$(A)_\Delta$ - DOUBLE SEQUENCE SPACES OF FUZZY NUMBERS VIA ORLICZ FUNCTION

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Abstract. The aim of this paper is to introduce and study a new concept of strong double $(A)_\Delta$-convergent sequence of fuzzy numbers with respect to an Orlicz function and also some properties of the resulting sequence spaces of fuzzy numbers are examined. In addition, we define the double (A, Δ)-statistical convergence of fuzzy numbers and establish some connections between the spaces of strong double $(A)_\Delta$-convergent sequence and double (A, Δ)-statistical convergent sequence.

1. Introduction and Background

Before proceeding let us recall a few concepts, which we shall use throughout this paper.

Definition 1.1. Let A denote a four dimensional summability method that maps the complex double sequence x into the double sequence Ax where the mn-th term of Ax is as follows:

$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}.$$

A two dimensional matrix transformation is said to be regular if it maps every convergent sequence into a convergent sequence with the same limit. Robison, in 1926 presented a four dimensional analog of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P-convergent is not necessarily bounded. Following this line, Robison and Hamilton presented a Silverman-Toeplitz type multidimensional characterization of regularity in [4] and [12]. The definition of the regularity for four dimensional matrices will be stated next, followed by the Robison-Hamilton characterization of the regularity of four dimensional matrices.

Definition 1.2. The four dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit.

Theorem 1.3. *The four dimensional matrix A is RH-regular if and only if*

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RH_1: P-$\lim_{m,n} a_{m,n,k,l} = 0$ *for each* k *and* l;
RH_2: P-$\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;
RH_3: P-$\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each* l;
RH_4: P-$\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ *for each* k;
RH_5: $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ *is P-convergent; and*
RH_6: *there exist positive numbers* A *and* B *such that* $\sum_{k,l>B} |a_{m,n,k,l}| < A$.

Recall in [5] that an Orlicz function $M : [0,\infty) \to [0,\infty)$ is continuous, convex, non-decreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

Subsequently Orlicz function was used to define sequence spaces by Parashar and B. Choudhary [10] and others. An Orlicz function M can always be represented in the following integral form: $M(x) = \int_0^x p(t)dt$ where p is the known the kernel of M, right differential for $t \geq 0$, $p(0) = 0$, $p(t) > 0$ for $t > 0$, p is non-decreasing and $p(t) \to \infty$ as $t \to \infty$.

Since the theory of fuzzy numbers has been widely studied, it is impossible to find either a commonly accepted definition or a fixed notion. We therefore begin by introducing some notions and definitions which will be used throughout.

A fuzzy real number X is a fuzzy set on R , i.e., a mapping $X : R \to$ **I=[0,1]**, associating each real number t with its grade of membership $X(t)$.

A fuzzy real number X is said to be upper semi-continuous if for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon))$ is open in the usual topology of R, for all $a \in I$. If there exists $t \in R$ such that $X(t) = 1$, then the fuzzy real number X is called normal.

A fuzzy number X is said to be convex if $X(t) \geq X(s) \wedge X(r) = min(X(s), X(r))$ where $s < t < r$. The class of all upper semi-continuous, normal, convex fuzzy real numbers is denoted by $R(I)$ and throughout the article, by a fuzzy real number we mean a number belongs to $R(I)$.

The additive identity and multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

Let $C(R^n) = \{A \subset R^n : \text{A compact and convex}\}$. The space $C(R^n)$ has a linear structure induced by the operations

$$A + B = \{a + b : \text{ a } \in \text{ A, b } \in \text{B}\}$$

and

$$\lambda A = \{\lambda a : \ \lambda \in \text{ A } \}$$

for $A, B \in C(R^n)$ and $\lambda \in R$. The Hausdorff distance between A and B of $C(R^n)$ is defined as

$$\delta_\infty(A,\ \text{B}) = max\left\{ \underset{a\in A}{sup}\, \underset{b\in B}{inf} \|a-b\|\,,\ \ \underset{b\in B}{sup}\, \underset{a\in A}{inf} \|a-b\| \ \right\}$$

where $\|.\|$ denotes the usual Euclidean norm in R^n . It is well known that $(C(R^n), \delta_\infty)$ is a complete (not separable) metric space. Let $X : C(R^n) \to I$. The $\alpha-$ level set for a fuzzy number X, which is normal, convex and upper -semi continuous is given as

$$[X]^\alpha = \{t \in R^n : X(t) \geq \alpha\}$$

for $0 < \alpha \leq 1$ and it is a nonempty compact convex, subset of R^n. The 0-level set is the closure of $\{t \in R^n : X(t) > 0\}$.

The linear structure of $L(R^n)$ induces addition $X+Y$ and scaler multiplication λX, $\lambda \in R$, in terms of α-level sets, by

$$[X+Y]^\alpha = [X]^\alpha + [Y]^\alpha$$

and

$$[\lambda X]^\alpha = \lambda [X]^\alpha$$

for each $0 \leq \alpha \leq 1$.

Define for each $1 \leq q < \infty$

$$d_q(X,Y) = \left\{ \int_0^1 \delta_\infty (X^\alpha, Y^\alpha)^q \, d\alpha \right\}^{\frac{1}{q}}$$

and $d_\infty(X,Y) = \sup\limits_{0\leq\alpha\leq1} \delta_\infty (\mathrm{X}^\alpha, \mathrm{Y}^\alpha)$. Clearly $d_\infty(X,Y) = \lim\limits_{q\to\infty} d_q(\mathrm{X}, \mathrm{Y})$ with $d_q \leq d_r$ if $q \leq r$. Moreover $(C(R^n), d_q)$ is a complete, separable and locally compact metric space [1].

Throughout the paper, d will denote d^q with $1 \leq q < \infty$.

A fuzzy double sequence is a double infinite array of fuzzy numbers. We denote a fuzzy double sequence by (X_{mn}), where X_{mn}'s are fuzzy numbers for each $m, n \in \mathbb{N}$. Let $s^{''}$ denote the set of all double sequences of fuzzy numbers.

The concepts of fuzzy set and fuzzy set operations were first introduced by Zadeh [24]. Subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets, such as fuzzy topological spaces, similarity relations and fuzzy ordering, fuzzy measures of fuzzy events and fuzzy mathematical programming.

Statistical convergence of single sequences of fuzzy numbers was first deduced by Savas and Nuray [8]. Since the set of all real numbers can be embedded in the set of fuzzy numbers, statistical convergence in reals can be considered as a special case of those fuzzy numbers. However, since the set of fuzzy numbers is partially ordered and does not carry a group structure, most of the results known for the sequences of real numbers may not be valid for fuzzy setting. Therefore this theory should not be considered as a trivial extension of what has been known in real case.

We give the following definitions for fuzzy double sequences.

Definition 1.4. [11] A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be convergent in the Pringsheim's sense or P- convergent to a fuzzy number X_0, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$d(X_{kl}, X_0) < \epsilon \quad \text{for} \quad k, l > n_0,$$

and we denote by $P - limX = X_0$. The fuzzy number X_0 is called the Pringsheim limit of (X_{kl}).

Let $c^2(F)$ denote the set of all double convergent sequences of fuzzy numbers.

Definition 1.5. [21] A double sequence $X = (X_{kl})$ of fuzzy numbers is bounded if there exists a positive number M such that $d(X_{kl}, \bar{0}) < M$ for all k and l. We will denote the set of all bounded double sequences by $l_\infty^{''}(F)$.

Savas [13] introduced and discussed convergent double sequences of fuzzy numbers and showed that the set of all convergent double sequences of fuzzy numbers is complete.

In this paper we introduce and study the concept of strong double $(A)_\Delta$-summability with respect to an Orlicz function and also some properties of this sequence space are examined.
Before we state our main results, first we shall present the following definition by combining a four dimensional matrix transformation A and Orlicz function.

2. Main Results

Definition 2.1. Let M be an Orlicz function and $A = (a_{m,n,k,l})$ be a nonnegative RH-regular summability matrix method. We now present the following sets of double sequence spaces:

$$\omega_0^{''}(A, M, p)_\Delta(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

$$\omega^{''}(A, M, p)_\Delta(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

and

$$\omega_\infty^{''}(A, M, p)_\Delta(F) = \left\{ X \in s^{''} : \sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \right\},$$

where

$$\Delta_{11} X_{mn} = X_{mn} - X_{m+1,n} - X_{m,n+1} + X_{m+1,n+1} \text{ and}$$

$$\bar{0}(t) := \begin{cases} 1, & \text{if } \quad t = 0, \\ 0, & \quad \text{otherwise.} \end{cases}$$

Let us consider a few special cases of the above sets .

(1) If $M(x) = x$, for all $x \in [0, \infty)$, then we have

$$\omega_0^{''}(A, p)_\Delta(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} d(\Delta_{11} X_{k,l}, \bar{0})^{p_{k,l}} = 0 \right\},$$

$$\omega^{''}(A, p)_\Delta(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} d(\Delta_{11} X_{k,l}, X_0)^{p_{k,l}} = 0, \right\},$$

and

$$\omega_\infty^{''}(A, p)_\Delta(F) = \left\{ X \in s^{''} : \sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} d(\Delta_{11} X_{k,l}, \bar{0})^{p_{k,l}} < \infty \right\}.$$

(2) If $p_{k,l} = 1$ for all (k,l), then we have

$$\omega_0^{''}(A,M)_\Delta(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11}X_{k,l},\bar{0})}{\rho} \right) \right] = 0, \text{ for some } \rho > 0 \right\},$$

$$\omega^{''}(A,M)_\Delta(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11}X_{k,l},X_0)}{\rho} \right) \right] = 0, \text{ for some } \rho > 0 \right\},$$

and

$$\omega_\infty^{''}(A,M)_\Delta(F) = \left\{ X \in s^{''} : \sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11}X_{k,l},\bar{0})}{\rho} \right) \right] < \infty, \text{ for some } \rho > 0 \right\}.$$

(3) If we take $M(x) = x$, for all $x \in [0,\infty)$ and $p_{k,l} = 1$ for all (k,l), then we have

$$\omega_0^{''}(A)_\Delta(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} d(\Delta_{11}X_{k,l},\bar{0}) = 0 \right\},$$

$$\omega^{''}(A)_\Delta(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} d(\Delta_{11}X_{k,l},X_0) = 0, \right\},$$

and

$$\omega_\infty^{''}(A)_\Delta(F) = \left\{ X \in s^{''} : \sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} d(\Delta_{11}X_{k,l},\bar{0}) < \infty \right\}.$$

(4) If we take $A = (C,1,1)$, we have

$$\omega^{''}(M,p)_\Delta^0(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M\left(\frac{d(\Delta_{11}X_{k,l},\bar{0})}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

$$\omega^{''}(M,p)_\Delta(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M\left(\frac{d(\Delta_{11}X_{k,l},X_0)}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0, \right\},$$

and

$$\omega^{''}(M,p)_\Delta^\infty(F) = \left\{ X \in s^{''} : \sup_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M\left(\frac{d(\Delta_{11}X_{k,l},\bar{0})}{\rho} \right) \right]^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \right\}.$$

(5) If we take $A = (C,1,1)$ and $p_{k,l} = 1$ for all (k,l), then we have

$$\omega^{''}(M)_\Delta^0(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M\left(\frac{d(\Delta_{11}X_{k,l},\bar{0})}{\rho} \right) \right] = 0, \text{ for some } \rho > 0 \right\},$$

$$\omega^{''}(M)_\Delta(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M\left(\frac{d(\Delta_{11}X_{k,l},X_0)}{\rho} \right) \right] = 0, \text{ for some } \rho > 0, \right\},$$

and

$$\omega^{''}(M)_\Delta^\infty(F) = \left\{ X \in s^{''} : \sup_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M\left(\frac{d(\Delta_{11}X_{k,l},\bar{0})}{\rho} \right) \right] < \infty, \right\}.$$

(6) If we take $A = (C, 1, 1)$, $M(x) = x$, for all $x \in [0, \infty)$, and $p_{k,l} = 1$ for all (k, l), then we have

$$\omega_{\Delta}^{''0}(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} d(\Delta_{11} X_{k,l}, \bar{0}) = 0 \right\},$$

$$\omega_{\Delta}^{''}(F) = \left\{ X \in s^{''} : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} d(\Delta_{11} X_{k,l}, X_0) = 0, \right\},$$

and

$$\omega_{\Delta}^{''\infty}(F) = \left\{ X \in s^{''} : \sup_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} d(\Delta_{11} X_{k,l}, \bar{0}) < \infty \right\}.$$

(7) Let us consider the following notations and definitions. The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty,$$
$$l_0 = 0, h_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty,$$

and let $h_{r,s} = h_r h_s$, $\theta_{r,s}$ is determined by $I_{r,s} = \{(i, j) : k_{r-1} < i \leq k_r \ \&\ l_{s-1} < j \leq l_s\}$. If we take

$$a_{r,s,k,l} = \begin{cases} \frac{1}{\bar{h}_{r,s}}, & \text{if } (k,l) \in I_{r,s}; \\ 0 & \text{otherwise.} \end{cases}$$

We write

$$\omega^{''}(\theta, M, p)_{\Delta}^{0}(F) = \left\{ X \in s^{''} : P - \lim_{r,s} \frac{1}{\bar{h}_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

$$\omega^{''}(\theta, M, p)_{\Delta}(F) = \left\{ X \in s^{''} : P - \lim_{r,s} \frac{1}{\bar{h}_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} = 0, \right\},$$

and

$$\omega^{''}(\theta, M, p)_{\Delta}^{\infty}(F) = \left\{ X \in s^{''} : \sup_{r,s} \frac{1}{\bar{h}_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \right\}.$$

(8) As a final illustration let

$$a_{i,j,k,l} = \begin{cases} \frac{1}{\bar{\lambda}_{i,j}}, & \text{if } k \in I_i = [i - \lambda_i + 1, i] \text{ and } l \in I_j = [j - \lambda_j + 1, j] \\ 0, & \text{otherwise} \end{cases}$$

where we shall denote $\bar{\lambda}_{i,j}$ by $\lambda_i \mu_j$. Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and $\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 0$ and $\mu_{j+1} \leq \mu_j + 1, \mu_1 = 0$. Then our definition reduces to the following

$$\omega^{''}(\lambda, M, p)_{\Delta}^{0}(F) = \left\{ X \in s^{''} : P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k \in I_i, l \in I_j} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

$$\omega^{''}(\lambda, M, p)_{\Delta}(F) = \left\{ X \in s^{''} : P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k \in I_i, l \in I_j} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

and

$$\omega^{''}(\bar{\lambda}, M, p)_\Delta^\infty(F) = \left\{ X \in s^{''} : \sup_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k \in I_i, l \in I_j} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \right\}.$$

The following inequalities will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \leq \sup_{k,l} p_{k,l} = H$ and let $C = \max\{1, 2^{H-1}\}$. Then for the factorable sequences $\{a_k\}$ and $\{b_k\}$ in the complex plane, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq C(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}).$$

Theorem 2.2. *If M is an Orlicz function, then $\omega_0^{''}(A, M, p)_\Delta(F) \subset \omega^{''}(A, M, p)_\Delta(F) \subset \omega_\infty^{''}(A, M, p)_\Delta(F)$.*

Proof. The inclusion $\omega_0^{''}(A, M, p)_\Delta(F) \subset \omega^{''}(A, M, p)_\Delta(F)$ is easy and so is omitted. If we choose X in $\omega^{''}(A, M, p)_\Delta(F)$, then there exists some positive number ρ_1 such that

$$P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho_1} \right) \right]^{p_{k,l}} = 0.$$

Define $\rho = 2\rho_1$. Since M is nondecreasing and convex, we obtain the following:

$$\begin{aligned}
& \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} \\
\leq & \sum_{k,l=0,0}^{\infty,\infty} \frac{1}{2^{p_{k,l}}} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho_1} \right) \right. \\
+ & \left. M\left(\frac{d(X_0, \bar{0})}{\rho_1} \right) \right]^{p_{k,l}} \\
\leq & \; C \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho_1} \right) \right]^{p_{k,l}} \\
+ & \; C \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M \left[\left(\frac{d(X_0, \bar{0})}{\rho_1} \right) \right]^{p_{k,l}} \\
\leq & \; C \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho_1} \right) \right]^{p_{k,l}} \\
+ & \; C \max\left\{ 1, \left[M\left(\frac{d(X_0, \bar{0})}{\rho_1} \right) \right]^H \right\} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}.
\end{aligned}$$

Since A is RH-regular, we are granted that $X \in \omega_\infty^{''}(A, M, p)_\Delta(F)$ and this completes the proof. □

Theorem 2.3. (1) *If $0 < \inf p_{k,l} \leq p_{k,l} < 1$, then*

$$\omega^{''}(A, M, p)_\Delta(F) \subset \omega^{''}(A, M)_\Delta(F)$$

(2) *If $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$, then*

$$\omega^{''}(A, M)_\Delta(F) \subset \omega^{''}(A, M, p)_\Delta(F)$$

Proof. (1) Let $X \in \omega^{''}(A, M, p)(F)$, since $0 < \inf p_{k,l} \leq 1$, we have

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho}\right) \leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}}$$

and hence $X \in \omega^{''}(A, M, p)_{\Delta}(F)$.

(2) Let $p_{k,l} \geq 1$ for each (k, l) and $\sup_{k,l} p_{k,l} < \infty$. Let $X \in \omega^{''}(A, M)_{\Delta}(F)$. Then for each $0 < \epsilon < 1$ there exists a positive integer n_0 such that

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho}\right) \leq \epsilon < 1$$

for all $m, n \geq n_0$. This implies that

$$\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}} \leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho}\right).$$

Thus $X \in \omega^{''}(A, M, p)_{\Delta}(F)$. □

The following corollary follows immediately from the above theorem.

Corollary 2.4. Let $A = (C, 1, 1)$ double Cesàro matrix and let M be an Orlicz function.

(1) If $0 < \inf p_{k,l} \leq p_{k,l} < 1$, then $\omega^{''}(M, p)_{\Delta}(F) \subset \omega^{''}(M)_{\Delta}(F)$.
(2) If $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$, then $\omega^{''}(M)_{\Delta}(F) \subset \omega^{''}(M, p)_{\Delta}(F)$.

3. Double (A, Δ)-statistical Convergence

Natural density was generalized by Freedman and Sember in [3] by replacing C_1 with a nonnegative regular summability matrix $A = (a_{n,k})$. Thus, if K is a subset of N then the A-density of K is given by $\delta_A(K) = \lim_n \sum_{k \in K} a_{n,k}$ if the limit exists.

A real number sequence x is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k < n : |x_k - L| \geq \epsilon\}| = 0,$$

where by $k < n$ we mean that $k = 0, 1, 2, ..., n$ and the vertical bars indicate the number of elements in the enclosed set. In this case, we write $st_1 - \lim x = L$ or $x_k \to L(st_1)$. Statistical convergence is a generalization of the usual notion of convergence for real valued sequences that parallels the usual theory of convergence. The idea of statistical convergence was first introduce by Fast [2]. Today, statistical convergence has become one of the most active area of research in the field of summability theory.

Before we present the new definition and the main theorems, we shall state a few known results. The following definition was presented by Nuray and Savaş [14] for single sequence of fuzzy numbers . A sequence X is said to be statistically convergent or s-convergent to X_0, if for every $\epsilon > 0$

$$\lim_n \frac{1}{n}\Big|\{k < n : d(X_k, X_0) \geq \epsilon\}\Big| = 0,$$

where the vertical bars indicate the numbers of elements in the enclosed set. In this case we write $s - \lim X = L$ or $X_k \to X_0(s)$.

Let $K \subset \mathbb{N} \times \mathbb{N}$ be a two-dimensional set to positive integers and let $K(m,n)$ be the number of (k,l) in K such that $k \leq n$ and $l \leq m$.

In case the double sequence $\frac{K(m,n)}{mn}$ has a limit in the Pringsheim sense then we say that K has a double natural density as

$$P - \lim_{m,n} \frac{K(m,n)}{mn} = \delta^2(K).$$

Let $K \subset \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers, then the A-density of K is given by

$$\delta_A^2(K) = P - \lim_{m,n} \sum_{(k,l)\in K} a_{m,n,k,l}$$

provided that the limit exists.

Recently Savas and Mursaleen [19] introduced the idea of statistical convergence for double sequence of fuzzy numbers as follows:

Definition 3.1. A double sequence $X = (X_{kl})$ of fuzzy numbers is said to be *statistically convergent* to X_0 provided that for each $\epsilon > 0$

$$P - \lim_{m,n} \frac{1}{nm} \left|\{(k,l); k \leq m \text{ and } l \leq n : d(X_{kl}, X_0) \geq \epsilon\}\right| = 0.$$

In this case, we write $st_2 - \lim_{k,l} X_{k,l} = X_0$ and denote the set of all double statistically convergent sequences of fuzzy numbers by $st^2(F)$.
We now have

Definition 3.2. A double sequence X of fuzzy numbers is said to be (A, Δ)-statistically convergent to L if for every positive ϵ

$$\delta_A^2\left(\{(k,l) : d(\Delta_{11} X_{k,l}, X_0) \geq \epsilon\}\right) = 0.$$

In this case we write $X_{k,l} \to X_0(s^2(A,\Delta)(F))$ or $s^2(A,\Delta)(F) - \lim X = X_0$ and

$$s^2(A,\Delta)(F) = \{X : \exists X_0 \in R(I), s^2(A,\Delta)(F) - \lim X = X_0\}.$$

If $A = (C,1,1)$ then $(s^2(A,\Delta))(F)$ reduces to $(s^2(\Delta))(F)$ which is defined as follows: A double sequence X of fuzzy numbers is said to be (Δ)-statistically convergent to X_0, if for every positive $\epsilon > 0$ the set

$$P - \lim_{m,n} \frac{1}{nm} \left|\{(k,l); k \leq m \text{ and } l \leq n : d(\Delta_{1,1} X_{kl}, X_0) \geq \epsilon\}\right| = 0.$$

In this case we write $s^2(\Delta)(F) - \lim X = X_0$. If we take

$$a_{r,s,k,l} = \begin{cases} \frac{1}{h_{r,s}}, & \text{if } k \in I_r = (k_{r-1}, k_r] \text{ and } l \in I_s = (l_{s-1}, l_s] \\ 0 & \text{otherwise ,} \end{cases}$$

where the double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ and $\bar{h}_{r,s}$ are defined above. Then our definition reduces to the following: A double fuzzy numbers sequence X is said to be lacunary (θ, Δ)-statistically convergent to X_0, if for every positive $\epsilon > 0$ the set

$$P - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : d(\Delta_{11} X_{k,l}, X_0) \geq \epsilon\}| = 0.$$

Finally, if we write

$$a_{i,j,k,l} = \begin{cases} \frac{1}{\lambda_{i,j}}, & \text{if } k \in I_i = [i - \lambda_i + 1, i] \text{ and } l \in I_j = [j - \lambda_j + 1, j]; \\ 0, & \text{otherwise ,} \end{cases}$$

Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ be defined as above. A double fuzzy numbers sequence X is said to be $(\bar{\lambda}, \Delta)$-statistically convergent to X_0, if for every positive $\epsilon > 0$ the set

$$P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} |\{k \in I_i \text{ and } l \in I_j : d(\Delta_{11} X_{k,l}, X_0) \geq \epsilon\}| = 0.$$

Theorem 3.3. *If M is an Orlicz function and $0 < h = \inf_{k,l} p_{k,l} \leq p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$, then $\omega^{''}(A, M, p)_\Delta(F) \subset s^2(A, \Delta)(F)$.*

Proof. If $X \in \omega^{''}(A, M, p)_\Delta(F)$, then there exists $\rho > 0$ such that

$$P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} = 0.$$

Then, we obtain for a given $\epsilon > 0$ and let $\epsilon_1 = \frac{\epsilon}{\rho}$ that

$$\begin{aligned}
&\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \\
= &\sum_{k,l=0,0; d(\Delta_{11} X_{k,l}, X_0) \geq \epsilon}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \\
+ &\sum_{k,l=0,0; d(\Delta_{11} X_{k,l}, X_0) < \epsilon}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \\
\geq &\sum_{k,l=0,0; d(\Delta_{11} X_{k,l}, X_0) \geq \epsilon}^{\infty,\infty} a_{m,n,k,l} \left[M\left(\frac{d(\Delta_{11} X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} \\
\geq &\sum_{k,l=0,0; d(\Delta_{11} X_{k,l}, X_0) \geq \epsilon}^{\infty,\infty} a_{m,n,k,l} \min\{[M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H, \\
\geq &\left(\min\left\{ [M(\epsilon_1)]^h, [M(\epsilon_1)]^H \right\} \right) \sum_{k,l=0,0; d(\Delta_{11} X_{k,l}, X_0) \geq \epsilon}^{\infty,\infty} a_{m,n,k,l} \\
\geq &\min\{[M(\epsilon_1)]^h, [M(\epsilon_1)]^H\} \delta_A^2 \left(\{(k,l) : d(X_{k,l}, X_0) \geq \epsilon\} \right).
\end{aligned}$$

Hence $X \in s^2(A, \Delta)(F)$. □

Theorem 3.4. *Let M be an Orlicz function and $X = (X_{kl})$ be a bounded sequence of fuzzy numbers and $0 < h = \inf_{k,l} p_{k,l} \leq p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$, then $s^2(A,\Delta)(F) \subset \omega''(A, M, p)_\Delta(F)$.*

Proof. Suppose that $X \in l''_\infty(F)$ and $X_{k,l} \to X_0(s^2(A,\Delta))(F)$. Since $X \in l''_\infty(F)$, there is a constant $K > 0$ such that $d(\Delta X_{k,l}, \bar{0}) < K$ for all k, l . Given $\varepsilon > 0$ we have

$$
\begin{aligned}
&\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{d(\Delta_{11}X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}}\\
= &\sum_{k,l=0,0;d(\Delta_{11}X_{k,l},X_0)\geq\epsilon}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{d(\Delta_{11}X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}}\\
+ &\sum_{k,l=0,0;d(\Delta_{11}X_{k,l},X_0)<\epsilon}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{d(\Delta_{11}X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}}\\
\leq &\sum_{k,l=0,0;d(\Delta_{11}X_{k,l},X_0)\geq\epsilon}^{\infty,\infty} a_{m,n,k,l}\max\left\{\left[M\left(\frac{K}{\rho}\right)\right]^h, \left[M\left(\frac{K}{\rho}\right)\right]^H\right\}\\
+ &\sum_{k,l=0,0;d(\Delta_{11}X_{k,l},X_0)<\epsilon}^{\infty,\infty} a_{m,n,k,l}\left[M\left(\frac{\epsilon}{\rho}\right)\right]^{p_{k,l}}\\
\leq &\ \delta_A^2\left(\{(k,l) : d(\Delta_{11}X_{k,l}, X_0) \geq \epsilon\}\right)\max\left\{[M(T)]^h, [M(T)]^H\right\}\\
+ &\ \max\left\{\left[M\left(\frac{\epsilon}{\rho}\right)\right]^h, \left[M\left(\frac{\epsilon}{\rho}\right)\right]^H\right\}, \frac{K}{\rho} = T.
\end{aligned}
$$

Thus $X \in \omega''(A, M, p)_\Delta(F)$. □

If we let $A = (C, 1, 1)$, $M(X) = X$, and $p_{k,l} = 1$ for each (k, l) we have the following corollary:

Corollary 3.5. (i) If $X_{k,l} \to L(\omega'')_\Delta(F)$ then $X_{k,l} \to L(s^2(\Delta))(F)$.

(ii) If $X \in l''_\infty(\Delta)(F)$ and $X_{k,l} \to L(s^2(\Delta)(F))$ then $X_{k,l} \to L(\omega'')_\Delta(F)$.

(iii) $s^2(\Delta)(F) \cap l''_\infty(\Delta)(F) = (\omega'')_\Delta(F) \cap l''_\infty(\Delta)(F)$ where $l''_\infty(\Delta)(F) =: \{X : (\Delta_{11}X_{k,l}) \in l''_\infty(F)\}$.

Remark 3.6. In corollary 3.5, if $X_{k,l} \to L(s^2(\Delta))$ which is unbounded then $X_{k,l} \to L(\omega'')_\Delta(F)$ may not be true. This follows from the following example.

Example 3.7. *Consider the sequence $(\triangle_{1,1}X_{kl})$ in $R(I)$ defined as follows: For $l = i^2$, $i \in \mathbb{N}$ and for all $k \in \mathbb{N}$,*

$$
\triangle_{1,1}X_{kl}(t) := \begin{cases} 1 + \frac{t}{\sqrt{n}}, & \textit{for} \quad -\sqrt{n} \leq t \leq 0,\\ 1 - \frac{t}{\sqrt{n}}, & \textit{for} \quad 0 \leq t \leq \sqrt{n},\\ 0, & \quad \textit{otherwise}. \end{cases}
$$

For $l \neq i^2$, $i \in \mathbb{N}$ *and for all* $k \in \mathbb{N}$,

$$\triangle_{1,1} X_{kl} = \bar{0}.$$

The sequence $(\triangle_{1,1} X_{kl})$ *defined as above is unbounded and* $s^2(\Delta)$*- convergent to* $\bar{0}$. *But it is easy to see that it is not* $(\omega^{''})_{\Delta}$*-summable to* $\bar{0}$.

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