in S} \circ \varphi \circ \mu_r^T = [s]_{s \in S} \circ \mu_r^S = r = [t]_{t \in T} \circ \mu_r^T$ for $r \in S$ and $[s]_{s \in S} \circ \varphi \circ \mu_{0_{AB}}^T = [s]_{s \in S} \circ 0_{A \bigoplus_{s \in S} A} = 0_{AB} = [t]_{t \in T} \circ \mu_{0_{AB}}^T$, imply $[s]_{s \in S} \circ \varphi = [t]_{t \in T}$. Lastly, $\bigvee S = [s]_{s \in S} \circ \Delta = [s]_{s \in S} \circ \varphi \circ \Delta' = [t]_{t \in T} \circ \Delta' = \bigvee T$. To show $\bigsqcup S = \bigsqcup T$, define a $\mathbf{C}$-morphism $\bigoplus_{s \in S} B \xrightarrow{\psi} \bigoplus_{t \in T} B$ by $\psi \circ \mu_s^S = \mu_s^T$ for every $s \in S$, and dualize the above path.

Suppose now we have some $\mathcal{S} \in \mathcal{P}(\mathcal{P}(\mathbf{C}(A, B)))$, with the additional property of $0_{AB} \notin \bigcup \mathcal{S}$. Given $s \in S \in \mathcal{S}$, define $\mathcal{S} \xrightarrow{f_s^S} \mathbf{C}(A, B)$ by

$$f_s^S(T) = \begin{cases} s, & T = S \\ 0_{AB}, & T \neq S \end{cases}$$

and let $F = \{f_s^S \,|\, s \in S \in \mathcal{S}\}$. We show that $\bigsqcup_{S \in \mathcal{S}}(\bigvee S) = \bigvee_{f \in F}(\bigsqcup_{S \in \mathcal{S}} f(S))$. With Δ_S standing for the $\mathbf{C}$-morphism $A \xrightarrow{\Delta_S} \bigoplus_{s \in S} A$, we obtain $\bigsqcup_{S \in \mathcal{S}}(\bigvee S) = \nabla \circ \langle [s]_{s \in S} \circ \Delta_S \rangle_{S \in \mathcal{S}}$ and $\bigvee_{f \in F}(\bigsqcup_{S \in \mathcal{S}} f(S)) = [\bigsqcup_{S \in \mathcal{S}} f(S)]_{f \in F} \circ \Delta = [\nabla \circ \langle f(S) \rangle_{S \in \mathcal{S}}]_{f \in F} \circ \Delta = \nabla \circ [\langle f(S) \rangle_{S \in \mathcal{S}}]_{f \in F} \circ \Delta = \nabla \circ \langle [f(S)]_{f \in F} \rangle_{S \in \mathcal{S}} \circ \Delta = \nabla \circ \langle [f(S)]_{f \in F} \circ \Delta \rangle_{S \in \mathcal{S}}$ by Lemma 4.8. It is enough to get $[s]_{s \in S} \circ \Delta_S = [f(S)]_{f \in F} \circ \Delta$ for every $S \in \mathcal{S}$.

Define a $\mathbf{C}$-morphism $\bigoplus_{f \in F} A \xrightarrow{\varphi} \bigoplus_{s \in S} A$ by $\pi_s^S \circ \varphi = \pi_{f_s^S}^F$ for every $s \in S$. Moreover, given $f \in F$ let $A \xrightarrow{\psi_f} \bigoplus_{s \in S} A$ be defined as

$$\psi_f = \begin{cases} 0_{A \bigoplus_{s \in S} A}, & f(S) = 0_{AB} \\ \mu_{f(S)}^S, & f(S) \neq 0_{AB}. \end{cases}$$

Given $s \in S$,

$$\pi_s^S \circ \varphi \circ \mu_f^F = \pi_{f_s^S}^F \circ \mu_f^F = \begin{cases} 1_A, & f = f_s^S \\ 0_A, & f \neq f_s^S \end{cases} = \begin{cases} 1_A, & f(S) = s \neq 0_{AB} \\ 0_A, & \text{otherwise} \end{cases} =$$

$$\begin{cases} \pi_s^S \circ \mu_{f(S)}^S, & f(S) \neq 0_{AB} \\ 0_A, & f(S) = 0_{AB} \end{cases} = \pi_s^S \circ \psi_f$$

and therefore $\varphi \circ \mu_f^F = \psi_f$. It follows that $\pi_s^S \circ \varphi \circ \Delta = \pi_{f_s^S}^F \circ \Delta = 1_A = \pi_s^S \circ \Delta_S$ for every $s \in S$, implying $\varphi \circ \Delta = \Delta_S$. On the other hand,

$$[s]_{s \in S} \circ \varphi \circ \mu_f^F = [s]_{s \in S} \circ \psi_f = \begin{cases} [s]_{s \in S} \circ \mu_{f(S)}^S, & f(S) \neq 0_{AB} \\ 0_{AB}, & f(S) = 0_{AB} \end{cases} =$$

$$\begin{cases} f(S), & f(S) \neq 0_{AB} \\ 0_{AB}, & f(S) = 0_{AB} \end{cases} = f(S) = [f(S)]_{f \in F} \circ \mu_f^F$$

for every $f \in F$, implying $[s]_{s \in S} \circ \varphi = [f(S)]_{f \in F}$. On the last step, $[s]_{s \in S} \circ \Delta_S = [s]_{s \in S} \circ \varphi \circ \Delta = [f(S)]_{f \in F} \circ \Delta$.

Suppose now we are given an arbitrary $\mathcal{S} \in \mathcal{P}(\mathcal{P}(\mathbf{C}(A, B)))$. Since 0_{AB} influences neither $\bigvee$ nor $\bigsqcup$, $\bigsqcup_{S \in \mathcal{S}}(\bigvee S) = \bigsqcup_{S \in \mathcal{S}}(\bigvee S^0) = \bigvee_{f \in F}(\bigsqcup_{S \in \mathcal{S}} f(S^0))$ with $S^0 = S \backslash \{0_{AB}\}$. In particular, for every $T \subseteq \mathbf{C}(A, B)$, $\bigsqcup T = \bigsqcup_{t \in T} \bigvee \{t\} = \bigvee_{f \in F} \bigsqcup_{t \in T} f(\{t\}^0) = \bigvee_{r \in T^0} \bigsqcup_{t \in T} f_r^{\{r\}}(\{t\}^0) = \bigvee_{r \in T^0} \bigsqcup \{r\} = \bigvee T^0 = \bigvee T$. It now follows that $\bigvee \{\bigvee S \mid S \in \mathcal{S}\} = \bigvee_{f \in F}(\bigvee_{S \in \mathcal{S}} f(S^0)) = \bigvee \{\bigvee_{S \in \mathcal{S}} f_t^T(S^0) \mid 0_{AB} \neq t \in T \in \mathcal{S}\} = \bigvee \{\bigvee \{t\} \mid 0_{AB} \neq t \in T \in \mathcal{S}\} = \bigvee \{t \mid 0_{AB} \neq t \in T \in \mathcal{S}\} = \bigvee (\bigcup \mathcal{S})^0 = \bigvee \bigcup \mathcal{S}$, proving the desired property.

Having shown that $(\mathbf{C}(A, B), \bigvee)$ is a $\bigvee$-semilattice, the only thing left for us to verify is distributivity over morphism composition and that is quite easy. Given $S \subseteq \mathbf{C}(A, B)$, $\varphi \in \mathbf{C}(D, A)$ and $\psi \in \mathbf{C}(B, C)$, $\psi \circ \bigvee S = \psi \circ [s]_{s \in S} \circ \Delta = [\psi \circ s]_{s \in S} \circ \Delta = \bigvee_{s \in S}(\psi \circ s)$ as well as $(\bigvee S) \circ \varphi = (\bigsqcup S) \circ \varphi = \nabla \circ \langle s \rangle_{s \in S} \circ \varphi = \nabla \circ \langle s \circ \varphi \rangle_{s \in S} = \bigsqcup_{s \in S}(s \circ \varphi) = \bigvee_{s \in S}(s \circ \varphi)$. $\qquad \square$

Theorem 4.9 provides some important consequences, one of them generalizing [44, Theorem 40.13].

Corollary 4.10. *For every pointed category* $\mathbf{C}$ *with products or coproducts, the following are equivalent:*

(1) *there exists a quantaloid structure on* $\mathbf{C}$*;*
(2) *there exists a unique quantaloid structure on* $\mathbf{C}$*;*
(3) $\mathbf{C}$ *has biproducts.*

Corollary 4.11. *For every pointed variety* $\mathbf{A}$*, the following are equivalent:*

(1) *there exists a quantaloid structure on* $\mathbf{A}$*;*
(2) *there exists a unique quantaloid structure on* $\mathbf{A}$*;*
(3) $\mathbf{A}$ *has biproducts.*

Briefly speaking, every pointed category with biproducts is a quantaloid and that brings the enormous advantage of encoding the highly non-categorical $\bigvee$-semilattice

structure on hom-sets with the purely categorical concept of biproduct. On the other hand, a question rises immediately, whether quantaloids are precisely the pointed categories with biproducts. It appears, that this is not always the case, J. Rosický showing in [96, Proposition 1] the following result.

Definition 4.12. A *quantaloid homomorphism* is a functor $\mathbf{Q} \xrightarrow{F} \mathbf{Q}'$ such that on hom-sets it induces a $\bigvee$-semilattice homomorphism $\mathbf{Q}(A, B) \to \mathbf{Q}'(F(A), F(B))$. **Qtlds** is the (quasi)category of quantaloids and their homomorphisms. **BQtlds** is the full subcategory of **Qtlds** of all quantaloids with biproducts.

Proposition 4.13. BQtlds *is a reflective subcategory of* **Qtlds**.

Proof. The reflection is given by *quantaloids of matrices* [94, Proposition 2.3.2]. □

It follows that the existence of biproducts provides you with something more than just the structure of $\bigvee$-semilattice. Interestingly enough, this extra property is precisely the point (missed in [86, 88, 90]) which clarifies the essence of powerset operators. The next two sections will investigate the matter thoroughly.

5. Functors Induced by Biproducts

In the last section we proved that biproducts provide a very convenient way of encoding the quantaloid structure of a given category. In this section we go even further, showing that they are capable of producing two functors which lift the powerset operators. The new functorial description, apart from bringing into light their real essence, makes it much easier to investigate their properties. To be in line with the results, currently accepted in the fuzzy community under the influence of [86, 87, 88, 90], we shape the new functors according to a rather specific pattern.

Lemma 5.1. *Given a pointed category* $\mathbf{D}$ *with biproducts, every subcategory* $\mathbf{C}$ *of* $\mathbf{D}$ *provides two functors:*

(1) $\mathbf{Set} \times \mathbf{C} \xrightarrow{(-)^{\rightarrow}} \mathbf{D}$ *defined by* $((X, A) \xrightarrow{(f,\varphi)} (Y, B))^{\rightarrow} = \bigoplus_{x \in X} A \xrightarrow{(f,\varphi)^{\rightarrow}} \bigoplus_{y \in Y} B$, *with* $(f, \varphi)^{\rightarrow}$ *given by commutativity for every* $x \in X$ *of the diagram*

$$
\begin{array}{ccc}
A & \xrightarrow{\mu_x^X} & \bigoplus_{x \in X} A \\
{\scriptstyle \varphi} \downarrow & & \downarrow {\scriptstyle (f,\varphi)^{\rightarrow}} \\
B & \xrightarrow[\mu_{f(x)}^Y]{} & \bigoplus_{y \in Y} B.
\end{array}
$$

For every $y \in Y$, $\pi_y^Y \circ (f, \varphi)^{\rightarrow} = \varphi \circ (\bigvee\{\pi_x^X \mid f(x) = y\})$.

(2) $\mathbf{Set} \times \mathbf{C}^{op} \xrightarrow{(-)^{\leftarrow}} \mathbf{D}^{op}$ *defined by* $((X, A) \xrightarrow{(f,\varphi)} (Y, B))^{\leftarrow} = \bigoplus_{y \in Y} B \xrightarrow{(f,\varphi)^{\leftarrow}} \bigoplus_{x \in X} A$, *with* $(f, \varphi)^{\leftarrow}$ *given by commutativity for every* $x \in X$ *of the*

diagram

$$\begin{array}{ccc} \bigoplus_{y\in Y} B & \xrightarrow{\pi^Y_{f(x)}} & B \\ {\scriptstyle(f,\varphi)^{\leftarrow}}\big\downarrow & & \big\downarrow{\scriptstyle\varphi^{op}} \\ \bigoplus_{x\in X} A & \xrightarrow{\pi^X_x} & A. \end{array}$$

For every $y \in Y$, $(f,\varphi)^{\leftarrow} \circ \mu^Y_y = (\bigvee\{\mu^X_x \mid f(x)=y\}) \circ \varphi^{op}$.

Moreover, for every pair $A \overset{\varphi}{\underset{\psi^{op}}{\rightleftarrows}} B$ of $\mathbf{D}$-morphisms, equivalent are:

    I. $\varphi \circ \psi^{op} \leqslant 1_B$ *and* $1_A \leqslant \psi^{op} \circ \varphi$;

    II. $(f,\varphi)^{\rightarrow} \circ (f,\psi)^{\leftarrow} \leqslant 1_{\bigoplus_{y\in Y} B}$ *and* $1_{\bigoplus_{x\in X} A} \leqslant (f,\psi)^{\leftarrow} \circ (f,\varphi)^{\rightarrow}$ *for every map $X \overset{f}{\to} Y$.*

If $(X,A) \xrightarrow{(f,\varphi)} (Y,B)$ is a morphism in $\mathbf{Set} \times \mathbf{C}^{op}$ (resp. $\mathbf{Set} \times \mathbf{C}$) with $X \overset{f}{\to} Y$ a bijection, then $(f,\varphi)^{\leftarrow} = (f^{-1},\varphi^{op})^{\rightarrow}$ (resp. $(f,\varphi)^{\rightarrow} = (f^{-1},\varphi^{op})^{\leftarrow}$).

Proof. Notice that according to Corollary 4.10, $\mathbf{D}$ is an *ordered category*, i.e., its hom-sets are partially ordered sets, with composition of morphisms being monotone.

Ad (1)&(2). To verify that the first correspondence $(-)^{\rightarrow}$ defines a functor, it will be enough to check preservation of composition. Given $\mathbf{Set} \times \mathbf{C}$-morphisms $(X,A) \xrightarrow{(f,\varphi)} (Y,B) \xrightarrow{(g,\psi)} (Z,C)$, commutativity for every $x \in X$ of the diagram

implies $((g,\psi) \circ (f,\varphi))^{\rightarrow} = (g,\psi)^{\rightarrow} \circ (f,\varphi)^{\rightarrow}$. Moreover, given $y \in Y$,

$$\pi^Y_y \circ (f,\varphi)^{\rightarrow} = \pi^Y_y \circ (f,\varphi)^{\rightarrow} \circ (\bigvee_{x\in X}(\mu^X_x \circ \pi^X_x)) = \bigvee_{x\in X}(\pi^Y_y \circ (f,\varphi)^{\rightarrow} \circ \mu^X_x \circ \pi^X_x) =$$

$$\bigvee_{x\in X}(\pi^Y_y \circ \mu^Y_{f(x)} \circ \varphi \circ \pi^X_x) = \begin{cases} \bigvee\{\varphi \circ \pi^X_x \mid f(x)=y\}, & y \in \{f(x) \mid x \in X\} \\ 0_{(\bigoplus_{x\in X} A)B}, & y \notin \{f(x) \mid x \in X\} \end{cases} =$$

$$\varphi \circ (\bigvee\{\pi^X_x \mid f(x)=y\}).$$

The case of $(-)^{\leftarrow}$ can be obtained dually.

Ad (I) $\Rightarrow$ (II). Straightforward calculations show that $(f,\varphi)^{\rightarrow} \circ (f,\psi)^{\leftarrow} = (f,\varphi)^{\rightarrow} \circ (\bigvee_{x\in X}\mu^X_x \circ \pi^X_x) \circ (f,\psi)^{\leftarrow} = \bigvee_{x\in X}((f,\varphi)^{\rightarrow} \circ \mu^X_x \circ \pi^X_x \circ (f,\psi)^{\leftarrow}) = \bigvee_{x\in X}(\mu^Y_{f(x)} \circ \varphi \circ \psi^{op} \circ \pi^Y_{f(x)}) \leqslant \bigvee_{x\in X}(\mu^Y_{f(x)} \circ \pi^Y_{f(x)}) \leqslant \bigvee_{y\in Y}(\mu^Y_y \circ \pi^Y_y) = 1_{\bigoplus_{y\in Y} B}$ and $(f,\psi)^{\leftarrow} \circ$

$(f, \varphi)^{\rightarrow} = (\bigvee_{x \in X}(\mu_x^X \circ \pi_x^X)) \circ (f, \psi)^{\leftarrow} \circ (f, \varphi)^{\rightarrow} \circ (\bigvee_{z \in X}(\mu_z^X \circ \pi_z^X)) = \bigvee_{x \in X} \bigvee_{z \in X}(\mu_x^X \circ \pi_x^X \circ (f, \psi)^{\leftarrow} \circ (f, \varphi)^{\rightarrow} \circ \mu_z^X \circ \pi_z^X) \geqslant \bigvee_{x \in X}(\mu_x^X \circ \pi_x^X \circ (f, \psi)^{\leftarrow} \circ (f, \varphi)^{\rightarrow} \circ \mu_x^X \circ \pi_x^X) = \bigvee_{x \in X}(\mu_x^X \circ \psi^{op} \circ \pi_{f(x)}^Y \circ \mu_{f(x)}^Y \circ \varphi \circ \pi_x^X) = \bigvee_{x \in X}(\mu_x^X \circ \psi^{op} \circ \varphi \circ \pi_x^X) \geqslant \bigvee_{x \in X}(\mu_x^X \circ \pi_x^X) = 1_{\bigoplus_{x \in X} A}.$

Ad (II) $\Rightarrow$ (I). For the unique map $\mathbf{1} \xrightarrow{!} \mathbf{1}$, $(!, \varphi)^{\rightarrow} = \varphi$ and $(!, \psi)^{\leftarrow} = \psi^{op}$.

The last statement follows from the fact that $\pi_x^X \circ (f, \varphi)^{\leftarrow} = \varphi^{op} \circ \pi_{f(x)}^Y = \varphi^{op} \circ (\bigvee\{\pi_y^Y \mid f^{-1}(y) = x\}) = \pi_x^X \circ (f^{-1}, \varphi^{op})^{\rightarrow}$ for every $x \in X$, implies $(f, \varphi)^{\leftarrow} = (f^{-1}, \varphi^{op})^{\rightarrow}$. $\qquad\square$

Corollary 5.2. *Given a pointed category* $\mathbf{D}$ *with biproducts, every* $\mathbf{D}$-*object* A *provides two functors:*

(1) **Set** $\xrightarrow{(-)_A^{\rightarrow}} \mathbf{D}$ *defined by* $(X \xrightarrow{f} Y)_A^{\rightarrow} = \bigoplus_{x \in X} A \xrightarrow{f_A^{\rightarrow}} \bigoplus_{y \in Y} A$, *with* $f_A^{\rightarrow}$ *given by commutativity for every* $x \in X$ *of the diagram*

$$
\begin{array}{ccc}
A & \xrightarrow{\mu_x^X} & \bigoplus_{x \in X} A \\
& \searrow^{\mu_{f(x)}^Y} & \downarrow^{f_A^{\rightarrow}} \\
& & \bigoplus_{y \in Y} A.
\end{array}
$$

For every $y \in Y$, $\pi_y^Y \circ f_A^{\rightarrow} = \bigvee\{\pi_x^X \mid f(x) = y\}$.

(2) **Set** $\xrightarrow{(-)_A^{\leftarrow}} \mathbf{D}^{op}$ *defined by* $(X \xrightarrow{f} Y)_A^{\leftarrow} = \bigoplus_{y \in Y} A \xrightarrow{f_A^{\leftarrow}} \bigoplus_{x \in X} A$, *with* $f_A^{\leftarrow}$ *given by commutativity for every* $x \in X$ *of the diagram*

$$
\begin{array}{ccc}
\bigoplus_{y \in Y} A & & \\
\downarrow^{f_A^{\leftarrow}} & \searrow^{\pi_{f(x)}^Y} & \\
\bigoplus_{x \in X} A & \xrightarrow{\pi_x^X} & A.
\end{array}
$$

For every $y \in Y$, $f_A^{\leftarrow} \circ \mu_y^Y = \bigvee\{\mu_x^X \mid f(x) = y\}$.

Moreover, $f_A^{\rightarrow} \circ f_A^{\leftarrow} \leqslant 1_{\bigoplus_{y \in Y} A}$ *and* $1_{\bigoplus_{x \in X} A} \leqslant f_A^{\leftarrow} \circ f_A^{\rightarrow}$ *for every map* $X \xrightarrow{f} Y$. *Given a bijective map* $X \xrightarrow{f} Y$, $(f^{-1})_A^{\leftarrow} = f_A^{\rightarrow}$ *and* $(f^{-1})_A^{\rightarrow} = f_A^{\leftarrow}$.

To provide the intuition for the just constructed functors, consider the following example.

Example 5.3. Let $\mathbf{D}$ be the category $\mathbf{U}Q\text{-}\mathbf{Mod}$ of u-Q-modules. Every two-element u-$\mathbf{2}$-module $\mathbf{2}$ provides the following two functors, which lift the traditional crisp powerset operators (we again do not distinguish between a set and its characteristic map):

1. **Set** $\xrightarrow{f^{\rightarrow}}$ **CSLat**$(\bigvee)$ given by $(X \xrightarrow{f} Y)^{\rightarrow} = \mathcal{P}(X) \xrightarrow{f^{\rightarrow}} \mathcal{P}(Y)$, with $f^{\rightarrow}$ defined by

$$(f^{\rightarrow}(S))(y) = \pi_y^Y \circ f^{\rightarrow}(S) = (\bigvee\{\pi_x^X \mid f(x) = y\})(S) = \bigvee\{\pi_x^X(S) \mid f(x) = y\} =$$

$$\bigvee\{S(x) \mid f(x) = y\} = \begin{cases} \top, & y \in \{f(x) \mid x \in X\} \\ \bot, & y \notin \{f(x) \mid x \in X\} \end{cases}$$

   and therefore $f^{\rightarrow}(S) = \{f(s) \mid s \in S\}$;

2. **Set** $\xrightarrow{f^{\leftarrow}}$ **LoCSLat**$(\bigvee)$ given by $(X \xrightarrow{f} Y)^{\leftarrow} = \mathcal{P}(Y) \xrightarrow{f^{\leftarrow}} \mathcal{P}(X)$, with $f^{\leftarrow}$ defined by

$$(f^{\leftarrow}(T))(x) = \pi_x^X \circ f^{\leftarrow}(T) = \pi_{f(x)}^Y(T) = T(f(x)) = \begin{cases} \top, & f(x) \in T \\ \bot, & f(x) \notin T \end{cases}$$

   and therefore $f^{\leftarrow}(T) = \{x \mid f(x) \in T\}$.

Every u-Q-module A provides the following two functors, which lift the fixed-basis powerset operators of L. A. Zadeh [122]:

3. **Set** $\xrightarrow{f_A^{\rightarrow}}$ **UQ-Mod** given by $(X \xrightarrow{f} Y)_A^{\rightarrow} = A^X \xrightarrow{f_A^{\rightarrow}} A^Y$, with $f_A^{\rightarrow}$ defined by $(f_A^{\rightarrow}(\alpha))(y) = \pi_y^Y \circ f_A^{\rightarrow}(\alpha) = (\bigvee\{\pi_x^X \mid f(x) = y\})(\alpha) = \bigvee\{\pi_x^X(\alpha) \mid f(x) = y\} = \bigvee\{\alpha(x) \mid f(x) = y\}$;

4. **Set** $\xrightarrow{f_A^{\leftarrow}}$ **LoUQ-Mod** given by $(X \xrightarrow{f} Y)_A^{\leftarrow} = A^Y \xrightarrow{f_A^{\leftarrow}} A^X$, with $f_A^{\leftarrow}$ defined by $(f_A^{\leftarrow}(\beta))(x) = \pi_x^X \circ f_A^{\leftarrow}(\beta) = \pi_{f(x)}^Y(\beta) = (\beta \circ f)(x)$.

Every subcategory **C** of **UQ-Mod** provides the following two functors (recall the composition map $\langle - \rangle$ from Introduction), which lift the variable-basis approach of S. E. Rodabaugh [86, 88, 90]:

5. **Set** $\times$ **C** $\xrightarrow{(-)^{\rightarrow}}$ **UQ-Mod** given by $((X, A) \xrightarrow{(f,\varphi)} (Y, B))^{\rightarrow} = A^X \xrightarrow{(f,\varphi)^{\rightarrow}} B^Y$, with $(f, \varphi)^{\rightarrow}$ defined by $((f, \varphi)^{\rightarrow}(\alpha))(y) = \pi_y^Y \circ (f, \varphi)^{\rightarrow}(\alpha) = (\varphi \circ (\bigvee\{\pi_x^X \mid f(x) = y\}))(\alpha) = \varphi \circ (\bigvee\{\pi_x^X(\alpha) \mid f(x) = y\}) = \varphi(\bigvee\{\alpha(x) \mid f(x) = y\}) = (\langle\varphi\rangle \circ f_A^{\rightarrow}(\alpha))(y)$;

6. **Set** $\times$ **C**^{op} $\xrightarrow{(-)^{\leftarrow}}$ **LoUQ-Mod** given by $((X, A) \xrightarrow{(f,\varphi)} (Y, B))^{\leftarrow} = B^Y \xrightarrow{(f,\varphi)^{\leftarrow}} A^X$, with $(f, \varphi)^{\leftarrow}$ defined by $((f, \varphi)^{\leftarrow}(\beta))(x) = \pi_x^X \circ (f, \varphi)^{\leftarrow}(\beta) = \varphi^{op} \circ \pi_{f(x)}^Y(\beta) = (\varphi^{op} \circ \beta \circ f)(x)$.

Example 5.3 justifies the most important approaches to powerset theories currently popular in the fuzzy community. Moreover, the new categorical presentation clarifies the nature of these operators, namely, the forward one corresponding to coproducts and the backward one to products of objects in categories. In particular, the partial order induced by biproducts on hom-sets provides the Galois connection used by S. E. Rodabaugh on Step (2) of his construction. It is the language of categories which allows one to look beyond this poset adjunction into the realm of biproducts.

6. **Copowers Versus Free Objects in Constructs**

The last section clarified the nature of poslat powerset theories completely. An attentive reader, however, will immediately notice one important point left untouched. The problem is related to the fixed-basis powerset operator. Up to now we have shown two ways of obtaining its expression, i.e., either through a particular monad on **Set** (Meta-Theorem 3.25), or employing the technique of biproducts (Example 5.3). The reader may still remember our result on disguising the partial order on hom-sets of a given category with the help of biproducts (Corollary 4.10). This section is bound to show the same trick with the monad, constructed from the adjunction of Corollary 3.17, raising even more the (already much appreciated) influence of biproducts in powerset theories. Our result is based on [3, Exercise 10R], the simple proof of which we provide for the sake of both the reader and the machinery exploited.

Lemma 6.1. *Let $(\mathbf{C}, |-|)$ be a construct such that $|-|$ is representable by an object A. For every set X and every $\mathbf{C}$-object B, the following conditions are equivalent:*

(1) *B is a free object over X;*
(2) *B is an Xth copower of A.*

Proof. *Ad* $(1) \Rightarrow (2)$. Since $|-|$ is representable by A, there exists a natural isomorphism $|-| \xrightarrow{\tau} \hom(A, -)$. Moreover, since B is free over X, there exists a universal arrow $X \xrightarrow{\eta_X} |B|$. Thus every $x \in X$ provides a $\mathbf{C}$-morphism $A \xrightarrow{\mu_x} B = A \xrightarrow{\tau_B \circ \eta_X(x)} B$. We show that $\mathcal{C} = ((\mu_x)_{x \in X}, B)$ is an Xth copower of A. Given a $\mathbf{C}$-sink $\mathcal{S} = (A \xrightarrow{\varphi_x} C)_{x \in X}$, there exists a map $X \xrightarrow{g_{\mathcal{S}}} \hom(A, C)$ defined by $g_{\mathcal{S}}(x) = \varphi_x$. The fact provides a $\mathbf{C}$-morphism $B \xrightarrow{\varphi} C$, given by commutativity of the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\quad \eta_X \quad} & |B| \\
{\scriptstyle g_{\mathcal{S}}} \downarrow & & \downarrow {\scriptstyle |\varphi|} \\
\hom(A, C) & \xrightarrow{\quad \tau_C^{-1} \quad} & |C|.
\end{array}
$$

Moreover, since τ is a natural transformation, the next diagram commutes as well:

$$
\begin{array}{ccc}
|B| & \xrightarrow{\quad \tau_B \quad} & \hom(A, B) \\
{\scriptstyle |\varphi|} \downarrow & & \downarrow {\scriptstyle \hom(A, \varphi)} \\
|C| & \xrightarrow{\quad \tau_C \quad} & \hom(A, C).
\end{array}
$$

Putting two diagrams together, for every $x \in X$, one obtains: $\varphi \circ \mu_x = \varphi \circ \tau_B \circ \eta_I(x) = \hom(A, \varphi) \circ \tau_B(\eta_I(x)) = \tau_C \circ |\varphi|(\eta_X(x)) = \tau_C \circ \tau_C^{-1} \circ g_{\mathcal{S}}(x) = \varphi_x$. Given another $\mathbf{C}$-morphism $B \xrightarrow{\psi} C$ with the same property, $\tau_C \circ |\psi| \circ \eta_X(x) = \hom(A, \psi) \circ \tau_B(\eta_X(x)) = \psi \circ \tau_B \circ \eta_X(x) = \psi \circ \mu_x = \varphi_x = \tau_C \circ |\varphi| \circ \eta_X(x)$ and therefore $|\psi| \circ \eta_X(x) = |\varphi| \circ \eta_X(x)$ for every $x \in X$, yielding $\psi = \varphi$.

Ad $(2) \Rightarrow (1)$. If $\mathcal{C} = ((\mu_x)_{x \in X}, B)$ is an Xth copower of A, then with the just introduced notations in mind, we define a map $X \xrightarrow{\eta_X} |B| = X \xrightarrow{g_{\mathcal{C}}} \hom(A, B) \xrightarrow{\tau_B^{-1}}$

$|B|$ and show that it is the universal arrow. Given a map $X \xrightarrow{f} |C|$, the bijection $|C| \xrightarrow{\tau_C} \hom(A,C)$ provides a $\mathbf{C}$-sink $\mathcal{S} = (A \xrightarrow{\varphi_x = \tau_C \circ f(x)} B)_{x \in X}$ and therefore a $\mathbf{C}$-morphism $B \xrightarrow{\varphi} C$ such that $\varphi \circ \mu_x = \varphi_x$ for every $x \in X$. Thus, given $x \in X$, $\tau_C \circ |\varphi| \circ \eta_X(x) = \hom(A,\varphi) \circ \tau_B \circ \tau_B^{-1} \circ g_{\mathcal{C}}(x) = \varphi \circ \mu_x = \varphi_x = \tau_C \circ f(x)$ implies $|\varphi| \circ \eta_X(x) = f(x)$. Moreover, if another $\mathbf{C}$-morphism $B \xrightarrow{\psi} C$ has the same property, then $\varphi_x = \tau_C \circ f(x) = \tau_C \circ |\psi| \circ \eta_X(x) = \hom(A,\psi) \circ \tau_B \circ \tau_B^{-1} \circ g_{\mathcal{C}}(x) = \psi \circ \mu_x$ for every $x \in X$ and therefore $\varphi = \psi$. □

An important consequence of Lemma 6.1 is provided by the following corollary. The reader is advised to recall the functor $(-)_A^{\rightarrow}$ of Corollary 5.2.

Corollary 6.2. *Let $(\mathbf{C}, |-|)$ be a pointed construct which has biproducts. The following are equivalent:*

(1) *$\mathbf{C}$ has an object A free over a one-element set;*
(2) *the underlying functor $|-|$ is representable by A;*
(3) *$(-)_A^{\rightarrow}$ is a left adjoint to $|-|$.*

If $\mathbf{C}$ is an equationally presentable variety, then each of the above-mentioned items implies equivalence of $\mathbf{C}$ to a monadic construct.

Proof. For $(1) \Leftrightarrow (2)$ recall that in a construct, an object A is free over a singleton set iff A represents the forgetful functor [3, Example 8.23(1)(b)]. For $(3) \Rightarrow (1)$ notice that $f_A^{\rightarrow}(\mathbf{1}) = A$. Moreover, the last statement of the corollary follows from Proposition 3.19.

Ad $(2) \Rightarrow (3)$. Combination of the machinery for obtaining a left adjoint functor from universal arrows, given in the proof of Corollary 3.10, with their explicit description provided in Lemma 6.1, gives a functor $\mathbf{Set} \xrightarrow{F} \mathbf{C}$ with $F(X \xrightarrow{f} Y) = \bigoplus_{x \in X} A \xrightarrow{Ff} \bigoplus_{y \in Y} A$. Moreover, $Ff \circ \mu_x^X = \tau_{A^Y} \circ (\tau_{A^Y}^{-1} \circ g_{\mathcal{C}_Y} \circ f)(x) = \mu_{f(x)}^Y$ for every $x \in X$, yields $Ff = f_A^{\rightarrow}$. □

Following Corollary 6.2, every equationally presentable pointed variety with biproducts and a free algebra over a one-element set is equivalent to a monadic construct, the monad in question being generated by the fixed-basis forward powerset operator of Corollary 5.2. The question rises how does it relate to the respective operator of L. A. Zadeh? The answer can be rather tricky in fact, since in general there is no way to transfer the $\bigvee$-semilattice structure from hom-sets to the points of algebras in question. Luckily for us, there is a free algebra over a one-element set and that changes the situation dramatically. For the sake of convenience, we introduce the following notation [86, 88, 90]: given a set X and an object A of a strictly concrete category $\mathbf{C}$ (recall Remark 3.18), the constant member of A^X having value a is denoted by $\underline{a}$; if A is free over X, then the respective lift of a map $X \xrightarrow{f} |B|$ to a $\mathbf{C}$-morphism $A \to B$ is denoted by $\overline{f}$; we also fix a singleton $\mathbf{1} = \{0\}$.

Lemma 6.3. *Suppose $(\mathbf{C}, |-|)$ is a strictly concrete category which is a quantaloid. If $\mathbf{C}$ has an object A free over $\mathbf{1}$, then every $\mathbf{C}$-object B can be equipped with*

a $\bigvee$-semilattice structure preserved by **C**-morphisms. Moreover, for every pair $B \overset{\varphi}{\underset{\psi}{\rightrightarrows}} C$ of **C**-morphisms, $\varphi \leqslant \psi$ iff $\varphi(b) \leqslant \psi(b)$ for every $b \in B$.

Proof. By the assumption, there exists a natural isomorphism $| - | \overset{\tau}{\to} \hom(A, -)$ and thus for every $S \subseteq B$, we define $\bigvee^B S = \tau_B^{-1}(\bigvee_{s \in S} \tau_B(s))$, equipping B with a $\bigvee$-semilattice structure. Given a **C**-morphism $B \overset{\varphi}{\to} C$, it follows that $\varphi(\bigvee^B S) = \varphi \circ \tau_B^{-1}(\bigvee_{s \in S} \tau_B(s)) = \tau_C^{-1} \circ \hom(A, \varphi)(\bigvee_{s \in S} \tau_B(s)) = \tau_C^{-1}(\varphi \circ (\bigvee_{s \in S} \tau_B(s))) = \tau_C^{-1}(\bigvee_{s \in S}(\varphi \circ \tau_B(s))) = \tau_C^{-1}(\bigvee_{s \in S}(\hom(A, \varphi) \circ \tau_B(s))) = \tau_C^{-1}(\bigvee_{s \in S}(\tau_C \circ \varphi(s))) = \bigvee_{s \in S}^C \varphi(s)$ (see the diagram for τ in the proof of Lemma 6.1). The explicit expression for $\bigvee^B S$ is provided by $|\bigvee_{s \in S} \overline{\underline{s}}| \circ \eta_{\mathbf{1}}(0)$.

Given **C**-morphisms $B \overset{\varphi}{\underset{\psi}{\rightrightarrows}} C$ with $\varphi \leqslant \psi$, every $b \in B$ yields $\varphi(b) \leqslant \varphi(b) \vee \psi(b) = \tau_C^{-1}(\tau_C(\varphi(b)) \vee \tau_C(\psi(b))) = \tau_C^{-1}(\hom(A, \varphi) \circ \tau_B(b) \vee \hom(A, \psi) \circ \tau_B(b)) = \tau_C^{-1}(\varphi \circ \tau_B(b) \vee \psi \circ \tau_B(b)) = \tau_C^{-1}(\psi \circ \tau_B(b)) = \tau_C^{-1}(\hom(A, \psi) \circ \tau_B(b)) = \tau_C^{-1}(\tau_C \circ \psi(b)) = \psi(b)$, i.e., $\varphi(b) \leqslant \psi(b)$. On the other hand, given $b \in B$, $\varphi(b) \leqslant \psi(b)$ implies $\varphi(b) \vee \psi(b) = \psi(b)$ implies $\tau_C^{-1}(\varphi \circ \tau_B(b) \vee \psi \circ \tau_B(b)) = \tau_C^{-1} \circ \psi \circ \tau_B(b)$ implies $(\varphi \vee \psi) \circ \tau_B(b) = \psi \circ \tau_B(b)$ implies $\hom(A, \varphi \vee \psi) \circ \tau_B(b) = \hom(A, \psi) \circ \tau_B(b)$ and therefore, altogether, $\hom(A, \varphi \vee \psi) = \hom(A, \psi)$. If A is a **C**-separator, then $\hom(A, -)$ is faithful [3, Proposition 7.12] and thus $\varphi \vee \psi = \psi$. To show that A has the required property, let $D \overset{\phi_1}{\underset{\phi_2}{\rightrightarrows}} E$ be distinct **C**-morphisms. Then there exists $d \in D$ with $\phi_1(d) \neq \phi_2(d)$ and therefore $\phi_1 \circ \overline{d}(\eta_{\mathbf{1}}(0)) \neq \phi_2 \circ \overline{d}(\eta_{\mathbf{1}}(0))$, yielding $\phi_1 \circ \overline{d} \neq \phi_2 \circ \overline{d}$. $\square$

Corollary 6.4. *Suppose* $(\mathbf{C}, | - |)$ *is a strictly concrete pointed category which has biproducts and an object A free over* $\mathbf{1}$. *Given a map* $X \overset{f}{\to} Y$, $(f_A^{\to}(\alpha))(y) = \bigvee\{\alpha(x) \,|\, f(x) = y\}$ *for every* $\alpha \in A^X$ *and every* $y \in Y$.

Proof. By Corollary 6.2, $(-)_A^{\to}$ is a left adjoint to $| - |$ and therefore **C**-products are concrete. Using the explicit description of the action of $(-)_A^{\to}$ on morphisms described in Corollary 5.2 and the point-wise $\bigvee$-semilattice structure of Lemma 6.3, one obtains $(f_A^{\to}(\alpha))(y) = (\pi_y^Y \circ f_A^{\to})(\alpha) = (\bigvee\{\pi_x^X \,|\, f(x) = y\})(\alpha) = \bigvee\{\pi_x^X(\alpha) \,|\, f(x) = y\} = \bigvee\{\alpha(x) \,|\, f(x) = y\}$. $\square$

The results just obtained give rise to an important meta-mathematical consequence, which improves significantly the respective one, stated by us earlier in Meta-Theorem 3.25.

Meta-Theorem 6.5. Given an equationally presentable pointed variety **A** which has biproducts and an algebra A free over a one-element set, there exists a monad on **Set**, the algebras of which generate the fixed-basis forward powerset operator $(-)_A^{\to}$ in the sense of L. A. Zadeh.

7. **Variety-based Powerset Operators and Their Induced Theories**

In the previous sections we have investigated thoroughly the properties of powerset operators touching both the traditional and the fuzzy approaches as well as looking somewhat beyond their scope in the realm of varieties. Following our current trend on developing a purely catalg outlook on fuzzy mathematics started in [99] and later on continued in [101, 102, 104, 110, 114], in this section we present a variety-based setting for powerset theories illustrating it by two examples coming from the area of topology. The advantage gained is the fact that the catalg framework ultimately erases the border between the traditional and the fuzzy developments, bringing into light a theory deemed to underline the algebraic essence of the whole (and not only fuzzy) mathematics, thus propagating algebra as the main driving force of modern exact sciences. Briefly speaking, algebra is at the bottom of everything.

We begin by outlining our general framework, i.e., postulating the requirements on the already fixed variety **A** which will allow us to develop a fruitful theory potentially rich on applications. Analyzing the already obtained results, we come to the conclusion of postulating the following definition (recall the notion of reduct introduced in Definition 2.1).

Definition 7.1. A variety **A** is called *convenient* provided that there exists an Ω'-reduct **B** of **A** satisfying the following properties:

(1) **B** is equationally presentable;
(2) **B** is pointed and has biproducts;
(3) **B** has an algebra free over a one-element set.

Example 7.2. Every variety having the category **U**Q**-Mod** as a reduct (e.g., the ones of Definitions 2.3, 3.1, 3.4) is convenient.

Example 7.3. The functor **CSLat**$(\bigvee)$ $\xrightarrow{F}$ **CSLat**$(\bigwedge)$ defined by the formula $F((A, \leqslant) \xrightarrow{\varphi} (B, \leqslant)) = (A, \leqslant^{op}) \xrightarrow{\varphi} (B, \leqslant^{op})$ is a concrete isomorphism (cf. [3, Exercise 5N]) and therefore every variety with **CSLat**$(\bigwedge)$ as a reduct (e.g., those of Definitions 2.2, 2.4) is convenient.

Counterexample 7.4. The category **Set** considered as a variety is *not* convenient.

An attentive reader will notice immediately that the notion of convenience provides a more flexible approach than the restricted one of those varieties only which satisfy the conditions on **B** in Definition 7.1. On the other hand, the approach still has some drawbacks which will be mentioned at their appearance. Finally, the term "convenient" is borrowed from the field of category theory, where it is widely-accepted to call categorical settings adjusted for a particular problem by convenient (e.g., *convenient topology* of [77]).

On the next step, we choose the ground category for our theory. Unlike the approaches, currently accepted in the fuzzy community and stimulated by such "fuzzy guru" as S. E. Rodabaugh [86, 88, 90] and P. Eklund [30], we fix two ground categories instead of one (recall that "**Lo**" stands for the dual).

Definition 7.5. Given a convenient variety $\mathbf{A}$, the *ground categories* for the variety-based powerset theories are fixed to $\mathbf{Set} \times \mathbf{A}$ and $\mathbf{Set} \times \mathbf{LoA}$.

Definition 7.5 reveals a crucial difference of our approach from the already mentioned standard ones. The main reason for the step is the necessity of having a separate ground category for both forward and backward powerset operator, which arises from our general framework of varieties.

We have arrived now at the turning point in our theory, namely, at the definition of the powerset operators themselves. Notice that we cannot obtain them directly from Lemma 5.1 since our variety $\mathbf{A}$ may lack biproducts, and therefore a more sophisticated machinery is employed (recall that every reduct is equipped with a concrete functor $|| - ||$).

Definition 7.6. Given a subcategory $\mathbf{C}$ of a convenient variety $\mathbf{A}$,

(1) the *forward variety-based powerset operator w.r.t.* $\mathbf{C}$ is the functor $\mathbf{Set} \times \mathbf{C} \xrightarrow{(-)\overrightarrow{\mathbf{C}}} \mathbf{B}$ defined by $((X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2))\overrightarrow{\mathbf{C}} = ((X_1, ||A_1||) \xrightarrow{(f,||\varphi||)} (X_2, ||A_2||))\overrightarrow{\mathbf{B}}$, where $(-)\overrightarrow{\mathbf{B}}$ is the functor given by Item (1) of Lemma 5.1;

(2) the *backward variety-based powerset operator w.r.t.* $\mathbf{C}$ is the functor $\mathbf{Set} \times \mathbf{C}^{op} \xrightarrow{(-)\overleftarrow{\mathbf{C}}} \mathbf{LoB}$ given by $((X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2))\overleftarrow{\mathbf{C}} = ((X_1, ||A_1||) \xrightarrow{(f,||\varphi||)} (X_2, ||A_2||))\overleftarrow{\mathbf{B}}$, where $(-)\overleftarrow{\mathbf{B}}$ is the functor given by Item (2) of Lemma 5.1.

The reader should be aware that the object part of the just defined powerset operators actually lies in $\mathbf{A}$ (resp. $\mathbf{LoA}$), whereas the action on morphisms can potentially produce something strictly out of the scope of the categories in question. Also notice that we have started with the variable-basis approach leaving the fixed-one aside. It is easy to see that the subcategory $\mathbf{S}_A$ for an algebra A (recall the respective notation of Remark 2.5) provides the missing part.

Having introduced the powerset operators, we are going to consider some of their properties which will make their use considerably easier. We begin with the more transparent case of the backward operator.

Lemma 7.7. *Given a subcategory $\mathbf{C}$ of a convenient variety $\mathbf{A}$ and a $(\mathbf{Set} \times \mathbf{C}^{op})$-morphism $(X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2)$, $((X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2))\overleftarrow{\mathbf{C}} = C_2^{X_2} \xrightarrow{(f,\varphi)\overleftarrow{\mathbf{C}}} C_1^{X_1}$ with $(f,\varphi)\overleftarrow{\mathbf{C}}(\beta) = \varphi^{op} \circ \beta \circ f$. Moreover, $(f,\varphi)\overleftarrow{\mathbf{C}}$ is an $\mathbf{A}$-homomorphism and therefore the codomain of $(-)\overleftarrow{\mathbf{C}}$ is $\mathbf{LoA}$.*

Proof. The explicit formula for the action of $(-)\overleftarrow{\mathbf{C}}$ on morphisms can be obtained precisely as in Item (6) of Example 5.3. To show that $(f,\varphi)\overleftarrow{\mathbf{C}}$ is in $\mathbf{A}$, notice that given $\lambda \in \Lambda$, $\beta_i \in C_2^{X_2}$ for $i \in n_\lambda$ and $x_1 \in X_1$, $((f,\varphi)\overleftarrow{\mathbf{C}} \circ \omega_\lambda^{C_2^{X_2}}(\langle \beta_i \rangle_{n_\lambda}))(x_1) = (\varphi^{op} \circ \omega_\lambda^{C_2^{X_2}}(\langle \beta_i \rangle_{n_\lambda}) \circ f)(x_1) = \varphi^{op} \circ \omega_\lambda^{C_2}(\langle \beta_i \circ f(x_1) \rangle_{n_\lambda}) = \omega_\lambda^{C_1}(\langle \varphi^{op} \circ \beta_i \circ f(x_1) \rangle_{n_\lambda}) = \omega_\lambda^{C_1}(\langle ((f,\varphi)\overleftarrow{\mathbf{C}}(\beta_i))(x_1) \rangle_{n_\lambda}) = (\omega_\lambda^{C_1^{X_1}}(\langle (f,\varphi)\overleftarrow{\mathbf{C}}(\beta_i) \rangle_{n_\lambda}))(x_1)$. $\square$

Lemma 7.7 provides the reader with the functor $\mathbf{Set} \times \mathbf{C}^{op} \xrightarrow{(-)\overleftarrow{\mathbf{C}}} \mathbf{LoA}$ justifying once more the approach of S. E. Rodabaugh [86, 88, 90] as well as our previous developments of [100, 103, 104, 110]. Moreover, our general variety-based setting

has the merit of the results proved being valid for *all* varieties in the scope of the respective requirements and not just for a particular variety of, e.g., s-quantales.

The case of the forward powerset operator is not so cheerful as that enjoyed by its counterpart, the reason being quite simple. The structure of $\bigvee$-semilattice enriching the objects of a strictly concrete category $\mathbf{C}$ in Lemma 6.3 relates perfectly to its morphisms. On the other hand, since the $\mathbf{C}$-structure on objects was not specified explicitly, we never considered its compatibility with $\bigvee$. This, however, is the unlucky point that makes all the trouble. Our current setting of varieties provides an explicit description of the structure of algebra and therefore it is possible to relate it to $\bigvee$, the following definition doing the job.

Definition 7.8. Given a subcategory $\mathbf{C}$ of a convenient variety $\mathbf{A}$, an operation ω_λ with $n_\lambda \in \Omega \setminus \Omega'$ is called $\bigvee_{\mathbf{C}}$-*compatible* provided that every $\mathbf{C}$-object C satisfies the identity $\bigvee_{j \in J} \omega_\lambda^A(\langle c_{ij} \rangle_{n_\lambda}) = \omega_\lambda^A(\langle \bigvee_{j \in J} c_{ij} \rangle_{n_\lambda})$ for every subset $\{c_{ij} \mid i \in n_\lambda, j \in J\} \subseteq C$. $\bigvee_{\mathbf{A}}$-compatibility is called $\bigvee$-*compatibility*. $\mathbf{A}_{\bigvee_{\mathbf{C}}}$ is the (non-full) subcategory of $\mathbf{A}$ with objects precisely those of $\mathbf{A}$, and morphisms those $\bigvee$-preserving maps $|A_1| \xrightarrow{f} |A_2|$ which are $\mathbf{B}$-homomorphisms and preserve $\bigvee_{\mathbf{C}}$-compatible operations.

The reader should be aware that the condition of $\bigvee$-compatibility is a rather restrictive one. For example, the meet-operation $\wedge$ in $\mathbf{Frm}$ is not $\bigvee$-compatible. On the other hand, the involution $(-)^*$ of an is-quantale (Definition 2.3) is $\bigvee$-preserving and therefore provides an example of a $\bigvee$-compatible operation.

Lemma 7.9. *Given a subcategory $\mathbf{C}$ of a convenient variety $\mathbf{A}$ and a $(\mathbf{Set} \times \mathbf{C})$-morphism $(X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2)$, $((X_1, C_1) \xrightarrow{(f,\varphi)} (X_2, C_2))_{\mathbf{C}}^{\rightarrow} = C_1^{X_1} \xrightarrow{(f,\varphi)_{\mathbf{C}}^{\rightarrow}} C_2^{X_2}$ with $((f,\varphi)_{\mathbf{C}}^{\rightarrow}(\alpha))(x_2) = \bigvee\{\varphi \circ \alpha(x_1) \mid f(x_1) = x_2\}$. Moreover, $(f,\varphi)_{\mathbf{C}}^{\rightarrow}$ is an $\mathbf{A}_{\bigvee_{\mathbf{C}}}$-morphism, and every operation ω_λ, $n_\lambda \in \Omega \setminus \Omega'$ preserved by all maps of the form $(f,\varphi)_{\mathbf{C}}^{\rightarrow}$ is $\bigvee_{\mathbf{C}}$-compatible, that fixes the codomain of $(-)_{\mathbf{C}}^{\rightarrow}$ at precisely $\mathbf{A}_{\bigvee_{\mathbf{C}}}$.*

Proof. The explicit formula for the action of $(-)_{\mathbf{C}}^{\rightarrow}$ on morphisms can be obtained precisely as in Item (5) of Example 5.3, using the technique of Lemma 6.3 and Corollary 6.4. Moreover, as a $\mathbf{B}$-homomorphism, $(f,\varphi)_{\mathbf{C}}^{\rightarrow}$ is $\bigvee$-preserving by Lemma 6.3. Given $\lambda \in \Lambda$, $\alpha_i \in C_1^{X_1}$ for $i \in n_\lambda$ and $x_2 \in X_2$, $\bigvee$-compatibility of ω_λ yields

$$((f,\varphi)_{\mathbf{C}}^{\rightarrow} \circ \omega_\lambda^{C_1^{X_1}}(\langle \alpha_i \rangle_{n_\lambda}))(x_2) = \bigvee\{\varphi \circ (\omega_\lambda^{C_1^{X_1}}(\langle \alpha_i \rangle_{n_\lambda}))(x_1) \mid f(x_1) = x_2\} =$$

$$\bigvee\{\varphi \circ \omega_\lambda^{C_1}(\langle \alpha_i(x_1) \rangle_{n_\lambda}) \mid f(x_1) = x_2\} = \bigvee\{\omega_\lambda^{C_2}(\langle \varphi \circ \alpha_i(x_1) \rangle_{n_\lambda}) \mid f(x_1) = x_2\} =$$

$$\omega_\lambda^{C_2}(\langle \bigvee\{\varphi \circ \alpha_i(x_1) \mid f(x_1) = x_2\} \rangle_{n_\lambda}) = \omega_\lambda^{C_2}(\langle ((f,\varphi)_{\mathbf{C}}^{\rightarrow}(\alpha_i))(x_2) \rangle_{n_\lambda}) =$$

$$(\omega_\lambda^{C_2^{X_2}}(\langle (f,\varphi)_{\mathbf{C}}^{\rightarrow}(\alpha_i) \rangle_{n_\lambda}))(x_2).$$

On the other hand, to show that ω_λ is $\bigvee_{\mathbf{C}}$-compatible, notice that given a subset $\{c_{ij} \mid i \in n_\lambda, j \in J\}$ of a $\mathbf{C}$-object C, the $(\mathbf{Set} \times \mathbf{C})$-morphism $(J, C) \xrightarrow{(!, 1_C)} (\mathbf{1}, C)$ and the family $\{\alpha_i \mid i \in n_\lambda\} \subseteq C^J$ given by $\alpha_i(j) = c_{ij}$, provide the required identity. $\square$

Lemma 7.9 gives us the functor $\mathbf{Set} \times \mathbf{C} \xrightarrow{(-)_{\mathbf{C}}^{\rightarrow}} \mathbf{A}_{\bigvee_{\mathbf{C}}}$ justifying once more the approach of S. E. Rodabaugh [86, 88, 90]. On the other hand, it fills the gap in our previous theories [100, 103, 104, 110] based entirely on the backward powerset operator. Strikingly enough, up to now we have managed to proceed without any forward powerset operator being available. Moreover, in [101] we even postulated the problem on the importance of its fuzzification for developing the theory of fuzzy topology. Our current results and their future developments will hopefully clarify the matter.

All preliminaries done, we are ready to introduce the main point of the section, i.e., variety-based powerset theories. To be in line with the already obtained results, we will take the approach of [90, Definition 3.5] as a good example, with the ultimate view of removing its drawbacks.

Definition 7.10. Given a subcategory $\mathbf{C}$ of a convenient variety $\mathbf{A}$, a $\mathbf{C}$-*powerset theory* is the tuple $\mathsf{P} = (\mathbf{A}, \mathbf{C}, (-)_{\mathbf{C}}^{\rightarrow}, (-)_{\mathbf{C}}^{\leftarrow})$. The triple $\mathsf{P} = (\mathbf{A}, \mathbf{C}, (-)_{\mathbf{C}}^{\rightarrow})$ (resp. $\mathsf{P} = (\mathbf{A}, \mathbf{C}, (-)_{\mathbf{C}}^{\leftarrow})$) is called a *forward* (resp. *backward*) $\mathbf{C}$-*powerset theory*. The *underlying* or *ground theory* of a $\mathbf{C}$-powerset theory P is the tuple $|\mathsf{P}| = (\mathbf{A}, \mathbf{C}, |-| \circ (-)_{\mathbf{C}}^{\rightarrow}, |-| \circ (-)_{\mathbf{C}}^{\leftarrow})$.

Example 7.11. There is plenty of examples of powerset theories in the literature, some of which have already been encountered by an attentive reader in the paper. The following short list is by no means complete and is deemed to show the fruitfulness of our approach only, the exciting task of finding other justifications being left to the reader.

(1) $\mathsf{P} = (\mathbf{USQuant}, \mathbf{S_2}, (-)_{\mathbf{S_2}}^{\rightarrow}, (-)_{\mathbf{S_2}}^{\leftarrow})$ provides the standard crisp powerset theory used in mathematics. For the sake of shortness, from now on, it will be denoted by $\mathcal{P} = ((-)^{\rightarrow}, (-)^{\leftarrow})$.

(2) $\mathsf{P} = (\mathbf{CL}, \mathbf{S_2}, (-)_{\mathbf{S_2}}^{\rightarrow}, (-)_{\mathbf{S_2}}^{\leftarrow})$ provides a non-standard crisp powerset theory. For the sake of shortness, from now on, it will be denoted by $\wp = ((-)^{\dashrightarrow}, (-)^{\dashleftarrow})$.

(3) $\mathsf{P} = (\mathbf{Frm}, \mathbf{S}_I, (-)_{\mathbf{S}_I}^{\rightarrow}, (-)_{\mathbf{S}_I}^{\leftarrow})$ provides the fixed-basis fuzzy approach of L. A. Zadeh. For the sake of shortness, from now on, it will be denoted by $\mathcal{Z} = ((-)_I^{\rightarrow}, (-)_I^{\leftarrow})$.

(4) $\mathsf{P} = (\mathbf{UQuant}, \mathbf{S}_L, (-)_{\mathbf{S}_L}^{\rightarrow}, (-)_{\mathbf{S}_L}^{\leftarrow})$ provides the fixed-basis L-fuzzy approach of J. A. Goguen. For the sake of shortness, from now on, it will be denoted by $\mathcal{G} = ((-)_L^{\rightarrow}, (-)_L^{\leftarrow})$. The machinery can be generalized to an arbitrary convenient variety $\mathbf{A}$, providing the theory $\mathcal{G}_{\mathbf{A}}^A$ which incorporates all the previous items in one common fixed-basis framework.

(5) $\mathsf{P} = (\mathbf{Frm}, \mathbf{C}, (-)_{\mathbf{C}}^{\rightarrow}, (-)_{\mathbf{C}}^{\leftarrow})$ provides the variable-basis approach of S. E. Rodabaugh. For the sake of shortness, from now on, it will be denoted by $\mathcal{R}_1 = ((-)_{\mathbf{C}}^{\rightarrow}, (-)_{\mathbf{C}}^{\leftarrow})$.

(6) $\mathsf{P} = (\mathbf{SQuant}, \mathbf{C}, (-)_{\mathbf{C}}^{\rightarrow}, (-)_{\mathbf{C}}^{\leftarrow})$ provides a modernized variable-basis approach of S. E. Rodabaugh. For the sake of shortness, from now on, it will be denoted by $\mathcal{R}_2 = ((-)_{\mathbf{C}}^{\rightarrow}, (-)_{\mathbf{C}}^{\leftarrow})$. Sometimes we will use the notation $\mathcal{R}$ with the idea that any $i \in \{1, 2\}$ will do.

(7) $\mathsf{P} = (\mathbf{Hut}, \mathbf{Hut}, (-)_{\mathbf{Hut}}^{\leftarrow})$ provides the variable-basis approach of P. Eklund motivated by that of S. E. Rodabaugh. For the sake of shortness, from now on, it will be denoted by $\mathcal{E} = ((-)_{\mathbf{H}}^{\leftarrow})$. Notice that $\mathbf{Hut}$ [87] is the variety of complete, completely distributive lattices equipped with an order-reversing involution (the so-called *Hutton algebras*).

(8) $\mathsf{P} = (\mathbf{A}, \mathbf{C}, (-)_{\mathbf{C}}^{\leftarrow})$ provides our former variety-based approach. For the sake of shortness, from now on, it will be denoted by $\mathcal{S} = ((-)_{\mathbf{C}}^{\leftarrow})$.

It is worthwhile to make a comment on Example 7.11, namely, on a relation between the first two powerset theories $\mathcal{P}$ and $\wp$. A cunning reader can ask whether their underlying theories (the two induced functors on $\mathbf{Set}$) are essentially the same. Strikingly enough, the answer is negative, the following considerations clarifying the matter.

Lemma 7.12. *The underlying functor of* $\mathbf{CSLat}(\bigwedge)$ *has a left adjoint.*

Proof. Given a set X, the map $X \xrightarrow{\eta} \mathcal{P}(X)$ defined by $\eta(x) = X \backslash \{x\}$ is the required universal arrow, since every map $X \xrightarrow{f} |A|$ has a unique extension $\mathcal{P}(X) \xrightarrow{\overline{f}} A$ given by $\overline{f}(S) = \bigwedge_{x \notin S} f(x)$ such that $|\overline{f}| \circ \eta = f$. $\qquad\square$

Corollary 7.13. *There is an adjoint situation* $(\eta, \epsilon) \colon F \dashv\!\!\!-\!\!- | - | \colon \mathbf{CSLat}(\bigwedge) \to \mathbf{Set}$.

Proof. With the scheme proposed in the proof of Corollary 3.10 in mind, notice that given a map $X \xrightarrow{f} Y$, $F(X \xrightarrow{f} Y) = \mathcal{P}(X) \xrightarrow{Ff} \mathcal{P}(Y)$ with $Ff(S) = Y \backslash f^{\rightarrow}(X \backslash S) = \{y \in Y \mid f^{\leftarrow}(\{y\}) \subseteq S\}$. Given a $\bigwedge$-lattice A, $F|A| \xrightarrow{\epsilon_A} A$ is defined by $\epsilon_A(S) = \bigwedge(A \backslash S)$. $\qquad\square$

By our previous results on monadic generation of powerset theories (Meta-Theorem 6.5) we obtain that $(-)^{\dashrightarrow} = F$ (notice the shift from $\bigvee$ to $\bigwedge$). Moreover, the biproduct machinery of Corollary 5.2 provides us with the backward powerset operator $(-)^{\leftarrow\!-}$. Similarity in the construction of products in both $\mathbf{CSLat}(\bigvee)$ and $\mathbf{CSLat}(\bigwedge)$ yields $|-| \circ (-)^{\leftarrow\!-} = |-| \circ (-)^{\leftarrow}$. On the other hand, the following lemma shows that $|-| \circ (-)^{\dashrightarrow} \neq |-| \circ (-)^{\rightarrow}$.

Lemma 7.14. *The functors* $\mathbf{Set} \xrightarrow{|-|\circ(-)^{\dashrightarrow}} \mathbf{Set}$ *and* $\mathbf{Set} \xrightarrow{|-|\circ(-)^{\rightarrow}} \mathbf{Set}$ *are essentially different.*

Proof. For the unique map $\{x_1, x_2\} \xrightarrow{f} \{y\}$, $|f^{\rightarrow}|(\{x_1\}) = \{y\}$ and $|f^{\dashrightarrow}|(\{x_1\}) = \varnothing$ yield $|f^{\dashrightarrow}|(\{x_1\}) \subsetneqq |f^{\rightarrow}|(\{x_1\})$. On the other hand, for the map $\{x\} \xrightarrow{g} \{y_1, y_2\}$ with $g(x) = y_1$, $|f^{\rightarrow}|(\{x\}) = \{y_1\}$ and $|f^{\dashrightarrow}|(\{x\}) = \{y_1, y_2\}$ yield $|f^{\rightarrow}|(\{x\}) \subsetneqq |f^{\dashrightarrow}|(\{x\})$. $\qquad\square$

It follows that $\mathcal{P}$ and $\wp$ provide essentially different underlying powerset theories, the fact already noticed in [3, Exercise 3J], but never mentioned in any of [86, 88, 90]. Moreover, an interesting peculiarity of our biproduct approach can be seen here as follows. Up to now, we have always thought of the biproduct-induced operation on hom-sets of a category as a kind of join-operation $\bigvee$. An attentive

reader, however, can easily see that its characterization in Lemma 3.21 (Items (1), (2′)) can be applied to meet-operation $\bigwedge$ as well (cf. [3, Example 5N]), i.e., we can freely interchange the operations, based on the already pointed out (Example 7.3) isomorphism between $\mathbf{CSLat}(\bigvee)$ and $\mathbf{CSLat}(\bigwedge)$, which simply takes the dual of the order in question. In particular, one can replace $\bigvee$ by $\bigwedge$ in Meta-Theorem 6.5. Everything is based on the (obvious) fact [55] that enrichment in isomorphic categories provides isomorphic categories. On the other hand, the powerset theories obtained from such categories can be completely different, motivating the next definition.

Definition 7.15. The *dual* P^{op} of a powerset theory P is the theory obtained from the dual partial order on hom-sets of the reduct $\mathbf{B}$ of the convenient variety $\mathbf{A}$.

Example 7.16. The powerset theories $\mathcal{P}$ and $\wp$ of Example 7.11 are dual.

The notion of powerset theory introduced, an immediate question rises on whether there are real applications for the new concept, apart from it serving as a nice framework for the existing approaches to powersets. To justify the notion, we will provide two examples stemming from the field of topology and motivated by our previous exertions on the topic. The first example shows a variety-based or catalg approach to the well-known notion of *topological space*. As can be seen from Introduction, fuzzification of the concept has occupied the scientific mind right from the start of fuzzy mathematics. The next definition should provide a uniting common framework for many (but not all) of the existing approaches. Stimulated by our recent research on composite topology [102, 103], the definition is formulated in an slightly unusual way. The reader is advised to recall the notion of *product category* [3, 44], to notice the use of composition of powerset operators (in the definition of continuity) mentioned in Example 7.11 as well as to refresh in his memory the notation for constant maps introduced just before Lemma 6.3.

Definition 7.17. Given a set-indexed family $\boldsymbol{P} = (\mathsf{P}_i)_{i \in I}$ of $\mathbf{C}_i$-powerset theories together with a family $\mathcal{R} = ((\| - \|_i, \mathbf{R}_i))_{i \in I}$ of reducts of $\mathbf{A}_i$, we denote $\mathbf{C} = \prod_{i \in I} \mathbf{C}_i$, and introduce the category $\mathbf{Top}(\boldsymbol{P}, \mathcal{R})$ whose objects (called $\mathbf{C}$-*composite topological spaces*) are triples $E = (\mathrm{pt}\, E, \Sigma\, E = ((\Sigma\, E)_i)_{i \in I}, \Omega\, E = ((\Omega\, E)_i)_{i \in I})$ with $(\mathrm{pt}\, E, \Sigma\, E)$ in $\mathbf{Set} \times \mathbf{C}^{op}$ and $(\Omega\, E)_i$ (called the *ith topology* on X, with the whole $\Omega\, E$ coined as $\mathbf{C}$-*composite topology* on X) a subalgebra of $\|(\Sigma\, E)_i^{\mathrm{pt}\, E}\|_i$ for every $i \in I$. Morphisms (called $\mathbf{C}$-*composite continuous maps*) $E_1 \xrightarrow{f} E_2$ are $(\mathbf{Set} \times \mathbf{C}^{op})$-morphisms $(\mathrm{pt}\, E_1, \Sigma\, E_1) \xrightarrow{f = (\mathrm{pt}\, f, \Sigma\, f = ((\Sigma\, f)_i)_{i \in I})} (\mathrm{pt}\, E_2, \Sigma\, E_2)$ which satisfy the condition $((\mathrm{pt}\, f, (\Sigma\, f)_i)_{\mathbf{C}_i}^{\leftarrow})^{\rightarrow}((\Omega\, E_2)_i) \subseteq (\Omega\, E_1)_i$ for every $i \in I$. The underlying functor to the ground category $\mathbf{Set} \times \mathbf{C}^{op}$ is denoted by $|-|$.

A $\mathbf{C}$-composite topological space E is called *stratified* provided that $\{\underline{a} \mid a \in (\Sigma\, E)_i\} \subseteq (\Omega\, E)_i$ for every $i \in I$. $\mathbf{STop}(\boldsymbol{P}, \mathcal{R})$ is the full subcategory of $\mathbf{Top}(\boldsymbol{P}, \mathcal{R})$ of all stratified spaces.

For the sake of simplicity, the category $(\mathbf{S})\mathbf{Top}(\boldsymbol{P}, \mathcal{R})$ induced by a single theory P is denoted by $(\mathbf{S})\mathbf{Top}(\mathsf{P}, \mathcal{R})$. The notation $\mathcal{R}$ for the family of reducts consisting of the identity functors is omitted.

Example 7.18. Similar to the case of powerset theories, there are many examples in modern mathematics illustrating the concept of variety-based topology. The

following provides a short extract to give the feeling of their abundance and the fruitfulness of the new unifying framework.

(1) **Top**$(\mathcal{P})$ provides the standard crisp topological theory.

(2) **Top**$(\wp)$ provides the theory of standard crisp closure spaces. For convenience of the reader, we recall from [5, 6] the definition of the concept, using the backward part of the standard powerset theory $\mathcal{P}$, which is precisely that of $\wp$.

- A *closure space* is a pair $(X, \mathcal{F})$, with X a set and $\mathcal{F} \subseteq \mathcal{P}(X)$ such that
    (a) $\varnothing \in \mathcal{F}$;
    (b) if $F_i \in \mathcal{F}$ for $i \in I$, then $\bigcap_{i \in I} F_i \in \mathcal{F}$.
- Given closure spaces $(X_1, \mathcal{F}_1)$ and $(X_2, \mathcal{F}_2)$, a map $X_1 \xrightarrow{f} X_2$ is called *continuous* provided that $f^{\leftarrow}(F) \in \mathcal{F}_1$ for every $F \in \mathcal{F}_2$.

(3) **Top**$((\mathcal{P})_{i \in \{1,2\}})$ provides the standard theory of bitopology [56, 91].

(4) **Top**$(\mathcal{Z})$ provides the fixed-basis theory of fuzzy topological spaces developed by C. L. Chang.

(5) **STop**$(\mathcal{Z})$ provides the fixed-basis theory of stratified fuzzy topological spaces of R. Lowen.

(6) **Top**$(\mathcal{G})$ provides the fixed-basis theory of L-fuzzy topological spaces of J. A. Goguen.

(7) **Top**$(\mathcal{R})$ provides the variable-basis theory of fuzzy topological spaces of S. E. Rodabaugh.

(8) **Top**$((\mathcal{G}^A_{\mathbf{SQuant}})_{i \in \{1,2\}})$ provides the fixed-basis fuzzy bitopology of S. E. Rodabaugh [91]. The shift to variable-basis (never mentioned in [91]) is then given by **Top**$((\mathcal{R})_{i \in \{1,2\}})$.

(9) **Top**$(\mathcal{E}, \mathbf{Frm})$ provides the variable-basis topology of P. Eklund motivated by that of S. E. Rodabaugh.

(10) **Top**$(\mathcal{S})$ provides our former variety-based approach.

(11) **Top**$((\mathcal{S}_i)_{i \in I})$ provides a particular instance of our recent approach to composite topological theories.

Some remarks are due to the last item of Example 7.18. In [101, 102, 103] we started to develop a more abstract approach to catalg topology motivated by recent achievements in the field, e.g., *generalized topological spaces* of M. Demirci [24, 25]. The framework is based on the notion of *topological theory* suggested by the respective concepts of [90, Definition 3.7] and [3, Exercise 22B]. The main idea is to generalize the backward powerset operator replacing the ground category $\mathbf{Set} \times \mathbf{C}^{op}$ with an arbitrary category $\mathbf{X}$. It will be the topic of our future research to compare these both approaches, with an ultimate goal to develop powerset theories for the ground categories more general than $\mathbf{Set}$, e.g., topos-like.

A remark on the second item of Example 7.18 also seems to be advisable. The theory of crisp closure spaces is based on the standard powerset theory $\mathcal{P}$. Our approach replaces the forward powerset operator with something completely different, posing the question on the new properties of the theory obtained.

An attentive reader will grasp immediately that all the above-mentioned topological theories actually never use forward powerset operators in their definitions (application of the standard forward powerset operator in Definition 7.17 is a cosmetic one which can be easily avoided). In fact, as was already mentioned, it is possible to develop a rich theory of topology without employing the concept at all. At one moment we even have doubted the necessity of such an operator in (at least) fuzzy topology. The next definition, however, is deemed to show that the notion is still useful, paving the way for its future developments. For the sake of shortness, from now on we omit the term "$\mathbf{C}$-composite" in the topological stuff. Moreover, the reader should recall that every category shares the same objects with its dual, differing on morphism only.

Definition 7.19. Given a set-indexed family $\boldsymbol{P}$ of powerset theories and two ground category morphisms $(X_1, (C_{1_i})_{i \in I}) \underset{g}{\overset{f}{\rightrightarrows}} (X_2, (C_{2_i})_{i \in I})$ with f in $\mathbf{Set} \times \mathbf{C}^{op}$ and g in $\mathbf{Set} \times \mathbf{C}$, the pair (f, g) is called an *adjunction* with f (resp. g) the *upper* (resp. *lower*) *adjoint*, provided that $\mathrm{pt}\, f = \mathrm{pt}\, g$ and $(\Sigma g)_i \circ (\Sigma f)_i^{op} \leqslant 1_{C_{2_i}}$, $1_{C_{1_i}} \leqslant (\Sigma f)_i^{op} \circ (\Sigma g)_i$ for every $i \in I$.

Given an adjunction (f, g), Lemma 5.1 implies $(\mathrm{pt}\, g, (\Sigma g)_i)_{\mathbf{C}_i}^{\rightarrow} \circ (\mathrm{pt}\, f, (\Sigma f)_i)_{\mathbf{C}_i}^{\leftarrow} \leqslant 1_{C_{2_i}^{X_2}}$ and $1_{C_{1_i}^{X_1}} \leqslant (\mathrm{pt}\, f, (\Sigma f)_i)_{\mathbf{C}_i}^{\leftarrow} \circ (\mathrm{pt}\, g, (\Sigma g)_i)_{\mathbf{C}_i}^{\rightarrow}$. Also notice that the terminology used is motivated by the adjunctions between partially ordered sets [35, Section 0-3].

Definition 7.20. Given a category $\mathbf{Top}(\boldsymbol{P}, \mathcal{R})$, a ground category morphism $|E_1| \overset{f}{\rightarrow} |E_2|$ is a *homeomorphism* provided that f is a $(\mathbf{Set} \times \mathbf{C}^{op})$-isomorphism, and both f and f^{-1} are continuous. f is called *open* provided that it has a lower adjoint g such that $((\mathrm{pt}\, g, (\Sigma g)_i)_{\mathbf{C}_i}^{\rightarrow})^{\rightarrow}((\Omega\, E_1)_i) \subseteq (\Omega\, E_2)_i$ for every $i \in I$.

The next lemma contains a simple generalization of the standard characterization of homeomorphisms.

Lemma 7.21. *Given a category* $\mathbf{Top}(\boldsymbol{P}, \mathcal{R})$, *for every ground category morphism* $|E_1| \overset{f}{\rightarrow} |E_2|$, *equivalent are:*

(1) *f is a homeomorphism;*
(2) *f is a $(\mathbf{Set} \times \mathbf{C}^{op})$-isomorphism which is continuous and open.*

Proof. Ad (1) $\Rightarrow$ (2). Since f is a ground category isomorphism, we define $g = (\mathrm{pt}\, f, ((\Sigma f^{-1})_i^{op})_{i \in I})$. The openness condition follows by Lemma 5.1 from the fact that $(\mathrm{pt}\, g, (\Sigma g)_i)_{\mathbf{C}_i}^{\rightarrow} = (\mathrm{pt}\, f, (\Sigma f^{-1})_i^{op})_{\mathbf{C}_i}^{\rightarrow} = (\mathrm{pt}\, f^{-1}, (\Sigma f^{-1})_i)_{\mathbf{C}_i}^{\leftarrow}$, the latter being a continuous map for every $i \in I$.

Ad (2) $\Rightarrow$ (1). The only challenge is the continuity of $|E_2| \xrightarrow{f^{-1}} |E_1|$. Since f is open, it has a lower adjoint g which by Definition 7.19 and the properties of adjunctions for partially ordered sets of [35, Section 0-3] has the form $(\mathrm{pt}\, f, ((\Sigma f^{-1})_i^{op})_{i \in I})$. By Lemma 5.1, $(\mathrm{pt}\, f^{-1}, (\Sigma f^{-1})_i)_{\mathbf{C}_i}^{\leftarrow} = (\mathrm{pt}\, f, (\Sigma f^{-1})_i^{op})_{\mathbf{C}_i}^{\rightarrow} = (\mathrm{pt}\, g, (\Sigma g)_i)_{\mathbf{C}_i}^{\rightarrow}$, the latter map having the required property for every $i \in I$. $\square$

The reader is invited to find other justifications for the forward powerset operator in topology. For our own part, we are going to present another example of

application of powerset theories which is based on the notion of *topological system*. The concept was introduced by S. Vickers [118] to provide a single framework for treating both topological spaces and their underlying algebraic structures - *locales* (introduced by J. Isbell [51] and studied thoroughly by P. T. Johnstone [53]). The notion was further developed by S. Abramsky and S. Vickers [2] to quantale modules (already known to the reader) as well as by P. Resende [81, 82] to *observable transition systems* coined also as *tropological systems*. Several attempts were made to replace topological spaces in the framework of systems by their fuzzy counterparts, the most significant ones being due to J. T. Denniston, S. E. Rodabaugh and C. Guido [27, 38, 40], who considered functorial relationships between lattice-valued topology and topological systems. The main result obtained was the slogan "fuzzy needs fuzzy", i.e., fuzzy topology on one side should be backed by fuzzy topological systems on the other. Following the line, the concept of *lattice-valued topological system* over locales was introduced in [26], significantly simplifying some results of [27]. For example, one can easily embed the category of lattice-valued topological spaces into the respective one of lattice-valued topological systems. Motivated by the results, we introduced in [100, 109] the concept of *variety-based topological system* using the above-mentioned trick with algebras and relying exclusively on the backward powerset operator. The following is deemed to present the full framework, the notion of system itself being made composite.

Definition 7.22. Given a set-indexed family $\boldsymbol{P} = (\mathsf{P}_i)_{i \in I}$ of $\mathbf{C}_i$-powerset theories together with a family $\mathcal{R} = (|| - ||_i, \mathbf{R}_i)_{i \in I}$ of reducts of $\mathbf{A}_i$, we denote $\mathbf{C} = \prod_{i \in I} \mathbf{C}_i$, $\mathbf{R} = \prod_{i \in I} \mathbf{R}_i$, and introduce the category $\mathbf{TopSys}(\boldsymbol{P}, \mathcal{R})$ whose objects (called $\mathbf{C}$-*composite topological systems*) are tuples $D = (\operatorname{pt} D, \Sigma D = ((\Sigma D)_i)_{i \in I}, \Omega D = ((\Omega D)_i)_{i \in I}, (\kappa_i)_{i \in I})$ with $(\operatorname{pt} D, \Sigma D, \Omega D)$ in $\mathbf{Set} \times \mathbf{C}^{op} \times \mathbf{R}^{op}$ and $(\Omega D)_i \xrightarrow{\kappa_i} ||(\Sigma D)_i^{\operatorname{pt} D}||_i$ an $\mathbf{R}_i$-morphism (called the *ith satisfaction map* on $(\operatorname{pt} D, \Sigma D, \Omega D)$, with the whole $(\kappa_i)_{i \in I}$ coined as $\mathbf{C}$-*composite satisfaction map* on $(\operatorname{pt} D, \Sigma D, \Omega D))$ for every $i \in I$. Morphisms (called $\mathbf{C}$-*composite continuous maps*) $D_1 \xrightarrow{f} D_2$ are $\mathbf{Set} \times \mathbf{C}^{op} \times \mathbf{R}^{op}$-morphisms $(\operatorname{pt} D_1, \Sigma D_1, \Omega D_1) \xrightarrow{f = (\operatorname{pt} f, \Sigma f = ((\Sigma f))_{i \in I}, \Omega f = ((\Omega f)_i)_{i \in I})} (\operatorname{pt} D_2, \Sigma D_2, \Omega D_2)$ making for every $i \in I$ the diagram

$$
\begin{array}{ccc}
(\Omega D_2)_i & \xrightarrow{\;\;(\Omega f)_i^{op}\;\;} & (\Omega D_1)_i \\
{\scriptstyle \kappa_{2_i}} \downarrow & & \downarrow {\scriptstyle \kappa_{1_i}} \\
||(\Sigma D_2)_i^{\operatorname{pt} D_2}||_i & \xrightarrow{\;\;||(\operatorname{pt} f, (\Sigma f)_i)_{\mathbf{C}_i}^{\leftarrow}||_i\;\;} & ||(\Sigma D_1)_i^{\operatorname{pt} D_1}||_i
\end{array}
$$

commute. The underlying functor to the ground category $\mathbf{Set} \times \mathbf{C}^{op} \times \mathbf{R}^{op}$ is denoted by $| - |$.

A $\mathbf{C}$-composite topological system D is called $\mathbf{C}$-*composite state property system* provided that κ_i is injective for every $i \in I$. $\mathbf{StPrSys}(\boldsymbol{P}, \mathcal{R})$ is the full subcategory of $\mathbf{TopSys}(\boldsymbol{P}, \mathcal{R})$ of all state property systems.

For the sake of brevity, we introduce the following notations:

(1) if $\mathcal{R} = ((\mathbf{A}_i, 1_{\mathbf{A}_i}))_{i \in I}$, then $\mathbf{TopSys}(\boldsymbol{P}, \mathcal{R})$ (resp. $\mathbf{StPrSys}(\boldsymbol{P}, \mathcal{R})$) is short-ened to $\mathbf{TopSys}(\boldsymbol{P})$ (resp. $\mathbf{StPrSys}(\boldsymbol{P})$);

(2) the category $\mathbf{TopSys}(\boldsymbol{P}, \mathcal{R})$ (resp. $\mathbf{StPrSys}(\boldsymbol{P}, \mathcal{R})$) induced by a single theory P is denoted by $\mathbf{TopSys}(\mathsf{P}, \mathcal{R})$ (resp. $\mathbf{StPrSys}(\mathsf{P}, \mathcal{R})$).

Example 7.23. There is plenty of examples in the literature of different approaches to the notion of topological system. The following lists the most prominent ones.

(1) $\mathbf{TopSys}(\mathcal{G}^2_{\mathbf{Frm}})$ provides the classical theory of topological systems of S. Vickers.

(2) $\mathbf{StPrSys}(\mathcal{G}^2_{\mathbf{CL}})$ provides the classical theory of state property systems of D. Aerts [4, 5, 6], the notion serving as the basic mathematical structure in the Geneva-Brussels approach to foundations of physics.

(3) $\mathbf{TopSys}(\mathcal{R}_1)$ provides the variable-basis approach developed by J. T. Denniston, A. Melton and S. E. Rodabaugh (never stipulated for $\mathcal{R}_2$).

(4) $\mathbf{TopSys}(\mathcal{S})$ provides our former variety-based approach.

(5) $\mathbf{StPrSys}(\mathcal{S})$ provides our variety-based generalization of the notion of state property system.

(6) $\mathbf{TopSys}(\mathcal{G}, \mathbf{Set})$ provides the theory of *Chu spaces* over a fixed set [10, 76] extremely useful in category theory and its applications. In particular, every small category $\mathbf{D}$ embeds fully into some $\mathbf{TopSys}(\mathcal{G}, \mathbf{Set})$ [76, Theorem 3.3]. A Chu space is called *context* in Formal Concept Analysis (FCA) [34], but "Chu" carries with it the notion of morphism, to form a category. On the other hand, FCA provides the notion of *concept*, intrinsic to a Chu space. A relation between these viewpoints on the topic was considered in, e.g., [58] bringing forward their possible fuzzification.

The reader should be aware that we used an alternative description of topological systems provided by J. T. Denniston *et al.* in [26, Definition 33]. Also notice that up to now we have already developed a rather successful theory of systems [101, 102, 105, 106, 107] without employing the forward powerset operator. It will be a real challenge to find an application of the latter notion in the theory.

8. Conclusion and Open Problems

In the paper we have presented a catalg approach to the operations of taking the image and preimage of (fuzzy) sets, coined as forward and backward powerset operators. The results obtained describe the operations in terms of two functors, each of them having its own ground category. Being slightly unusual on one hand, the approach boasts manyfold advantage on the other.

Firstly, we clarify completely the (long pondered on) relation between the operators in question, showing that the forward one corresponds to coproducts and the backward one to products in categories. In particular, the existence of biproducts provides both of them. Moreover, categorical duality between products and coproducts underlies the duality between the respective powerset operators, in a way that none of them implies the existence of the other or can be defined through the other (apart from small categories, where the existence of products coincides

with the existence of coproducts, in which case the category itself is equivalent to a complete lattice [44, Theorem 18.22]). It is the *machinery* of their definition that provides the duality in question, and categorical biproduct is precisely the missing point that brings it into light.

Secondly, our catalg approach shows a framework for incorporating many of the existing powerset theories under the common roof of varieties, opening the possibility of studying them simultaneously, every obtained result applicable to each of the theories. Moreover, our common setting paves the way for studying interrelations between individual powerset theories. More precisely, one can (and should) consider their category defined as follows.

Definition 8.1. Given powerset theories P_1 and P_2, a *powerset theory homomorphism* is a pair of functors $\mathbf{C}_1 \xrightarrow{F} \mathbf{C}_2$, $\mathbf{B}_1 \xrightarrow{G} \mathbf{B}_2$ making the diagrams

$$
\begin{array}{ccc}
\mathbf{Set} \times \mathbf{C}_1 & \xrightarrow{1_{\mathbf{Set}} \times F} & \mathbf{Set} \times \mathbf{C}_2 \\
{\scriptstyle (-)\overrightarrow{\mathbf{C}}_1} \downarrow & & \downarrow {\scriptstyle (-)\overrightarrow{\mathbf{C}}_2} \\
\mathbf{B}_1 & \xrightarrow{G} & \mathbf{B}_2
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{Set} \times \mathbf{C}_1^{op} & \xrightarrow{1_{\mathbf{Set}} \times F^{op}} & \mathbf{Set} \times \mathbf{C}_2^{op} \\
{\scriptstyle (-)\overleftarrow{\mathbf{C}}_1} \downarrow & & \downarrow {\scriptstyle (-)\overleftarrow{\mathbf{C}}_2} \\
\mathbf{B}_1 & \xrightarrow{G} & \mathbf{B}_2
\end{array}
$$

commute. **PwsThr** is the category of powerset theories and their homomorphisms.

Notice that a powerset theory homomorphism $\mathsf{P}_1 \xrightarrow{(F,G)} \mathsf{P}_2$ makes the diagram

$$
\begin{array}{ccc}
\mathbf{C}_1 & \xrightarrow{F} & \mathbf{C}_2 \\
{\scriptstyle ||-||} \downarrow & & \downarrow {\scriptstyle ||-||} \\
\mathbf{B}_1 & \xrightarrow{G} & \mathbf{B}_2
\end{array}
$$

commute as well (use the full embedding $\mathbf{C}_i \overset{E_i}{\hookrightarrow} \mathbf{Set} \times \mathbf{C}_i$ defined by $E_i(C_1 \xrightarrow{\varphi} C_2) = (\mathbf{1}, C_1) \xrightarrow{(!,\varphi)} (\mathbf{1}, C_2)$ and the fact that $(-)\overrightarrow{\mathbf{C}}_i \circ E_i = ||-||$ for $i \in \{1, 2\}$; also notice that the procedure is applicable to the $(-)^{op}$ case as well). It will be the topic of our further research to consider the properties of the new category. The reader, however, should be aware that there already exists another approach to powerset theories developed by S. E. Rodabaugh in [90, Definition 3.5], which is even more categorical but less algebraic than ours. More precisely, S. E. Rodabaugh uses a more general ground category than **Set** (actually an arbitrary category **K**), fixing at the same time the variety **SQuant** of s-quantales as the codomain of the powerset operators obtained. A new ground category necessitates a list of additional requirements on powerset operators having a significant drawback of being not well-motivated by examples. Moreover, the machinery is based on partial order, never mentioning biproducts and therefore failing to penetrate deep into the nature of powerset theories. By our opinion, the variety-based approach of the current paper is better appropriate for use in applications. On the other hand, a purely catalg framework should be void of any dependency on points and thus in future we will try to provide a framework incorporating both approaches.

The third and probably the most important advantage is the fact that our setting provides a framework for incorporating almost all of the most important existing topological theories including that of topological systems. The above-mentioned theories of S. E. Rodabaugh, however capable of doing the same job in many cases, appear to be useless when the underlying algebras are not a particular kind of s-quantales. For example, his framework never includes the construct of closure spaces which clearly has a strong topological flavor. The advantage grows when one comes to topological systems. Here we are even capable of getting outside mathematics into the realm of physics, no wonder since both sciences are much algebra-reliant. It is precisely the language of varieties that allows ones to see the forest (the structure of system) and not the individual trees (topological systems, state property systems, Chu spaces, *etc.*). One special remark is due to the new composite framework. It was motivated by our recent study on (fuzzy) bitopology, where we undermined significantly the claim of S. E. Rodabaugh [91] on categorical redundancy of classical bitopology in mathematics. The case is similar to the situation with quantale modules (Corollary 3.14 and Problem 3.15), however, instead of a concrete isomorphism (or even an equivalence) S. E. Rodabaugh has constructed a functor with just good properties. Our composite framework revealed its deficiency [102, 103], motivating the claim that (fuzzy) bitopology still deserves to be studied on its own. Inspired by the result, at the end of the last section we decided to apply the same composite framework to the notion of topological system that was never done before. The new setting of powerset theories will help us to investigate the topic from another perspective that will be the point of our further research.

Inspired by the well-known observation that every new theory always brings new problems, we would like to end the section (and the paper) with a brief outlook on some of them, taking their origin from algebra, category theory and topology. In particular, we have already said that our framework of varieties includes many of the currently popular (fuzzy) topological theories, but not all of them. The famous notion of (L, M)-*fuzzy topology* of T. Kubiak and A. Šostak [62] started by U. Höhle [46] (see also the approach of C. Guido [39]) is temporarily excluded. The reason is its reliance on a different kind of powerset operators being developed by, e.g., C. Guido, C. De Mitri and A. Frascella [23, 32, 33, 39]. Moreover, S. E. Rodabaugh [90, Definition 3.8] even provided a framework suitable for incorporating the approach in his powerset theories. Their above-mentioned deficiencies, however, still preserved, it seems to be highly advisable to replace the order-based setting with the biproduct-oriented one, giving rise to our first problem.

Problem 8.2. Is it possible to develop biproduct-oriented powerset theories which can incorporate the case of (L, M)-fuzzy topologies?

Our second problem concerns the challenging topic of properties of powerset theories. The following provides a list of them (made of different results of [3]) for the underlying powerset theory of $\mathcal{P}$ (Example 7.11).

(1) Both $|-| \circ (-)^{\rightarrow}$ and $|-| \circ (-)^{\leftarrow}$ are embeddings that are not full.

(2) There exists the powerset operator **Set** $\xrightarrow{|-| \circ (-)^{\dashrightarrow}}$ **Set** that is different from $|-| \circ (-)^{\rightarrow}$ (Lemma 7.14).

(3) $|-|\circ(-)^{\leftarrow}$ is representable (by any two-element set), but $|-|\circ(-)^{\rightarrow}$ is not representable.

(4) $|-|\circ(-)^{\rightarrow}$ preserves neither products (of pairs), nor coproducts (of pairs), nor equalizers, nor coequalizers.

(5) $|-|\circ(-)^{\rightarrow}$ has neither left nor right adjoint, but $\mathbf{Set}^{op} \xrightarrow{(|-|\circ(-)^{\leftarrow})^{op}} \mathbf{Set}$ has a left adjoint.

The problem now is rather easy to state and, hopefully, as much easy to solve.

Problem 8.3. Which of the above-mentioned properties are applicable to the original powerset theory $\mathcal{P}$? (Seems easy but provides you with a flavor of the challenge.) What about other powerset theories and their respective underlying ones?

The last problem concerns the approach to *fuzzy uniform spaces* through the concept of the so-called *uniform operator* [48, 50, 57, 89]. The underlying idea is to replace the classical way of defining a uniformity on a set X as a collection of binary relations on X [120, Definition 35.2] satisfying ceratin properties, through a family of elements of $(\mathbf{CSLat}(\bigvee))(\mathcal{P}(X), \mathcal{P}(X))$ (called *uniform operators*) subjected to certain requirements. The possibility of such a shift is based on the well-known one-to-one correspondence between binary relations R on X and $\bigcup$-preserving maps $\mathcal{P}(X) \xrightarrow{f} \mathcal{P}(X)$, given by the rules:

$$R \mapsto f_R \text{ with } f_R(S) = \{x \in X \mid xRs \text{ for some } s \in S\},$$
$$f \mapsto R_f \text{ with } xR_f y \text{ iff } x \in f(\{y\}).$$

During fuzzification, one replaces $\mathcal{P}(X)$ with a suitable fuzzy powerset obtaining fuzzy relations and fuzzy uniform operators. In particular, in [89] S. E. Rodabaugh provided foundations for the operators in question and their induced uniformities. One can easily notice that the above correspondence between relations and $\bigcup$-preserving maps is a consequence of the fact that the *Kleisli category* of the powerset monad on the category $\mathbf{Set}$ (Definition 3.20) is precisely the category $\mathbf{SetRel}$ of sets (as objects) and binary relations (as morphisms) [3, Exercise 20B(c)]. The result was generalized in [112, Proposition 4.12] as follows.

Definition 8.4. Given a u-quantale Q, Q-$\mathbf{SetRel}$ is the category whose objects are sets and whose morphisms (called Q-*relations*) $X \xrightarrow{R} Y$ are maps $X \times Y \xrightarrow{R} Q$, with the composition $X \xrightarrow{S \circ R} Z$ of R and $Y \xrightarrow{S} Z$ defined by $S \circ R(x, z) = \bigvee_{y \in Y} S(y, z) \otimes R(x, y)$, and the identity 1_X on X given by

$$1_X(x, y) = \begin{cases} e, & x = y \\ \bot, & x \neq y. \end{cases}$$

Proposition 8.5. Q-$\mathbf{SetRel}$ *is isomorphic to the Kleisli category of the Q-powerset monad on* $\mathbf{Set}$.

Being sufficiently simple to prove, Proposition 8.5 gives rise to a whole bunch of problems.

Problem 8.6. Is it possible to generalize Proposition 8.5 to our current biproduct-oriented context? (Recall, e.g., the monad from Meta-Theorem 6.5.)

Problem 8.7. Is it possible to develop the theory of variety-based uniformities similar to the respective one of topological spaces?

Problem 8.8. It is well-known that the category of variety-based topological spaces embeds into the category of variety-based topological systems as a full (regular mono)-coreflective subcategory [101]. What kind of relation exists (or could potentially exist) between variety-based uniform spaces and systems? (Notice that a classical topological space is *uniformizable* (its topology comes from a uniformity) iff it is completely regular (Tychonoff) [120, Theorem 38.2].)

The problems posed in the manuscript will be addressed to in our forthcoming papers on the topic.

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